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Procedia Structural Integrity 78 (2026) 1879–1886

Structural Integrity Procedia

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XX ANIDIS Conference

Analytical formulation for biaxial curvature domains of rectangular reinforced concrete sections

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Abstract

This study proposes a novel closed-form analytical model for defining the biaxial ultimate curvature domains of rectangular reinforced concrete (RC) sections under combined axial load and biaxial bending. The model is derived through a hybrid approach that couples high-fidelity fiber-based sectional analysis with data-driven calibration. A comprehensive parametric database, covering a wide range of geometric configurations, reinforcement ratios, and material properties, is generated and used to calibrate a super-ellipse-based formulation for the curvature domain. The key innovation lies in the derivation of an explicit expression for the super-ellipse shape exponent, obtained via a multi-population Genetic Programming (GP) symbolic regression algorithm. This expression accounts for the combined effects of axial load, reinforcement layout, and reinforcement ratio, thereby improving accuracy compared to existing models that rely on simplified assumptions. The resulting closed-form solution efficiently captures the direction-dependent inelastic deformation capacity of RC sections without requiring computationally intensive numerical analyses. Its applicability is demonstrated through extensive validation against the numerical database, showing excellent agreement. The proposed model offers structural engineers a practical tool for ductility verification and plastic hinge calibration, advancing performance-based seismic design methodologies.

Keywords: biaxial bending; curvature domain; reinforced concrete, cross-section, genetic programming, optimization, machine learning.

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10.1016/j.prostr.2025.12.239

1. Introduction

The assessment of the inelastic deformation capacity of reinforced concrete (RC) columns is a cornerstone of performance-based seismic design. Under axial load and biaxial bending, the response of RC columns is direction-dependent, and the section's deformation capacity can be significantly reduced compared to that along its principal axes. The definition of biaxial curvature domains is essential for a reliable assessment of deformation capacity at critical cross-sections. However, despite its importance, international standards like ACI 318 (2008) and Eurocode 2 (2005) offer limited guidance on this topic, focusing primarily on strength prediction while lacking specific tools or recommendations for determining the biaxial curvature of RC sections. On the other hand, Eurocode 2 (2005) and the Italian Technical Code NTC 2018, require a curvature ductility (μ_ϕ) verification as function of the design axial load N_{Ed}:

$$\mu_\phi = \mu_\phi(N_{Ed}) \geq \mu_{\phi,Ed} \quad (1)$$

without specifying in which direction the curvature ductility should be verified to be compared with the demand $\mu_{\phi,Ed}$, which depends on the adopted behavior factor q_0. Another issue is related to the definition of the concentrated plasticity models for nonlinear analysis, which would require a direction dependent calibration of the plastic hinges to properly approximate the inelastic response of the columns in biaxial bending.

The scientific literature has partially addressed this topic. Some studies (Di Ludovico et al. 2008, Fossetti and Papia 2012, Campione et al. 2016) provided a numerical determination of the ultimate curvature ductility domains. These studies concluded that the curvature domains are concave for low and moderate axial load levels, therefore the elliptic approximation as suggested by Bresler (1960) lead to unconservative estimates. Colajanni et al. (2012) proposed as super-ellipse equation for the ultimate curvature domain based on the ultimate curvatures $\phi_{uo,x}$ and $\phi_{uo,y}$ along the main orthogonal axes:

$$\left(\frac{\phi_{ux}}{\phi_{uo,x}}\right)^\beta + \left(\frac{\phi_{uy}}{\phi_{uo,y}}\right)^\beta = 1 \quad (2)$$

In Eq. (2) ϕ_{ux} and ϕ_{uy} are the components of the ultimate curvature along a generic direction, while the β exponent is a coefficient depending on the dimensionless axial load (ν).

$$\beta = 0.7 + 0.75(\nu - 0.1) \qquad 0.1 \leq \nu \leq 0.5 \quad (3)$$

The primary drawback is that the empirical correlation for the shape exponent often relies solely on the axial load level, neglecting the influence of other critical parameters, such as the reinforcement ratio, the aspect ratio of the cross section and rebars layout. This simplification limits the model's general validity.

