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A probabilistic analysis of selected notions of iterated conditioning under coherence

Lydia Castronovo [a,*,1,2], Giuseppe Sanfilippo [b,*,1,2]

[a] Dep. of Mathematics and Computer Sciences, Physical Sciences and Earth Sciences, University of Messina, Messina, Italy
[b] Department of Mathematics and Computer Science, University of Palermo, Palermo, Italy

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ABSTRACT

It is well known that basic conditionals satisfy some desirable basic logical and probabilistic properties, such as the compound probability theorem. However checking the validity of these becomes trickier when we switch to compound and iterated conditionals. Herein we consider de Finetti's notion of conditional both in terms of a three-valued object and as a conditional random quantity in the betting framework. We begin by recalling the notions of conjunction and disjunction among conditionals in selected trivalent logics. Then we analyze the notions of iterated conditioning in the frameworks of the specific three-valued logics introduced by Cooper-Calabrese, by de Finetti, and by Farrel. By computing some probability propagation rules we show that the compound probability theorem and other important properties are not always preserved by these formulations. Then, for each trivalent logic we introduce an *iterated conditional* as a suitable random quantity which satisfies the compound prevision theorem as well as some other desirable properties. We also check the validity of two generalized versions of Bayes' Rule for iterated conditionals. We study the p-validity of generalized versions of *Modus Ponens* and *two-premise centering* for iterated conditionals. Finally, we observe that *all* the basic properties *are satisfied* within the framework of iterated conditioning followed in recent papers by Gilio and Sanfilippo in the setting of conditional random quantities.

1. Introduction

The study of conditionals (typically expressed by "if – then" sentences), compound conditionals (which are obtained by combining conditionals with the logical operators such as "and", "or", "not"), and iterated conditionals (where both antecedent and consequent are conditionals) is a relevant research topic in many fields, such as philosophy of science, psychology of uncertain reasoning, probability theory, and conditional logics. See for example, [1,3,9,15,17,18,22,23,32,42,45,48,53,55–57,63,64]. Compound and iterated conditionals are largely used in natural language to describe decisions and inferences based on incomplete or uncertain information. Thus, the issue of how to interpret and combine conditionals so to represent human rational reasoning realistically is a key question in the AI community. Recently there has been a growing interest in using conditionals in the field of AI and of knowledge representation. See [10,21,26,28,35,44,46,54].

* Corresponding authors.
E-mail addresses: lydia.castronovo@studenti.unime.it (L. Castronovo), giuseppe.sanfilippo@unipa.it (G. Sanfilippo).

[1] Both authors equally contributed to this work.
[2] Also affiliated with INdAM-GNAMPA, Italy.

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Among the several tools for managing conditional uncertainty, the theory of subjective probability allows one to evaluate uncertainty on a conditional by means of a conditional probability assessment, interpreted as a degree of belief. In the subjective theory, the probability $P(E)$, that an individual in a given state of uncertain knowledge attributes to an event E, is a measure of one's degree of belief in the occurrence of E. In order to assess consistent probabilities operationally, de Finetti proposed a coherence principle based on a suitable betting scheme ([19]). In this framework, probability is assessed only for the events involved, and coherence amounts to the avoidance of a "Dutch Book". All the classical properties of a finitely additive probability follow from the coherence principle. This approach is therefore more flexible than the formalist set-theoretic characterization of probability because it is not necessary to give probability values to each event of a given Boolean algebra. The subjective theory has been extended to conditionals by means of conditional probability assessments.

In this framework, given two events A and B, the conditional *if A then B* is represented not by the material conditional event *not-A or B*, but rather by a conditional event designated as $B|A$. This is a three-valued object which can be assessed as *true*, *false*, or *void* ([20]). In this formulation, the probability of the conditional *if A then B* is interpreted by the conditional probability $P(B|A)$. Subjective conditional probability is primitive, it does not require an unconditional probability assessment and hence it can be properly defined even if the conditioning event has probability zero (see Remark 1). In particular an (unconditional) event B coincides with the conditional event $B|\Omega$ and hence $P(B) = P(B|\Omega)$.

Trivalent logics have usually been used to combine conditionals, as found in [1,5,16,20,24,25]. Conjoined and disjoined conditionals are interpreted in these logics as three-valued objects. However, such interpretations lead to the invalidity of some basic logical properties. Then, some basic probabilistic properties are not preserved when logical operations among conditionals belong to a trivalent logic ([40,59]). In this paper we will study different definitions of iterated conditioning by showing that similar problems arise both when iterated conditionals are represented as three-valued objects and when they depend on conjoined conditionals defined in trivalent logics. As a consequence, human-like reasoning about conditionals under uncertainty cannot properly be formalized in such terms.

A different approach to compound and iterated conditionals has been followed in [45], and in [52] relevant to the definition of conjunction. A related study has been developed in the setting of coherence in the recent papers [34,35,37,39]. In this approach, conjoined, disjoined, and iterated conditionals are defined not as three-valued objects, but rather as suitable conditional random quantities having possibly more than three values in the unit interval $[0,1]$. A betting interpretation governs these objects in a coherent-setting. An attractive advantage of this approach is that all the basic algebraic and probabilistic properties are preserved. These include De Morgan's Laws and Fréchet-Hoeffding bounds. For a synthesis see [39].

A way to build a Boolean algebra of conditionals satisfying suitable properties has been introduced in [28]. In this context, the atomic structure defines the compound conditionals at a formal algebraic level. A general theory of compound conditionals in the framework of conditional random quantities can be framed in such a structure [26]. Then based on a particular extension of a full conditional probability to this Boolean structure ([28]), some probabilistic properties of compound conditionals coincide with results obtained under coherence in the framework of conditional random quantities ([27]).

Conjunctions and disjunctions among conditionals have been introduced and studied quite commonly in three-valued logics ([1,2,5,8,12,13,16,20,24,25,42]). Moreover, de Finetti in 1935 ([20]) proposed a three-valued logic for conditional events by introducing suitably defined notions of conjunction and disjunction. These coincide with features of Kleen-Lukasiewicz-Heyting logic [13] (see also [47]). Further, Calabrese ([5]) and Cooper ([16]) introduced an algebra of conditionals by using the notions of quasi conjunction and quasi disjunction, similarly studied by Adams ([1]). In his trivalent logic de Finetti introduced an operation of iterated conditioning called "subordination" and denoted here by $|_{dF}$. This respects the requirement that, among other properties, the Import-Export principle ([52]) is satisfied (29). Farrell also introduced an operation of iterated conditioning (denoted here by $|_F$) in his trivalent logic ([25]), which uses the same notions of conjunction and disjunction as de Finetti. This also satisfies the Import-Export principle. Cooper and Calabrese also introduced an operation of iterated conditioning (denoted here by $|_C$) in their trivalent logic, which satisfies the Import-Export principle as well.

We should recall that the validity of such a principle, jointly with the requirement of preserving classical probabilistic properties, leads to the well-known Lewis' triviality results [51]. However, the notion of iterated conditioning studied in the framework of conditional random quantities under coherence avoids the Lewis' triviality results because the Import-Export Principle is not satisfied, even satisfying basic probabilistic properties (see [34,60,62]). This operation of iterated conditioning, denoted here by $|_{gs}$, is based on the notion of the conjunction of two conditionals defined as a conditional random quantity ($\wedge_{gs}$) by means of the following structure ([60]) $\square|\bigcirc = \square \wedge \bigcirc + \mathbb{P}(\square|\bigcirc)\bar{\bigcirc}$, where $\mathbb{P}$ is the symbol of prevision, $\square$ and $\bigcirc$ are (indicators of) conditional events, and $\square \wedge \bigcirc$ is a conditional random quantity with value in $[0,1]$. In the framework of subjective probability, the prevision $\mu = \mathbb{P}(\square|\bigcirc)$ represents the amount that you agree to pay, knowing that you will receive the random quantity $\square \wedge \bigcirc + \mathbb{P}(\square|\bigcirc)\bar{\bigcirc}$. Moreover, by the linearity property of a coherent prevision and by observing that $\bar{\bigcirc} = 1 - \bigcirc$, from the previous structure it follows that $\mathbb{P}(\square|\bigcirc) = \mathbb{P}(\square \wedge \bigcirc) + \mathbb{P}(\square|\bigcirc)[1 - \mathbb{P}(\bigcirc)]$, that is $\mathbb{P}(\square \wedge \bigcirc) = \mathbb{P}(\square|\bigcirc)\mathbb{P}(\bigcirc)$. This last equation is a generalized version of the compound probability theorem, whose standard version is $P(A \wedge B) = P(B|A)P(A)$. Indeed, when $\square$ and $\bigcirc$ are two events A and B, in a conditional bet on $B|A$, the structure above allows to numerically interpret the indicator of the conditional event $B|A$ as the random win $A \wedge B + P(B|A)\bar{A}$, which takes value 1, or 0, or $P(B|A)$, according to whether AB is true, or $A\bar{B}$ is true, or $\bar{A}$ is true ([30,49] see also [34]). Then, when $\square$ denotes $B|K$, $\bigcirc$ denotes $A|H$ (and hence $\bar{\bigcirc}$ denotes $\overline{A|H} = \bar{A}|H$), and $\square \wedge \bigcirc$ is the conjunction $(B|K) \wedge_{gs} (A|H)$, the iterated conditional $(B|K)|_{gs}(A|H)$ is defined as $(B|K) \wedge_{gs} (A|H) + \mu(\bar{A}|H)$, where $\mu = \mathbb{P}[(B|K)|_{gs}(A|H)]$. In addition, $\square$ and $\bigcirc$ can be also replaced by conjoined conditionals and hence the previous structure allows to introduce a more general notion of iterated conditional $\square|\bigcirc$ as done in [38]. The purpose of this paper is to investigate some of the basic properties valid for events and conditional events with a view to different operations of iterated conditioning. Indeed, things get more problematic when we replace events with conditional events and we move to the properties of iterated conditionals. We recall four selected notions of conjunction in trivalent logics: Kleene-Lukasiewicz-Heyting-de Finetti ($\wedge_K$), Lukasiewicz ($\wedge_L$), Bochvar-Kleene ($\wedge_B$), and Sobocinski or quasi conjunction ($\wedge_S$). After recalling some logical and probabilistic results in the trivalent logics, we study basic properties for the notions of iterated conditioning introduced by Cooper-Calabrese ($|_C$), by de Finetti ($|_{dF}$), and by Farrell ($|_F$), respectively. For each of them we also compute some sets of coherent assessments on families of conditional events involving iterated conditionals. This study is based on a geometrical approach for coherence checking of conditional probability assessments, which allows zero probability for conditioning events ([30]). Then, we observe that none of these operations of iterated conditioning preserves the compound probability theorem. By exploiting the structure $\square|\bigcirc = \square \wedge \bigcirc + \mathbb{P}(\square|\bigcirc)\bar{\bigcirc}$, for each conjunction among the four selected trivalent logics we introduce, in the framework of conditional random quantity, a suitable notion of iterated conditioning ($|_K$, $|_L$, $|_B$ and $|_S$). We observe that all of them satisfy the compound prevision theorem, we check the validity of some other basic properties, the Import-Export principle and two generalized versions of the Bayes' Rule for iterated conditioning. Then, we study the p-validity of generalized versions of Modus Ponens and two-premise centering for iterated conditionals. Finally, we remark that, among the selected iterated conditionals, $|_{gs}$ is the only one which satisfies all the basic properties.

The paper is organized as follows. In Section 2 we recall some basic notions and results which concern coherence, conditional random quantities, trivalent logics and logical operations among conditional events. We also give two examples on the geometrical interpretation of coherence. We end the section by recalling some logical and probabilistic properties satisfied by events and conditional events and we generalize them to compound and iterated conditionals. Then, we check the validity of those selected properties for the iterated conditioning defined by Cooper-Calabrese ($|_C$, Section 3), by de Finetti ($|_{dF}$, Section 4) and by Farrell ($|_F$, Section 5). In order to check some probabilistic properties, we also compute sets of coherent assessments on suitable families of conditional events. In Section 6, we recall some results on $|_{gs}$ and we introduce and study the iterated conditioning $|_K, |_L, |_B$, and $|_S$ in the framework of conditional random quantities under coherence. We also consider the validity of two generalized versions of Bayes' Rule. In Section 7, for selected operations of iterated conditioning we check the p-validity of two generalized inference rules: Modus Ponens and two-premise centering. We also consider two examples of natural language. Finally, in Section 8 we illustrate a brief summary on the validity of the basic properties, we give some conclusions and an outlook to future work. To improve the readability of the paper we put some proofs in Appendix A.

This paper is a revised and expanded version of the conference paper [11]. We reorganized the structure of the conference paper and we added proofs, examples, and new results. In particular, we expanded Section 2 by also adding two new examples. We included the proofs in Section 3 and in Section 4. We added several new results in Section 6 (Theorems 12–18). We also added Section 5 and Section 7.

2. Preliminary notions, results, and basic properties

In this section we first recall some preliminary notions and results on coherence, conditional events and conditional random quantities. We also give some examples and we deepen some aspects of conditional random quantities. Then, we recall the logical operations among conditional events in selected trivalent logics and in the framework of conditional random quantities. Finally, we recall some basic logical and probabilistic properties satisfied by events and conditional events and we rewrite them for compound and iterated conditionals.

2.1. Events, conditional events, conditional random quantities and coherence

An event A is a two-valued logical entity which is either *true*, or *false*. We use the same symbol to refer to an event and its indicator, which can take value 1, or 0, according to whether, the event is true, or false, respectively. We denote by Ω the sure event and by $\emptyset$ the impossible one. We denote by $A \wedge B$ (resp., $A \vee B$), or simply by AB, the conjunction (resp., disjunction) of A and B. By $\bar{A}$ we denote the negation of A. We simply write $A \subseteq B$ to denote that A logically implies B, i.e., $AB = A$. Given two events A and H, with $H \neq \emptyset$, the conditional event $A|H$ is a three-valued logical entity which is *true*, or *false*, or *void*, according to whether AH is true, or $\bar{A}H$ is true, or $\bar{H}$ is true, respectively. Notice that conditional events are three-valued logical entities and hence they are not in general events. However, as already observed, a conditional event $A|H$ reduces to the (unconditional) event A, when H is the sure event Ω, i.e. $A|\Omega = A$. We recall that, given any conditional event $A|H$, it holds that $AH|H = A|H$. Moreover, the negation $\overline{A|H}$ is defined as $\overline{A|H} = \bar{A}|H$. Given two conditional events $A|H$ and $B|K$, we say that $A|H$ logically implies $B|K$, denoted by $A|H \subseteq B|K$, if and only if AH logically implies BK and $\bar{B}K$ logically implies $\bar{A}H$, that is ([41]),

$$A|H \subseteq B|K \iff AH \subseteq BK \text{ and } \bar{B}K \subseteq \bar{A}H. \tag{1}$$

In the betting framework of subjective probability, to assess $P(A|H) = x$ amounts to say that, for every real number s, you are willing to pay an amount sx and to receive s, or 0, or sx, according to whether AH is true, or $\bar{A}H$ is true, or $\bar{H}$ is true (the bet is called off), respectively. Hence, for the random gain $G = sH(A - x)$, the possible values are $s(1 - x)$, or $-sx$, or 0, according to whether AH is true, or $\bar{A}H$ is true, or $\bar{H}$ is true, respectively.

We denote by X a *random quantity*, that is an uncertain real quantity, which has a well-determined but unknown value. In this paper we assume that X has a finite set of possible values. Given any event $H \neq \emptyset$, agreeing to the betting metaphor, if you assess that the prevision of "*X conditional on H*" (or short: "*X given H*"), $\mathbb{P}(X|H)$, is equal to μ, this means that for any given real number s you are willing to pay an amount $s\mu$ and to receive sX, or $s\mu$, according to whether H is true, or false (bet called off), respectively. The random gain is

$$G = s(XH + \mu\overline{H}) - s\mu = sH(X - \mu). \tag{2}$$

In particular, when X is (the indicator of) an event A, then

$$\mathbb{P}(X|H) = P(A|H). \tag{3}$$

Given a prevision function $\mathbb{P}$ defined on an arbitrary family $\mathcal{K}$ of finite conditional random quantities, consider a finite subfamily $\mathcal{F} = \{X_1|H_1, \dots, X_n|H_n\} \subseteq \mathcal{K}$ and the vector $\mathcal{M} = (\mu_1, \dots, \mu_n)$, where $\mu_i = \mathbb{P}(X_i|H_i)$ is the assessed prevision for the conditional random quantity $X_i|H_i$, $i \in \{1, \dots, n\}$. With the pair $(\mathcal{F}, \mathcal{M})$ we associate the random gain $G = \sum_{i=1}^n s_i H_i(X_i - \mu_i)$. We denote by $\mathcal{G}_{\mathcal{H}_n}$ the set of possible values of G restricted to $\mathcal{H}_n = H_1 \vee \dots \vee H_n$. Then, the notion of coherence is defined as below.

Definition 1. The function $\mathbb{P}$ defined on $\mathcal{K}$ is *coherent* if and only if $\forall n \geq 1$, $\forall s_1, \dots, s_n$, $\forall \mathcal{F} = \{X_1|H_1, \dots, X_n|H_n\} \subseteq \mathcal{K}$, it holds that: $\min \mathcal{G}_{\mathcal{H}_n} \leq 0 \leq \max \mathcal{G}_{\mathcal{H}_n}$.

In other words, $\mathbb{P}$ on $\mathcal{K}$ is incoherent (i.e., it is not coherent), if and only if there exists a finite combination of n bets such that, after discarding the case where all the bets are called off, the values of the random gain are all positive or all negative. In the particular case where $\mathcal{K}$ is a family of conditional events, by recalling (3), then Definition 1 becomes the well-known definition of coherence for a conditional probability function, denoted as P. In this case, for a finite subfamily of conditional events $\mathcal{F} = \{E_1|H_1, \dots, E_n|H_n\}$, we denote by $\mathcal{P} = (p_1, \dots, p_n)$, with $p_i = P(E_i|H_i)$, $i = 1, \dots, n$, the restriction of P to $\mathcal{F}$. We observe that, for the checking of coherence of the probability assessment p on a conditional event $A|H$, only the cases in which the bet is not called off are considered. Then, we do not consider objects $A|H$ with $H = \emptyset$, that is conditional events with an impossible conditioning event, because in this case a bet on $A|\emptyset$ can only be called off. Given a conditional event $A|H$, with $H \neq \emptyset$, if $\emptyset \neq AH \neq H$, then any value $P(A|H) \in [0,1]$ is a coherent assessment for $A|H$. Coherence requires that $P(AH|H) = P(H|H) = 1$ (resp., $P(A|H) = P(\emptyset|H) = 0$) when $AH = H$ (resp., $AH = \emptyset$).

2.2. Geometrical interpretation of coherence

Given a family $\mathcal{F} = \{X_1|H_1, \dots, X_n|H_n\}$ of n conditional random quantities, for each $i \in \{1, \dots, n\}$ we denote by $\{x_{i1}, \dots, x_{ir_i}\}$ the set of possible values of X_i when H_i is true; then, we set $A_{ij} = (X_i = x_{ij})$, $i = 1, \dots, n$, $j = 1, \dots, r_i$. We observe that, for each i, the family $\{A_{i1}H_i, \dots, A_{ir_i}H_i, \overline{H}_i\}$, or equivalently $\{(X_i = x_{i1}), \dots, (X_i = x_{ir_i}), \overline{H}_i\}$, is a partition of the sure event Ω, with $A_{ij}H_i = A_{ij}$ and $\bigvee_{j=1}^{r_i} A_{ij} = H_i$. Then,

$$\Omega = (A_{11} \vee \dots \vee A_{1r_1} \vee \overline{H}_1) \wedge \dots \wedge (A_{n1} \vee \dots \vee A_{nr_n} \vee \overline{H}_n). \tag{4}$$

By applying the distributive property, the expression in (4) becomes a disjunction of $(r_1+1)\cdots(r_n+1)$ conjunctions. The non-impossible conjunctions are the constituents, or possible elementary outcomes, generated by the family $\mathcal{F}$. We denote by C_0 the constituent $\overline{H}_1 \cdots \overline{H}_n$ (if non-impossible) and we denote by $C_1, \dots, C_m$ the (remaining) constituents which logically imply $\mathcal{H}_n = H_1 \vee \dots \vee H_n$, that is $C_h \subseteq \mathcal{H}_n$, $h = 1, \dots, m$. Of course, $C_0, C_1, \dots, C_m$ form a partition of the sure event Ω.

With each C_h, $h \in \{1, \dots, m\}$, we associate a vector $Q_h = (q_{h1}, \dots, q_{hn})$, where $q_{hi} = x_{ij}$ if $C_h \subseteq A_{ij}$, $j = 1, \dots, r_i$, while $q_{hi} = \mu_i$ if $C_h \subseteq \overline{H}_i$; with C_0 we associate $Q_0 = \mathcal{M} = (\mu_1, \dots, \mu_n)$. Denoting by $\mathcal{I}$ the convex hull of $Q_1, \dots, Q_m$, the condition $\mathcal{M} \in \mathcal{I}$ amounts to the existence of a vector $(\lambda_1, \dots, \lambda_m)$ such that: $\sum_{h=1}^m \lambda_h Q_h = \mathcal{M}$, $\sum_{h=1}^m \lambda_h = 1$, $\lambda_h \geq 0$, $\forall h$; in other words, $\mathcal{M} \in \mathcal{I}$ is equivalent to the solvability of the system (Σ), associated with $(\mathcal{F}, \mathcal{M})$,

$$(\Sigma) \quad \sum_{h=1}^m \lambda_h q_{hi} = \mu_i\,,\ i \in \{1, \dots, n\}\,, \sum_{h=1}^m \lambda_h = 1,\ \ \lambda_h \geq 0\,,\ \ h \in \{1, \dots, m\}\,. \tag{5}$$

Given the assessment $\mathcal{M} = (\mu_1, \dots, \mu_n)$ on $\mathcal{F} = \{X_1|H_1, \dots, X_n|H_n\}$, let $\mathcal{S}$ be the set of solutions $\Lambda = (\lambda_1, \dots, \lambda_m)$ of system (Σ). We point out that the solvability of system (Σ) is a necessary (but not sufficient) condition for coherence of $\mathcal{M}$ on $\mathcal{F}$. When (Σ) is solvable, that is $\mathcal{S} \neq \emptyset$, we define:

$$\begin{aligned} &\Phi_i(\Lambda) = \Phi_i(\lambda_1, \dots, \lambda_m) = \textstyle\sum_{r: C_r \subseteq H_i} \lambda_r,\ i \in \{1, \dots, n\},\ \Lambda \in \mathcal{S}; \\ &M_i = \max_{\Lambda \in \mathcal{S}}, \Phi_i(\Lambda),\ i \in \{1, \dots, n\}\,; \\ &I_0 = \{i\ :\ M_i = 0\}\,, \mathcal{F}_0 = \textstyle\bigcup_{i \in I_0} \{X_i|H_i\}. \end{aligned} \tag{6}$$

We also denote by $\mathcal{M}_0$ the sub-assessment of $\mathcal{M}$ on the sub-family $\mathcal{F}_0$. For what concerns the probabilistic meaning of I_0, it holds that $i \in I_0$ if and only if the (unique) coherent extension of $\mathcal{M}$ to $H_i|\mathcal{H}_n$ is zero. Then, the following theorem can be proved (see e.g., [37])

Theorem 1. *A conditional prevision assessment $\mathcal{M} = (\mu_1, \dots, \mu_n)$ on the family $\mathcal{F} = \{X_1|H_1, \dots, X_n|H_n\}$ is coherent if and only if the following conditions are satisfied: (i) the system (Σ) defined in (5) is solvable; (ii) if $I_0 \neq \emptyset$, then $\mathcal{M}_0$ is coherent.*

Let $\mathcal{S}'$ be a nonempty subset of the set of solutions $\mathcal{S}$ of system (Σ). We denote by I_0' the set I_0 defined as in (6), where $\mathcal{S}$ is replaced by $\mathcal{S}'$, that is

$$I_0' = \{i\ :\ M_i' = 0\},\ \text{where}\ M_i' = \max_{\Lambda \in \mathcal{S}'} \Phi_i(\Lambda)\,,\ \ i \in \{1, \dots, n\}. \tag{7}$$

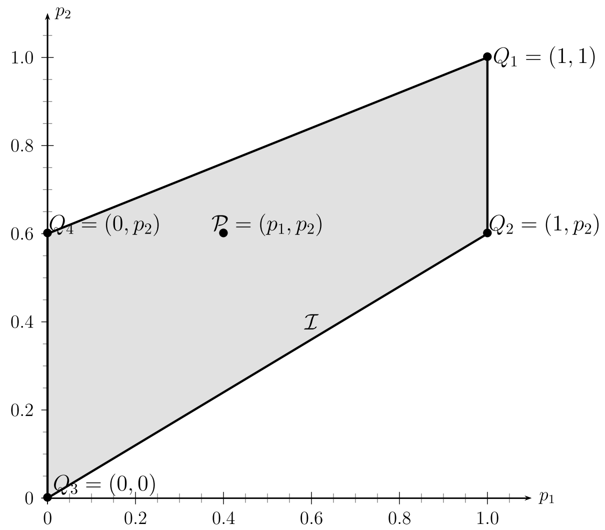

Fig. 1. Convex hull of the points Q_1, Q_2, Q_3, Q_4 associated with the pair $(\mathcal{F}, \mathcal{P})$, where $\mathcal{F} = \{B|K, B|(K \wedge (\overline{H} \vee A))\}$ and $\mathcal{P} = (p_1, p_2)$. In the figure the numerical values are: $p_1 = 0.4$ and $p_2 = 0.6$.

Moreover, we denote by $(\mathcal{F}'_0, \mathcal{M}'_0)$ the pair associated with I'_0. Then, we obtain

Theorem 2. *The assessment $\mathcal{M}$ on $\mathcal{F}$ is coherent if and only if the following conditions are satisfied: (i) the system (Σ) associated with the pair $(\mathcal{F}, \mathcal{M})$ is solvable; (ii) if $I'_0 \neq \emptyset$, then $\mathcal{M}'_0$ is coherent.*

Of course, the previous results can be used in the case of a probability assessment, which will be denoted by $\mathcal{P}$, on a family of conditional events $\mathcal{F}$. More precisely, given a family $\mathcal{F} = \{E_1|H_1, \dots, E_n|H_n\}$ of n conditional events, we observe that, for each i, the family $\{E_iH_i, \bar{E}_iH_i, \overline{H}_i\}$ is a partition of Ω. Then, $\Omega = \bigwedge_{i=1}^{n}(E_iH_i \vee \bar{E}_iH_i \vee \overline{H}_i)$. By applying the distributive property it follows that Ω can also be written as the disjunction of 3^n logical conjunctions, some of which may be impossible. The remaining ones, denoted by $C_0, C_1, \dots, C_m$, where $C_0 = \overline{H}_1 \cdots \overline{H}_n$ (if non-impossible), are the constituents generated by $\mathcal{F}$. Of course, $m + 1 \leq 3^n$, with $m + 1 = 3^n$ when, for example, the events in $\mathcal{F}$ are logically independent. Let $\mathcal{P} = (p_1, \dots, p_n)$ be a probability assessment on $\mathcal{F}$. In this case, for each constituent C_h, $h = 1, \dots, m$, it holds that $Q_h = (q_{h1}, \dots, q_{hn})$, where $q_{hi} = 1$, or 0, or p_i, according to whether $C_h \subseteq E_iH_i$, or $C_h \subseteq \bar{E}_iH_i$, or $C_h \subseteq \overline{H}_i$. The point $Q_0 = \mathcal{P}$ is associated with C_0. In the following example, which will be useful in Section 3 to prove Theorem 5, we illustrate how the constituents and the associated points are generated in order to check the coherence of a probability assessment.

