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On some questions on selectively highly divergent spaces

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Abstract

A topological space X is selectively highly divergent (SHD) if for every sequence of non-empty open sets $\{U_n : n \in \omega\}$ of X, we can find points $x_n \in U_n$, for every $n < \omega$ such that the sequence $\{x_n : n \in \omega\}$ has no convergent subsequences. In this note we answer four questions related to this notion that were asked by Jiménez-Flores, Ríos-Herrejón, Rojas-Sánchez and Tovar-Acosta.

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1. Introduction

In this note we consider a class of spaces recently studied in [4].

Definition 1.1. A topological space X is *selectively highly divergent* (SHD from here for short) if for every sequence of non-empty open subsets $\{U_n : n < \omega\}$ of X, we can find $x_n \in U_n$ such that the sequence $\{x_n : n < \omega\}$ has no convergent subsequence.

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Clearly, if a topological space X has a point of countable character, then it cannot be SHD, in particular no metrizable space is SHD.

Nice examples of SHD spaces are the compact Hausdorff space $\omega^* = \beta\omega \setminus \omega$ and the countable regular maximal space M described in [1]. These spaces however are strictly stronger than SHD because they do not contain non-trivial convergent sequences. In general, a selectively higly divergent space may have plenty of convergent sequences: a compact Hausdorff space of this kind is $\omega^* \times I$, while a countable regular one is $M \times \mathbb{Q}$.

The property of being selectively highly divergent is much stronger than being not sequentially compact. An easy example of non-sequentially compact space which is not SHD is the space $Z = \omega^* \oplus I$. Note that the space Z has an open subset which is sequentially compact, and one may suspect that a space having no non-empty open sequentially compact subspace should be SHD, but this is not the case.

Example 1.2. A compact Hausdorff space with no non-empty sequentially compact subspace which is not SHD.

Proof. Let $X = (\omega^* \times \omega) \cup \{p\}$, where $\omega^* \times \omega$ with the product topology is an open subspace of X, while a local base at p is the collection $\{(\omega^* \times [n, \omega[) \cup \{p\} : n < \omega\}$. □

If in Definition 1.1 we consider only constant sequences of open sets, i.e. $U_n = U$ for each $n < \omega$, then we see that a SHD space has the property that every non-empty open set contains a sequence with no subsequences converging in X. We may call a space with this property *highly divergent* (HD for short). Using this terminology, Example 1.2 provides an example of a compact Hausdorff HD space which is not SHD.

In [4] the authors formulated various questions about selectively highly divergent spaces. In our paper we will focus on four of them.

Question 1.3 ([4, Question 2])**.** *Is it true that if κ is an uncountable cardinal, then $X = \{0,1\}^\kappa$ is a SHD space?*

Question 1.4 ([4, Question 4])**.** *If X is Tychonoff, non-compact and SHD, does it hold that βX is SHD?*

Question 1.5 ([4, Question 5])**.** *Is the SHD property dense hereditary?*

Given a space X, let $\mathcal{F}[X]$ denote the Pixley-Roy hyperspace of X.

Question 1.6 ([4, Question 7])**.** *Is $\mathcal{F}[X]$ SHD whenever X is SHD and T_1?*

In the present note, we give a complete answer to Questions 1.3, 1.5 and 1.6 and a partial positive answer to Question 1.4.

All spaces are assumed to be T_1. For undefined notions, we refer the reader to [3] and [5].

2. The main results

We begin by presenting a complete answer to Question 1.3.

Recall that a collection $\mathcal{S}$ of subsets of ω is a splitting family if for every infinite subset $A \subseteq \omega$ there is an element $S \in \mathcal{S}$ satisfying $|S \cap A| = |A \setminus S| = \omega$. The smallest cardinality of a splitting family on ω is the splitting number $\mathfrak{s}$. It turns out that $\omega_1 \leq \mathfrak{s} \leq \mathfrak{c}$.

Theorem 2.1. *The space 2^κ is selectively highly divergent if and only if $\kappa \geq \mathfrak{s}$.*

Proof. If $\kappa < \mathfrak{s}$, then 2^κ is sequentially compact (see [2], Theorem 6.1). So, if 2^κ is SHD, then we should have $\kappa \geq \mathfrak{s}$. To complete the proof, we need to show that $\kappa \geq \mathfrak{s}$ implies that 2^κ is SHD. Since 2^κ is homeomorphic to $2^\mathfrak{s} \times 2^\kappa$, taking into account that any product having a SHD factor is SHD (see Theorem 1 in [4]), it suffices to prove that $2^\mathfrak{s}$ is selectively highly divergent.

Let $\mathcal{S}$ be a splitting family on ω of size $\mathfrak{s}$ and fix an indexing $\mathcal{S} = \{S_\alpha : \alpha < \mathfrak{s}\}$ in such a way that every element of $\mathcal{S}$ appears in the list $\mathfrak{s}$-many times.