The current study addresses the aforementioned limitations by proposing a generalized closed-form model for the biaxial ultimate curvature domain of rectangular RC sections. The methodology integrates high-fidelity numerical analysis with a data-driven machine learning strategy. A comprehensive dataset of ultimate curvature domains is first generated using a robust fiber-based model across a wide range of geometric and mechanical properties. The key innovation lies in deriving the formula for the super-ellipse shape exponent (β) to be used in Eq. (2). To find a close-form equation for this parameter a multipopulation Genetic Programming (GP) symbolic regression algorithm is employed. This technique automatically derives an explicit analytical expression for the exponent as a function of all relevant parameters, including axial load, reinforcement ratio, and rebar layout. The proposed model provides a simple closed-form solution that is able to capture the directionality of the ultimate curvature domain. Its application is suitable for ductility verification and calibration of plastic hinge models to be used for nonlinear analyses (e.g., Sberna et al. (2025a), Di Trapani et al. (2022a), Di Trapani et al. (2023), Sberna et al. (2022), Di Trapani et al. (2022c), Sberna et al. (2025b)).

2. Theoretical definition of biaxial ultimate curvature

The biaxial ultimate curvature domain represents the envelope of the maximum curvature ϕ_u that the cross-section can sustain for every possible orientation α of the neutral axis. The domain is defined point-by-point by enforcing the translational equilibrium of the cross-section between the external axial force (N_{ext}) and the internal forces resulting from the integration of normal stresses. (Fig. 1a). Under the assumption of linear distribution of the normal strain (Euler-Bernoulli hypothesis), the equilibrium equation is defined as:

$$N_{ext} = N_{int} = \int_0^{x_c} \sigma_c dA + \sum_{i=1}^{n_{bars}} A_{s,i} \cdot \sigma_{s,i} \tag{4}$$

In Eq. (4) σ_c is the compressive stress of the concrete at the generic infinitesimal area dA, and $A_{s,i}$ and $\sigma_{s,i}$, are the area of the i-th rebar and its corresponding stress, respectively.

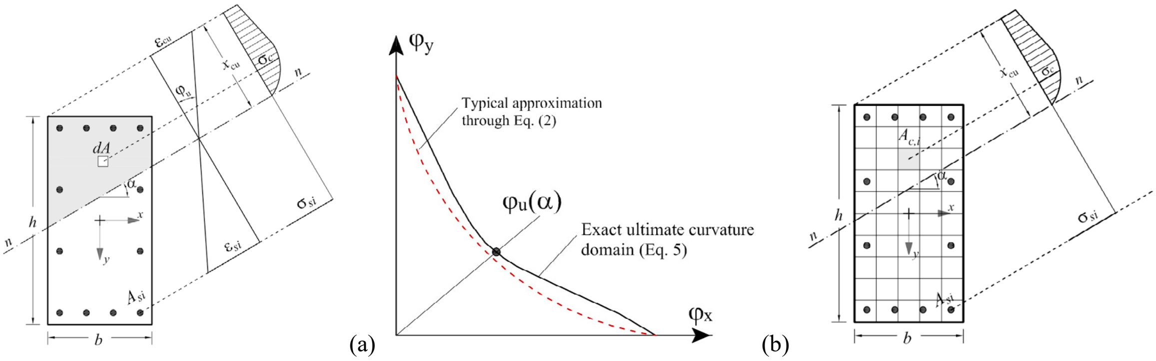

Fig. 1. (a) Strain and stress distributions in a RC section at ultimate limit states showing the curvature direction, neutral axis orientation, (b) schematic representation of the comparison between the exact ultimate curvature domain (solid line) derived from numerical analysis and its analytical approximation using a super-ellipse (dashed red line), (c) fiber-based discretization of the cross-section used for the numerical solution.

The ultimate limit state is defined by the attainment of the ultimate compressive strain of concrete (ε_{cu}) at the most compressed fiber, so once the neutral axis position (x_c) is determined from Eq. (4) the curvature at the generic bending direction $\phi_u(\alpha)$ is determined as:

$$\phi_u(\alpha) = \frac{\varepsilon_{cu}}{x_{cu}} \tag{5}$$

that is decomposed in the *x-y* reference system as:

$$\phi_u(\alpha) = \sqrt{\phi_{ux}^2(\alpha) + \phi_{uy}^2(\alpha)} \tag{6}$$

The complete ultimate curvature domain is defined by repeating this procedure for a representative number of orientations (α) of the neutral axis. A sample of the typical shape obtained for the curvature domain is illustrated in Fig. 1b, together with the shape of the typical approximation curve obtainable from the use of Eq. (2). The solution of Eq. (4) is difficult in closed form, therefore a numerical approach with a fiber discretization of the cross-section (Fig. 1c) is convenient in these cases, as illustrated in detail in the following sections.