Example 1. Let $\mathcal{F} = \{E_1|H_1, E_2|H_2\} = \{B|K, B|(K \wedge (\overline{H} \vee A))\}$, where A, B, H, K are four logically independent events, and let $\mathcal{P} = (p_1, p_2)$ be a probability assessment on $\mathcal{F}$. We verify that $\mathcal{P}$ is coherent for every $(p_1, p_2) \in [0, 1]^2$. It holds that

$$
\begin{aligned}
&\Omega = (E_1H_1 \vee \bar{E}_1H_1 \vee \overline{H}_1) \wedge (E_2H_2 \vee \bar{E}_2H_2 \vee \overline{H}_2) = \\
&= [BK \vee \bar{B}K \vee \bar{K}] \wedge [(BK \wedge (\overline{H} \vee A)) \vee (\bar{B}K \wedge (\overline{H} \vee A)) \vee (\bar{K} \vee \bar{A}H)] = \\
&= (BK \wedge (\overline{H} \vee A)) \vee \emptyset \vee \bar{A}HBK \vee \emptyset \vee (\bar{B}K \wedge (\overline{H} \vee A)) \vee \bar{A}H\bar{B}K \vee \emptyset \vee \emptyset \vee (\bar{K} \vee \bar{A}H\bar{K}) = \\
&= C_1 \vee C_2 \vee C_3 \vee C_4 \vee C_0,
\end{aligned}
$$

where the constituents are

$$
\begin{aligned}
&C_1 = BK \wedge (\overline{H} \vee A) = AHBK \vee \overline{H}BK,\ C_2 = \bar{A}HBK, C_3 = \bar{B}K \wedge (\overline{H} \vee A) = AH\bar{B}K \vee \overline{H}\,\bar{B}K, \\
&C_4 = \bar{A}H\bar{B}K,\ C_0 = \bar{K} \vee \bar{A}H\bar{K} = \bar{K}.
\end{aligned}
$$

The points Q_h's associated with the pair $(\mathcal{F}, \mathcal{P})$ are

$$
Q_1 = (1, 1),\ Q_2 = (1, p_2),\ Q_3 = (0, 0),\ Q_4 = (0, p_2),\ Q_0 = \mathcal{P} = (p_1, p_2).
$$

We denote by $\mathcal{I}$ the convex hull of points Q_1, Q_2, Q_3, Q_4 (see Fig. 1). The system (Σ) in (5) associated with the pair $(\mathcal{F}, \mathcal{P})$ is

$$
\begin{cases}
\lambda_1 + \lambda_2 = p_1, \\
\lambda_1 + p_2\lambda_2 + p_2\lambda_4 = p_2, \\
\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1, \\
\lambda_i \geq 0,\ \forall i = 1, \dots, 4.
\end{cases}
\tag{8}
$$

We observe that $\mathcal{P} = (p_1, p_2) = p_1Q_2 + (1 - p_1)Q_4$ and hence the system (8) is solvable and a solution is $\Lambda = (0, p_1, 0, 1 - p_1)$, with $p_1 \in [0, 1]$.

By considering the function ϕ as defined in (6), it holds that

$$\phi_1(\Lambda) = \sum_{h:C_h\subseteq H} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1,$$
$$\phi_2(\Lambda) = \sum_{h:C_h\subseteq (K\wedge(A\vee \overline{H}))} \lambda_h = \lambda_1 + \lambda_3 = 0.$$

We set $\mathcal{S}' = \{\Lambda\}$ and we get $\mathcal{I}'_0 = \{2\}$. We observe that the sub-assessment p_2 on $B|(K \wedge (\overline{H} \vee A))$ is coherent for every $p_2 \in [0,1]$. Thus, by Theorem 2, the assessment $\mathcal{P} = (p_1, p_2)$ on $\mathcal{F}$ is coherent $\forall (p_1, p_2) \in [0,1]^2$.

The next example, which is related to the compound prevision theorem listed in Section 2.5, illustrates how to check coherence for (conditional) prevision assessments.

Example 2. [3][Compound prevision theorem] Let $\mathcal{F} = \{XH, X|H, H\}$, where H is an event, with $H \neq \emptyset$, $H \neq \Omega$, and X is a finite random quantity. We denote by $\{x_1, \dots, x_n\}$ the possible values of X when H is true. We also set $x' = \min\{x_1, \dots, x_n\}$ and $x'' = \max\{x_1, \dots, x_n\}$. Let $\mathcal{M} = (z, \mu, p)$ be a prevision assessment on $\mathcal{F}$. We verify that $\mathcal{M}$ is coherent if and only if $\mu \in [x', x'']$, $p \in [0,1]$, and $z = \mu p$. Of course, coherence requires that $p \in [0,1]$. It holds that

$$\Omega = C_1 \vee \cdots \vee C_n \vee C_{n+1},$$

where the constituents are

$$C_1 = (X = x_1), \cdots, C_n = (X = x_n),\ C_{n+1} = \overline{H}.$$

The points Q_h's associated with the pair $(\mathcal{F}, \mathcal{M})$ are

$$Q_1 = (q_{11}, q_{12}, q_{13}) = (x_1, x_1, 1), \dots,\ Q_n = (q_{n1}, q_{n2}, q_{n3}) = (x_n, x_n, 1),$$
$$Q_{n+1} = (q_{(n+1)1}, q_{(n+1)2}, q_{(n+1)3}) = (0, \mu, 0).$$

We denote by $\mathcal{I}$ the convex hull of points $Q_1, \dots, Q_{n+1}$. The system (Σ) in (5) associated with the pair $(\mathcal{F}, \mathcal{M})$ is

$$\begin{cases} \lambda_1 x_1 + \cdots + \lambda_n x_n = z, \\ \lambda_1 x_1 + \cdots + \lambda_n x_n + \lambda_{n+1}\mu = \mu, \\ \lambda_1 + \cdots + \lambda_n = p, \\ \lambda_1 + \cdots + \lambda_{n+1} = 1, \\ \lambda_i \geq 0,\ i = 1, \dots, n+1. \end{cases} \iff \begin{cases} \lambda_1 x_1 + \cdots + \lambda_n x_n = z, \\ z = \mu p, \\ \lambda_1 + \cdots + \lambda_n = p, \\ \lambda_{n+1} = 1 - p, \\ \lambda_i \geq 0,\ i = 1, \dots, n+1. \end{cases} \tag{9}$$

We distinguish two cases: (i) $p = 0$; and $p \in (0,1]$.
Case (i). System (9) is solvable only if $z = 0$, with the unique solution given by $\Lambda = (\lambda_1, \dots, \lambda_n, \lambda_{n+1}) = (0, \dots, 0, 1)$. By considering the function ϕ as defined in (6), it holds that

$$\phi_1(\Lambda) = \phi_3(\Lambda) = \sum_{h:C_h\subseteq \Omega} \lambda_h = \lambda_1 + \cdots + \lambda_{n+1} = 1 > 0$$

and

$$\phi_2(\Lambda) = \sum_{h:C_h\subseteq H} \lambda_h = \lambda_1 + \cdots + \lambda_n = p = 0.$$

Then $I_0 = \{2\}$. We consider the sub-assessment $\mathcal{M}_0 = (\mu)$ on $\mathcal{F}_0 = \{X|H\}$. It can be easily proved that $\mu = \mathbb{P}(X|H)$ is coherent if and only if $\mu \in [x', x'']$. Then, by Theorem 1, the assessment $\mathcal{M} = (0, \mu, 0)$ on $\mathcal{F}$ is coherent if and only if $\mu \in [x', x'']$.
Case (ii). In this case System (9) becomes

$$\begin{cases} z = \mu p, \\ \mu = \frac{z}{p} = \frac{\lambda_1 x_1 + \cdots + \lambda_n x_n}{\lambda_1 + \cdots + \lambda_n}, \\ \lambda_1 + \cdots + \lambda_n = p, \\ \lambda_{n+1} = 1 - p, \\ \lambda_i \geq 0,\ i = 1, \dots, n+1, \end{cases} \tag{10}$$

which is solvable when $z = \mu p$ and $\mu \in [x', x'']$, because μ is a convex linear combination of $\{x_1, \dots, x_n\}$ with weights $\frac{\lambda_1}{\lambda_1+\cdots+\lambda_n}, \cdots, \frac{\lambda_n}{\lambda_1+\cdots+\lambda_n}$. For each solution Λ of System (10) it holds that $\phi_1(\Lambda) = \phi_3(\Lambda) = 1 > 0$ and $\phi_2(\Lambda) = p > 0$. Then, as $I_0 = \emptyset$, by Theorem 1 the assessment (z, μ, p), with $p \in (0,1]$ is coherent if and only if $\mu \in [x', x'']$ and $z = \mu p$.

[3] This example was inspired by Angelo Gilio's talk "On Coherence and Conditionals" presented at the workshop "Reasoning and uncertainty: probabilistic, logical, and psychological perspectives", Regensburg, August 9-10, 2022.

Therefore $\mathcal{M}$ is coherent if and only if $\mu \in [x', x'']$, $p \in [0,1]$, and $z = \mu p$. We now show that, in agreement to Definition 1, an incoherent assessment leads to the existence of a combination of bets where the values of the random gain are all positive or all negative (Dutch Book). We set $\mathcal{M} = (z, \mu, p)$, with $p \in [0,1]$, $\mu \in [x', x'']$ and $z \neq \mu p$. We observe that the points $Q_1, \dots, Q_{n+1}$ belong to the plane $\pi : -\mathcal{X} + \mathcal{Y} + \mu\mathcal{Z} = \mu$, where $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are the axes coordinates. Moreover, $\mathcal{M}$ does not belong to the convex hull $\mathcal{I}$ because, as $z \neq \mu p$, System (9) is not solvable. Now, by considering the function $f(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) = -\mathcal{X} + \mathcal{Y} + \mu\mathcal{Z} - \mu$, we observe that for each constant k the equation $f(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) = k$ represents a plane which is parallel to π and coincides with π when $k = 0$. Then, $f(Q_1) = \cdots = f(Q_{n+1}) = 0$. Moreover, $f(Q_h) - f(\mathcal{M}) = z - \mu p \neq 0$, $h = 1, \dots, n+1$. We recall that for $h = 1, \dots, n+1$, the value g_h, of the random gain

$$G = s_1(XH - z) + s_2 H(X - \mu) + s_3(H - p)$$

associated to the constituent C_h is

$$g_h = s_1(q_{h1} - z) + s_2 H(q_{h2} - \mu) + s_3(q_{h3} - p).$$

We observe that, by setting the stakes $s_1 = -1$, $s_2 = 1$, $s_3 = -\mu$, it holds that $g_h = f(Q_h) - f(\mathcal{M}) = z - \mu p \neq 0$, $h = 1, \dots, n+1$. Therefore, $\min g_h \cdot \max g_h > 0$, when $s_1 = -1$, $s_2 = 1$, $s_3 = -\mu$, that is a combination of bets where the values of the random gain are all positive or all negative.

Remark 1. By Example 2 coherence requires that (*compound prevision theorem*)

$$\mathbb{P}[XH] = \mathbb{P}[X|H]P(H). \tag{11}$$

In particular, when X is (the indicator of) an event E, Equation (11) becomes (*compound probability theorem*)

$$P(E \wedge H) = P(E|H)P(H). \tag{12}$$

Moreover, under logical independence of E and H, Example 2 shows that $P(E|H)$ coincides with the ratio $\frac{P(E \wedge H)}{P(H)}$ when $P(H) > 0$, and $P(E|H)$ can be any value in $[0,1]$ when $P(H) = 0$. Then, differently from the "standard" approach where $P(E|H)$ is defined only when $P(H) > 0$ by the ratio $\frac{P(E \wedge H)}{P(H)}$, in the coherence-based approach the conditional probability $P(E|H)$ is a primitive notion which is properly defined even if $P(H) = 0$.

2.3. Numerical interpretation of a conditional random quantity

Given a conditional event $A|H$, with $P(A|H) = x$, the indicator of $A|H$, denoted by the same symbol, is the following random quantity (see, e.g., [30,49])

$$A|H = AH + x\overline{H} = AH + x(1 - H) = \begin{cases} 1, & \text{if } AH \text{ is true}, \\ 0, & \text{if } \bar{A}H \text{ is true}, \\ x, & \text{if } \overline{H} \text{ is true}. \end{cases} \tag{13}$$

Notice that, by the linearity property of a coherent prevision, it holds that

$$\mathbb{P}[AH + x\overline{H}] = xP(H) + xP(\overline{H}) = x = P(A|H).$$

Then, in a conditional bet on $A|H$, the indicator in (13) represents the *random win* that you receive when you pay the amount $P(A|H) = x \in [0,1]$. Indeed, you receive 1 (you win), or 0 (you lose), or x (the bet is called off), according to whether AH is true, or $\bar{A}H$ is true, or $\overline{H}$ is true, respectively.

We also observe that, the value x of the random quantity $A|H$ (subjectively) depends on the assessed probability $P(A|H) = x$. When $H \subseteq A$ (i.e., $AH = H$), it holds that $P(A|H) = 1$; then, for the indicator $A|H$, when $H \subseteq A$, it holds that $A|H = AH + x\overline{H} = H + \overline{H} = 1$. Similarly, if $AH = \emptyset$, as $P(A|H) = 0$, it follows that $A|H = 0 + 0\overline{H} = 0$. For the indicator of the negation of $A|H$, as $P(\bar{A}|H) = 1 - P(A|H)$, it holds that $\bar{A}|H = 1 - A|H$. Given two conditional events $A|H$ and $B|K$, for every coherent assessment (x, y) on $\{A|H, B|K\}$, it holds that ([39, formula (15)])

$$AH + x\overline{H} \leq BK + y\bar{K} \iff \text{either } A|H \subseteq B|K, \text{ or } AH = \emptyset, \text{ or } K \subseteq B,$$

that is, between the numerical values of $A|H$ and $B|K$, under coherence it holds that

$$A|H \leq B|K \iff \text{either } A|H \subseteq B|K, \text{ or } AH = \emptyset, \text{ or } K \subseteq B. \tag{14}$$

Of course, the relation $A|H \leq B|K$ requires that $P(A|H) = x \leq y = P(B|K)$. Then,

$$P(A|H) \leq P(B|K)\ \forall \text{coherent } P \iff \text{either } A|H \subseteq B|K, \text{ or } AH = \emptyset, \text{ or } K \subseteq B, \tag{15}$$

which, when $H = K = \Omega$, reduces to

$P(A) \le P(B) \forall$coherent $P \iff A \subseteq B \iff A \le B$.

By following the approach given in [14,34,50], once a coherent assessment $\mu = \mathbb{P}(X|H)$ is specified, the conditional random quantity $X|H$ (is not looked at as the restriction to H, but) is defined as X, or μ, according to whether H is true, or $\overline{H}$ is true; that is,

$$X|H = XH + \mu\overline{H}. \tag{16}$$

As shown in (16), given any random quantity X and any event $H \neq \emptyset$, in the framework of subjective probability, in order to define $X|H$ we just need to specify the value μ of the conditional prevision $\mathbb{P}(X|H)$. Indeed, once the value μ is specified, the object $X|H$ is (subjectively) determined. We observe that (16) is consistent because

$$\mathbb{P}[XH + \mu\overline{H}] = \mathbb{P}[XH] + \mu P(\overline{H}) = \mathbb{P}[X|H]P(H) + \mu P(\overline{H}) = \mu P(H) + \mu P(\overline{H}) = \mu.$$

By (16), the random gain associated with a bet on $X|H$ can be represented as $G = s(X|H - \mu)$, that is G is the difference between what you receive, $sX|H$, and what you pay, $s\mu$. In what follows, for any given conditional random quantity $X|H$, we assume that, when H is true, the set of possible values of X is finite. In this case we say that $X|H$ is a finite conditional random quantity. Denoting by $\{x_1, \dots, x_r\}$ the set of possible values of X restricted to H and by setting $A_j = (X = x_j)$, $j = 1, \dots, r$, it holds that $\bigvee_{j=1}^r A_j = H$ and $X|H = XH + \mu\overline{H} = x_1A_1 + \dots + x_rA_r + \mu\overline{H}$.

The result below ([34, Theorem 4]) shows that if two conditional random quantities $X|H$, $Y|K$ coincide when $H \vee K$ is true, then $X|H$ and $Y|K$ also coincide when $H \vee K$ is false, and hence $X|H$ coincides with $Y|K$ in all cases.

Theorem 3. *Given any events $H \neq \emptyset$ and $K \neq \emptyset$, and any random quantities X and Y, let Π be the set of the coherent prevision assessments $\mathbb{P}[X|H] = \mu$ and $\mathbb{P}[Y|K] = \nu$.*
(i) Assume that, for every $(\mu, \nu) \in \Pi$, $X|H = Y|K$ when $H \vee K$ is true; then $\mu = \nu$ for every $(\mu, \nu) \in \Pi$.
(ii) For every $(\mu, \nu) \in \Pi$, $X|H = Y|K$ when $H \vee K$ is true if and only if $X|H = Y|K$.

Remark 2. Theorem 3 has been generalized in [35, Theorem 6] by replacing the symbol "=" by "≤" in statements (i) and (ii). In other words, if $X|H \le Y|K$ when $H \vee K$ is true, then $\mathbb{P}[X|H] \le \mathbb{P}[Y|K]$ and hence $X|H \le Y|K$ in all cases.

2.4. Trivalent logics, logical operations of conditionals and conditional random quantities

We recall some notions of conjunction among conditional events in some trivalent logics: Kleene-Lukasiewicz-Heyting conjunction ($\wedge_K$), or de Finetti conjunction ([20]); Lukasiewicz conjunction ($\wedge_L$); Bochvar internal conjunction, or Kleene weak conjunction ($\wedge_B$); Sobocinski conjunction, or quasi conjunction ($\wedge_S$). In all these definitions the result of the conjunction is still a conditional event with set of truth values {true, false, void} (see, e.g., [12,13]). We also recall the notions of conjunction among conditional events, $\wedge_{gs}$, introduced as a suitable conditional random quantity in a betting-scheme context ([34,35], see also [45,52]). We list below in an explicit way the five conjunctions and the associated disjunctions obtained by De Morgan's law ([40]):

1. $(A|H) \wedge_K (B|K) = AHBK|(AHBK \vee \overline{A}H \vee \overline{B}K) = AHBK|(HK \vee \overline{A}H \vee \overline{B}K)$,
   $(A|H) \vee_K (B|K) = (AH \vee BK)|(\bar{A}H\bar{B}K \vee AH \vee BK)$;
2. $(A|H) \wedge_L (B|K) = AHBK|(AHBK \vee \overline{A}\,H \vee \overline{B}K \vee \overline{H}\,\overline{K})$,
   $(A|H) \vee_L (B|K) = (AH \vee BK)|(\bar{A}H\bar{B}K \vee AH \vee BK \vee \overline{H}\,\overline{K})$;
3. $(A|H) \wedge_B (B|K) = AHBK|HK$,
   $(A|H) \vee_B (B|K) = (A \vee B)|HK$;
4. $(A|H) \wedge_S (B|K) = ((AH \vee \overline{H}) \wedge (BK \vee \overline{K}))|(H \vee K)$,
   $(A|H) \vee_S (B|K) = (AH \vee BK)|(H \vee K)$;
5. $(A|H) \wedge_{gs} (B|K) = (AHBK + P(A|H)\overline{H}BK + P(B|K)AH\bar{K})|(H \vee K)$,
   $(A|H) \vee_{gs} (B|K) = (AH \vee BK + P(A|H)\overline{H}\,\bar{B}K + P(B|K)\bar{A}H\bar{K})|(H \vee K)$.

The operations above are all commutative and associative. By setting $P(A|H) = x$, $P(B|K) = y$, $P((A|H) \wedge_i (B|K)) = z_i$, $i \in \{K, L, B, S\}$, and $\mathbb{P}[(A|H) \wedge_{gs} (B|K)] = z_{gs}$, based on (13) and on (16) the conjunctions $(A|H) \wedge_i (B|K)$, $i \in \{K, L, B, S, gs\}$ can be also looked at as random quantities with set of possible value illustrated in Table 1. A similar interpretation can also be given for the associated disjunctions. In Table 2 we list the lower and upper bounds for the coherent extensions $z_i = P((A|H) \wedge_i (B|K))$, $i \in \{K, L, B, S\}$ ([59]) and $z_{gs} = \mathbb{P}[(A|H) \wedge_{gs} (B|K)]$ ([34]) of the given assessment $P(A|H) = x$ and $P(B|K) = y$. Notice that, differently from conditional events which are three-valued objects, the conjunction $(A|H) \wedge_{gs} (B|K)$ (and the associated disjunction) is no longer a three-valued object, but a five-valued object with values in $[0, 1]$. In betting terms, the prevision $z_{gs} = \mathbb{P}[(A|H) \wedge_{gs} (B|K)]$ represents the amount you agree to pay, with the proviso that you will receive the random quantity $AHBK + x\overline{H}BK + yAH\bar{K}$, if $H \vee K$ is true, z_{gs} if $\overline{H}\,\bar{K}$ is true. In other words by paying z_{gs} you receive: 1, if both conditional events are true; 0, if at least one of the conditional events is false; the probability of the conditional event that is void if one conditional event is void and the other one is true; the amount z_{gs} you paid if both conditional events are void. The notion of conjunction $\wedge_{gs}$ (and disjunction $\vee_{gs}$) among conditional events has been generalized to the case of n conditional events in [35]. For some applications see, e.g., [60,62]. Developments of this approach to general compound conditionals have been given in [26]. Differently from the other notions of

Table 1
Numerical values (of the indicator) of the conjunctions $(A|H) \wedge_i (B|K)$, $i \in \{K, L, B, S, gs\}$. The triplet (x, y, z_i) denotes a coherent assessment on $\{A|H, B|K, (A|H) \wedge_i (B|K)\}$.

	$A\|H$	$B\|K$	$\wedge_K$	$\wedge_L$	$\wedge_B$	$\wedge_S$	$\wedge_{gs}$
$AHBK$	1	1	1	1	1	1	1
$AH\overline{B}K$	1	0	0	0	0	0	0
$AH\overline{K}$	1	y	z_K	z_L	z_B	1	y
$\overline{A}HBK$	0	1	0	0	0	0	0
$\overline{A}H\overline{B}K$	0	0	0	0	0	0	0
$\overline{A}H\overline{K}$	0	y	0	0	z_B	0	0
$\overline{H}BK$	x	1	z_K	z_L	z_B	1	x
$\overline{H}\overline{B}K$	x	0	0	0	z_B	0	0
$\overline{H}\overline{K}$	x	y	z_K	0	z_B	z_S	z_{gs}

Table 2
Lower and upper bounds for the coherent extensions $z_i = P((A|H) \wedge_i (B|K))$, $i \in \{K, L, B, S\}$ and $z_{gs} = \mathbb{P}[(A|H) \wedge_{gs} (B|K)]$ of the given assessment $P(A|H) = x$ and $P(B|K) = y$.

Conjunction	Lower bound	Upper bound
$(A\|H) \wedge_K (B\|K)$	$z'_K = 0$	$z''_K = \min\{x, y\}$
$(A\|H) \wedge_L (B\|K)$	$z'_L = 0$	$z''_L = \min\{x, y\}$
$(A\|H) \wedge_L (B\|K)$	$z'_B = 0$	$z''_B = 1$
$(A\|H) \wedge_S (B\|K)$	$z'_S = \max\{x + y - 1, 0\}$	$z''_S = \begin{cases} \frac{x+y-2xy}{1-xy}, & \text{if } (x,y) \neq (1,1) \\ 1, & \text{if } (x,y) = (1,1) \end{cases}$

conjunctions, $\wedge_{gs}$ preserves the classical logical and probabilistic properties valid for unconditional events (see, e.g., [39]). In particular, the Fréchet-Hoeffding bounds, i.e., the lower and upper bounds $z' = \max\{x + y - 1, 0\}, z'' = \min\{x, y\}$, obtained under logical independence in the unconditional case for the coherent extensions $z = P(AB)$ of $P(A) = x$ and $P(B) = y$, when A and B are replaced by $A|H$ and $B|K$, are only satisfied by z_{gs} (see Table 2).

2.5. *Some basic properties and import-export principle*

In this section, after recalling some basic logical and probabilistic properties satisfied by events and conditional events, we rewrite them for compound and iterated conditionals, by replacing events with conditional events. We also recall the Import-Export principle and its connection with the Lewis' triviality results.

Given two events A and B, with $A \neq \emptyset$, it is well-known the validity of following properties

1. $B|A = AB|A$;
2. $AB \leq B|A$ and hence $P(AB) \leq P(B|A)$;
3. $P(AB) = P(B|A)P(A)$ (*compound probability theorem*, see Remark 1);
4. if A and B are logically independent, by setting $P(A) = x$ and $P(B) = y$, the extension $\mu = P(B|A)$ is coherent if and only if $\mu \in [\mu', \mu'']$, where (see, e.g. [62, Theorem 6])

$$\mu' = \begin{cases} \frac{\max\{x+y-1,0\}}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases} \quad \mu'' = \begin{cases} \frac{\min\{x,y\}}{x}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases} \tag{17}$$

By replacing events A, B by conditional events $A|H, B|K$, and for the compound conditionals the symbol of probability P by the symbol of prevision $\mathbb{P}$, the properties 1, 2, 3, and 4 become:

P1. $(B|K)|(A|H) = [(A|H) \wedge (B|K)]|(A|H)$;
P2. $(A|H) \wedge (B|K) \leq (B|K)|(A|H)$ and hence $\mathbb{P}[(A|H) \wedge (B|K)] \leq \mathbb{P}[(B|K)|(A|H)]$;
P3. $\mathbb{P}[(A|H) \wedge (B|K)] = \mathbb{P}[(B|K)|(A|H)]P(A|H)$ (*compound formula for iterated conditional*);
P4. if A, B, H, K are logically independent events, denoting $P(A|H) = x$ and $P(B|H) = y$, the extension μ on $(B|K)|(A|H)$ is coherent if and only if $\mu \in [\mu', \mu'']$, where μ' and μ'' are given in formula (17).