Recall that a base for the topology of 2^κ consists of the sets $[\sigma]$, where $\sigma \in Fin(\kappa, 2)$ is a partial function whose domain is a finite subset of κ and $[\sigma] = \{x \in 2^\kappa : \sigma \subseteq x\}$. Let $\{U_n : n < \omega\}$ be a family of non-empty open subsets of $2^\mathfrak{s}$ and for each n choose a partial function $\sigma_n : \mathfrak{s} \to 2$ such that $[\sigma_n] \subseteq U_n$.

For each n let $x_n \in 2^\mathfrak{s}$ be the point defined as follows. If $\alpha \in dom(\sigma_n)$, then let $x_n(\alpha) = \sigma_n(\alpha)$; if $\alpha \in \mathfrak{s} \setminus dom(\sigma_n)$, then let $x_n(\alpha) = 1$ when $n \in S_\alpha$ and $x_n(\alpha) = 0$ when $n \notin S_\alpha$. Of course, we have $x_n \in [\sigma_n] \subseteq U_n$.

We claim that the sequence $\{x_n : n < \omega\}$ does not have convergent subsequences. Assume by contradiction that the subsequence $\{x_n : n \in A\}$ converges to a point p. Since the family $\mathcal{S}$ is splitting, there exists $S \in \mathcal{S}$ such that $|A \cap S| = |A \setminus S| = \omega$. Since the set $\bigcup\{dom(\sigma_n) : n < \omega\}$ is countable and S appears in the list $\{S_\alpha : \alpha < \mathfrak{s}\}$ $\mathfrak{s}$-many times, we may find $\gamma \in \mathfrak{s} \setminus \bigcup\{dom(\sigma_n) : n \in \omega\}$ such that $S_\gamma = S$. Now, since the sequence $\{x_n : n \in A \cap S\}$ converges to p and $x_n(\gamma) = 1$ for each $n \in A \cap S$, we must have $p(\gamma) = 1$. But even the sequence $\{x_n : n \in A \setminus S\}$ converges to p and hence we must also have $p(\gamma) = 0$. As this is a contradiction, the proof is complete. □

Theorem 2.1 will help us answer Question 1.5 in the negative.

Example 2.2. A compact Hausdorff SHD space with a dense subspace which is not SHD.

Proof. Let $X = 2^\mathfrak{c}$. Theorem 2.1 sais that X is selectively highly divergent. Let Y be the Σ-product of $2^\mathfrak{c}$, that is $Y = \{x \in X : |x^{-1}(1)| \leq \omega\}$, with the topology induced from X. Then Y is a dense subset of X: Since in Y every countable set is contained in a copy of the Cantor set, we immediately see that Y is sequentially compact. Thus, Y is a dense subspace of X which is not selectively higly divergent. □

We now give a partial answer to Question 1.4. Recall that a set $A \subseteq X$ is C^*-embedded in X if every bounded real valued continuous function defined on A can be continuously extented to the whole of X. The Tietze-Urysohn theorem implies that every closed subspace of a normal space is C^*-embedded.

Theorem 2.3. *Let X be a Tychonoff SHD space. If every closed copy of the discrete space ω is C^*- embedded, then βX is SHD.*

Proof. Let $\{U_n : n < \omega\}$ be a sequence of non-empty open sets of βX. Since X is SHD, we may pick points $x_n \in U_n \cap X$ in such a way that $\{x_n : n < \omega\}$ does not have subsequences which are convergent in X. We claim that $\{x_n : n < \omega\}$ does not have convergent subsequences even in βX.

Assume by contradiction that the sequence $\{x_n : n \in A\}$ converges to a point $p \in \beta X$. Clearly, we should have $p \in \beta X \setminus X$. But then, the set $\{x_n : n \in A\}$ is closed and discrete in X. Split A in the union of two infinite subsets B and C and define $f : \{x_n : n \in A\} \to [0, 1]$ by letting $f(x_n) = 0$ in $n \in B$ and $f(x_n) = 1$ if $n \in C$. Since the set $\{x_n : n \in A\}$ is C^*- embedded, we may continuously extend f to a function $f : X \to [0, 1]$.The next step is to extend f to a continuous function $g : \beta X \to [0, 1]$. Since $\{x_n : n \in A\}$ converges to p, we should have $g(p) \in \overline{\{g(x_n) : n \in B\}} = \overline{\{f(x_n) : n \in B\}} = \{0\}$, i. e. $g(p) = 0$. The same argument shows that $g(p) \in \overline{\{f(x_n) : n \in C\}} = \{1\}$, i. e. $g(p) = 1$. As this is a contradiction, the proof is complete. □

We may mention a couple of corollaries.