3. Proposed machine learning framework for the closed form expression of β exponent

The closed-form model for the biaxial ultimate curvature domain is developed using the three-step computational framework illustrated in Figure 2. In Step 1 an extensive database of ultimate curvature domains is generated using fiber-based sectional analysis for a wide spectrum of RC sections with varying geometrical and mechanical properties. In Step 2, each numerically computed domain, $\hat{\phi}(\alpha)$, is approximated by a super-ellipse equation, $\phi^*(\alpha \mid \beta)$ based on

Eq. (2). An optimal value of the shape exponent, β_{OPT} , is determined for each section by solving the following optimization problem, which minimizes the relative error between the numerical solution and its approximation:

$$\beta_{OPT} = argmin_{\beta>0}\left\{\sum_{\alpha\in[0,90^{\circ}]}\left|\frac{\hat{\phi}(\alpha)-\phi^{*}(\alpha\,|\,\beta)}{\hat{\phi}(\alpha)}\right|\right\} \tag{7}$$

This process will create a new database linking the properties of each section to its optimal shape exponent. Finally, in Step 3, a multi-population Genetic Programming (GP) algorithm performs symbolic regression (Fernandez et al. 2003) on this dataset. This technique is based on the principles of Genetic Algorithms (GA), and it is used for calibrating empirical equation of β. A similar approach can be found in Sirotti et al. (2021) and Di Trapani et al. (2022b). The GP algorithm automatically searches for and derives a single, explicit closed-form expression of β_{ML} that can approximate the curvature domain when applied to Eq. (2) as a function of the most influential physical and mechanical parameters of the RC cross-section.

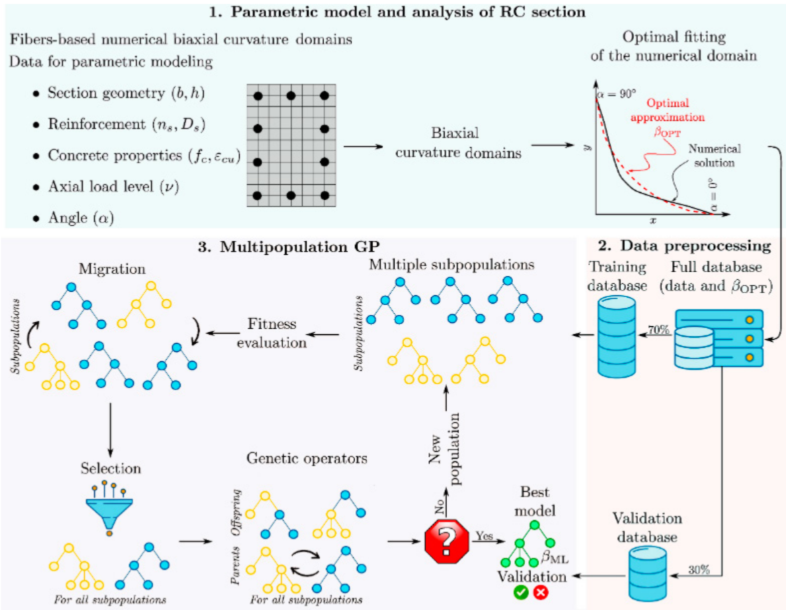

Fig. 2. Machine-learning-aided framework for the closed-form calculation of the biaxial curvature domains.