We will show in Example 3, Example 4, and Section 7 that when some of the properties P1-P4 are not satisfied, some counterintuitive probabilistic results are obtained. These are avoided by a suitable notion of iterated conditioning which satisfies properties P1-P4. Another classical property that can be checked for the iterated conditional is the Import-Export principle. Given three events B, K, A, with $AK \neq \emptyset$, we say that the Import-Export principle ([52]) is satisfied for the iterated conditional $(B|K)|A$, if

$$(B|K)|A = B|AK. \tag{18}$$

We also recall that the validity of the Import-Export principle (together with the validity of the total probability theorem) could lead to the counter-intuitive consequences related to Lewis' triviality results ([51], see also [60]). Indeed, assuming that the total probability theorem holds for iterated conditionals, that is

$$P(C|A) = P((C|A) \wedge C) + P((C|A) \wedge \bar{C}) = P((C|A)|C)P(C) + P((C|A)|\bar{C})P(\bar{C}),$$

if the Import-Export principle is valid, by applying (18) and by observing that $P(C|AC) = 1$ and $P(C|A\bar{C}) = 0$, it follows that

$$P(C|A) = P(C|AC)P(C) + P(C|A\bar{C})P(\bar{C}) = P(C), \tag{19}$$

which of course is not valid in general for conditional events. Then, the non validity of the Import-Export principle may avoid Lewis' triviality results.

In Sections 3, 4, 5, we will check the validity of the previous properties for notions of compound and iterated conditionals introduced in different trivalent logics as suitable conditional events (in these cases, in properties P2 and P4, the symbol of prevision $\mathbb{P}$ is replaced by the symbol of probability P, because the involved objects are conditional events). Then, in Section 6 we will check the basic properties for the iterated conditionals, defined as conditional random quantities, built using the structure $\square|\bigcirc = \square \wedge \bigcirc + \mathbb{P}(\square|\bigcirc)\bar{\bigcirc}$ and the different notions of conjunction in trivalent logic recalled in Section 2.4.

3. The iterated conditional in the trivalent logic of Cooper-Calabrese

In this section, in the framework of a trivalent logic, we study the validity of the Import-Export principle and of the properties P1-P4 for the notion of iterated conditional, here denoted by $(B|K)|_C(A|H)$, studied by Cooper ([16]) and by Calabrese ([5], see also [6,7]). We also give two results on the set of all coherent assessments for suitable families of conditional events which include the iterated conditional $(B|K)|_C(A|H)$.

We recall that the notions of conjunction and disjunction of conditionals used by Cooper and Calabrese coincide with $\wedge_S$ and $\vee_S$, respectively.[4]

Definition 2. Given any pair of conditional events $A|H$ and $B|K$, the iterated conditional $(B|K)|_C(A|H)$ is defined as the following conditional event

$$(B|K)|_C(A|H) = B|(K \wedge (\overline{H} \vee A)). \tag{20}$$

We observe that in (20) the conditioning event is the conjunction of the conditioning event K of the consequent $B|K$ and the material conditional $\overline{H} \vee A$ associated with the antecedent $A|H$.

Remark 3 *(Import-Export principle for $|_C$).* By applying Definition 2 with $H = \Omega$, it holds that

$$(B|K)|_C A = ABK|AK = B|AK, \tag{21}$$

which shows that the Import-Export principle is satisfied by $|_C$.

3.1. Property P1

We observe that

$$\begin{aligned} ((A|H) \wedge_S (B|K))|_C(A|H) &= ((AHBK \vee AH\bar{K} \vee \overline{H}BK)|(H \vee K))|_C(A|H) = \\ &= (AHBK \vee AH\bar{K} \vee \overline{H}BK)|(AK \vee \overline{H}K \vee AH\bar{K}). \end{aligned} \tag{22}$$

From (20) and (22) it follows that $(B|K)|_C(A|H) \neq ((A|H) \wedge_S (B|K))|_C(A|H)$. Indeed, as illustrated by Table 3, when the constituent $AH\bar{K}$ is true, it holds that $(B|K)|_C(A|H)$ is void, while $((A|H) \wedge_S (B|K))|_C(A|H)$ is true. Then, property P1 is not satisfied by the pair $(\wedge_S, |_C)$.

3.2. Property P2

From Table 3 we also obtain that property P2 is not satisfied by $(\wedge_S, |_C)$. Indeed, under logical independence, when $AH\bar{K}$ is true, it holds that $(A|H) \wedge_S (B|K)$ is true, while $(B|K)|_C(A|H)$ is void. Then, since $AH \neq \emptyset$, $K \nsubseteq B$, and $((A|H) \wedge_S (B|K)) \nsubseteq (B|K)|_C(A|H)$, from (14) it follows that $((A|H) \wedge_S (B|K)) \nleq (B|K)|_C(A|H)$.

[4] Note that also Cantwell ([8]) defines the iterated conditional as Cooper and Calabrese, but, unlike them, in his trivalent logic conjunction and disjunction are defined by $\wedge_K$ and $\vee_K$, respectively.

Table 3
Truth values of $(A|H) \wedge_S (B|K)$, $(B|K)|_C(A|H)$, and $((A|H) \wedge_S (B|K))|_C(A|H)$.

C_h	$(A|H) \wedge_S (B|K)$	$(B|K)|_C(A|H)$	$((A|H) \wedge_S (B|K))|_C(A|H)$
$AHBK \vee \overline{H}BK$	True	True	True
$AH\bar{B}K \vee \overline{H}\,\bar{B}K$	False	False	False
$AH\bar{K}$	True	Void	True
$\bar{A}H$	False	Void	Void
$\overline{H}\,\bar{K}$	Void	Void	Void

Table 4
Constituents and points Q_h's associated with $\mathcal{F} = \{A|H, (B|K)|_C(A|H), (A|H) \wedge_S (B|K)\}$ and $\mathcal{P} = (x, y, z)$.

	C_h	$A|H$	$(B|K)|_C(A|H)$	$(A|H) \wedge_S (B|K)$	Q_h
C_1	$AHBK$	1	1	1	Q_1
C_2	$\bar{A}H$	0	y	0	Q_2
C_3	$\overline{H}BK$	x	1	1	Q_3
C_4	$AH\bar{B}K$	1	0	0	Q_4
C_5	$\overline{H}\,\bar{B}K$	x	0	0	Q_5
C_6	$AH\bar{K}$	1	y	1	Q_6
C_0	$\overline{H}\,\bar{K}$	x	y	z	Q_0

3.3. Property P3

Now we focus our attention on the following result regarding the coherence of a probability assessment on $\{A|H, (B|K)|_C(A|H), (A|H) \wedge_S (B|K)\}$ (Theorem 4), then we use this result in order to check the validity of property P3 for the pair $(\wedge_S, |_C)$.

Theorem 4. *Let A, B, H, K be any logically independent events. A probability assessment $\mathcal{P} = (x, y, z)$ on the family of conditional events $\mathcal{F} = \{A|H, (B|K)|_C(A|H), (A|H) \wedge_S (B|K)\}$ is coherent if and only if $(x, y) \in [0, 1]^2$ and $z \in [z', z'']$, where $z' = xy$ and $z'' = max(x, y)$.*

Proof. The constituents C_h's and the points Q_h's associated with the assessment $\mathcal{P} = (x, y, z)$ on $\mathcal{F}$ are (see also Table 4)

$$C_1 = AHBK, C_2 = \bar{A}HBK \vee \bar{A}H\bar{B}K \vee \bar{A}H\bar{K} = \bar{A}H, C_3 = \overline{H}BK, C_4 = AH\bar{B}K,$$
$$C_5 = \overline{H}\,\bar{B}K, C_6 = AH\bar{K}, C_0 = \overline{H}\,\bar{K},$$

and

$$Q_1 = (1, 1, 1),\ Q_2 = (0, y, 0),\ Q_3 = (x, 1, 1), Q_4 = (1, 0, 0),$$
$$Q_5 = (x, 0, 0),\ Q_6 = (1, y, 1), \mathcal{P} = Q_0 = (x, y, z).$$

The system (Σ) in (5) associated with the pair $(\mathcal{F}, \mathcal{P})$ becomes

$$\begin{cases} \lambda_1 + x\lambda_3 + \lambda_4 + x\lambda_5 + \lambda_6 = x, \\ \lambda_1 + y\lambda_2 + \lambda_3 + y\lambda_6 = y, \\ \lambda_1 + \lambda_3 + \lambda_6 = z, \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = 1, \\ \lambda_i \geq 0,\ i = 1, \dots, 6. \end{cases} \quad (23)$$

Lower bound We first prove that the assessment (x, y, xy) is coherent for every $(x, y) \in [0, 1]^2$. Then, in order to prove that $z' = xy$ is the lower bound for $z = P((A|H) \wedge_S (B|K))$, we verify that the assessment (x, y, z) is not coherent when $z < z' = xy$.

We observe that $\mathcal{P} = (x, y, xy) = xyQ_1 + (1 - x)Q_2 + x(1 - y)Q_4$, so a solution of (23) is given by $\Lambda = (xy, 1 - x, 0, x(1 - y), 0, 0)$. Then, by considering the function ϕ as defined in (6), it holds that

$$\phi_1(\Lambda) = \textstyle\sum_{h:C_h \subseteq H} \lambda_h = \lambda_1 + \lambda_2 + \lambda_4 + \lambda_6 = xy + (1 - x) + x(1 - y) = 1 > 0,$$
$$\phi_2(\Lambda) = \textstyle\sum_{h:C_h \subseteq (A \vee \overline{H}) \vee K} \lambda_h = \lambda_1 + \lambda_3 + \lambda_4 + \lambda_5 = xy + x(1 - y) = x,$$
$$\phi_3(\Lambda) = \textstyle\sum_{h:C_h \subseteq H \vee K} \lambda_h = \lambda_1 + \cdots + \lambda_6 = 1 > 0.$$

Let $\mathcal{S}' = \{(xy, 1 - x, 0, x(1 - y), 0, 0)\}$ denote a subset of the set $\mathcal{S}$ of all solutions of (23). We have that $M_1' = 1$, $M_2' = x$, $M_3' = 1$ (as defined in (7)). We distinguish two cases: (i) $x > 0$, (ii) $x = 0$. In the case (i) we get $M_1' > 0$, $M_2' > 0$, $M_3' > 0$ and then $I_0' = \emptyset$. By Theorem 1, the assessment (x, y, xy) is coherent $\forall (x, y) \in [0, 1]^2$. In the case (ii) we have that $M_1' > 0$, $M_2' = 0$, $M_3' > 0$, hence $I_0' = 2$. We observe that the sub-assessment $\mathcal{P}_0' = y$ on $\mathcal{F}_0' = \{(B|K)|_C(A|H)\}$ is coherent for every $y \in [0, 1]$. Then, by Theorem 1, the assessment (x, y, xy) on $\mathcal{F}$ is coherent $\forall (x, y) \in [0, 1]^2$.

In order to verify that $z' = xy$ is the lower bound for z, we observe that the points Q_1, Q_2, Q_4 belong to the plane $\pi : yX + Y - Z = y$, where X, Y, Z are the axes coordinates.

Now, by considering the function $f(X,Y,Z) = yX + Y - Z$, we observe that for each constant k the equation $f(X,Y,Z) = k$ represents a plane which is parallel to π and coincides with π when $k = y$. We also observe that $f(Q_1) = f(Q_2) = f(Q_4) = y$, $f(Q_3) = f(Q_5) = xy \leq y$ and $f(Q_6) = f(1, y, 1) = y + y - 1 = 2y - 1 \leq y$.

Then, for every $\mathcal{P} = \sum_{h=1}^{6} \lambda_h Q_h$, with $\sum_{h=1}^{6} \lambda_h = 1$ and $\lambda_h \geq 0$, that is $\mathcal{P} \in \mathcal{I}$, it holds that $f(\mathcal{P}) = f(\sum_{h=1}^{6} \lambda_h Q_h) = \sum_{h=1}^{6} \lambda_h f(Q_h) \leq y$. On the other side, given $a > 0$, by considering $\mathcal{P} = (x, y, xy - a)$ it holds that $f(\mathcal{P}) = y + a > y$ and hence $\mathcal{P} = (x, y, xy - a) \notin \mathcal{I}$. Therefore, for any given $a > 0$ the assessment $(x, y, xy - a)$ is not coherent because $(x, y, xy - a) \notin \mathcal{I}$. Then the lower bound of $z = P((A|H) \wedge_S (B|K))$ is $z' = xy$.

Upper bound. We verify that the assessment $(x, y, \max(x, y))$ on $\mathcal{F}$ is coherent for every $(x, y) \in [0,1]^2$. Moreover, we show that $z'' = \max(x, y)$ is the upper bound for $z = P((A|H) \wedge_S (B|K))$ by showing that any assessment (x, y, z) on $\mathcal{F}$ with $(x, y) \in [0,1]^2$ and $z > \max\{x, y\}$ is not coherent. We distinguish two cases: (i) $x \geq y$, (ii) $x < y$.
(i) We have that $\max(x, y) = x$ and hence

$$\mathcal{P} = (x, y, x) = (1 - x)Q_2 + xQ_6.$$

Then, the vector $\Lambda = (0, 1 - x, 0, 0, 0, x)$ is a solution of (23). Moreover, it holds that

$$\phi_1(\Lambda) = \sum_{h: C_h \subseteq H} \lambda_h = \lambda_1 + \lambda_2 + \lambda_4 + \lambda_6 = 1 - x + x = 1 > 0,$$
$$\phi_2(\Lambda) = \sum_{h: C_h \subseteq (K \wedge (A \vee \bar{H}))} \lambda_h = \lambda_1 + \lambda_3 + \lambda_4 + \lambda_5 = 0,$$
$$\phi_3(\Lambda) = \sum_{h: C_h \subseteq (H \vee K)} \lambda_h = \lambda_1 + \cdots + \lambda_6 = 1 > 0.$$

Let $\mathcal{S}' = \{(0, 1 - x, 0, 0, 0, x)\}$ denote a subset of the set $\mathcal{S}$ of all solutions of (23). We have that $M_1' = 1$, $M_2' = 0$, $M_3' = 1$. It follows that $\mathcal{I}_0' = 2$. As the sub-assessment $\mathcal{P}_0' = y$ on $\mathcal{F}_0' = \{(B|K)|_C(A|H)\}$ is coherent $\forall y \in [0,1]$, by Theorem 1, it follows that the assessment $(x, y, \max(x, y))$ is coherent.

In order to verify that $z'' = \max(x, y) = x$ is the upper bound for z, we observe that if $\max\{x, y\} = 1$, then $(1, y, z)$ with $z > 1$ is incoherent. Let us assume that $y \leq x < 1$. We observe that the points Q_2, Q_3, Q_6 belong to the plane $\pi : X + \frac{1-x}{1-y}Y - Z = \frac{y(1-x)}{1-y}$, where X, Y, Z are the axes coordinates.

Now, by considering the function $f(X,Y,Z) = X + \frac{1-x}{1-y}Y - Z - \frac{y(1-x)}{1-y}$, we observe that $f(Q_1) = 1 - x > 0$, $f(Q_2) = f(Q_3) = f(Q_6) = 0$, $f(Q_4) = 1 - y\frac{1-x}{1-y} \geq 0$, $f(Q_5) = \frac{x-y}{1-y} \geq 0$. Then, for every $\mathcal{P} = \sum_{h=1}^{6} \lambda_h Q_h$, with $\sum_{h=1}^{6} \lambda_h = 1$ and $\lambda_h \geq 0$, that is $\mathcal{P} \in \mathcal{I}$, it holds that $f(\mathcal{P}) = f(\sum_{h=1}^{6} \lambda_h Q_h) = \sum_{h=1}^{6} \lambda_h f(Q_h) \geq 0$. On the other side, given $z > x$, by considering $\mathcal{P} = (x, y, z)$ it holds that $f(\mathcal{P}) = x - z < 0$ and hence $\mathcal{P} = (x, y, z) \notin \mathcal{I}$. Therefore, for any given $z > x$ the assessment (x, y, z) is not coherent because $(x, y, z) \notin \mathcal{I}$. Then the upper bound on z is $z'' = z = \max\{x, y\}$.
(ii) In this case $\max(x, y) = y$. We prove that the assessment $(x, y, \max(x, y))$ is coherent. We observe that

$$(x, y, y) = yQ_3 + (1 - y)Q_5.$$

Then, the vector $\Lambda = (0, 0, y, 0, 1 - y, 0)$ is a solution of (23). We have

$$\phi_1(\Lambda) = \sum_{h: C_h \subseteq H} \lambda_h = 0; \;\; \phi_2(\Lambda) = \sum_{h: C_h \subseteq (K \wedge (A \vee \bar{H}))} \lambda_h = 1 > 0; \;\; \phi_3(\Lambda) = \sum_{h: C_h \subseteq (H \vee K)} \lambda_h = 1 > 0.$$

Let $\mathcal{S}' = \{(0, 0, y, 0, 1 - y, 0)\}$ denote a subset of the set $\mathcal{S}$ of all solutions of (23). We have that $M_1' = 0$, $M_2' = 1$, $M_3' = 1$. It follows that $\mathcal{I}_0' = 1$. The sub-assessment x on $\{A|H\}$ is coherent $\forall x \in [0,1]$. Then, by Theorem 1, the assessment $(x, y, \max(x, y))$ on $\mathcal{F}$ is coherent $\forall (x, y) \in [0,1]^2$.

In order to verify that $z'' = \max(x, y) = y$ is the upper bound for z, we observe that if $\max\{x, y\} = 1$, then $(x, 1, z)$ with $z > 1$ is incoherent. Let us assume that $x \leq y < 1$. We observe that the points Q_3, Q_5, Q_6 belong to the plane $\pi : \frac{1-y}{1-x}X - \frac{x(1-y)}{1-x} + Y - Z = 0$, where X, Y, Z are the axes coordinates.

Now, by considering the function $f(X,Y,Z) = \frac{1-y}{1-x}X - \frac{x(1-y)}{1-x} + Y - Z$, we observe that $f(Q_1) = f(Q_4) = 1 - y > 0$, $f(Q_2) = \frac{y-x}{1-x} > 0$ $f(Q_3) = f(Q_5) = f(Q_6) = 0$. Then, for every $\mathcal{P} = \sum_{h=1}^{6} \lambda_h Q_h$, with $\sum_{h=1}^{6} \lambda_h = 1$ and $\lambda_h \geq 0$, that is $\mathcal{P} \in \mathcal{I}$, it holds that $f(\mathcal{P}) = f(\sum_{h=1}^{6} \lambda_h Q_h) = \sum_{h=1}^{6} \lambda_h f(Q_h) \geq 0$. On the other side, given $z > y$, by considering $\mathcal{P} = (x, y, z)$ it holds that $f(\mathcal{P}) = y - z < 0$ and hence $\mathcal{P} = (x, y, z) \notin \mathcal{I}$. Therefore, for any given $z > y$ the assessment (x, y, z) is not coherent because $(x, y, z) \notin \mathcal{I}$. Then the upper bound on z is $z'' = y = \max\{x, y\}$.

Finally, for each given $(x, y) \in [0,1]$, as (x, y, z') and (x, y, z'') are coherent, by the Fundamental theorem of probability, any assessment (x, y, z) with $z \in [z', z'']$ is coherent too. Then, the assessment (x, y, z) on $\mathcal{F}$ is coherent for every $(x, y) \in [0,1]^2$ and $z \in [z', z'']$. □

Remark 4 *(Property P3)*. From Theorem 4 it follows that any probability assessment $(x, y, z) \in [0,1]^3$ on $\mathcal{F} = \{A|H, (B|K)|_C(A|H), (A|H) \wedge_S (B|K)\}$, for which $z = xy$, is coherent. In other words, any assessment which satisfies property P3 is coherent. For instance, the assessment $(1, 0, 0)$ on $\mathcal{F}$ is coherent. However, when $xy < \max\{x, y\}$, there are coherent probability assessments which do not satisfy property P3. Indeed, any assessment (x, y, z), with $xy < z \le \max\{x, y\}$, is coherent. Then, property P3 is not satisfied in general by the pair $(\wedge_S, |_C)$. For instance the assessment $(1,0,1)$ on $\mathcal{F}$ is coherent. However, the assessment $(1,0,1)$, on $\{A, B|A, AB\}$ is not coherent because does not satisfy the compound probability theorem.

We illustrate in the following example some counterintuitive (probabilistic) aspects of the invalidity of properties P1, P2, and P3.

Example 3. Let two independent random quantities X and Y be given. We assume that $X \in \{1,\dots,6\}$, with $P(X=h)=\frac{1}{6}$, $h=1,\dots,6$, and $Y \in \{0,1\}$, with $P(Y=0)=P(Y=1)=\frac{1}{2}$. We set $A=(X\ge 3)$, $H=(X\ge 2)$, $B=(Y=1)$, and $K=(Y=0)$. From (20), as $B=\bar{K}$, it follows that

$$(B|K)|_C(A|H)=\bar{K}|(K\wedge(\overline{H}\vee A))=\emptyset|(K\wedge(\overline{H}\vee A)). \tag{24}$$

Moreover, since

$$(A|H)\wedge_S(B|K)=(\overline{H}\vee AH)\wedge(\bar{K}\vee BK)|(H\vee K)=(AH\bar{K})|(H\vee K), \tag{25}$$

and $(H\vee K)\wedge(\overline{H}\vee A)=AH\vee\overline{H}K$, it holds that

$$\begin{aligned}&((A|H)\wedge_S(B|K))|_C(A|H)=(AH\bar{K}|(H\vee K))|_C(A|H)=AH\bar{K}|((H\vee K)\wedge(\overline{H}\vee A))=\\&=AH\bar{K}|(AH\vee\overline{H}K).\end{aligned} \tag{26}$$

From (24) and (26) it follows that $((A|H)\wedge_S(B|K))|_C(A|H)\neq(B|K)|_C(A|H)$, and hence property P1 is not satisfied. Concerning the probabilistic aspects we observe that $P((B|K)|_C(A|H))=0$, while

$$P(((A|H)\wedge_S(B|K))|_C(A|H))=P(AH\bar{K}|(AH\vee\overline{H}K))=\frac{P(A\bar{K})}{P(AH)+P(\overline{H}K)}=\frac{\frac{4}{6}\frac{1}{2}}{\frac{4}{6}+\frac{1}{6}\frac{1}{2}}=\frac{4}{9}\neq 0.$$

From (24) and (25) it holds that $(A|H)\wedge_S(B|K)\nleq(B|K)|_C(A|H)$ and hence property P2 is not satisfied. Moreover,

$$P((A|H)\wedge_S(B|K))=P(AH\bar{K}|(H\vee K))=\frac{P(A\bar{K})}{P(H\vee K)}=\frac{\frac{4}{6}\cdot\frac{1}{2}}{\frac{5}{6}+\frac{1}{2}-\frac{5}{6}\cdot\frac{1}{2}}=\frac{4}{11}>0=P((B|K)|_C(A|H)).$$

We also obtain (the counterintuitive result) that the probability of the conjunction is greater than the probability of one of the two conjuncts, indeed it holds that $P((A|H)\wedge_S(B|K))=\frac{4}{11}>0=P(B|K)$. Therefore,

$$0=P(A|H)P((B|K)|_C(A|H))\neq P((A|H)\wedge_S(B|K))=\frac{4}{11}$$

and hence property P3 is not satisfied.

3.4. Property P4

We check the validity of property P4 for the iterated conditioning $|_C$ by studying the set of all coherent probability assessments on the family $\{A|H, B|K, (B|K)|_C(A|H)\}$ (Theorem 5).

Theorem 5. *Let A, B, H, K be any logically independent events. The probability assessment $\mathcal{P}=(x,y,z)$ on the family of conditional events $\mathcal{F}=\{A|H, B|K, (B|K)|_C(A|H)\}$ is coherent for every $(x,y,z)\in[0,1]^3$.*

Proof. We recall that $(B|K)|_C(A|H)=B|(K\wedge(\overline{H}\vee A))$. Then, the constituents C_h's and the points Q_h's associated with the assessment $\mathcal{P}=(x,y,z)$ on $\mathcal{F}=\{A|H, B|K, B|(K\wedge(\overline{H}\vee A))\}$ are (see also Table 5)

$$\begin{aligned}&C_1=AHBK,\ C_2=AH\bar{B}K,\ C_3=AH\bar{K},\ C_4=\bar{A}HBK,\ C_5=\bar{A}H\bar{B}K,\\&C_6=\bar{A}H\bar{K},\ C_7=\overline{H}BK,\ C_8=\overline{H}\,\bar{B}K,\ C_0=\overline{H}\,\bar{K},\end{aligned}$$

and

$$\begin{aligned}&Q_1=(1,1,1),Q_2=(1,0,0),Q_3=(1,y,z),Q_4=(0,1,z),Q_5=(0,0,z),\\&Q_6=(0,y,z),Q_7=(x,1,1),Q_8=(x,0,0),\mathcal{P}=Q_0=(x,y,z).\end{aligned}$$

We denote by $\mathcal{I}$ the convex hull of points $Q_1,\dots,Q_8$ (see Fig. 2). The system (Σ) in (5) associated with the pair $(\mathcal{F},\mathcal{P})$ becomes

$$\begin{cases}\lambda_1+\lambda_2+\lambda_3+x\lambda_7+x\lambda_8=x,\\ \lambda_1+y\lambda_3+\lambda_4+y\lambda_6+\lambda_7=y,\\ \lambda_1+z\lambda_3+z\lambda_4+z\lambda_5+z\lambda_6+\lambda_7=z,\\ \lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5+\lambda_6+\lambda_7+\lambda_8=1,\\ \lambda_i\ge 0,\ i=1,\dots,8.\end{cases} \tag{27}$$

Table 5
Constituents and points Q_h's associated with $\mathcal{F} = \{A|H, B|K, (B|K)|_C(A|H)\}$ and $\mathcal{P} = (x, y, z)$.