Corollary 2.4. *If X is a normal SHD space, then βX is SHD.*

Corollary 2.5. *If X is a countable Tychonoff SHD space, then βX is SHD.*

So, we see that βM is SHD.

Example 2.2 already shows that the HD property is not dense hereditary. We now describe another example which involves the Čech-Stone compactification. Let us consider the space $\beta\mathbb{Q}$. It is clear that $\mathbb{Q}$ is dense and far to be higly divergent. We check that $\beta\mathbb{Q}$ is HD. To this end, let U be a non-empty open subset of $\beta\mathbb{Q}$ and take a non-empty open set V such that $\overline{V} \subseteq U$. The set $V \cap \mathbb{Q}$ contains a closed copy A of the discrete space ω. Since A is C^*-embedded in $\mathbb{Q}$, we have that $\overline{A} \subseteq U$ is homeomorphic to $\beta\omega$ and so every non-trivial sequence in $\overline{A} \subseteq U$ has no convergent subsequences in $\beta\mathbb{Q}$.

Notice that $\beta\mathbb{Q}$ is not SHD because it is first countable at each point $q \in \mathbb{Q}$. So, $\beta\mathbb{Q}$ is another compact Hausdorff HD space which is not SHD. However, the space X given in Example 1.2 is of different nature because every dense set D of X is higly divergent. To check this, let U be a non-empty open set in the subspace D and fix an open set V of X such that $U = V \cap D$. There is some $n \in \omega$ such that $V \cap \omega^* \times \{n\} \neq \varnothing$ and so even $V \cap \omega^* \times \{n\} \cap D = U \cap \omega^* \times \{n\} \neq \varnothing$. Since the latter set is infinite, we may fix an infinite set $\{x_n : n < \omega\}$ in it. $\{x_n : n < \omega\}$ is a sequence in U with no subsequences converging in $\omega^* \times \{n\}$ and so a fortiori in D.

We finish by giving a complete answer to Question 1.6. Given a space X, the Pixley-Roy topology on X is the space $\mathcal{F}[X] = [X]^{<\omega}$ equipped with the topology generated by sets of the form $[F, U] = \{G \in \mathcal{F}(X) : F \subset G \subset U\}$, where F is a finite subset of X and U is an open subset of X.

The authors of [4] proved that if X is an SHD space whose every countable subset is closed and discrete (this hypothesis is verified, in particular if X is a P-space), then $\mathcal{F}[X]$ is also SHD, and asked whether this is true in general.

Theorem 2.6. *Let X be any SHD space. Then $\mathcal{F}[X]$ is also SHD.*

Proof. Let $\mathcal{U}$ be a countable sequence of non-empty open subsets of $\mathcal{F}[X]$. Without loss of generality we can assume that $\mathcal{U}$ is made up of basic open sets and thus we can enumerate $\mathcal{U}$ as $\{[F_n, U_n] : n < \omega\}$, where $F_n \in \mathcal{F}[X]$ and U_n is a non-empty open subset of X. By the SHD property of X we can pick a point $x_n \in U_n$, for every $n < \omega$ such that $\{x_n : n < \omega\}$ has no converging subsequence. Define $G_n = F_n \cup \{x_n\}$. Then $G_n \in [F_n, U_n]$, for every $n < \omega$. We claim that $\{G_n : n < \omega\}$ has no converging subsequence. Suppose that this is not the case and let $\{G_{n_k} : k < \omega\}$ be a subsequence converging to some point $G \in \mathcal{F}[X]$. That induces a subsequence $\{x_{n_k} : k < \omega\}$ of $\{x_n : n < \omega\}$ in the space X. Moreover, fix an enumeration $\{y_i : 1 \leq i \leq p\}$ of the set G.

Since $S_0 = \{x_{n_k} : k < \omega\}$ does not converge to y_1 then there are an infinite subset S_1 of S_0 and an open neighbourhood U_1 of y_1 such that $U_1 \cap S_1 = \varnothing$. Now, since S_1 does not converge to y_2, there are an infinite subset S_2 of S_1 and an open neighbourhood U_2 of y_2 such that $U_2 \cap S_2 = \varnothing$. Continuing in this way we can construct a decreasing sequence of infinite sets $\{S_i : 0 \leq i \leq p\}$ and a sequence of open sets $\{U_i : 1 \leq i \leq p\}$ such that $y_i \in U_i$ and $U_i \cap S_i = \varnothing$, for every $i \in \{1, \ldots, p\}$.

Notice that $U = \bigcup\{U_i : 1 \leq i \leq p\}$ is an open set which contains G and is disjoint from S_p. It follows that the set $[G, U]$ is an open neighbourhood of G in the Pixley-Roy topology which does not contain a tail of the sequence $\{G_{n_k} : k < \omega\}$ and that is a contradiction. □

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