4. Numerical evaluation of biaxial curvature domains and generation of the cross-section dataset

The solution of Eq. (4) is obtained using a numerical fiber-section approach, where the cross-section is discretized into a finite number of concrete fibers having area $A_{c,i}$ and stress $\sigma_{c,i}$ and one steel fiber per rebar (Fig. 1c). To numerically define the ultimate curvature domain for each selected inclination of the neutral axis the external force N_{ext} is equated to the internal axial force, N_{int} which results from the sum of the contributions of all the fibers:

$$N_{\text{int}} = \sum_{i=1}^{n_{\text{fib,conc}}} A_{\text{c,i}} \cdot \sigma_{\text{c,i}} + \sum_{i=1}^{n_{\text{bars}}} A_{s,i} \cdot \sigma_{s,i} \tag{8}$$

The position of the neutral axis is found through an iterative procedure that adjusts its depth until the internal force balances the applied external load, N_{ext} . Convergence is achieved when the residual force satisfies the condition $\Delta N = |N_{ext} - N_{int}| \le \varepsilon_{tol}$, where ε_{tol} is a predefined strict tolerance. Using this numerical procedure, a comprehensive parametric database was generated to serve as the ground truth for training the GP algorithm. The database was systematically created to cover a broad spectrum of realistic design scenarios. It includes a total of 227 unique cross-sections with varying dimensions and reinforcement arrangements, as summarized in Table 1.

Table 1. Geometrical and mechanical properties of the reinforced concrete cross-sections included in the database.

Property	Values set or range
b [mm]	{300, 400, 500}
h [mm]	{300, 400, 500, 600, 700, 800}
f_c [MPa]	{14, 20, 25, 30, 35}
ε_{cu} [-]	{0.005, 0.01}
$Ø_{reb}$ [mm]	{14, 16, …, 32}

Five concrete classes with compressive strengths (f_c) ranging from 14 to 35 MPa and two levels of ultimate strain (ε_{cu} = 0.005 and 0.01) to account for different strength confinement levels. The reinforcing steel was modelled with an elastic-perfectly plastic relationship, characterized by a yield strength of 450 MPa and an elastic modulus of 210 GPa. For each cross-section, analyses were performed for a range of dimensionless axial load levels (ν) from 0.1 to 0.6, in increments of 0.1. The base to height (*b/h*) aspect ratio, the total longitudinal reinforcement ratio (ρ) and the dimensionless longitudinal rebar interaxes along *x* and *y* were also varied. The aspect ratio (*b/h*) was limited to 1 because of the polar symmetry of the section. The cases with *b/h* > 1 are comprised in those with *b/h* ⩽ 1 by rotating the cross-section of 90°. Under this assumption the dimensionless rebar interaxes $l_x = i_x/(b-2\delta)$ and $l_y = i_y/(h-2\delta)$ are referred to the short (*b*) and long (*h*) sides respectively. This extensive parametric investigation resulted in a final dataset containing 13,620 ultimate curvature domains. Figure 7 illustrates the statistical distribution of the key non-dimensional geometric parameters within this database, confirming its diversity and representativeness for developing a general model.

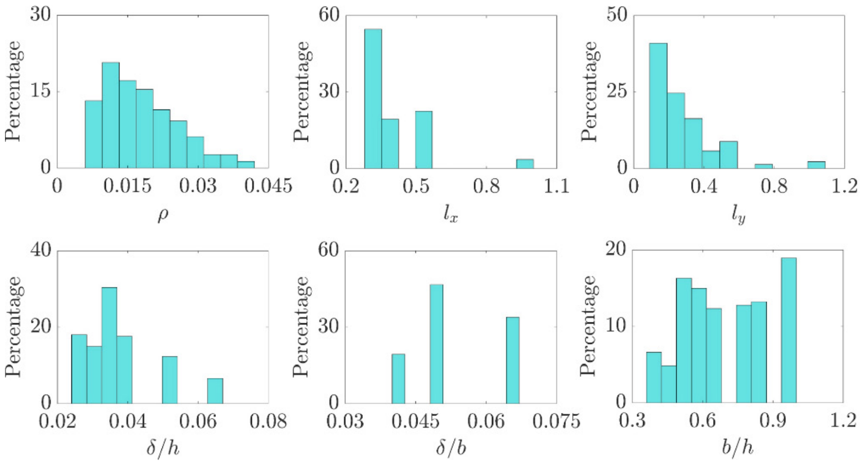

Fig. 3. Distribution of geometric parameters in the database used to calibrate the ultimate curvature domain model.