	C_h	$A\|H$	$B\|K$	$(B\|K)\|_C(A\|H)$	Q_h
C_1	$AHBK$	1	1	1	Q_1
C_2	$H\bar{B}K$	1	0	0	Q_2
C_3	$AH\bar{K}$	1	y	z	Q_3
C_4	$\bar{A}HBK$	0	1	z	Q_4
C_5	$\bar{A}H\bar{B}K$	0	0	z	Q_5
C_6	$\bar{A}H\bar{K}$	0	y	z	Q_6
C_7	$\bar{H}BK$	x	1	1	Q_7
C_8	$\bar{H}\bar{B}K$	x	0	0	Q_8
C_0	$\bar{H}\bar{K}$	x	y	z	Q_0

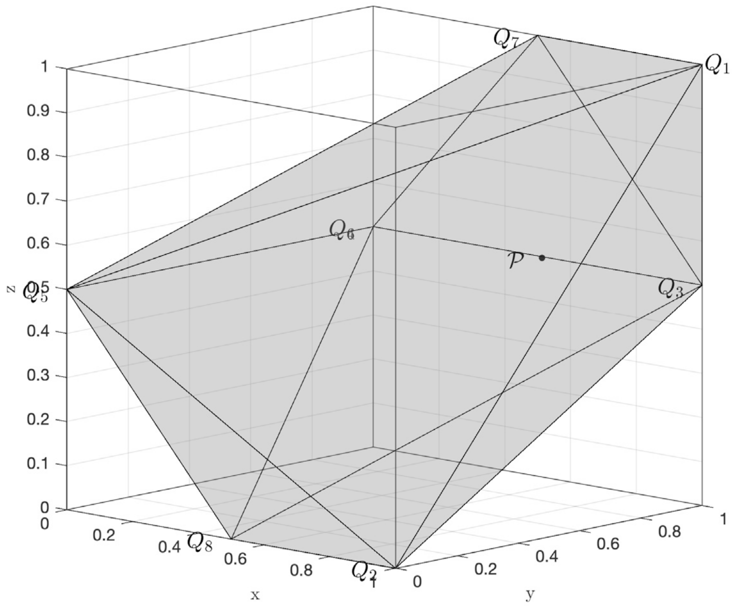

Fig. 2. Convex hull of the points Q_i for $i = 1, \ldots, 8$ associated with the pair $(\mathcal{F}, \mathcal{P})$, where $\mathcal{P} = (x, y, z)$ and $\mathcal{F} = \{A|H, B|K, (B|K)|_C(A|H)\}$. In the figure the numerical values are: $x = 0.5$, $y = 1$, $z = 0.5$.

We observe that $\mathcal{P}$ belongs to the segment with end points Q_3, Q_6; indeed $(x, y, z) = xQ_3 + (1 - x)Q_6 = x(1, y, z) + (1 - x)(0, y, z)$. The vector $\Lambda = (0, 0, x, 0, 0, 1 - x, 0, 0)$ is a solution of (27), with

$$\phi_1(\Lambda) = \sum_{h:C_h \subseteq H} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = 1 > 0,$$
$$\phi_2(\Lambda) = \sum_{h:C_h \subseteq K} \lambda_h = \lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 + \lambda_7 + \lambda_8 = 0,$$
$$\phi_3(\Lambda) = \sum_{h:C_h \subseteq ((A \vee \overline{H}) \wedge K)} \lambda_h = \lambda_1 + \lambda_2 + \lambda_7 + \lambda_8 = 0.$$

We set $\mathcal{S}' = \{\Lambda\}$ and we get $\mathcal{I}'_0 = \{2, 3\}$. So we obtain $\mathcal{F}'_0 = \{B|K, B|(K \wedge (\overline{H} \vee A))\}$ and $\mathcal{P}'_0 = (y, z)$. By recalling Example 1 the sub-assessment (y, z) on $\{B|K, B|(K \wedge (\overline{H} \vee A))\}$ is coherent for every $(y, z) \in [0, 1]^2$. Thus, by Theorem 2, the assessment (x, y, z) on $\mathcal{F}$ is coherent $\forall (x, y, z) \in [0, 1]^3$. $\square$

Remark 5 *(Property P4).* We observe that the probability propagation rule valid for unconditional events (Property P4) is no longer valid for Cooper-Calabrese's iterated conditional. Indeed, from Theorem 5, any probability assessment (x, y, z) on $\mathcal{F} = \{A|H, B|K, (B|K)|_C(A|H)\}$, with $(x, y, z) \in [0, 1]^3$ is coherent. For instance, the assessment $(1, 1, 0)$ on $\mathcal{F}$ is coherent, while it is not coherent on $\{A, B, B|A\}$.

We briefly sum up here the results obtained in this section (see also Table 11) for the iterated conditioning $|_C$. We verified that $|_C$ does not satisfy any of the desirable properties P1-P4 we considered in Section 2.5. More precisely, we verified that $(B|K)|_C(A|H) \neq ((A|H) \wedge_S (B|K))|_C(A|H)$. Thus, P1 does not hold. Furthermore, $((A|H) \wedge_S (B|K)) \nleq (B|K)|_C(A|H)$. So P2 does not hold either. Still further, $P(A|H)P((B|K)|_C(A|H)) \neq P((A|H) \wedge_S (B|K))$ for some probability P. Thus, P3 does not hold. Finally, we proved

Table 6
Truth table of $(A|H)\wedge_K(B|K)$, $(B|K)|_{dF}(A|H)$, and $((A|H)\wedge_K(B|K))|_{dF}(A|H)$.

C_h	$(A|H)\wedge_K(B|K)$	$(B|K)|_{dF}(A|H)$	$((A|H)\wedge_K(B|K))|_{dF}(A|H)$
$AHBK$	True	True	True
$AH\bar{B}K$	False	False	False
$AH\bar{K}\vee\bar{H}BK\vee\bar{H}\bar{K}$	Void	Void	Void
$\bar{A}H\vee\bar{H}\bar{B}K$	False	Void	Void

that every assessment $(x,y,z)\in[0,1]^3$ on $\{A|H,B|K,(B|K)|_C(A|H)\}$ is coherent. From this it follows that P4 does not hold. We also observed that Import-Export Principle is satisfied.

4. The iterated conditional in the trivalent logic of de Finetti

In this section we analyze the notion of iterated conditional introduced by de Finetti in [20]. After recalling that the notion of conjunction and disjunction of conditionals introduced by de Finetti in [20] coincide with $\wedge_K$ and $\vee_K$ (see Section 2.4), we check the validity of the Import-Export principle and of the properties P1-P4 for this iterated conditional in the corresponding trivalent logic.

Definition 3. Given any pair of conditional events $A|H$ and $B|K$, de Finetti iterated conditional, denoted by $(B|K)|_{dF}(A|H)$, is defined as

$$(B|K)|_{dF}(A|H)=B|(AHK). \tag{28}$$

Remark 6 *(Import-Export principle for $|_{dF}$).* By applying Definition 3 with $H=\Omega$, it holds that

$$(B|K)|_{dF}A=B|AK, \tag{29}$$

which shows that the Import-Export principle [52] is satisfied by $|_{dF}$. Then, from (21), it follows that

$$(B|K)|_C A=(B|K)|_{dF}A=B|AK.$$

4.1. Property P1

To check property P1 we observe that from (28) it holds that

$$\begin{aligned}&((A|H)\wedge_K(B|K))|_{dF}(A|H)=(AHBK|(HK\vee\bar{A}H\vee\bar{B}K))|_{dF}(A|H)=\\&AHBK|(AHK\vee AH\bar{B}K)=AHBK|AHK=(B|K)|_{dF}(A|H).\end{aligned} \tag{30}$$

Then, property P1 is satisfied by the pair $(\wedge_K,|_{dF})$ (see also Table 6).

4.2. Property P2

From Table 6 we also observe that relation P2 is satisfied by $(\wedge_K,|_{dF})$. Indeed, according to (1), if $(A|H)\wedge_K(B|K)$ is true, then $(B|K)|_{dF}(A|H)$ is true; if $(B|K)|_{dF}(A|H)$ is false, then $(A|H)\wedge_K(B|K)$ is false. Thus, since $(A|H)\wedge_K(B|K)\subseteq(B|K)|_{dF}(A|H)$, from (14) it follows that $(A|H)\wedge_K(B|K)\le(B|K)|_{dF}(A|H)$.

4.3. Property P3

We consider now the following result regarding the coherence of a probability assessment on $\{A|H,(B|K)|_{dF}(A|H),(A|H)\wedge_K(B|K)\}$ (Theorem 6) in order to check the validity of property P3 for the pair $(\wedge_K,|_{dF})$.

Theorem 6. *Let A, B, H, K be any logically independent events. A probability assessment $\mathcal{P}=(x,y,z)$ on the family of conditional events $\mathcal{F}=\{A|H,(B|K)|_{dF}(A|H),(A|H)\wedge_K(B|K)\}$ is coherent if and only if $(x,y)\in[0,1]^2$ and $z\in[z',z'']$, where $z'=0$ and $z''=xy$.*

Proof. See Appendix A.1. ☐

Remark 7 *(Property P3).* From Theorem 6 any probability assessment (x,y,z) on $\mathcal{F}=\{A|H,(B|K)|_{dF}(A|H),(A|H)\wedge_K(B|K)\}$, with $(x,y)\in[0,1]^2$ and $0\le z\le xy$, is coherent. Then, any assessment which satisfies property P3 is coherent. Moreover, as $z=xy$ is not the unique coherent extension of the conjunction $(A|H)\wedge_K(B|K)$, the quantity $P((A|H)\wedge_K(B|K))$ could not coincide with the product $P((B|K)|_{dF}(A|H))P(A|H)$. For example, if we choose the probability assessment $\mathcal{P}=(1,1,0)$, we observe that $\mathcal{P}$ is coherent on $\mathcal{F}$ because $0\in[z',z'']=[0,1]$. However, we observe that $\mathcal{P}$ on $\{A,B|A,AB\}$ is not coherent because $P(AB)=0\neq P(B|A)P(A)$.

Then, property P3 is not satisfied by the pair $(\wedge_K,|_{dF})$.

Table 7
Truth table of $(A|H)\wedge_K(B|K)$, $(B|K)|_F(A|H)$, and $((A|H)\wedge_K(B|K))|_F(A|H)$.

C_h	$(A|H)\wedge_K(B|K)$	$(B|K)|_F(A|H)$	$((A|H)\wedge_K(B|K))|_F(A|H)$
$AHBK$	True	True	True
$AH\bar{B}K \vee \overline{H}\,\bar{B}K$	False	False	False
$AH\bar{K} \vee \overline{H}BK \vee \overline{H}\,\bar{K}$	Void	Void	Void
$\bar{A}H$	False	Void	Void

4.4. Property P4

To check the validity of property P4 for the iterated conditioning $|_{dF}$ we study the set of all coherent probability assessments on the family $\{A|H, B|K, (B|K)|_{dF}(A|H)\}$ (Theorem 7).

Theorem 7. *Let A, B, H, K be any logically independent events. The probability assessment $\mathcal{P}=(x,y,z)$ on the family of conditional events $\mathcal{F}=\{A|H, B|K, (B|K)|_{dF}(A|H)\}$ is coherent for every $(x,y,z)\in[0,1]^3$.*

Proof. See Appendix A.1. □

Remark 8 *(Property P4).* We observe that the probability propagation rule valid for unconditional events (property P4) is no longer valid for de Finetti's iterated conditional. Indeed, from Theorem 7, any probability assessment (x,y,z) on $\mathcal{F}=\{A|H, B|K, (B|K)|_{dF}(A|H)\}$, with $(x,y,z)\in[0,1]^3$ is coherent. For instance, the assessment $(1,1,0)$ is coherent on $\mathcal{F}$ but it is not coherent on $\{A, B, B|A\}$.

We briefly sum up here the results obtained in this section (see also Table 11) for the iterated conditioning $|_{dF}$. We verified that $|_{dF}$ satisfies both properties P1 and P2. Indeed, $(B|K)|_{dF}(A|H)=((A|H)\wedge_K(B|K))|_{dF}(A|H)$ and $((A|H)\wedge_K(B|K))\le(B|K)|_{dF}(A|H)$. However, the iterated conditioning $|_{dF}$ does not satisfy property P3, because $P(A|H)P((B|K)|_{dF}(A|H))\neq P((A|H)\wedge_K(B|K))$ for some probability P. Moreover, we showed that every assessment $(x,y,z)\in[0,1]^3$ on $\{A|H, B|K, (B|K)|_{dF}(A|H)\}$ is coherent and from this it follows that property P4 does not hold. We also observed that Import-Export Principle is satisfied by $|_{dF}$.

5. The iterated conditional in the trivalent logic of Farrell

In this section we consider another definition of iterated conditional as a suitable conditional event which was proposed by Farrell in [25, p. 385] (see also [24]). In his trivalent logic, the author uses $\wedge_K$ and $\vee_K$ as conjunction and disjunction of conditional events (see Section 2.4), respectively. We first recall the definition of the iterated conditional, then we check the validity of the Import-Export principle and of the properties P1-P4.

Definition 4. Given any pair of conditional events $A|H$ and $B|K$, Farrell iterated conditional, here denoted by $(B|K)|_F(A|H)$, is defined as the conditional event

$$(B|K)|_F(A|H) = AHBK|(AHBK \vee AH\bar{B}K \vee \overline{H}\,\bar{B}K). \tag{31}$$

Remark 9 *(Import-Export principle for $|_F$).* By applying (31) with $H=\Omega$, it holds that

$$(B|K)|_F A = ABK|(ABK \vee A\bar{B}K) = B|AK.$$

Then, as $(B|K)|_F A = B|AK$, the Import-Export principle is satisfied by $|_F$. Moreover, by recalling (21) and (29), it follows that

$$(B|K)|_F A = (B|K)|_C A = (B|K)|_{dF} A = B|AK.$$

5.1. Property P1

By recalling that $(A|H)\wedge_K(B|K) = AHBK|(AHBK \vee \bar{A}H \vee \bar{B}K)$, from (31) it follows that

$$\begin{aligned}((A|H)\wedge_K(B|K))|_F(A|H) &= (AHBK|(AHBK \vee \bar{A}H \vee \bar{B}K))|_F(A|H) = \\ AHBK|(AHBK \vee AH\bar{B}K \vee \overline{H}\,\bar{B}K) &= (B|K)|_F(A|H).\end{aligned} \tag{32}$$

Then, as $((A|H)\wedge_K(B|K))|_F(A|H) = (B|K)|_F(A|H)$, Property P1 is satisfied by the pair $(\wedge_K, |_F)$. This relation can also be obtained by observing that the truth values of $(B|K)|_F(A|H)$ and of $((A|H)\wedge_K(B|K))|_F(A|H)$ in Table 7 coincide.

5.2. Property P2

We observe that (see Table 7) the iterated conditional $(B|K)|_F(A|H)$ is true when the conjunction $(A|H) \wedge_K (B|K)$ is true; moreover, the conjunction $(A|H) \wedge_K (B|K)$ is false when the iterated conditional $(B|K)|_F(A|H)$ is false. Then, $(A|H) \wedge_K (B|K) \subseteq (B|K)|_F(A|H)$. Thus, it follows from (14) that $(A|H) \wedge_K (B|K) \le (B|K)|_F(A|H)$, which means that P2 is satisfied by $(\wedge_K, |_F)$.

5.3. Property P3

To check the validity of property P3 for $(B|K)|_F(A|H)$ we study the set of coherent probability assessments on the family $\mathcal{F} = \{A|H, (B|K)|_F(A|H), (A|H) \wedge_K (B|K)\}$.

Theorem 8. *Let A, B, H, K, be any logically independent events. A probability assessment $\mathcal{P} = (x, y, z)$ on the family of conditional events $\mathcal{F} = \{A|H, (B|K)|_F(A|H), (A|H) \wedge_K (B|K)\}$ is coherent if and only if $(x, y) \in [0, 1]^2$ and $z \in [z', z'']$, where $z' = 0$ and $z'' = T_0^H(x, y)$, where*

$$T_0^H(x,y) = \begin{cases} 0, & \text{if } x = 0 \text{ or } y = 0 \\ \frac{xy}{x+y-xy}, & \text{if } x \neq 0 \text{ and } y \neq 0, \end{cases}$$

is the Hamacher t-norm with parameter $\lambda = 0$.

Proof. See Appendix A.2. □

Remark 10 *(Property P3).* From Theorem 8 it follows that any probability assessment (x, y, z) on $\mathcal{F} = \{A|H, (B|K)|_F(A|H), (A|H) \wedge_K (B|K)\}$, with $(x, y) \in [0, 1]^2$ and $z = xy$, is coherent because $xy \in [z', z'']$, where $z' = 0$ and $z'' = T_0^H(x, y)$. Then, any assessment which satisfies property P3 is coherent. Moreover, as $z = xy$ is not the unique coherent extension, the quantity $P((A|H) \wedge_K (B|K))$ could not coincide with the product $P((B|K)|_F(A|H))P(A|H)$. For example, if we choose the probability assessment $\mathcal{P} = (1, 1, 0)$, we observe that $\mathcal{P}$ is coherent on $\mathcal{F}$, because $0 \in [z', z''] = [0, 1]$. However, we observe that the assessment $\mathcal{P} = (1, 1, 0)$ on $\{A, B|A, AB\}$ is not coherent, because $P(AB) = 0 \neq P(B|A)P(A) = 1$. Then, property P3 is not satisfied by the pair $(\wedge_K, |_F)$.

5.4. Property P4

To check the validity of property P4 for the iterated conditioning $|_F$ we study the set of all coherent probability assessments on the family $\{A|H, B|K, (B|K)|_F(A|H)\}$.

Theorem 9. *Let A, B, H, K be any logically independent events. The probability assessment $\mathcal{P} = (x, y, z)$ on the family of conditional events $\mathcal{F} = \{A|H, B|K, (B|K)|_F(A|H)\}$ is coherent for every $(x, y, z) \in [0, 1]^3$.*

Proof. See Appendix A.3. □

Remark 11 *(Property P4).* We observe that the probability propagation rule valid for unconditional events (property P4) is no longer valid for Farrell's iterated conditional. Indeed, from Theorem 9, any probability assessment (x, y, z) on $\mathcal{F} = \{A|H, B|K, (B|K)|_F(A|H)\}$, with $(x, y, z) \in [0, 1]^3$ is coherent. For instance, the assessment $(1, 1, 0)$ is coherent on $\mathcal{F}$ but it is not coherent on $\{A, B, B|A\}$.

We briefly sum up here the results obtained in this section (see also Table 11) for the iterated conditioning $|_F$. We verified that the $|_F$ satisfies both properties P1 and P2. Indeed, $(B|K)|_F(A|H) = ((A|H) \wedge_K (B|K))|_F(A|H)$ and $((A|H) \wedge_K (B|K)) \le (B|K)|_F(A|H)$. However, the iterated conditioning $|_F$ does not satisfy property P3, because $P(A|H)P((B|K)|_F(A|H)) \neq P((A|H) \wedge_K (B|K))$ for some probability P. Moreover, we showed that every assessment $(x, y, z) \in [0, 1]^3$ is coherent on $\{A|H, B|K, (B|K)|_F(A|H)\}$ and hence property P4 is not satisfied. Finally, we observed that Import-Export Principle is satisfied by $|_F$.

6. Iterated conditionals and compound prevision theorem

We observe that none of the iterated conditioning operations, $|_C$, $|_{dF}$, and $|_F$, studied in the previous sections satisfies the compound probability theorem P3. In this section, we consider iterated conditionals which, among other things, satisfy property P3. We first recall a structure used for defining, in the framework of conditional random quantities, the iterated conditioning $|_{gs}$. Then, we introduce four notions of iterated conditioning which are based on the same structure and on the four conjunctions of trivalent logics recalled in Section 2. For all these new objects we check the validity of the basic properties.

We recall that in [60] (see also [34]), by using the structure

$$\square|\bigcirc = \square \wedge \bigcirc + \mathbb{P}(\square|\bigcirc)\bar{\bigcirc}, \tag{33}$$

with $\square = B|K$, $\bigcirc = A|H \neq \emptyset$, and $\square \wedge \bigcirc = (B|K) \wedge_{gs} (A|H)$, the iterated conditional $(B|K)|_{gs}(A|H)$ has been defined as the following conditional random quantity

$$(B|K)|_{gs}(A|H) = (A|H) \wedge_{gs} (B|K) + \mu_{gs} (\bar{A}|H), \tag{34}$$

where $\mu_{gs} = \mathbb{P}[(B|K)|_{gs}(A|H)]$. We underline that when $\square = A, \bigcirc = H$ formula (33) reduces to formula (13). We also recall that ([34])

$$\mathbb{P}[(A|H) \wedge_{gs} (B|K)] = \mathbb{P}[(B|K)|_{gs}(A|H)]P(A|H). \tag{35}$$

Moreover, in [38, Definition 7], by exploiting the structure (33), the iterated conditioning $|_{gs}$ has been extended to the case of conjoined conditionals. More precisely, denoting by $\mathcal{C}(\mathcal{F}_1)$ (resp., $\mathcal{C}(\mathcal{F}_2)$) the $\wedge_{gs}$-conjunction of the conditional events in a finite family of conditional events $\mathcal{F}_1$ (resp., $\mathcal{F}_2$), the iterated conditional $\mathcal{C}(\mathcal{F}_2)|_{gs}\mathcal{C}(\mathcal{F}_1)$ has been defined as

$$\mathcal{C}(\mathcal{F}_2)|_{gs}\mathcal{C}(\mathcal{F}_1) = \mathcal{C}(\mathcal{F}_2) \wedge_{gs} \mathcal{C}(\mathcal{F}_1) + \mu(1 - \mathcal{C}(\mathcal{F}_1)) = \mathcal{C}(\mathcal{F}_1 \cup \mathcal{F}_2) + \mu(1 - \mathcal{C}(\mathcal{F}_1)), \tag{36}$$

where $\mu = \mathbb{P}[\mathcal{C}(\mathcal{F}_2)|_{gs}\mathcal{C}(\mathcal{F}_1)]$. In addition, it holds [38, Equation (8)] that

$$\mathbb{P}[\mathcal{C}(\mathcal{F}_2) \wedge_{gs} \mathcal{C}(\mathcal{F}_1)] = \mathbb{P}[\mathcal{C}(\mathcal{F}_2)|_{gs}\mathcal{C}(\mathcal{F}_1)]\mathbb{P}[\mathcal{C}(\mathcal{F}_1)]. \tag{37}$$

Formulas (35) and (37) generalize the compound probability theorem recalled in (12).

We now introduce the different notions of iterated conditioning, beyond $|_{gs}$, obtained with the structure (33), by using the trivalent logic conjunctions $\wedge_K, \wedge_L, \wedge_B$, and $\wedge_S$, recalled in Section 2. Among other things, we will show that a formula like (35) still holds for each of these new objects.

Definition 5. Given two conditional events $A|H$, $B|K$, with $AH \neq \emptyset$, for each $i \in \{K, L, B, S, gs\}$, we define the iterated conditional $(B|K)|_i(A|H)$ as

$$(B|K)|_i(A|H) = (A|H) \wedge_i (B|K) + \mu_i (\bar{A}|H), \tag{38}$$

where $\mu_i = \mathbb{P}[(B|K)|_i(A|H)]$.

Remark 12. In a betting framework, by setting $\mathbb{P}[(B|K)|_i(A|H)] = \mu_i$, $i \in \{K, L, B, S, gs\}$, then for every real number s you agree to pay an amount $s\mu_i$ in order to receive the random quantity $s \cdot (B|K)|_i(A|H)$. The associated random gain is

$$G = s[(B|K)|_i(A|H) - \mu_i] = s[(A|H) \wedge_i (B|K) + \mu_i (\bar{A}|H) - \mu_i]. \tag{39}$$

We observe that, for each $i \in \{K, L, S, gs\}$, if $\bar{A}|H$ is true, as $(A|H) \wedge_i (B|K) = 0$ (see Table 1), it follows that $(B|K)|_i(A|H) = \mu_i$, for every μ_i, and hence $G = s(\mu_i - \mu_i) = 0$. In other words, for $i \in \{K, L, S, gs\}$ the bet on $(B|K)|_i(A|H)$ is called off when the antecedent $A|H$ is false. Concerning $i = B$, if $\bar{A}HK$ is true, as $(A|H) \wedge_B (B|K) = 0$ (see Table 1), it follows that $(B|K)|_B(A|H) = \mu_B$ and hence $G = 0$. Then, the bet on $(B|K)|_B(A|H)$ is called off when the antecedent $A|H$ is false and $B|K$ is not void. As we will see in Remark 14, for each iterated conditional $(B|K)|_i(A|H)$, $i \in \{K, L, B, S, gs\}$, there are other cases where the bet is called off, i.e. $(B|K)|_i(A|H) = \mu_i$.

Now, we check the validity of properties P1–P4, introduced in Section 2.5, for each iterated conditioning $|_i$, $i \in \{K, L, B, S, gs\}$.

6.1. *Property P1*

In the next result we show that each iterated conditioning $|_i$, $i \in \{K, L, B, S\}$, satisfies property P1.

Theorem 10. *Given two conditional events $A|H$, $B|K$, with $AH \neq \emptyset$, it holds that*

$$((A|H) \wedge_i (B|K))|_i(A|H) = (B|K)|_i(A|H), \quad i \in \{K, L, B, S\}. \tag{40}$$

Proof. Let $i \in \{K, L, B, S\}$ be given. We set $\mathbb{P}[(B|K)|_i(A|H)] = \mu_i$ and $\mathbb{P}[((A|H) \wedge (B|K))|_i(A|H)] = \nu_i$. By Definition 5, as $(A|H) \wedge_i (A|H) \wedge_i (B|K) = (A|H) \wedge_i (B|K)$, it holds that

$$((A|H) \wedge_i (B|K))|_i(A|H) = (A|H) \wedge_i (B|K) + \nu_i(\bar{A}|H). \tag{41}$$

Then, as $(B|K)|_i(A|H) = (A|H) \wedge_i (B|K) + \mu_i(\bar{A}|H)$, in order to prove (40) it is enough to verify that $\nu_i = \mu_i$. We observe that $((A|H) \wedge_i (B|K))|_i(A|H) - (B|K)|_i(A|H) = (\nu_i - \mu_i)(\bar{A}|H)$, where $\nu_i - \mu_i = \mathbb{P}[((A|H) \wedge_i (B|K))|_i(A|H) - (B|K)|_i(A|H)]$. By setting $P(A|H) = x$, it holds that

$$(\nu_i - \mu_i)(\bar{A}|H) = \begin{cases} 0, \text{ if } A|H = 1, \\ \nu_i - \mu_i, \text{ if } A|H = 0, \\ (\nu_i - \mu_i)(1-x), \text{ if } A|H = x, \ 0 < x < 1. \end{cases}$$

Notice that, in the betting scheme, $\nu_i - \mu_i$ is the amount to be paid in order to receive the random amount $(\nu_i - \mu_i)(\bar{A}|H)$. Then, by coherence, $\nu_i - \mu_i$ must be a linear convex combination of the possible values of $(\nu_i - \mu_i)(\bar{A}|H)$, by discarding the cases where the bet is called off, that is the cases where you receive back the paid amount $\nu_i - \mu_i$, whatever $\nu_i - \mu_i$ is. In other words, coherence requires that $\nu_i - \mu_i$ must belong to the convex hull of the set $\{0, (\nu_i - \mu_i)(1-x)\}$, that is $\nu_i - \mu_i = \alpha \cdot 0 + (1-\alpha)(\nu_i - \mu_i)(1-x)$, for some $\alpha \in [0,1]$. Then, as $0 < x < 1$, we observe that the previous equality holds if and only if $\nu_i - \mu_i = 0$, that is $\nu_i = \mu_i$. Therefore, equality (40) holds. □

We recall that $|_{gs}$ satisfies property P1. Indeed, for the generalized iterated conditional (36) it holds that ([38, Theorem 5])

$$\mathcal{C}(\mathcal{F}_2)|_{gs}\mathcal{C}(\mathcal{F}_1) = \mathcal{C}(\mathcal{F}_1 \cup \mathcal{F}_2)|_{gs}\mathcal{C}(\mathcal{F}_1). \tag{42}$$

Then, by applying (42) with $\mathcal{F}_1 = \{A|H\}$, $\mathcal{F}_2 = \{B|K\}$, it follows that

$$(B|K)|_{gs}(A|H) = ((A|H) \wedge_{gs} (B|K))|_{gs}(A|H).$$

Thus, property P1 is satisfied by each iterated conditioning $|_i \in \{|_K, |_L, |_B, |_S, |_{gs}\}$.

6.2. Property P2

We show that each iterated conditioning $|_i$, $i \in \{K, L, B, S, gs\}$, satisfies property P2. We observe that, coherence requires $\mu_i \geq 0$, $i \in \{K, L, B, S, gs\}$ (see Remark 13 in Section 6.3). Then, for each $i \in \{K, L, B, S, gs\}$, as $(A|H) \wedge_i (B|K) \leq (A|H) \wedge_i (B|K) + \mu_i(\bar{A}|H)$, from (38) it follows that

$$(A|H) \wedge_i (B|K) \leq (B|K)|_i(A|H), \tag{43}$$

and hence $\mathbb{P}[(A|H) \wedge_i (B|K)] \leq \mathbb{P}[(B|K)|_i(A|H)]$.

6.3. Property P3

We already recalled in equation (35) that the iterated conditioning $|_{gs}$ satisfies P3. Then, by setting $z_{gs} = \mathbb{P}[(A|H) \wedge_{gs} (B|K)]$, $\mu_{gs} = \mathbb{P}[(B|K)|_{gs}(A|H)]$, and $x = P(A|H)$ it holds that $z_{gs} = x\mu_{gs}$. By exploiting the structure (33), we show below that P3 is also valid for $|_K, |_L, |_B$, and $|_S$. Indeed, by the linearity property of a coherent prevision, from (38) we obtain that

$$\begin{aligned} \mu_i &= \mathbb{P}[(B|K)|_i(A|H)] = P((B|K) \wedge_i (A|H)) + \mu_i P(\bar{A}|H) = \\ &= P((B|K) \wedge_i (A|H)) + \mu_i P(\bar{A}|H) = z_i + \mu_i(1-x), \end{aligned} \tag{44}$$

where $x = P(A|H)$ and $z_i = P((A|H) \wedge_i (B|K))$, $i \in \{K, L, B, S\}$. As $\mu_i = z_i + \mu_i(1-x)$, it follows that

$$z_i = \mu_i x, \ \ i \in \{K, L, B, S\}.$$

Therefore, coherence requires that

$$\mathbb{P}[(A|H) \wedge_i (B|K)] = \mathbb{P}[(B|K)|_i(A|H)]P(A|H), \ i \in \{K, L, B, S, gs\}, \tag{45}$$

which states that the compound prevision formula for iterated conditionals (property P3) is valid for each iterated conditioning $|_i$, $i \in \{K, L, B, S, gs\}$.

Remark 13. Notice that, as $P(A|H) \geq 0$ and $\mathbb{P}[(A|H) \wedge_i (B|K)] \geq 0$ for $i \in \{K, L, B, S, gs\}$, from (45) it follows that $\mathbb{P}[(B|K)|_i(A|H)] \geq 0$, $i \in \{K, L, B, S, gs\}$.

We set $P(A|H) = x$, $P(B|K) = y$, $P((A|H) \wedge_i (B|K)) = z_i$, $i \in \{K, L, B, S\}$, $\mathbb{P}[(A|H) \wedge_i (B|K)] = z_{gs}$, and $\mathbb{P}[(B|K)|_i(A|H)] = \mu_i$, $i \in \{K, L, B, S, gs\}$. Then, based on Definition 5 and on the compound prevision theorem, for each $|_i$, $i \in \{K, L, B, S, gs\}$, we obtain the random quantities $(B|K)|_i(A|H)$ illustrated below.

- $(|_K)$. We recall that $(A|H) \wedge_K (B|K) = AHBK|(HK \vee \overline{A}H \vee \overline{B}K)$, then we have

$$(B|K)|_K(A|H) = AHBK|(HK \vee \overline{A}H \vee \overline{B}K) + \mu_K(\bar{A}|H) = \\ = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } AH\bar{B}K \text{ is true,} \\ z_K, & \text{if } AH\bar{K} \text{ is true,} \\ z_K + \mu_K(1-x), & \text{if } \overline{H}BK \vee \overline{H}\,\bar{K} \text{ is true,} \\ \mu_K(1-x), & \text{if } \overline{H}\,\bar{B}K \text{ is true,} \\ \mu_K, & \text{if } \bar{A}H \text{ is true.} \end{cases} \tag{46}$$

From (45), coherence requires that $z_K = x\mu_K$ and $z_K + \mu_K(1-x) = \mu_K$. Then, we obtain

$$(B|K)|_K(A|H) = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } AH\bar{B}K \text{ is true,} \\ x\mu_K, & \text{if } AH\bar{K} \text{ is true,} \\ \mu_K(1-x), & \text{if } \overline{H}\,\bar{B}K \text{ is true,} \\ \mu_K, & \text{if } \bar{A}H \vee \overline{H}BK \vee \overline{H}\,\bar{K} \text{ is true.} \end{cases} \tag{47}$$

- $(|_L)$. We recall that $(A|H) \wedge_L (B|K) = AHBK|(AHBK \vee \overline{A}\,H \vee \overline{B}K \vee \overline{H}\,\overline{K})$, then we have

$$(B|K)|_L(A|H) = AHBK|(AHBK \vee \overline{A}\,H \vee \overline{B}K \vee \overline{H}\,\overline{K}) + \mu_L(\bar{A}|H) = \\ = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } AH\bar{B}K \text{ is true,} \\ z_L, & \text{if } AH\bar{K} \text{ is true,} \\ z_L + \mu_L(1-x), & \text{if } \overline{H}BK \text{ is true,} \\ \mu_L(1-x), & \text{if } \overline{H}\,\bar{B}K \vee \overline{H}\,\bar{K} \text{ is true,} \\ \mu_L, & \text{if } \bar{A}H \text{ is true.} \end{cases} \tag{48}$$

From (45), coherence requires that $z_L = x\mu_L$ and $z_L + \mu_L(1-x) = \mu_L$. Then, we obtain

$$(B|K)|_L(A|H) = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } AH\bar{B}K \text{ is true,} \\ x\mu_L, & \text{if } AH\bar{K} \text{ is true,} \\ \mu_L(1-x), & \text{if } \overline{H}\,\bar{B}K \vee \overline{H}\,\bar{K} \text{ is true,} \\ \mu_L, & \text{if } \bar{A}H \vee \overline{H}BK \text{ is true.} \end{cases} \tag{49}$$

- $(|_B)$. We recall that $(A|H) \wedge_B (B|K) = AHBK|BK$, then we have

$$(B|K)|_B(A|H) = AHBK|HK + \mu_B(\bar{A}|H) = \\ = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } AH\bar{B}K \text{ is true,} \\ z_B, & \text{if } AH\bar{K} \text{ is true,} \\ z_B + \mu_B, & \text{if } \bar{A}H\bar{K} \text{ is true,} \\ z_B + \mu_B(1-x), & \text{if } \overline{H} \text{ is true,} \\ \mu_B, & \text{if } \bar{A}HBK \vee \bar{A}H\bar{B}K \text{ is true.} \end{cases} \tag{50}$$

From (45), coherence requires that $z_B = x\mu_B$ and $z_B + \mu_B(1-x) = \mu_B$. Then, we obtain

$$(B|K)|_B(A|H) = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } AH\bar{B}K \text{ is true,} \\ x\mu_B, & \text{if } AH\bar{K} \text{ is true,} \\ \mu_B(1+x), & \text{if } \bar{A}H\bar{K} \text{ is true,} \\ \mu_B, & \text{if } \overline{H} \vee \bar{A}HBK \vee \bar{A}H\bar{B}K \text{ is true.} \end{cases} \tag{51}$$

Table 8
Numerical values of $(B|K)|_i(A|H)$, $i \in \{K, L, B, S, gs\}$. We denote $P(A|H) = x$, $P(B|K) = y$, and $\mathbb{P}[(B|K)|_i(A|H)] = \mu_i$, $i \in \{K, L, B, S, gs\}$.

	$(B\|K)\|_K(A\|H)$	$(B\|K)\|_L(A\|H)$	$(B\|K)\|_B(A\|H)$	$(B\|K)\|_S(A\|H)$	$(B\|K)\|_{gs}(A\|H)$
$AHBK$	1	1	1	1	1
$AH\overline{B}K$	0	0	0	0	0
$AH\overline{K}$	$x\mu_K$	$x\mu_L$	$x\mu_B$	1	y
$\overline{A}HBK$	μ_K	μ_L	μ_B	μ_S	μ_{gs}
$\overline{A}H\overline{B}K$	μ_K	μ_L	μ_B	μ_S	μ_{gs}
$\overline{A}H\overline{K}$	μ_K	μ_L	$\mu_B(1+x)$	μ_S	μ_{gs}
$\overline{H}BK$	μ_K	μ_L	μ_B	$1+\mu_S(1-x)$	$x+\mu_{gs}(1-x)$
$\overline{H}\,\overline{B}K$	$\mu_K(1-x)$	$\mu_L(1-x)$	μ_B	$\mu_S(1-x)$	$\mu_{gs}(1-x)$
$\overline{H}\,\overline{K}$	μ_K	$\mu_L(1-x)$	μ_B	μ_S	μ_{gs}

- $(|_S)$. We recall that $(A|H) \wedge_S (B|K) = ((AH \vee \overline{H}) \wedge (BK \vee \overline{K}))$, then we have

$$(B|K)|_S(A|H) = ((AH \vee \overline{H}) \wedge (BK \vee \overline{K}))|(H \vee K) + \mu_S(\bar{A}|H) = \tag{52}$$
$$= \begin{cases} 1, & \text{if } AHB \vee AH\bar{K} \text{ is true,} \\ 0, & \text{if } AH\bar{B}K \text{ is true,} \\ 1+\mu_S(1-x), & \text{if } \overline{H}BK \text{ is true,} \\ \mu_S(1-x), & \text{if } \overline{H}\,\bar{B}K \text{ is true,} \\ z_S + \mu_S(1-x), & \text{if } \overline{H}\,\bar{K} \text{ is true,} \\ \mu_S & \text{if } \bar{A}H \text{ is true.} \end{cases}$$

From (45), coherence requires that $z_S = x\mu_S$ and $z_S + \mu_S(1-x) = \mu_S$. Then, we obtain

$$(B|K)|_S(A|H) = \begin{cases} 1, & \text{if } AHB \vee AH\bar{K} \text{ is true,} \\ 0, & \text{if } AH\bar{B}K \text{ is true,} \\ 1+\mu_S(1-x), & \text{if } \overline{H}BK \text{ is true,} \\ \mu_S(1-x), & \text{if } \overline{H}\,\bar{B}K \text{ is true,} \\ \mu_S & \text{if } \bar{A}H \vee \overline{H}\,\bar{K} \text{ is true.} \end{cases} \tag{53}$$

- $(|_{gs})$. We recall that $(A|H) \wedge_{gs} (B|K) = (AH \vee BK + x\overline{H}\,\bar{B}K + y\bar{A}H\bar{K})|(H \vee K)$, then we have ([34])

$$(B|K)|_{gs}(A|H) = (AHBK + x\overline{H}BK + yAH\bar{K})|(H \vee K) + \mu_{gs}(\bar{A}|H) = \tag{54}$$
$$= \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } AH\bar{B}K \text{ is true,} \\ y, & \text{if } AH\bar{K} \text{ is true,} \\ x+\mu_{gs}(1-x), & \text{if } \overline{H}BK \text{ is true,} \\ \mu_{gs}(1-x), & \text{if } \overline{H}\,\bar{B}K \text{ is true,} \\ z_{gs} + \mu_{gs}(1-x), & \text{if } \overline{H}\,\bar{K} \text{ is true,} \\ \mu_{gs}, & \text{if } \bar{A}H \text{ is true.} \end{cases}$$

From (45), coherence requires that $z_{gs} = x\mu_{gs}$ and $z_{gs} + \mu_{gs}(1-x) = \mu_{gs}$. Then, we recall that

$$(B|K)|_{gs}(A|H) = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } AH\bar{B}K \text{ is true,} \\ y, & \text{if } AH\bar{K} \text{ is true,} \\ x+\mu_{gs}(1-x), & \text{if } \overline{H}BK \text{ is true,} \\ \mu_{gs}(1-x), & \text{if } \overline{H}\,\bar{B}K \text{ is true,} \\ \mu_{gs}, & \text{if } \bar{A}H \vee \overline{H}\,\bar{K} \text{ is true.} \end{cases} \tag{55}$$

In Table 8 we summarize the possible values of the iterated conditionals $(B|K)|_i(A|H)$, $i \in \{K, L, B, S, gs\}$.

Remark 14. In addition to the cases considered in Remark 12, based on property P3 (see Table 8), for each $i \in \{K, L, B, S, gs\}$, there are other situations where $(B|K)|_i(A|H) = \mu_i$, for every μ_i, and hence the associated bet is called off. For example, the iterated conditional $(B|K)|_K(A|H)$ coincides with μ_K not only when $\bar{A}|H$ is true, but also when $\overline{H}BK$, or $\overline{H}\,\bar{K}$, is true. Moreover, a bet on these iterated conditionals could also be called off for values of x as well. For instance, $(B|K)|_K(A|H)$ reduces to μ_K, when $x=1$ and $AH\bar{K}$ is true, or when $x=0$ and $\overline{H}\,\bar{B}K$ is true.

6.4. Property P4

In this section, for each $i \in \{K, L, B, S\}$, we find the set of all coherent assessments on the family $\{A|H, B|K, (A|H) \wedge_i (B|K), (B|K)|_i(A|H)\}$. Moreover, for each $i \in \{K, L, B, S\}$, we determine the interval of coherent extensions on $(B|K)|_i(A|H)$ and we check the validity of property P4. Then, we recall that property P4 is satisfied by $|_{gs}$.

The iterated conditioning $|_K$.

Theorem 11. *Let A, B, H, K be any logically independent events. The set Π of all the coherent assessments (x, y, z, μ) on the family $\mathcal{F} = \{A|H, B|K, (A|H) \wedge_K (B|K), (B|K)|_K(A|H)\}$ is $\Pi = \Pi' \cup \Pi''$, where $\Pi' = \{(x, y, z, \mu) : x \in (0,1], y \in [0,1], z \in [z', z''], \mu = \frac{z}{x}\}$ with $z' = 0$, $z'' = min\{x, y\}$, and $\Pi'' = \{(0, y, 0, \mu) : (y, \mu) \in [0,1]^2\}$.*

Proof. It is well-known that the assessment (x, y) on $\{A|H, B|K\}$ is coherent for every $(x, y) \in [0,1]^2$. By Table 2, the assessment $z = P((A|H) \wedge_K (B|K))$ is a coherent extension of (x, y) if and only if $z \in [z', z'']$ where $z' = 0$ and $z'' = min\{x, y\}$. Assuming $x > 0$, from (45), it holds that $\mu = \frac{z}{x}$. Then, every $(x, y, z, \mu) \in \Pi'$ is coherent, i.e., $\Pi' \subseteq \Pi$. Of course, if $x > 0$ and $(x, y, z, \mu) \notin \Pi'$, then the assessment (x, y, z, μ) is not coherent, i.e. $(x, y, z, \mu) \notin \Pi$.

Let us consider now the case $x = 0$, so that $z' = 0$ and $z'' = 0$. We show that the assessment $(0, y, 0, \mu)$ is coherent if and only if $(y, \mu) \in [0,1]^2$, that is $\Pi'' \subseteq \Pi$.

As $x = 0$, from (47), it holds that

$$(B|K)|_K(A|H) = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } AH\bar{B}K \vee AH\bar{K} \text{ is true,} \\ \mu, & \text{if } \bar{A}H \vee \overline{H}BK \vee \overline{H}\,\bar{B}K \vee \overline{H}\,\bar{K} \text{ is true,} \end{cases} \tag{56}$$

that is $(B|K)|_K(A|H) = AHBK + \mu(\bar{A} \vee \overline{H})$. Moreover we have that $AHBK + \mu(\bar{A} \vee \overline{H})$ and the conditional event $BK|AH = AHBK + \eta(\bar{A} \vee \overline{H})$, where $\eta = P(BK|AH)$, coincide when AH is true. Then, from Theorem 3, it follows that $\mu = \eta$ and hence $(B|K)|_K(A|H)$ and $BK|AH$ should also coincide when AH is false. Thus, $(B|K)|_K(A|H)$ and $BK|AH$ coincide in all cases. Therefore $\mathcal{F} = \{A|H, B|K, (A|H) \wedge_K (B|K), (B|K)|_K(A|H)\} = \{A|H, B|K, AHBK|(AHBK \vee \overline{A}H \vee \overline{B}K), BK|AH\}$ and $\mathcal{H}_4 = H \vee K \vee (AHBK \vee \overline{A}H \vee \overline{B}K) \vee AH = H \vee K$. The constituents C_h's and the points Q_h's associated with $(\mathcal{F}, \mathcal{P})$, where $\mathcal{P} = (0, y, 0, \mu)$ are the following:

$$\begin{aligned} &C_1 = AHBK, C_2 = AH\bar{B}K, C_3 = AH\bar{K}, C_4 = \bar{A}HBK, C_5 = \bar{A}H\bar{B}K, \\ &C_6 = \bar{A}H\bar{K}, C_7 = \overline{H}BK, C_8 = \overline{H}\,\bar{B}K, C_0 = \overline{H}\,\bar{K}, \end{aligned}$$

and

$$\begin{aligned} &Q_1 = (1,1,1,1), Q_2 = (1,0,0,0), Q_3 = (1,y,0,0), Q_4 = (0,1,0,\mu), Q_5 = (0,0,0,\mu), \\ &Q_6 = (0,y,0,\mu), Q_7 = (0,1,0,\mu), Q_8 = (0,0,0,\mu), Q_0 = (0,y,0,\mu). \end{aligned}$$

The constituents contained in $\mathcal{H}_4 = H \vee K$ are $C_1, \dots, C_8$. The system (Σ) in (5) associated with the pair $(\mathcal{F}, \mathcal{P})$ is

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = 0, \\ \lambda_1 + y\lambda_3 + \lambda_4 + y\lambda_6 + \lambda_7 = y, \\ \lambda_1 = 0, \\ \lambda_1 + \mu\lambda_4 + \mu\lambda_5 + \mu\lambda_6 + \mu\lambda_7 + \mu\lambda_8 = \mu \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 + \lambda_8 = 1, \\ \lambda_i \geq 0, \; i = 1, \dots, 8. \end{cases} \tag{57}$$

We observe that, as $y \in [0,1]$, $\mathcal{P}$ belongs to the segment with end points Q_4, Q_5 because $\mathcal{P} = yQ_4 + (1-y)Q_5$. Then, $\Lambda = (0,0,0,y,1-y,0,0,0)$ is a solution of (57) with

$$\begin{aligned} &\phi_1(\Lambda) = \textstyle\sum_{h: C_h \subseteq H} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = 1 > 0, \\ &\phi_2(\Lambda) = \textstyle\sum_{h: C_h \subseteq K} \lambda_h = \lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 + \lambda_7 + \lambda_8 = 1 > 0, \\ &\phi_3(\Lambda) = \textstyle\sum_{h: C_h \subseteq (AHBK \vee \overline{A}H \vee \overline{B}K)} \lambda_h = \lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_8 = 1 > 0, \\ &\phi_4(\Lambda) = \textstyle\sum_{h: C_h \subseteq AH} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 = 0. \end{aligned}$$

Let $\mathcal{S}' = \{(0,0,0,y,1-y,0,0,0)\}$ denote a subset of the set $\mathcal{S}$ of all solutions of (57). We have that $M'_1 = M'_2 = M'_3 = 1$ and $M'_4 = 0$ (as defined in (7)). It follows that $I'_0 = \{4\}$. As the sub-assessment $\mathcal{P}'_0 = \mu$ on $\mathcal{F}'_0 = \{BK|AH\}$ is coherent $\forall \mu \in [0,1]$, by Theorem 1, it follows that the assessment $(0, y, 0, \mu)$ on $\mathcal{F} = \{A|H, B|K, (A|H) \wedge_K (B|K), (B|K)|_K(A|H)\}$ is coherent for every $(y, \mu) \in [0,1]^2$, that is $(0,y,0,\mu)\in\Pi''$. Thus $\Pi''\subseteq\Pi$. Of course, if $(0,y,z,\mu)\notin\Pi''$ the assessment $(0,y,z,\mu)$ is not coherent and hence $(0,y,z,\mu)\notin\Pi$. Therefore $\Pi=\Pi'\cup\Pi''$. □

Based on Theorem 11, we obtain

Theorem 12. *Let A,B,H,K be any logically independent events. Given any assessment $(x,y)\in[0,1]^2$ on $\{A|H,B|K\}$, for the iterated conditional $(B|K)|_K(A|H)$ the extension $\mu_K=\mathbb{P}((B|K)|_K(A|H))$ is coherent if and only if $\mu_K\in[\mu'_K,\mu''_K]$, where*

$$\mu'_K=0,\ \ \mu''_K=\begin{cases}\min\left\{1,\frac{y}{x}\right\}, & \text{if } x>0;\\ 1, & \text{if } x=0.\end{cases} \tag{58}$$

Proof. Of course, the assessment (x,y) on $\{A|H,B|K\}$ is coherent. Assume that $x>0$. We simply write μ instead of μ_K. From Theorem 11, it follows that the set of all coherent assessments (x,y,z,μ) on $\mathcal{F}=\{A|H,B|K,(A|H)\wedge_K(B|K),(B|K)|_K(A|H)\}$ is $\Pi'=\{(x,y,z,\mu):0<x\le 1,0\le y\le 1,z'\le z\le z'',\mu=\frac{z}{x}\}$, where $z'=0$ and $z''=\min\{x,y\}$ (2). Then, μ is a coherent extension of (x,y) if and only if $\mu\in[\mu',\mu'']$, where $\mu'=0$ and $\mu''=\frac{z''}{x}=\min\left\{1,\frac{y}{x}\right\}$. Assume that $x=0$. From Theorem 11, it follows that the set of all coherent assessments (x,y,z,μ) on $\mathcal{F}=\{A|H,B|K,(A|H)\wedge_K(B|K),(B|K)|_K(A|H)\}$ is $\Pi''=\{(0,y,0,\mu):(y,\mu)\in[0,1]^2\}$. Then, μ is a coherent extension to $(B|K)|_K(A|H)$ of $(0,y)$ on $\{A|H,B|K\}$ if and only if $\mu\in[\mu',\mu'']$, where $\mu'=0$ and $\mu''=1$. □

Remark 15. We notice that the lower and upper bounds μ'_K and μ''_K for $(B|K)|_K(A|H)$, given in (58), do not coincide with the lower and upper bound μ' and μ'' for $B|A$, given in (17). In particular, the extension $\mu_K=0$ to $(B|K)|_K(A|H)$ of $(1,1)$ on $\{A|H,B|K\}$ is coherent because, as $\mu'_K=0$, the assessment $(1,1,0)$ on $\{A|H,B|K,(B|K)|_K(A|H)\}$ is coherent. However, $\mu=0$ is not a coherent extension to $B|A$ of $(1,1)$ on $\{A,B\}$, because, as $\mu'=\frac{\max\{1+1-1,0\}}{1}=1$, the assessment $(1,1,0)$ on $\{A,B,B|A\}$ is not coherent. Therefore, property P4 is not satisfied by $|_K$.

In the following example we show that the invalidity of property P4 leads to some counterintuitive aspects.

Example 4. In a random experiment we are supposed to pick a ball from one of two bags, U and V, both containing white balls. We do not know which of the two bags is selected. We set $H=$"the ball is picked from the bag U" (and hence $\overline{H}=$"the ball is picked from the bag V") and $A=$"the drawn ball is white". Of course $P(A|H)=P(A|\overline{H})=1$. Based on (46) it holds that

$$\begin{aligned}&(A|\overline{H})|_K(A|H)=(A|\overline{H})\wedge_K(A|H)+\mu_K(\bar{A}|H)=AH\overline{H}|(H\overline{H}\vee\bar{A}H\vee\bar{A}\,\overline{H})+\mu_K(\bar{A}|H)=\\&=\emptyset|\bar{A}+\mu_K(\bar{A}|H)=0+\mu_K(\bar{A}|H)=0,\end{aligned}$$

because, by coherence, $\mu_K=0$. Then, $\mathbb{P}[(A|\overline{H})|_K(A|H)]=0$ is the unique coherent extension of $P(A|H)=P(A|\overline{H})=1$. Thus, when using $|_K$, the iterated conditional *if the drawn ball is white if picked from U, then it is white if picked from V* has a prevision of 0, even if both conditionals, *the drawn ball is white if picked from U* and *the drawn ball is white if picked from V*, have probability 1.

The iterated conditioning $|_L$. We obtain results similar to the case $|_K$. Indeed we have

Theorem 13. *Let A, B, H, K be any logically independent events. The set Π of all the coherent assessments (x,y,z,μ) on the family $\mathcal{F}=\{A|H,B|K,(A|H)\wedge_L(B|K),(B|K)|_L(A|H)\}$ is $\Pi=\Pi'\cup\Pi''$, where $\Pi'=\{(x,y,z,\mu):x\in(0,1],y\in[0,1],z\in[z',z''],\mu=\frac{z}{x}\}$ with $z'=0$, $z''=min\{x,y\}$, and $\Pi''=\{(0,y,0,\mu):(y,\mu)\in[0,1]^2\}$.*

Proof. See Appendix A.4. □

Based on Theorem 13, we obtain

Theorem 14. *Let A,B,H,K be any logically independent events. Given a coherent assessment (x,y) on $\{A|H,B|K\}$, for the iterated conditional $(B|K)|_L(A|H)$ the extension $\mu_L=\mathbb{P}((B|K)|_L(A|H))$ is coherent if and only if $\mu_L\in[\mu'_L,\mu''_L]$, where*

$$\mu'_L=0,\ \ \mu''_L=\begin{cases}\min\left\{1,\frac{y}{x}\right\}, & \text{if } x>0;\\ 1, & \text{if } x=0.\end{cases} \tag{59}$$

Proof. The proof is the same as in Theorem 12 where K is replaced by L and Theorem 11 is replaced by Theorem 13. □

Remark 16. Theorem 14 shows that the interval of coherent extensions $[\mu'_L,\mu''_L]$ for $\mu_L=\mathbb{P}[(B|K)|_L(A|H)]$ does not coincide with the interval $[\mu',\mu'']$ of coherent assessment for $\mu=P(B|A)$ where μ' and μ'' are as in (17). Thus, property P4 is not satisfied by $|_L$.