5. Calibration and validation of the proposed capacity model

Previous studies (Campione et al. 2016) revealed that the domain's shape transitions from concave to convex as the axial load level ν increases. To provide more accurate prediction of the shape exponent β which will be referred to as β_{ML}, the dataset was divided, and two distinct formulations were developed: one for low axial loads $(\nu \leq 0.20)$ and another for higher levels $(0.20 \leq \nu \leq 0.60)$. The genetic programming procedure finally limited the number of variables for β_{ML} to the three: longitudinal reinforcement ratio ρ, horizontal reinforcement ratio l_x, and the normalized axial load ν, leading to the following final expression for β_{ML} :

$$\beta_{\text{ML}} = \begin{cases} 1.37 - \left(\sqrt{0.02\nu + 0.11\rho} - 0.19\log l_x\right)^2 - 0.036\left(l_x - \log\left(0.11\nu + 0.59\rho\right)\right)^2 & \upsilon < 0.20 \\ 0.85 + 0.10 l_x - 0.01\left(\dfrac{l_x}{\nu}\right)^2 + 0.30\nu^2 + 0.12\log\left(6.09\nu^2 + 2.82\dfrac{\rho}{\nu}\right) & 0.20 \le \nu \le 0.60 \end{cases} \tag{9}$$

In Fig. 4a the performance of the GP formulation is assessed by comparing the β_{ML} values predicted by the model (Eq. (9)) against the target optimal values, β_{OPT}. The proposed model showed quite good predictive capability, achieving a coefficient of determination $\left(R^2 = 0.94\right)$ for both the training and validation datasets. The plot also confirmed that nearly all predictions for the shape exponent fell within a $\pm 15\%$ error margin relative to the optimal values, validating the robustness of the derived formula. In Fig. 4b the performance of the model by Colajanni et al. (Eq. 3) is assessed clearly showing lower performance.

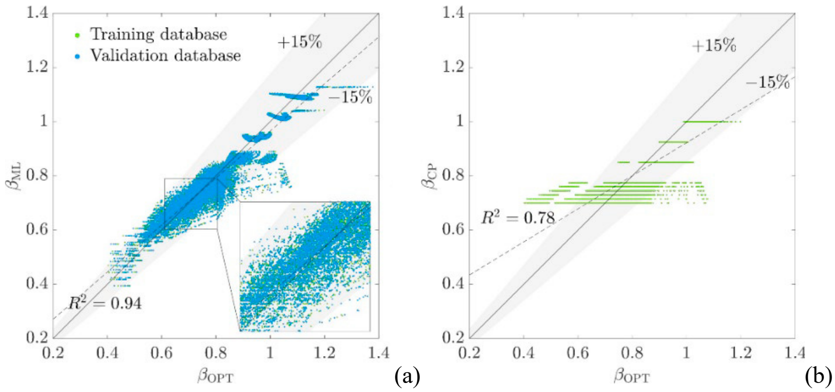

Fig. 4. Comparison between the optimal values of the exponent and corresponding predictions obtained by: (a) the proposed model (Eq. 9), (b) Colajanni et al. (2012) model (Eq. 3).

The sensitivity of Eq. (9) is illustrated in Fig. 5. The plots show that the value of β_{ML} for the ultimate curvature domain is dependent on ρ and ι_x for the lower axial loads levels. This sensitivity gradually vanishes as ν exceeds 0.20. Moreover, the continuity between the two expressions in Eq. (9) at the switching point $\left(\nu = 0.20\right)$ was inspected. The results show a small average difference of 3% between the formulations, a deviation considered negligible for practical applications given the overall accuracy gained.

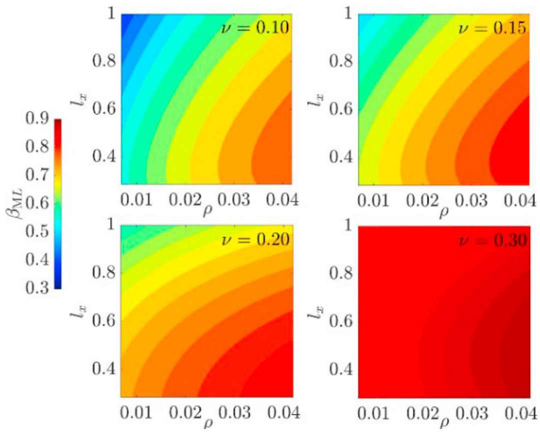

Fig. 5. Sensitivity analysis of the proposed closed-form approximation for the biaxial ultimate curvature domain