The iterated conditioning $|_B$. We continue by analyzing the set of coherent extensions for the iterated conditional $|_B$. We first show that, when we evaluate $\mathbb{P}[(B|K)|_B(A|H)]$, if $P(A|H)=0$, then coherence requires that $P((A|H)\wedge_B(B|K))=0$.

Remark 17. We recall that, given any $(x,y)\in[0,1]^2$, where $x=P(A|H)$ and $y=P(B|K)$, the interval of coherent extensions on $z_B=P((A|H)\wedge_B(B|K))$ is $[z'_B,z''_B]=[0,1]$ (Table 2). In particular it is coherent to assess $(0,y,z_B)$, with $z_B>0$, on $\{A|H,B|K,(A|H)\wedge_B(B|K)\}$. However, for the object $(B|K)|_B(A|H)$ coherence also requires that $z_B=\mu_B x$ (see (44)). Then, coherence requires that $z_B=0$ when we consider the assessment $(0,y,z_B,\mu_B)$ on $\{A|H,B|K,(A|H)\wedge_B(B|K),(B|K)|_B(A|H)\}$.

Theorem 15. *Let A, B, H, K be any logically independent events. The set Π of all the coherent assessments (x,y,z,μ) on the family $\mathcal{F}=\{A|H,B|K,(A|H)\wedge_B(B|K),(B|K)|_B(A|H)\}$ is $\Pi=\Pi'\cup\Pi''$, where $\Pi'=\{(x,y,z,\mu):x\in(0,1],y\in[0,1],z\in[z',z''],\mu=\frac{z}{x}\}$ with $z'=0$, $z''=1$, and $\Pi''=\{(0,y,0,\mu):(y,\mu)\in[0,1]^2\}$.*

Proof. See Appendix A.5. ☐

Theorem 16. *Let A,B,H,K be any logically independent events. Given a coherent assessment (x,y) on $\{A|H,B|K\}$, for the iterated conditional $(B|K)|_B(A|H)$ the extension $\mu_B=\mathbb{P}[(B|K)|_B(A|H)]$ is coherent if and only if $\mu_B\in[\mu'_B,\mu''_B]$, where*

$$\mu'_B=0,\ \ \mu''_B=\begin{cases}\frac{1}{x}, & \text{if } 0<x<1,\\ 1, & \text{if } x=0\vee x=1.\end{cases} \tag{60}$$

Proof. See Appendix A.6. ☐

Remark 18. We notice that the lower and upper bounds μ'_B and μ''_B for $(B|K)|_B(A|H)$, given in (60), do not coincide with the lower and upper bound μ' and μ'' for $B|A$, given in (17). In particular, the extension $\mu_B=\frac{1}{x}$ to $(B|K)|_B(A|H)$ of (x,y) on $\{A|H,B|K\}$, with $0<x<1$, is coherent. Indeed, as $\mu''_B=\frac{1}{x}$, from (60), the assessment $(x,y,\frac{1}{x})$ on $\{A|H,B|K,(B|K)|_B(A|H)\}$ is coherent. However, we recall that $\mu=\frac{1}{x}$ is not a coherent extension to $B|A$ of (x,y) on $\{A,B\}$, with $0<x<1$. Indeed, as $\frac{1}{x}>1\geq\mu''=min\{x,y\}$, from (17) the assessment $(x,y,\frac{1}{x})$ on $\{A,B,B|A\}$ is not coherent. Therefore, property P4 is not satisfied by $|_B$.

We also notice that, when $P(A|H)>0$, as $(A|H)\wedge_B(B|K)=AH|BK$, from (45) it holds that

$$\mathbb{P}[(B|K)|_B(A|H)]=\frac{P((A|H)\wedge_B(B|K))}{P(A|H)}=\frac{P(AH|BK)}{P(A|H)}. \tag{61}$$

We point out that the prevision of $(B|K)|_B(A|H)$ given in (61) coincides with the probability of the *super-conditional event* $B_K|A_H$, denoted by $P^*(B_K|A_H)$, introduced in [4, formula (20)]. As $P^*(B_K|A_H)$ can assume values greater than 1, in [4] it has been called *conditional hyper-probability*.

The iterated conditioning $|_S$. We first show that, when we evaluate $\mathbb{P}[(B|K)|_S(A|H)]$, if $P(A|H)=0$, then coherence requires that $P((A|H)\wedge_S(B|K))=0$.

Remark 19. We recall that, given any $(x,y)\in[0,1]^2$, where $x=P(A|H)$ and $y=P(B|K)$, the interval of coherent extensions on $z_S=P((A|H)\wedge_S(B|K))$ is $[z',z'']$ where (see Table 2)

$$z'=\max\{x+y-1,0\}\text{ and } z''=\begin{cases}\frac{x+y-2xy}{1-xy}, & \text{if } (x,y)\neq(1,1),\\ 1, & \text{if } (x,y)=(1,1).\end{cases}$$

In particular it is coherent to assess $z_S>0$ when $x=0$. However, for the object $(B|K)|_S(A|H)$ coherence also requires that $z_S=\mu_S x$ (see (44)). Then, coherence requires that $z_S=0$ when we consider the assessment $(0,y,z_S,\mu_S)$ on $\{A|H,B|K,(A|H)\wedge_S(B|K),(B|K)|_S(A|H)\}$.

Theorem 17. *Let A, B, H, K be any logically independent events. The set Π of all the coherent assessments (x,y,z,μ) on the family $\mathcal{F}=\{A|H,B|K,(A|H)\wedge_S(B|K),(B|K)|_S(A|H)\}$ is $\Pi=\Pi'\cup\Pi''$, where*

$$\begin{aligned}&\Pi'=\{(x,y,z,\mu):x\in(0,1],y\in[0,1],z\in[z',z''],\mu=\tfrac{z}{x}\},\\ &\text{with } z'=\max\{x+y-1,0\}, z''=\begin{cases}\frac{x+y-2xy}{1-xy}, & \text{if } (x,y)\neq(1,1),\\ 1, & \text{if } (x,y)=(1,1),\end{cases}\end{aligned} \tag{62}$$

and

$$\Pi''=\{(0,y,0,\mu):y\in[0,1],\mu\geq 0\}. \tag{63}$$

Proof. See Appendix A.7. ☐

Table 9
Interval $[\mu_i', \mu_i'']$ of coherent extensions of the assessment $(x, y) \in [0, 1]^2$ on $\{A|H, B|K\}$ to the iterated conditional $(B|K)|_i(A|H)$, $i \in \{C, dF, F, K, L, B, S, gs\}$, under the assumption that A, H, B, K are logically independent.

Iterated conditioning	Interval of coherent extensions
$\vert_C$	$[0, 1]$
$\vert_{dF}$	$[0, 1]$
$\vert_F$	$[0, 1]$
$\vert_K$	$\left[0, \begin{cases} \min\left\{1, \frac{y}{x}\right\}, & \text{if } x \neq 0; \\ 1, & \text{if } x = 0. \end{cases}\right]$
$\vert_L$	$\left[0, \begin{cases} \min\left\{1, \frac{y}{x}\right\}, & \text{if } x \neq 0; \\ 1, & \text{if } x = 0. \end{cases}\right]$
$\vert_B$	$\left[0, \begin{cases} \frac{1}{x}, & \text{if } 0 < x < 1, \\ 1, & \text{if } x = 0 \vee x = 1. \end{cases}\right]$
$\vert_S$	$\begin{cases} \left[\max\{\frac{x+y-1}{x}, 0\}, \begin{cases} \frac{x+y-2xy}{x(1-xy)}, & \text{if } (x, y) \neq (1, 1); \\ 1, & \text{if } (x, y) = (1, 1). \end{cases}\right] & \text{if } x \neq 0; \\ \mu \geq 0, & \text{if } x = 0; \end{cases}$
$\vert_{gs}$	$\left[\begin{cases} \max\{\frac{x+y-1}{x}, 0\}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases} \begin{cases} \min\{1, \frac{y}{x}\}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}\right]$

Remark 20. From Theorem 17 we observe that, when $x = 0$, the set of all coherent assessments (x, y, z, μ) on $\mathcal{F}$ is Π'' given in (63). Then, in this case, $\mu = \mathbb{P}[(B|K)|_S(A|H)] = \mathbb{P}[(AHBK + AH\bar{K} + (1 + \mu)\overline{H}BK)|(AH \vee \overline{H}BK)]$ is coherent for every value $\mu \geq 0$.

Based on Theorem 17, when $x > 0$, we obtain the following result on the lower and upper bounds for $\mathbb{P}[(B|K)|_S(A|H)]$.

Theorem 18. *Let A, B, H, K be any logically independent events. Given a coherent assessment (x, y) on $\{A|H, B|K\}$, with $x \neq 0$, for the iterated conditional $(B|K)|_S(A|H)$ the extension $\mu_S = \mathbb{P}[(B|K)|_S(A|H)]$ is coherent if and only if $\mu_S \in [\mu_S', \mu_S'']$, where*

$$\mu_S' = \max\{\frac{x + y - 1}{x}, 0\} \text{ and } \mu_S'' = \begin{cases} \frac{x+y-2xy}{x(1-xy)}, & \text{if} (x, y) \neq (1, 1); \\ 1, & \text{if } (x, y) = (1, 1). \end{cases} \tag{64}$$

Proof. See Appendix A.8. □

Remark 21. We observe that the lower and upper bounds μ_S' and μ_S'' for $(B|K)|_S(A|H)$, given in (64), do not coincide with the lower and upper bound μ' and μ'' for $B|A$, given in (17). Then, formula P4 is not satisfied by $|_S$. In particular the extension $\mu_S = \frac{1}{x}$ to $(B|K)|_S(A|H)$ of $(x, 1)$ on $\{A|H, B|K\}$, with $0 < x < 1$, is coherent because $\mu_S'' = \frac{1-x}{x(1-x)} = \frac{1}{x}$. However, the assessment $\mu = \frac{1}{x}$ is not a coherent extension on $\mu = P(B|A)$ of $P(A) = x$, $P(B) = 1$, because by (17) it holds that $\mu'' \leq 1 < \frac{1}{x}$. We also notice that in a bet on the iterated conditional $(B|K)|_S(A|H)$, by paying (the coherent assessment) $\mu_S = \frac{1}{x}$, with $x \in (0, 1)$, the amount 1, received when both the conditional events $A|H$ and $B|K$ are true, does not coincide with the maximum value of the random win. In other words, we obtain that the associated random gain $G = (B|K)|_S(A|H) - \mu_S$ is negative even when both the antecedent and the consequent of the iterated conditional are true. For instance, by setting $x = \frac{1}{3}$ and $\mu_S = \frac{1}{x} = 3$, if the event $\overline{H}\,\bar{B}K$ is true, then the iterated conditional $(B|K)|_S(A|H) = \frac{1}{x} - 1 = 3 - 1 = 2 > 1$ and hence $G = 2 - 3 = -1$.

The iterated conditioning $|_{gs}$. We recall that $|_{gs}$, unlike all the other iterated conditioning, satisfies property P4 ([62, Theorem 4]). Indeed, given a coherent assessment (x, y) on $\{A|H, B|K\}$, under logical independence, for the iterated conditional $(B|K)|_{gs}(A|H)$ the extension $\mu = \mathbb{P}[(B|K)|_{gs}(A|H)]$ is coherent if and only if $\mu \in [\mu_{gs}', \mu_{gs}'']$, where

$$\mu_{gs}' = \begin{cases} \max\{\frac{x+y-1}{x}, 0\}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases} \quad \mu_{gs}'' = \begin{cases} \min\{1, \frac{y}{x}\}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0, \end{cases}$$

which coincide with μ' and μ'' given in (17), respectively. As we can see from Table 9, only the iterated conditioning $|_{gs}$ satisfies property P4.

Table 10
Numerical values of $(B|K)|_i A$, $i \in \{K, L, B, S, gs\}$ and of the conditional event $B|AK$. We denote $x = P(A)$, $y = P(B|K)$, $z = P(B|AK)$ and $\mu_i = \mathbb{P}[(B|K)|_i A]$, $i \in \{K, L, B, S, gs\}$. We observe that $(B|K)|_i A \neq B|AK$, $i \in \{K, L, B, S, gs\}$.

	$(B\|K)\|_K A$	$(B\|K)\|_L A$	$(B\|K)\|_B A$	$(B\|K)\|_S A$	$(B\|K)\|_{gs} A$	$B\|AK$
ABK	1	1	1	1	1	1
$A\bar{B}K$	0	0	0	0	0	0
$A\bar{K}$	$x\mu_K$	$x\mu_L$	$x\mu_B$	1	y	z
$\bar{A}BK$	μ_K	μ_L	μ_B	μ_S	μ_{gs}	z
$\bar{A}\bar{B}K$	μ_K	μ_L	μ_B	μ_S	μ_{gs}	z
$\bar{A}\bar{K}$	μ_K	μ_L	$\mu_B(1+x)$	μ_S	μ_{gs}	z

6.5. Import-export principle

In this section we show that none of the iterated conditioning $|_K, |_L, |_B, |_S, |_{gs}$ satisfies the Import-Export principle. We recall that the Import-Export principle is valid when $(B|K)|A = B|AK$ (see equation (18)). We remind that, in agreement with [1,45] and differently from [52], for the iterated conditional $(B|K)|_{gs}(A|H)$ the Import-Export principle is not valid. As a consequence, as shown in [34] (see also [60,62]), Lewis' triviality results ([51]) are avoided by $|_{gs}$. For what concerns the iterated conditioning $|_K, |_L, |_B, |_S$, it holds that $(B|K)|_i A \neq B|AK$, $i \in \{K, L, B, S\}$, because there are some constituents where the two objects may assume different values (see Table 10). For instance, when $A\bar{K}$ is true and $z = P(B|AK) \neq 1$, it follows that $(B|K)|_S A = 1 \neq z = B|AK$. Then, none of the iterated conditioning $|_K, |_L, |_B, |_S, |_{gs}$ satisfies the Import-Export principle. However, we observe that Import-Export principle could be satisfied under some suitable logical relations among the events A, B, K. For instance, if $A\bar{K} = \emptyset$, it can be easily proved that $(B|K)|_i A = B|AK$, $i \in \{K, L, S, gs\}$. Thus, when $A\bar{K} = \emptyset$, the Import-Export principle is satisfied by $(B|K)|_K A$, $(B|K)|_L A$, $(B|K)|_S A$, and $(B|K)|_{gs} A$.

6.6. Generalized versions of Bayes' rule

In this section, by exploiting property P4, we analyze generalized versions of Bayes' Rule for the iterated conditioning $|_K, |_L, |_B, |_S$, and $|_{gs}$.

From (45) it follows that $\mathbb{P}[(B|K) \wedge_i (A|H)] = \mathbb{P}[(B|K)|_i(A|H)]P(A|H) = \mathbb{P}[(A|H)|_i(B|K)]P(B|K)$, $i \in \{K, L, B, S, gs\}$. Then, when $P(A|H) > 0$ it holds that

$$\mathbb{P}[(B|K)|_i(A|H)] = \frac{\mathbb{P}[(A|H)|_i(B|K)]P(B|K)}{P(A|H)}, \; i \in \{K, L, B, S, gs\}. \tag{65}$$

Formula (65) is a generalization of the following well-known version of Bayes's Rule

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}, \text{ if } P(A) > 0,$$

where the events A, B are replaced by the conditional events $A|H$, $B|K$, respectively, and the conditioning operator $|$ is replaced by the iterated conditioning $|_i$, $i \in \{K, L, B, S, gs\}$. We also recall that, given two events A and B, it holds that $A = AB \vee A\bar{B}$, and hence $P(A) = P(AB) + P(A\bar{B}) = P(A|B)P(B) + P(A|\bar{B})P(B)$. Then, we obtain the following version of Bayes' Rule

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|\bar{B})P(\bar{B})}. \tag{66}$$

We now check, for each iterated conditioning $|_i$, $i \in \{K, L, B, S, gs\}$, the validity of the following generalized version of formula (66)

$$\mathbb{P}[(B|K)|(A|H)] = \frac{\mathbb{P}[(A|H)|(B|K)]P(B|K)}{\mathbb{P}[(A|H)|(B|K)]P(B|K) + \mathbb{P}[(A|H)|(\bar{B}|K)]P(\bar{B}|K)}. \tag{67}$$

Based on (45), for each $i \in \{K, L, B, S, gs\}$, it follows that

$$\mathbb{P}[(A|H) \wedge_i (B|K)] + \mathbb{P}[(A|H) \wedge_i (\bar{B}|K)] = \mathbb{P}[(A|H)|_i(B|K)]P(B|K) + \mathbb{P}[(A|H)|_i(\bar{B}|K)]P(\bar{B}|K). \tag{68}$$

We recall that when A, B, and $\bar{B}$ in the equality $A = (A \wedge B) \vee (A \wedge \bar{B})$ are replaced by the conditional events $A|H, B|K$, and $\bar{B}|K$, respectively, and the conjunction $\wedge$ is replaced by $\wedge_i$, $i \in \{K, L, B, S\}$, it holds that the corresponding equality is not satisfied. Indeed we have that

$$(A|H) \neq [(A|H) \wedge_i (B|K)] \vee_i [(A|H) \wedge_i (\bar{B}|K)], \;\; i \in \{K, L, B, S\}.$$

More precisely, it holds that (see [40, Section 4.1])

- $[(A|H) \wedge_K (B|K)] \vee_K [(A|H) \wedge_K (\bar{B}|K)] = AHK|(AHK \vee \bar{A}H) \neq A|H$;
- $[(A|H) \wedge_L (B|K)] \vee_L [(A|H) \wedge_L (\bar{B}|K)] = AHK|(H \vee \bar{K}) \neq A|H$;
- $[(A|H) \wedge_B (B|K)] \vee_B [(A|H) \wedge_B (\bar{B}|K)] = A|(HK) \neq A|H$;

- $[(A|H) \wedge_S (B|K)] \vee_S [(A|H) \wedge_S (\bar{B}|K)] = (A \vee \overline{H})|(H \vee K) \neq A|H$.

Moreover, for each $i \in \{K, L, B, S\}$, we observe that the conditional probability $P(A|H)$ does not necessarily coincide with

$$P((A|H) \wedge_i (B|K)) + P((A|H) \wedge_i (\bar{B}|K)).$$

Indeed, it can be proved that the assessment $(1,0,0)$ on $\{A|H, (A|H) \wedge_i (B|K), (A|H) \wedge_i (\bar{B}|K)\}$, $i \in \{K, L, B\}$, as well as the assessment $(1,1,1)$ on $\{A|H, (A|H) \wedge_S (B|K), (A|H) \wedge_S (\bar{B}|K)\}$, is coherent. As a consequence, from (68), we obtain that the probability $P(A|H)$ does not necessarily coincide with

$$\mathbb{P}[(A|H)|_i(B|K)]P(B|K) + \mathbb{P}[(A|H)|_i(\bar{B}|K)]P(\bar{B}|K).$$

Thus, for each iterated conditioning $|_i$, $i \in \{K, L, B, S\}$, the generalized version of Bayes's Rule given in formula (67) is not satisfied. However, we recall that ([36])

$$(A|H) = (A|H) \wedge_{gs} (B|K) + (A|H) \wedge_{gs} (\bar{B}|K).$$

Then, by the linearity property of a coherent prevision and by (68), it follows that

$$\begin{aligned} P(A|H) &= \mathbb{P}[(A|H) \wedge_{gs} (B|K)] + \mathbb{P}[(A|H) \wedge_{gs} (\bar{B}|K)] = \\ &= \mathbb{P}[(A|H)|_{gs}(B|K)]P(B|K) + \mathbb{P}[(A|H)|_{gs}(\bar{B}|K)]P(\bar{B}|K). \end{aligned}$$

Hence, when $P(A|H) > 0$, it holds that

$$\mathbb{P}[(B|K)|_{gs}(A|H)] = \frac{\mathbb{P}[(A|H)|_{gs}(B|K)]P(B|K)}{\mathbb{P}[(A|H)|_{gs}(B|K)]P(B|K) + \mathbb{P}[(A|H)|_{gs}(\bar{B}|K)]P(\bar{B}|K)}. \quad (69)$$

Therefore, the generalization of the second version of Bayes' Rule only holds for $|_{gs}$ and does not hold for $|_K, |_L, |_B$, and $|_S$.

7. Generalized version of Modus Ponens and two premise centering

In this section, for selected definitions of the iterated conditioning with possible value in $[0,1]$, based on the probability propagation rules obtained in the previous sections and exploited for checking the validity of property P4 (see Table 9), we study the p-validity of the generalization of two inference rules: Modus Ponens and two-premise centering. We first recall the notions of p-consistency and p-entailment for conditional random quantities, which take values in a finite subset of $[0,1]$ ([62]). These concepts are based on the notions of p-consistency and p-entailment given for conditional events by Adams ([1]) and also studied in the setting of coherence in [31].

Definition 6. Let $\mathcal{F}_n = \{X_i|H_i,\ i = 1, \dots, n\}$ be a family of n conditional random quantities which take values in a finite subset of $[0,1]$. Then, $\mathcal{F}_n$ is *p-consistent* if and only if, the (prevision) assessment $(\mu_1, \mu_2, \dots, \mu_n) = (1, 1, \dots, 1)$ on $\mathcal{F}_n$ is coherent.

Definition 7. A p-consistent family $\mathcal{F}_n = \{X_i|H_i,\ i = 1, \dots, n\}$ *p-entails* a conditional random quantity $X|H$, which takes values in a finite subset of $[0,1]$, denoted by $\mathcal{F}_n \Rightarrow_p X|H$, if and only if for any coherent (prevision) assessment $(\mu_1, \dots, \mu_n, z)$ on $\mathcal{F}_n \cup \{X|H\}$: if $\mu_1 = \cdots = \mu_n = 1$, then $z = 1$.

We say that the inference from a p-consistent family of premises $\mathcal{F}_n$ to a conclusion $X|H$ is *p-valid* if and only if $\mathcal{F}_n$ p-entails $X|H$.

We recall that as the iterated conditionals $(B|K)|_C(A|H)$, $(B|K)|_{dF}(A|H)$, and $(B|K)|_F(A|H)$ are conditional events, their indicators take values in the interval [0,1]. Moreover, from Table 8 we observe that the iterated conditionals $(B|K)|_K(A|H)$, $(B|K)|_L(A|H)$ and $(B|K)|_{gs}(A|H)$ take values in $[0,1]$. On the other hand, the iterated conditionals $(B|K)|_B(A|H)$ and $(B|K)|_S(A|H)$ may take values outside the interval $[0,1]$. Thus, in order to examine the p-validity of generalized inference rules, we will only consider the iterated conditionals $(B|K)|_i(A|H)$, $i \in \{C, dF, F, K, L, gs\}$.

7.1. Modus Ponens

We recall that, given two events A and B the Modus Ponens inference, with (p-consistent) premise set $\{A, B|A\}$ and conclusion B, is p-valid ([65], see also [33,43]), that is

$$\{A, B|A\} \Rightarrow_p B. \quad (70)$$

For each $i \in \{C, dF, F, K, L, gs\}$, we will study the p-validity of the generalized version of Modus Ponens obtained when the events A, B are replaced by the conditional events $A|H, B|K$ and the conditional event $B|A$ is replaced by the iterated conditional $(B|K)|_i(A|H)$.[5] An example of this generalization is (see also [29,61]):

[5] The particular case where $K = \Omega$ and $|_i = |_{gs}$ has been studied in [61].

$$\overbrace{\textit{The cup is fragile if made of glass}}^{A|H}.$$

$$\text{If } \overbrace{\textit{the cup is fragile if made of glass}}^{A|H}, \text{ then } \overbrace{\textit{it breaks if dropped}}^{B|K}.$$

$$\text{Therefore, } \overbrace{\textit{the cup breaks if dropped}}^{B|K}.$$

Then, we check the p-validity of the generalized version of Modus Ponens where the premise set is $\{A|H, (B|K)|_i(A|H)\}$ and the conclusion is $B|K$, that is

$$P(A|H) = 1,\ \mathbb{P}[(B|K)|_i(A|H)] = 1 \implies P(B|K) = 1. \tag{71}$$

It is easy to verify (see Table 9) that the family $\{A|H, (B|K)|_i(A|H)\}$ is p-consistent, $i \in \{C, dF, F, K, L, gs\}$. We will show that the generalized version of Modus Ponens is p-valid for $|_i \in \{|_K, |_L, |_{gs}\}$. Indeed, in these cases from Table 9 it follows that $\mathbb{P}[(B|K)|_i(A|H)] \le \mu_i'' = \min\{1, \frac{y}{x}\} = y = P(B|K)$, when $P(A|H) = x = 1$. Then, $P(B|K)$ is necessarily equal to 1, when $P(A|H) = 1$ and $\mathbb{P}[(B|K)|_i(A|H)] = 1$ and hence formula (71) is satisfied. Thus,

$$\{A|H,\ (B|K)|_i(A|H)\} \Rightarrow_p B|K,\ \ i \in \{K, L, gs\}. \tag{72}$$

For $i \in \{C, dF, F\}$, as illustrated in Table 9, the interval of coherent extensions $[\mu_i', \mu_i'']$ on $(B|K)|_i(A|H)$ of the assessment $(1, 0)$ on $\{A|H, B|K\}$ coincides with the unit interval $[0, 1]$. In particular the assessment $(1, 1, 0)$ on $\{A|H, (B|K)|_i(A|H), (B|K)\}$ is coherent and hence formula (71) is not satisfied for $i \in \{C, dF, F\}$, that is

$$\{A|H,\ (B|K)|_i(A|H)\} \not\Rightarrow_p B|K,\ \ i \in \{C, dF, F\}. \tag{73}$$

Concerning the example above, from P(*the cup is fragile if made of glass*) $= 1$ and $\mathbb{P}$[If *the cup is fragile if made of glass,* then *it breaks if dropped*] $= 1$, it follows (as a natural result) that P(*the cup breaks if dropped*) $= 1$ when the iterated conditioning is interpreted as $|_K, |_L, |_{gs}$, because (71) is satisfied in these cases. However, the same conclusion does not follow (which is strange) when the iterated conditioning is interpreted as $|_C, |_{dF}$, and $|_F$ because (71) does not hold in these cases.

7.2. Two-premise centering

We recall that the inference of *two-premise centering*, that is inferring *if A then B* from the two separate premises A and B, is p-valid ([62]). Indeed, given two events A and B it holds that

$$\{A, B\} \Rightarrow_p B|A. \tag{74}$$

For each $i \in \{C, dF, F, K, L, gs\}$, we will study the p-validity of the generalized version of two-premise centering, where the unconditional events A, B are replaced by the conditional events $A|H$, $B|K$, respectively, and the conditional event $B|A$ is replaced by the iterated conditional $(B|K)|_i(A|H)$. An example of this generalization is

$$\overbrace{\textit{The cup is fragile if made of glass}}^{A|H}.$$

$$\overbrace{\textit{The cup breaks if dropped}}^{B|K}.$$

$$\text{Therefore, if } \overbrace{\textit{the cup is fragile if made of glass}}^{A|H}, \text{ then it } \overbrace{\textit{breaks if dropped}}^{B|K}.$$

For each i, we consider as premise set $\{A|H, B|K\}$, as conclusion the iterated conditional $(B|K)|_i(A|H)$ and we check the p-validity of the generalized version of two-premise centering, that is

$$P(A|H) = 1,\ P(B|K) = 1 \implies \mathbb{P}[(B|K)|_i(A|H)] = 1. \tag{75}$$

Of course, as the assessment $(x, y) = (1, 1)$ on $\{A|H, B|K\}$ is coherent, the premise set $\{A|H, B|K\}$ is p-consistent. We recall that generalized two-premise centering is p-valid for $|_{gs}$ ([62, Equation (13)]), that is

$$\{A|H, B|K\} \Rightarrow_p (B|K)|_{gs}(A|H). \tag{76}$$

The previous result can be also obtained from Table 9, by observing that the lower bound of the coherent extensions on $(B|K)|_{gs}(A|H)$ of the assessment $x = y = 1$ is $\mu_i' = \max\{\frac{x+y-1}{x}, 0\} = 1$. Moreover, for $i \in \{C, dF, F, K, L\}$, from Table 9 it follows that any value in $[0, 1]$ is a coherent extension on $(B|K)|_i(A|H)$ of $P(A|H) = P(B|K) = 1$. Thus, (75) is not satisfied by $|_C, |_{dF}, |_F, |_K, |_L$, and hence

Table 11
Properties P1-P4, the non validity of Import-Export Principle (No IE), and iterated conditionals in their own logic. The symbol ✓ means that the property is satisfied. The blank space means that the property is not satisfied.

	Property	$\vert_C$	$\vert_{dF}$	$\vert_F$	$\vert_K$	$\vert_L$	$\vert_B$	$\vert_S$	$\vert_{gs}$
No IE	$(B\vert K)\vert A \neq B\vert AK$				✓	✓	✓	✓	✓
P1	$(B\vert K)\vert(A\vert H) = [(A\vert H) \wedge (B\vert K)]\vert(A\vert H)$		✓	✓	✓	✓	✓	✓	✓
P2	$(A\vert H) \wedge (B\vert K) \leq (B\vert K)\vert(A\vert H)$		✓	✓	✓	✓	✓	✓	✓
P3	$\mathbb{P}[(A\vert H) \wedge (B\vert K)] = \mathbb{P}[(B\vert K)\vert(A\vert H)]P(A\vert H)$				✓	✓	✓	✓	✓
P4	Lower and upper bounds for $(B\vert K)\vert(A\vert H)$								✓

$$\{A|H, B|K\} \not\Rightarrow_p (B|K)|_i(A|H), \quad i \in \{C, dF, F, K, L\}. \tag{77}$$

Concerning the example above, from P(*the cup is fragile if made of glass*) $= 1$ and P(*the cup breaks if dropped*) $= 1$ it follows that $\mathbb{P}$[If *the cup is fragile if made of glass,* then *it breaks if dropped*] $= 1$, when the iterated conditioning is interpreted as $|_{gs}$, because the generalized two-premise centering is p-valid. However, the same conclusion does not follow when the iterated conditioning is interpreted as $|_C, |_{dF}, |_F, |_K, |_L$, because the generalized two-premise centering is p-invalid in these cases.

8. Conclusions

We recalled the trivalent logics of Kleene-Lukasiewicz-Heyting-de Finetti, Lukasiewicz, Bochvar-Kleene, and Sobociński and the notion of compound conditional as conditional random quantity. We considered four basic logical and probabilistic properties, P1-P4, valid for events and conditional events. We generalized them by replacing events A and B with conditional events $A|H$ and $B|K$ and we checked their validity and the validity of the Import-Export principle for selected notions of iterated conditioning. In particular, we studied the iterated conditioning introduced in trivalent logics by Cooper-Calabrese ($|_C$), de Finetti ($|_{dF}$), and Farrel ($|_F$), by also focusing on the numerical representation of the truth-values. We observed that the notions of conjunction and disjunction of conditional events used by Cooper and Calabrese coincide with $\wedge_S$ and $\vee_S$, respectively. Farrell and de Finetti defined two different structures of iterated conditioning in the same trivalent logic where conjunction and disjunction are $\wedge_K$ and $\vee_K$, respectively. We computed the set of coherent probability assessments on the families of events $\{A|H, (B|K)|_C(A|H), (A|H) \wedge_S (B|K)\}$, $\{A|H, B|K, (B|K)|_C(A|H)\}$, as well as $\{A|H, (B|K)|_i(A|H), (A|H) \wedge_K (B|K)\}$ and $\{A|H, B|K, (B|K)|_i(A|H)\}$ with $i \in \{dF, F\}$. In Table 11 we summarize the results for the properties given in Section 2.5. We observe that $|_C$, $|_{dF}$, and $|_F$ satisfy the Import-Export principle and none of these objects, defined in the framework of trivalent logics, satisfies the compound probability theorem (P3).

By exploiting the structure $\square|\bigcirc = \square \wedge \bigcirc + \mathbb{P}(\square|\bigcirc)\bar{\bigcirc}$ used in order to define $|_{gs}$ from the conjunction $\wedge_{gs}$, for each $i \in \{K, L, B, S\}$, we defined the iterated conditioning $|_i$ from the conjunction $\wedge_i$. We observed that the iterated conditionals $(B|K)|_i(A|H)$, $i \in \{K, L, B, S, gs\}$, are all conditional random quantities (not always in [0,1]) which satisfy the compound prevision theorem (property P3). We also noticed that properties P1, P2 and the non validity of the Import-Export principle are satisfied by $|_K, |_L, |_B, |_S$, and $|_{gs}$ (see Table 11). However, property P4 is not satisfied by $|_K, |_L, |_B$, and $|_S$. We observed that all the basic logical and probabilistic properties are satisfied only by the iterated conditioning $|_{gs}$. We also showed that a generalized version of the Bayes' Rule $P(B|A) = \frac{P(A|B)P(B)}{P(A)}$ for iterated conditionals is satisfied by $|_i$ with $i \in \{K, L, B, S, gs\}$. However, a generalized version of the Bayes' Rule in the following version $P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B)+P(A|\bar{B})P(\bar{B})}$ for iterated conditionals, only holds for $|_{gs}$ and does not hold for $|_K, |_L, |_B, |_S$.

We discussed the implications of the obtained results on the probability propagation rules from $\{A|H, B|K\}$ to the iterated conditional $(B|K)|_i(A|H)$, by studying the p-validity of the generalized versions of Modus Ponens and two-premise centering, which involve conditional events and iterated conditionals. We observed that for $|_B$ and $|_S$, as the iterated conditionals $(B|K)|_B(A|H)$ and $(B|K)|_S(A|H)$ can take value outside the interval [0,1], the study of p-validity is meaningless. For the remaining ones $|_C, |_{dF}, |_F, |_K, |_L$, and $|_{gs}$, we showed that only for the iterated conditioning $|_{gs}$ turns out that both generalized versions of the inference rules are p-valid. We also examined two examples of the generalized inference rules in natural language by observing that, when we adopt the notion of iterated conditioning $|_i$, with $|_i \in \{|_C, |_{dF}, |_F, |_K, |_L\}$, some results which are counterintuitive in commonsense reasoning can be obtained.

Therefore, based on the results illustrated above, only the iterated conditioning $|_{gs}$, which is based on the conjunction $\wedge_{gs}$ introduced in the framework of conditional random quantities, preserves all the basic logical and probabilistic properties; moreover, Lewis' triviality results are avoided in particular because the Import-Export Principle is not satisfied. However, some basic probabilistic properties are not preserved when the other notions of iterated conditioning are adopted. Then, the 'probability' of these iterated conditionals does not properly allow to represent uncertainty in conditional sentences of human reasoning and the study of uncertainty is necessary for understanding human and artificial rationality in general.

Then, the results obtained in this paper can be useful in AI in order to build a theory of formal reasoning which properly manages the uncertainty present in conditional or compound conditional sentences. Once it is described how the conditionals and the logical operations among them are interpreted, by means of our results, it is possible, for instance, to understand how a (coherent) agent propagates the uncertainty present in the conditionals $A|H$ and $B|K$ to the iterated conditional $(B|K)|_i(A|H)$. Then, it is crucial

Table A.12
Constituents and points Q_h's associated with $\mathcal{F} = \{A|H, (B|K)|_{dF}(A|H), (A|H) \wedge_K (B|K)\}$ and $\mathcal{P} = (x, y, z)$.

	C_h	$A\|H$	$(B\|K)\|_{dF}(A\|H)$	$(A\|H) \wedge_K (B\|K)$	Q_h
C_1	$AHBK$	1	1	1	Q_1
C_2	$\bar{A}H$	0	y	0	Q_2
C_3	$AH\bar{B}K$	1	0	0	Q_3
C_4	$\bar{H}\,\bar{B}K$	x	y	0	Q_4
C_5	$AH\bar{K}$	1	y	z	Q_5
C_0	$\bar{H}BK \vee \bar{H}\,\bar{K}$	x	y	z	Q_0

to know, for each iterated conditioning, which (desirable) basic logical and probabilistic properties are preserved. In particular, our results allow to know that when the iterated conditioning $|_i$ belongs to $\{|_C, |_{dF}, |_F, |_K, |_L\}$, as the two-premise centering is not p-valid, then the agent agrees with the following probabilistically non-informative inference: from $P(A|H) = 1$ and $P(B|K) = 1$ infer that every $\mathbb{P}[(B|K)|_i(A|H)] \in [0,1]$ is coherent.

As already done for the iterated conditioning $|_{gs}$ in [38], future work will concern the (possible) characterization of the p-entailment of Adams in the setting of coherence by means of the iterated conditioning $|_K, |_L, |_B$, and $|_S$. We will also study the different notions of iterated conditioning in the framework of nonmonotonic reasoning in System P and in other non-classical logics, like connexive logic ([58]). Finally, as done in [38] for the case of conjoined conditionals, in the more general theory of compound conditionals as suitable conditional random quantities ([10]), based on the structure $\square|\bigcirc = \square \wedge \bigcirc + \mathbb{P}(\square|\bigcirc)\bar{\bigcirc}$, future work for this group of research could be devoted to the study of the (extended) iterated conditioning $|_{gs}$ to the case where $\square$ and $\bigcirc$ are compound conditionals.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Appendix A

A.1. Proof of Theorem 6

Theorem 6. *Let A, B, H, K be any logically independent events. A probability assessment $\mathcal{P} = (x, y, z)$ on the family of conditional events $\mathcal{F} = \{A|H, (B|K)|_{dF}(A|H), (A|H) \wedge_K (B|K)\}$ is coherent if and only if $(x, y) \in [0,1]^2$ and $z \in [z', z'']$, where $z' = 0$ and $z'' = xy$.*

Proof. The constituents C_h's and the point Q_h's associated with the assessment $\mathcal{P} = (x, y, z)$ on $\mathcal{F}$ are (see also Table A.12)

$$C_1 = AHBK, C_2 = \bar{A}HBK \vee \bar{A}H\bar{B}K \vee \bar{A}H\bar{K} = \bar{A}H, C_3 = AH\bar{B}K, C_4 = \bar{H}\,\bar{B}K, C_5 = AH\bar{K}, C_0 = \bar{H}BK \vee \bar{H}\,\bar{K},$$

and

$$Q_1 = (1,1,1), Q_2 = (0,y,0), Q_3 = (1,0,0), Q_4 = (x,y,0), Q_5 = (1,y,z), \mathcal{P} = Q_0 = (x,y,z).$$

We denote by $\mathcal{I}$ the convex hull of points $Q_1, \dots, Q_5$ (see Fig. A.3). The system (Σ) in (5) associated with the pair $(\mathcal{F}, \mathcal{P})$ becomes

$$\begin{cases} \lambda_1 + \lambda_3 + x\lambda_4 + \lambda_5 = x, \\ \lambda_1 + y\lambda_2 + y\lambda_4 + y\lambda_5 = y, \\ \lambda_1 + z\lambda_5 = z, \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 1, \\ \lambda_i \geq 0, \ i = 1, \dots, 5. \end{cases} \tag{A.1}$$

Lower bound. We first prove that the assessment $(x, y, 0)$ is coherent for every $(x, y) \in [0,1]^2$. We observe that $\mathcal{P} = (x, y, 0) = Q_4$, so a solution of (A.1) is given by $\Lambda = (0, 0, 0, 1, 0)$.

Then, by considering the function ϕ as defined in (6), it holds that

$$\begin{aligned}
&\phi_1(\Lambda) = \textstyle\sum_{h:C_h \subseteq H} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_5 = 0,\\
&\phi_2(\Lambda) = \textstyle\sum_{h:C_h \subseteq (AHK)} \lambda_h = \lambda_1 + \lambda_3 = 0,\\
&\phi_3(\Lambda) = \textstyle\sum_{h:C_h \subseteq (AHBK \vee \bar{A}H \vee \bar{B}K)} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1 > 0.
\end{aligned}$$

Let $\mathcal{S}' = \{(0, 0, 0, 1, 0)\}$ denote a subset of the set $\mathcal{S}$ of all solutions of (A.1). We have that $M_1' = 0$, $M_2' = 0$, $M_3' = 1$ (as defined in (7)). Then $I_0' = \{1, 2\}$ and we set $\mathcal{K} = \mathcal{F}_0' = \{A|B, (B|K)|_{dF}(A|H)\}$ and $\mathcal{V} = \mathcal{P}_0' = (x, y)$. The constituents C_h's and the point Q_h's associated with the assessment $\mathcal{P}_0$ on $\mathcal{F}_0$ are

$$\begin{aligned}
&C_1 = AHBK, C_2 = \bar{A}HBK \vee \bar{A}H\bar{B}K \vee \bar{A}H\bar{K} = \bar{A}H,\\
&C_3 = AH\bar{B}K, C_4 = AH\bar{K}, C_0 = \overline{H}\,\bar{K} \vee \overline{H}BK \vee \overline{H}\,\bar{B}K = \overline{H},
\end{aligned}$$

and

$$Q_1 = (1, 1),\ Q_2 = (0, y),\ Q_3 = (1, 0),\ Q_4 = (1, y), \mathcal{P} = Q_0 = (x, y).$$

The system (Σ) in (5) associated with the pair $(\mathcal{K}, \mathcal{V})$ becomes

$$\begin{cases}
\lambda_1 + \lambda_3 + \lambda_4 = x,\\
\lambda_1 + y\lambda_2 + y\lambda_4 = y,\\
\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + = 1,\\
\lambda_i \geq 0,\ i = 1, \ldots, 4.
\end{cases} \tag{A.2}$$

We observe that $(x, y) = (1 - x)Q_2 + xQ_4$ so a solution of (A.2) is $\Lambda = (0, 1 - x, 0, x)$.

By considering the function ϕ as defined in (6), it holds that

$$\begin{aligned}
&\phi_1(\Lambda) = \textstyle\sum_{h:C_h \subseteq H} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1,\\
&\phi_2(\Lambda) = \textstyle\sum_{h:C_h \subseteq (AHK)} \lambda_h = \lambda_1 + \lambda_3 = 0.
\end{aligned}$$

Let $\mathcal{S}' = \{(0, 1 - x, 0, x)\}$ denote a subset of the set $\mathcal{S}$ of all solutions of (A.2). We have that $M_1' = 0$, $M_2' = 1$, (as defined in (7)). Then, the set I_0' associated with $(\mathcal{K}, \mathcal{V})$ is $I_0' = \{2\}$. We observe that the sub-assessment y on $\{(B|K)|_{dF}(A|H)\}$ is coherent for every $y \in [0, 1]$. Then, by Theorem 1, the assessment $(x, y, 0)$ on $\mathcal{F}$ is coherent $\forall (x, y) \in [0, 1]^2$.

Upper bound. We verify that the assessment (x, y, xy) on $\mathcal{F}$ is coherent for every $(x, y) \in [0, 1]^2$. Moreover, we show that $z'' = xy$ is the upper bound for $z = P((A|H) \wedge_K (B|K))$ by showing that any assessment (x, y, z) on $\mathcal{F}$ with $(x, y) \in [0, 1]^2$ and $z > xy$ is not coherent.

We observe that

$$(x, y, xy) = xyQ_1 + (1 - x)Q_2 + x(1 - y)Q_3.$$

Then, the vector $\Lambda = (xy, 1 - x, x(1 - y), 0, 0)$ is a solution of (A.1). Moreover, it holds that

$$\begin{aligned}
&\phi_1(\Lambda) = \textstyle\sum_{h:C_h \subseteq H} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_5 = 1 > 0,\\
&\phi_2(\Lambda) = \textstyle\sum_{h:C_h \subseteq (AHK)} \lambda_h = \lambda_1 + \lambda_3 = xy + 1 - x,\\
&\phi_3(\Lambda) = \textstyle\sum_{h:C_h \subseteq (AHBK \vee \bar{A}H \vee \bar{B}K)} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1 > 0.
\end{aligned}$$

Let $\mathcal{S}' = \{(xy, 1 - x, x(1 - y), 0, 0)\}$ denote a subset of the set $\mathcal{S}$ of all solutions of (A.1). We have that $M_1' = 1$, $M_2' = xy + 1 - x$, $M_3' = 1$ (as defined in (7)). We distinguish two cases: (i) $(x \neq 1) \vee (y \neq 0)$, (ii) $(x = 1) \wedge (y = 0)$. In the case (i) we get $M_1' > 0$, $M_2' > 0$, $M_3' > 0$ and hence $I_0' = \emptyset$. By Theorem 1, the assessment (x, y, xy) is coherent $\forall (x, y) \in [0, 1]^2$. In the case (ii) we get $M_1' > 0$, $M_2' = 0$, $M_3' > 0$, then $I_0' = \{2\}$. We observe that the sub-assessment $\mathcal{P}_0' = y$ on $\mathcal{F}_0' = \{(B|K)|_{dF}(A|H)\}$ is coherent for every $y \in [0, 1]$. Then, by Theorem 1, the assessment (x, y, xy) on $\mathcal{F}$ is coherent $\forall (x, y) \in [0, 1]^2$.

We verify that $z'' = xy$ is the upper bound for z, by showing that the assessment (x, y, z) is incoherent when $z > xy$.

Let $z > xy$. We distinguish the following cases: (a) $y \neq 0$; (b) $y = 0$.

Case (a). We observe that the points Q_1, Q_2, Q_3 belong to the plane $\pi : yX + Y - Z = y$, where X, Y, Z are the axes coordinates. We set $f(X, Y, Z) = yX + Y - Z$ and we observe that $f(\mathcal{P}) = f(x, y, z) = yx + y - z$. For each $h = 1, 2, 3, 4, 5$, we compute the difference $f(Q_h) - f(\mathcal{P})$. We have

$$\begin{aligned}
&f(Q_1) - f(\mathcal{P}) = f(Q_2) - f(\mathcal{P}) = f(Q_3) - f(\mathcal{P}) = z - xy;\\
&f(Q_4) - f(\mathcal{P}) = z;\ \ f(Q_5) - f(\mathcal{P}) = y - xy = y(1 - x);
\end{aligned}$$

We consider the sub-cases: (i) $x < 1$; (ii) $x = 1$.

(i) We recall that, by setting the stakes $s_1 = y$, $s_2 = 1$, $s_3 = -1$, it holds that $g_h = f(Q_h) - f(\mathcal{P})$, $h = 1, \ldots, 5$, where g_h is the value of the random gain G associated with the constituent $C_h \subseteq \mathcal{H}$. As $x < 1$, it follows that $g_h = f(Q_h) - f(\mathcal{P}) > 0$, $h = 1, \ldots, 5$. Thus, as

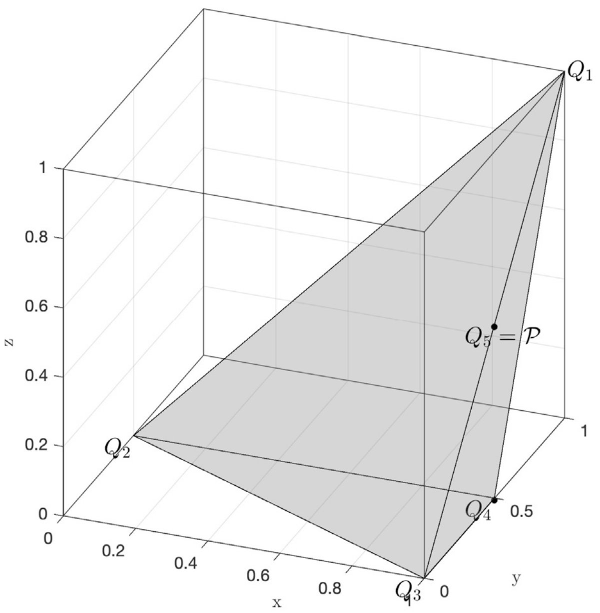

Fig. A.3. Convex hull of the points Q_1, Q_2, Q_3, Q_4, Q_5 associated with the pair $(\mathcal{F}, \mathcal{P})$, where $\mathcal{F} = \{A|H, (B|K)|_{dF}(A|H), (A|H) \wedge_K (B|K)\}$ and $\mathcal{P} = (x, y, z)$. In the figure the numerical values are: $x = 1$, $y = 0.5$, $z = 0.5$. Notice that $Q_5 = \mathcal{P}$.

there exist (s_1, s_2, s_3) such that $\min G_{\mathcal{H}} \max G_{\mathcal{H}} > 0$, it follows that the assessment (x, y, z) is not coherent when $z > xy$ and $x < 1$.
(ii). In this case, as $x = 1$, it follows that $\mathcal{P} = (x, y, z) = (1, y, z) = Q_5 \in \mathcal{I}$. Then, the vector $\Lambda = (0, 0, 0, 0, 1)$ is a solution of (A.1). Then, by considering the function ϕ as defined in (6), it holds that

$$\phi_1(\Lambda) = \sum_{h:C_h \subseteq H} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_5 = 1 > 0,$$
$$\phi_2(\Lambda) = \sum_{h:C_h \subseteq (AHK)} \lambda_h = \lambda_1 + \lambda_3 = 0,$$
$$\phi_3(\Lambda) = \sum_{h:C_h \subseteq (AHBK \vee \bar{A}H \vee \bar{B}K)} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0.$$

Then, we get $\mathcal{I}_0 \subseteq \{2, 3\}$. and we study the coherence of the sub-assessment $\mathcal{P}_0 = (y, z)$ on $\mathcal{F}_0 = \{(B|K)|_{dF}(A|H), (A|H) \wedge_K (B|K)\}$.

The constituents C_h's and the point Q_h's associated with the pair $(\mathcal{P}_0, \mathcal{F}_0)$ are

$$C_1 = AHBK,\ C_2 = \bar{A}H \vee \overline{H}\,\bar{B}K, C_3 = AH\bar{B}K,\ C_0 = \overline{H}\,\bar{K} \vee AH\bar{K} \vee \overline{H}BK,$$

with $C_h \subseteq \mathcal{H} = HBK \vee \bar{A}H \vee \bar{B}K$, $h = 1, 2, 3$. The associated points Q_h's are

$$Q_1 = (1, 1),\ Q_2 = (y, 0),\ Q_3 = (0, 0),\ \mathcal{P} = Q_0 = (x, y, z).$$

As shown in Fig. A.4 the convex hull $\mathcal{I}$ of the points Q_1, Q_2, and Q_3 is the triangle of vertices $(1, 1), (y, 0)$, and $(0, 0)$. We observe that $\mathcal{P}_0 = (y, z) \notin \mathcal{I}$ because $z > y$. Then, the sub-assessment $\mathcal{P}_0$ on $\mathcal{F}_0$ is not coherent. Therefore, by Theorem 1 the assessment $\mathcal{P} = (1, y, z)$ on $\mathcal{F}$ is not coherent too.

Notice that, if $y = 0$, $(0, z) \in \mathcal{I} \iff z = 0$, so if $z > 0$ the assessment is not coherent.

So in this case we have that the probability assessment is coherent if and only if $z \leq xy$.

Case (b). In this case $y = 0$. We have already analyzed the situation $(x = 1) \wedge (y = 0)$ in the precedent case, so let's assume $x \neq 1$.

We observe that the points Q_1, Q_2, Q_5 belong to the plane $\pi : zX + (1 - z)Y - Z = 0$, where X, Y, Z are the axes coordinates. We set $f(X, Y, Z) = zX + (1 - z)Y - Z$ and we observe that $f(\mathcal{P}) = f(x, y, z) = -(1 - x)z$. For each $h = 1, 2, 3, 4, 5$, we consider the quantity $f(Q_h) - f(P)$. Then, we obtain

$$f(Q_1) - f(\mathcal{P}) = z(1 - x);$$
$$f(Q_2) - f(\mathcal{P}) = z(1 - x);$$
$$f(Q_3) - f(\mathcal{P}) = z(2 - x);$$
$$f(Q_4) - f(\mathcal{P}) = z;$$
$$f(Q_5) - f(\mathcal{P}) = z(1 - x).$$

We recall that, by setting the stakes $s_1 = z$, $s_2 = 1 - z$, $s_3 = -1$, it holds that $g_h = f(Q_h) - f(\mathcal{P})$, $h = 1, \ldots, 5$, where g_h is the value of the random gain G associated with the constituent $C_h \subseteq \mathcal{H}$. As $x < 1$, it follows that $g_h = f(Q_h) - f(\mathcal{P}) > 0$, $h = 1, \ldots, 5$. Thus, as there exist (s_1, s_2, s_3) such that $\min G_{\mathcal{H}} \max G_{\mathcal{H}} > 0$, it follows that the assessment (x, y, z) is not coherent when $z > xy$ and $y = 0$.