A visual validation of the model is provided in Figure 6, where the exact reference biaxial ultimate curvature domains are plotted for a set of representative sections against the corresponding closed-form approximations obtained by combining Eq. (2) with Eq. (9) and that by Colajanni et al. (2012) combining Eq. (2) with Eq. (3). The various curves are obtained for ν values evenly distributed between 0.1 and 0.6. A good agreement is generally observed between the proposed model and the exact biaxial curvature domains. The plot also shows that as ν decreases, the approximation error increases, with the largest errors occurring for $\nu \leq 0.2$. This is due to the fact that the exact domain does not exhibit a consistent shape with the super-ellipse at the lowest axial loads, but rather it shifts between an inward- and an outward-curved geometry as the bending angle α varies. However, significant improvement is obtained at the lowest axial force levels by using the proposed β_{ML} rather than β_{CP} formulation by Colajanni et al. (2012).

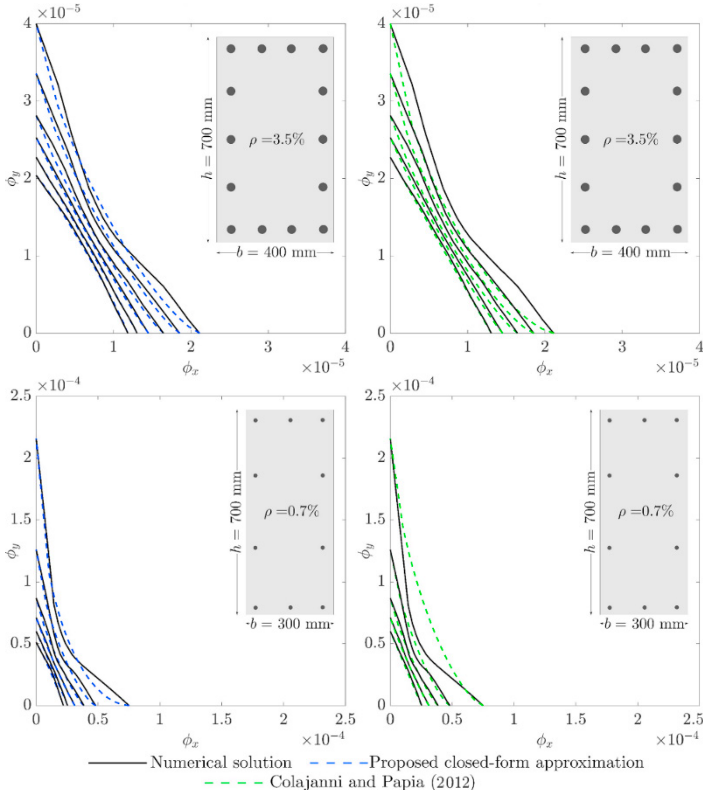

Fig. 6. Comparison of the proposed closed-form formulation for the biaxial ultimate curvature domain (left side) with that of Colajanni et al. (2012) (right side) for two different cross-sections.

Conclusions

This paper introduced a novel closed-form model for the biaxial ultimate curvature of rectangular RC sections. The model was developed by leveraging a large parametric database, generated via fiber-section numerical analyses, to calibrate an analytical expression using a Genetic Programming (GP) algorithm. The key innovation is an explicit equation for the super-ellipse shape exponent β which, unlike previous models that often rely on a single parameter, accounts for the combined influence of the axial load, the reinforcement ratio, and the reinforcement layout. This approach allows the proposed formulation to capture the direction-dependent response of the ultimate curvatures of RC cross-section, achieving a high coefficient of determination $\left(R^2 = 0.94\right)$ and demonstrating superior performance compared to existing formulations.

The resulting model provides a practical and reliable tool for structural design and assessment, bridging the gap between computationally expensive numerical models and oversimplified analytical rules, favoring a straightforward implementation into existing design workflows and automated structural assessment procedures.

Acknowledgements

This study was carried out within the RETURN Extended Partnership and received funding from the European Union Next-GenerationEU (National Recovery and Resilience Plan – NRRP, Mission 4, Component 2, Investment 1.3 – D.D. 1243 2/8/2022, PE0000005).

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