We conclude that $z'' = xy$ is the upper bound for z. □

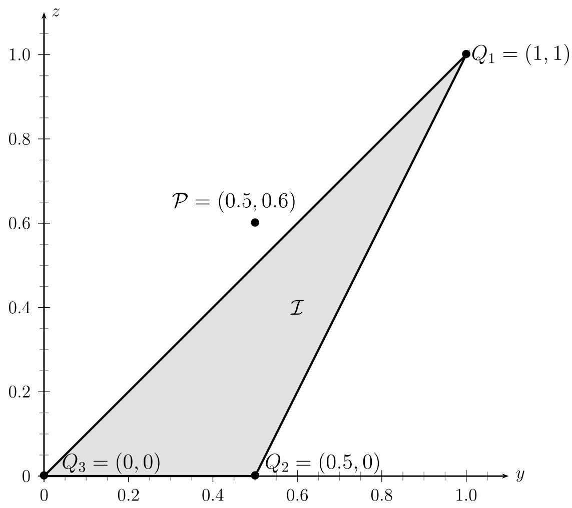

Fig. A.4. Convex hull of the points Q_1, Q_2, Q_3 associated with the pair $(\mathcal{P}_0, \mathcal{F}_0)$, where $\mathcal{P}_0 = (y, z)$ and $\mathcal{F}_0 = \{(B|K)|_{dF}(A|H), (A|H) \wedge_K (B|K)\}$. In the figure the numerical values are $y = 0.5$, $z = 0.6$, and $\mathcal{P}_0$ is denoted by $\mathcal{P}$.

Table A.13
Constituents and points Q_h's associated with $\mathcal{F} = \{A|H, B|K, (B|K)|_{dF}(A|H)\}$ and $\mathcal{P} = (x, y, z)$.

	C_h	$A\|H$	$B\|K$	$(B\|K)\|_{dF}(A\|H)$	Q_h
C_1	$AHBK$	1	1	1	Q_1
C_2	$AH\bar{B}K$	1	0	0	Q_2
C_3	$AH\bar{K}$	1	y	z	Q_3
C_4	$\bar{A}HBK$	0	1	z	Q_4
C_5	$\bar{A}H\bar{B}K$	0	0	z	Q_5
C_6	$\bar{A}H\bar{K}$	0	y	z	Q_6
C_7	$\bar{H}BK$	x	1	z	Q_7
C_8	$\bar{H}\bar{B}K$	x	0	z	Q_8
C_0	$\bar{H}\bar{K}$	x	y	z	Q_0

Proof of Theorem 7

Theorem 7. *Let A, B, H, K be any logically independent events. The probability assessment $\mathcal{P} = (x, y, z)$ on the family of conditional events $\mathcal{F} = \{A|H, B|K, (B|K)|_{dF}(A|H)\}$ is coherent for every $(x, y, z) \in [0, 1]^3$.*

Proof. The constituents C_h's and the points Q_h's associated with the assessment $\mathcal{P} = (x, y, z)$ on $\mathcal{F}$ are (see also Table A.13)

$$C_1 = AHBK, C_2 = AH\bar{B}K, C_3 = AH\bar{K}, C_4 = \bar{A}HBK, C_5 = \bar{A}H\bar{B}K,$$
$$C_6 = \bar{A}H\bar{K}, C_7 = \bar{H}BK, C_8 = \bar{H}\,\bar{B}K, C_0 = \bar{H}\,\bar{K},$$

and

$$Q_1 = (1,1,1), Q_2 = (1,0,0), Q_3 = (1,y,z), Q_4 = (0,1,z), Q_5 = (0,0,z),$$
$$Q_6 = (0,y,z), Q_7 = (x,1,z), Q_8 = (x,0,z), \mathcal{P} = Q_0 = (x,y,z).$$

The system (Σ) in (5) associated with the pair $(\mathcal{F}, \mathcal{P})$ becomes

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 + x\lambda_7 + x\lambda_8 = x, \\ \lambda_1 + y\lambda_3 + \lambda_4 + y\lambda_6 + \lambda_7 = y, \\ \lambda_1 + z\lambda_3 + z\lambda_4 + z\lambda_5 + z\lambda_6 + z\lambda_7 + z\lambda_8 = z, \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 + \lambda_8 = 1, \\ \lambda_i \geq 0, \ i = 1, \dots, 8. \end{cases} \tag{A.3}$$

We observe that $\mathcal{P}$ belongs to the segment with end points Q_3, Q_6, indeed $(x,y,z) = xQ_3 + (1-x)Q_6 = x(1,y,z) + (1-x)(0,y,z)$. The point $\mathcal{P}$ also belongs the segment with end points Q_7 and Q_8 because $(x,y,z) = yQ_7 + (1-y)Q_8 = y(x,1,z) + (1-y)(x,0,z)$. Then

$$(x,y,z) = \frac{x}{2}(1,y,z) + \frac{1-x}{2}(0,y,z) + \frac{y}{2}(x,1,z) + \frac{1-y}{2}(x,0,z),$$

so the vector $\Lambda = (0, 0, \frac{x}{2}, 0, 0, \frac{1-x}{2}, \frac{y}{2}, \frac{1-y}{2})$ is a solution of (A.3), with

Table A.14
Constituents and points Q_h's associated with $\mathcal{F} = \{A|H, (B|K)|_F(A|H), (A|H) \wedge_K (B|K)\}$ and $\mathcal{P} = (x, y, z)$.

	C_h	$A\vert H$	$(B\vert K)\vert_F(A\vert H)$	$(A\vert H)\wedge_K(B\vert K)$	Q_h
C_1	$AHBK$	1	1	1	Q_1
C_2	$AH\bar{B}K$	1	0	0	Q_2
C_3	$AH\bar{K}$	1	y	z	Q_3
C_4	$\bar{A}H$	0	y	0	Q_4
C_5	$\bar{H}\bar{B}K$	x	0	0	Q_5
C_0	$\bar{H}BK \vee \bar{H}\bar{K}$	x	y	z	Q_0

$$\begin{aligned}
&\phi_1(\Lambda) = \textstyle\sum_{h:C_h\subseteq H} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5\lambda_6 = \frac{1}{2} > 0,\\
&\phi_2(\Lambda) = \textstyle\sum_{h:C_h\subseteq K} \lambda_h = \lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 + \lambda_7 + \lambda_8 = \frac{1}{2},\\
&\phi_3(\Lambda) = \textstyle\sum_{h:C_h\subseteq AHK} \lambda_h = \lambda_1 + \lambda_2 = 0.
\end{aligned}$$

Let $\mathcal{S}' = \{(0, 0, \frac{x}{2}, 0, 0, \frac{1-x}{2}, \frac{y}{2}, \frac{1-y}{2})\}$ denote a subset of the set $\mathcal{S}$ of all solutions of (A.3). We have that $M_1' = \frac{1}{2}$, $M_2' = \frac{1}{2}$, $M_3' = 0$ (7). We get $\mathcal{I}_0' = \{3\}$. We observe that the sub-assessment z on $\{(B|K)|_{dF}(A|H)\}$ is coherent for every $z \in [0,1]$. Then, by Theorem 1, the assessment (x, y, z) on $\mathcal{F}$ is coherent $\forall (x, y, z) \in [0,1]^3$. □

A.2. *Proof of Theorem 8*

Theorem 8. *Let A, B, H, K, be any logically independent events. A probability assessment $\mathcal{P} = (x, y, z)$ on the family of conditional events $\mathcal{F} = \{A|H, (B|K)|_F(A|H), (A|H) \wedge_K (B|K)\}$ is coherent if and only if $(x, y) \in [0,1]^2$ and $z \in [z', z'']$, where $z' = 0$ and $z'' = T_0^H(x, y)$, where*

$$T_0^H(x,y) = \begin{cases} 0, & \text{if } x = 0 \text{ or } y = 0 \\ \frac{xy}{x+y-xy}, & \text{if } x \neq 0 \text{ and } y \neq 0, \end{cases}$$

is the Hamacher t-norm with parameter $\lambda = 0$.

Proof. We recall that $\{A|H, (B|K)|_F(A|H), (A|H) \wedge_K (B|K)\} = \{A|H, AHBK|(AHBK \vee AH\bar{B}K \vee \bar{H}\bar{B}K), AHBK|(AHBK \vee \bar{A}H \vee \bar{B}K)\}$. Then, $\mathcal{H}_3 = H \vee (AHBK \vee AH\bar{B}K \vee \bar{H}\bar{B}K) \vee (AHBK \vee \bar{A}H \vee \bar{B}K) = H \vee \bar{H}\bar{B}K$. The constituents C_h's and the points Q_h's associated with the assessment $\mathcal{P} = (x, y, z)$ on $\mathcal{F}$ are (see Table A.14)

$$C_1 = AHBK,\ C_2 = AH\bar{B}K,\ C_3 = AH\bar{K},\ C_4 = \bar{A}HBK \vee \bar{A}H\bar{B}K \vee \bar{A}H\bar{K} = \bar{A}H,\ C_5 = \bar{H}\bar{B}K,\ C_0 = \bar{H}BK \vee \bar{H}\bar{K},$$

and

$$Q_1 = (1,1,1),\ Q_2 = (1,0,0),\ Q_3 = (1,y,z),\ Q_4 = (0,y,0),\ Q_5 = (x,0,0),\ \mathcal{P} = Q_0 = (x,y,z).$$

The system (Σ) in (5) associated with the pair $(\mathcal{F}, \mathcal{P})$ becomes

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 + x\lambda_5 = x, \\ \lambda_1 + y\lambda_3 + y\lambda_4 = y, \\ \lambda_1 + z\lambda_3 = z, \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 1, \\ \lambda_i \geq 0,\ i = 1, \ldots, 5. \end{cases} \tag{A.4}$$

Upper bound. We first prove that the assessment $(x, y, 0)$ is coherent for every $(x, y) \in [0,1]^2$. We observe that $\mathcal{P} = (x, y, 0) = xQ_3 + (1-x)Q_4$, so a solution of (A.4) is given by $\Lambda = (0, 0, x, 1-x, 0)$.

Then, by considering the function ϕ as defined in (6), it holds that

$$\begin{aligned}
&\phi_1(\Lambda) = \textstyle\sum_{h:C_h\subseteq H} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1 > 0,\\
&\phi_2(\Lambda) = \textstyle\sum_{h:C_h\subseteq(AHBK\vee AH\bar{B}K\vee\bar{H}\bar{B}K)} \lambda_h = \lambda_1 + \lambda_2 + \lambda_5 = 0,\\
&\phi_3(\Lambda) = \textstyle\sum_{h:C_h\subseteq(AHBK\vee\bar{A}H\vee\bar{B}K)} \lambda_h = \lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 = 1 - x.
\end{aligned}$$

Let $\mathcal{S}' = \{(0, 0, x, 1-x, 0)\}$ denote a subset of the set $\mathcal{S}$ of all solutions of (A.4). We have that $M_1' = 1$, $M_2' = 0$, $M_3' = 1 - x$ (as defined in (7)). We distinguish two cases: (i) $x \neq 1$; (ii) $x = 1$.

- (i) In this case, $M_1' > 0$, $M_2' = 0$, $M_3' > 0$ and hence $I_0' = \{2\}$. We observe that the sub-assessment $\mathcal{P}_0' = y$ on $\mathcal{F}_0' = \{(B|K)|_F(A|H)\}$ is coherent for every $y \in [0,1]$. Then, by Theorem 2, the assessment $(x, y, 0)$ on $\mathcal{F}$ is coherent $\forall (x, y) \in [0,1]^2$;
- (ii) We have that $M_1' > 0$, $M_2' = M_3' = 0$ and hence $I_0' = \{2,3\}$. We set $\mathcal{K} = \mathcal{F}_0' = \{(B|K)|_F(A|H), (A|H) \wedge_K (B|K)\}$ and $\mathcal{V} = \mathcal{P}_0' = (y, 0)$. By Theorem 2, as (A.4) is solvable and $I_0' = \{2,3\}$, it is sufficient to check the coherence of the sub-assessment $\mathcal{V}$ on

$\mathcal{K}$ in order to check the coherence of $(x,y,0)$. We have that $\mathcal{H}_2 = (AHBK \vee AH\bar{B}K \vee \bar{H}\,\bar{B}K) \vee (AHBK \vee \bar{A}H \vee \bar{B}K) = AHBK \vee \bar{A}H \vee \bar{B}K$. The constituents C_h's and the point Q_h's associated with the assessment $\mathcal{V}$ on $\mathcal{K}$ are

$$C_1 = AHBK,\ C_2 = AH\bar{B}K \vee \bar{H}\,\bar{B}K,\ \ C_3 = \bar{A}H, C_0 = AH\bar{K} \vee \bar{H}\,\bar{K} \vee \bar{H}BK,$$

and

$$Q_1 = (1,1),\ Q_2 = (0,0),\ Q_3 = (y,0),\ \mathcal{P}'_0 = Q_0 = (y,0).$$

We have that $\mathcal{H}_2 = C_1 \vee \cdots \vee C_3 = AHBK \vee \bar{A}H \vee \bar{B}K$. The system (Σ) in (5) associated with the pair $(\mathcal{K},\mathcal{V})$ becomes

$$\begin{cases} \lambda_1 + y\lambda_3 = y, \\ \lambda_1 = 0, \\ \lambda_1 + \lambda_2 + \lambda_3 = 1, \\ \lambda_i \geq 0,\ i = 1,\dots,3. \end{cases} \tag{A.5}$$

We observe that $\mathcal{V} = (y,0) = Q_3$ so a solution of (A.5) is $\Lambda = (0,0,1)$.
By considering the function ϕ as defined in (6), it holds that

$$\phi_1(\Lambda) = \textstyle\sum_{h:C_h \subseteq (AHBK \vee AH\bar{B}K \vee \bar{H}\,\bar{B}K)} \lambda_h = \lambda_1 + \lambda_2 = 0,$$
$$\phi_2(\Lambda) = \textstyle\sum_{h:C_h \subseteq (AHBK \vee \bar{A}H \vee \bar{B}K)} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 = 1.$$

Let $\mathcal{S}' = \{(0,0,1)\}$ denote a subset of the set $\mathcal{S}$ of all solutions of (A.4). We have that $M'_1 = 0$, $M'_2 = 1$, (as defined in (7)). Then, the set I'_0 associated with $(\mathcal{K},\mathcal{V})$ is $I'_0 = 1$. We observe that the sub-assessment y on $\{(B|K)|_{dF}(A|H)\}$ is coherent for every $y \in [0,1]$. Then, by Theorem 2, the sub-assessment $(y,0)$ on $\mathcal{K}$ is coherent for every $y \in [0,1]$ and hence the assessment $(x,y,0)$ on $\mathcal{F}$ is coherent $\forall (x,y) \in [0,1]^2$. Thus, $z' = 0$ is the lower bound of $z = P((A|H) \wedge_K (B|K))$, for every $(x,y) \in [0,1]$ on $\{A|H, (B|K)|_F(A|H)\}$.

Upper bound. To study the upper bound z'' of $z = P((A|H) \wedge_K (B|K))$ we distinguish the following cases: (a) $x = 0$; (b) $y = 0$; (c) $x \neq 0$ and $y \neq 0$.
(a). In this case system (A.4) associated to the pair $(\mathcal{F},(0,y,z))$ becomes

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = 0, \\ \lambda_1 + y\lambda_3 + y\lambda_4 = y, \\ \lambda_1 + z\lambda_3 = z, \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 1, \\ \lambda_i \geq 0,\ i = 1,\dots,5. \end{cases} \Longleftrightarrow \begin{cases} \lambda_1 = \lambda_2 = \lambda_3 = 0, \\ y\lambda_4 = y \\ z = 0, \\ \lambda_4 + \lambda_5 = 1, \\ \lambda_i \geq 0,\ i = 4,5, \end{cases} \tag{A.6}$$

which is solvable only if $z = 0$. We have already verified that $\mathcal{P} = (x,y,0)$ is coherent on $\mathcal{F}$ for all $(x,y) \in [0,1]^2$, hence, in particular, $(0,y,0)$ is coherent. We observe that system (A.6) with $z > 0$ is not solvable, then the assessment $(0,y,z)$, with $z > 0$ is not coherent. Thus, $z'' = 0$ is the upper bound for z when $x = 0$ and $y \in [0,1]$.
(b). We recall that the pair $(\wedge_K, |_F)$ satisfies property P2, that is $z \leq y$. As $y = 0$, it follows that $z = 0$. We have already verified that $\mathcal{P} = (x,y,0)$ is coherent on $\mathcal{F}$ for all $(x,y) \in [0,1]^2$, hence, in particular, $(x,0,0)$ is coherent. We also observe that the assessment $(x,0,z)$, with $z > 0$ is not coherent, because the pair $(y = 0, z)$ violates property P2. Thus, $z'' = 0$ is the upper bound for z when $x \in [0,1]$ and $y = 0$.
(c) First of all, we verify that the assessment $\mathcal{P} = (x, y, \frac{xy}{x+y-xy})$ on $\mathcal{F}$ is coherent. Then, we verify that $z'' = \frac{xy}{x+y-xy}$ is the upper bound for z, by showing that (x,y,z), with $z > z''$, is incoherent for every $(x,y) \in [0,1]^2$.

We observe that

$$\mathcal{P} = (x, y, \frac{xy}{x+y-xy}) = \frac{xy}{x+y-xy}Q_1 + \frac{y(1-x)}{x+y-xy}Q_4 + \frac{x(1-y)}{x+y-xy}Q_5.$$

Then a solution of (A.4) is $\Lambda = (\frac{xy}{x+y-xy}, 0, 0, \frac{y(1-x)}{x+y-xy}, \frac{x(1-y)}{x+y-xy})$ and it holds that

$$\phi_1(\Lambda) = \textstyle\sum_{h:C_h \subseteq H} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = \frac{y}{x+y-xy},$$
$$\phi_2(\Lambda) = \textstyle\sum_{h:C_h \subseteq (AHBK \vee AH\bar{B}K \vee \bar{H}\,\bar{B}K)} \lambda_h = \lambda_1 + \lambda_2 + \lambda_5 = \frac{x}{x+y-xy},$$
$$\phi_3(\Lambda) = \textstyle\sum_{h:C_h \subseteq (AHBK \vee \bar{A}H \vee \bar{B}K)} \lambda_h = \lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 = 1 > 0.$$

Let $\mathcal{S}' = \{(\frac{xy}{x+y-xy}, 0, 0, \frac{y(1-x)}{x+y-xy}, \frac{x(1-y)}{x+y-xy})\}$ be a subset $\mathcal{S}$ of all solutions of (A.4). We have that $M'_1 > 0$, $M'_2 > 0$, $M'_3 > 0$ (as defined in (7)). Then, by Theorem 2 the assessment $(x, y, \frac{xy}{x+y-xy})$ is coherent for every $x \neq 0$ and $y \neq 0$.

Now we prove that in this case $z'' = \frac{xy}{x+y-xy}$ is the upper bound for z.

We distinguish two sub-cases: (i) $x = 1$, (ii) $x \neq 1$.
(i) We recall that the pair $(\wedge_K, |_F)$ satisfies property P2 and hence $z \leq y$. Then, when $x = 1$, as $\frac{xy}{x+y-xy} = y$, we have that any assessment $z > \frac{xy}{x+y-xy} = y$ is not coherent. Therefore, $z'' = \frac{xy}{x+y-xy} = y$ is the upper bound for $z = P((A|H) \wedge_K (B|K))$.

Table A.15
Constituents and points Q_h's associated with $\mathcal{F} = \{A|H, B|K, (B|K)|_F(A|H)\}$ and $\mathcal{P} = (x, y, z)$.

	C_h	$A\|H$	$B\|K$	$(B\|K)\|_F(A\|H)$	Q_h
C_1	$AHBK$	1	1	1	Q_1
C_2	$AH\bar{B}K$	1	0	0	Q_2
C_3	$AH\bar{K}$	1	y	z	Q_3
C_4	$\bar{A}HBK$	0	1	z	Q_4
C_5	$\bar{A}H\bar{B}K$	0	0	z	Q_5
C_6	$\bar{A}H\bar{K}$	0	y	z	Q_6
C_7	$\bar{H}BK$	x	1	z	Q_7
C_8	$\bar{H}\bar{B}K$	x	0	0	Q_8
C_0	$\bar{H}\bar{K}$	x	y	z	Q_0

(ii) We recall that in this case $x \in (0,1)$ and $y \neq 0$. We observe that the points Q_1, Q_4, Q_5 belong to the plane $\pi : yX + xY - (x + y - xy)Z = xy$, where X, Y, Z are the axes coordinates. We set $f(X,Y,Z) = yX + xY - (x+y-xy)Z - xy$, we choose $\mathcal{P} = (x,y,z)$ with $z > \frac{xy}{x+y-xy}$, i.e. $(x+y-xy)z - xy > 0$, and we observe that $f(\mathcal{P}) = xy - (x+y-xy)z$. For each $h = 1, \dots, 5$, we consider the quantity $f(Q_h) - f(P)$. Then, we obtain

$$
\begin{aligned}
&f(Q_1) - f(\mathcal{P}) = (x+y-xy)z - xy > 0;\\
&f(Q_2) - f(\mathcal{P}) = y(1-x) + (x+y-xy)z - xy > 0;\\
&f(Q_3) - f(\mathcal{P}) = y - xy = y(1-x) > 0;\\
&f(Q_4) - f(\mathcal{P}) = (x+y-xy)z - xy > 0;\\
&f(Q_5) - f(\mathcal{P}) = (x+y-xy)z - xy > 0.
\end{aligned}
\tag{A.7}
$$

We recall that, by setting the stakes $s_1 = y$, $s_2 = x$, $s_3 = -x - y + xy$, it holds that $g_h = f(Q_h) - f(\mathcal{P})$, $h = 1, \dots, 5$, where g_h is the value of the random gain G associated with the constituent $C_h \subseteq \mathcal{H}_3$. From (A.7) we obtain that $g_h = f(Q_h) - f(\mathcal{P}) > 0$, $h = 1, \dots, 5$. Thus, as there exist (s_1, s_2, s_3) such that $\min G_{\mathcal{H}} \max G_{\mathcal{H}} > 0$, it follows that the assessment (x, y, z) is not coherent when $z > \frac{xy}{x+y-xy}$.

We conclude that $z'' = \frac{xy}{x+y-xy}$ is the upper bound for z.

Therefore $z'' = T_0^H(x, y)$ is the upper bound for z, for every $(x, y) \in [0,1]^2$. □

A.3. Proof of Theorem 9

Theorem 9. *Let A, B, H, K be any logically independent events. The probability assessment $\mathcal{P} = (x, y, z)$ on the family of conditional events $\mathcal{F} = \{A|H, B|K, (B|K)|_F(A|H)\}$ is coherent for every $(x, y, z) \in [0,1]^3$.*

Proof. The constituents C_h's and the points Q_h's associated with the assessment $\mathcal{P} = (x, y, z)$ on $\mathcal{F}$ are (see also Table A.15)

$$
\begin{aligned}
&C_1 = AHBK, C_2 = AH\bar{B}K, C_3 = AH\bar{K}, C_4 = \bar{A}HBK, C_5 = \bar{A}H\bar{B}K,\\
&C_6 = \bar{A}H\bar{K}, C_7 = \bar{H}BK, C_8 = \bar{H}\,\bar{B}K, C_0 = \bar{H}\,\bar{K},
\end{aligned}
$$

and

$$
\begin{aligned}
&Q_1 = (1,1,1), Q_2 = (1,0,0), Q_3 = (1,y,z), Q_4 = (0,1,z), Q_5 = (0,0,z),\\
&Q_6 = (0,y,z), Q_7 = (x,1,z), Q_8 = (x,0,0), \mathcal{P} = Q_0 = (x,y,z).
\end{aligned}
$$

We observe that $C_1 \vee \dots \vee C_8 = H \vee K = \mathcal{H}_3$. The system (Σ) in (5) associated with the pair $(\mathcal{F}, \mathcal{P})$ becomes

$$
\begin{cases}
\lambda_1 + \lambda_2 + \lambda_3 + x\lambda_7 + x\lambda_8 = x,\\
\lambda_1 + y\lambda_3 + \lambda_4 + y\lambda_6 + \lambda_7 = y,\\
\lambda_1 + z\lambda_3 + z\lambda_4 + z\lambda_5 + z\lambda_6 + z\lambda_7 = z,\\
\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 + \lambda_8 = 1,\\
\lambda_i \geq 0, \;\; i = 1, \dots, 8.
\end{cases}
\tag{A.8}
$$

We observe that $\mathcal{P}$ belongs to the segment with end points Q_3, Q_6; indeed $(x, y, z) = xQ_3 + (1-x)Q_6 = x(1, y, z) + (1-x)(0, y, z)$. Then, the vector $\Lambda = (0, 0, x, 0, 0, 1-x, 0, 0)$ is a solution of (A.8), with

$$
\begin{aligned}
&\phi_1(\Lambda) = \textstyle\sum_{h:C_h \subseteq H} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = 1,\\
&\phi_2(\Lambda) = \textstyle\sum_{h:C_h \subseteq K} \lambda_h = \lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 + \lambda_7 + \lambda_8 = 0,\\
&\phi_3(\Lambda) = \textstyle\sum_{h:C_h \subseteq [AHBK \vee AH\bar{B}K \vee \bar{H}\,\bar{B}K]} \lambda_h = \lambda_1 + \lambda_2 + \lambda_8 = 0.
\end{aligned}
$$

Let $\mathcal{S}' = \{(0, 0, x, 0, 0, 1-x, 0, 0)\}$ be a subset $\mathcal{S}$ of all solutions of (A.8). We have that $M'_1 > 0$, $M'_2 = 0$, $M'_3 = 0$ (as defined in (7)). Then, $I'_0 = \{2, 3\}$. We check the coherence of the sub-assessment $\mathcal{P}'_0 = (y, z)$ on $\mathcal{F}'_0 = \{B|K, (B|K)|_F(A|H)\}$. The constituents C_h's and the point Q_h's associated with the assessment $\mathcal{P}'_0$ on $\mathcal{F}'_0$ are

$$C_1 = AHBK, C_2 = AH\bar{B}K \vee \overline{H}\,\bar{B}K,$$
$$C_3 = \bar{A}HBK \vee \overline{H}BK, C_4 = \bar{A}H\bar{B}K, C_0 = \bar{K},$$

and

$$Q_1 = (1,1),\ Q_2 = (0,0),\ Q_3 = (1,z),\ Q_4 = (0,z), \mathcal{P} = Q_0 = (y,z).$$

We observe that $C_1 \vee \cdots \vee C_4 = K = \mathcal{H}_2$. The system (Σ) in (5) associated with the pair $(\mathcal{F}_0, \mathcal{P}_0)$ becomes

$$\begin{cases} \lambda_1 + \lambda_3 = y, \\ \lambda_1 + z\lambda_3 + z\lambda_4 = z, \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1, \\ \lambda_i \geq 0,\ i = 1, \dots, 4. \end{cases} \tag{A.9}$$

We observe that $(y,z) = yQ_3 + (1-y)Q_4$ so a solution of ((A.9) is $\Lambda_0 = (0,0,y,1-y)$. By considering the function ϕ' it holds that

$$\phi_1(\Lambda) = \sum_{h:C_h \subseteq K} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1,$$
$$\phi_2(\Lambda) = \sum_{h:C_h \subseteq [AHBK \vee AH\bar{B}K \vee \overline{H}\,\bar{B}K]} \lambda_h = \lambda_1 + \lambda_2 = 0.$$

Let $S' = \{(0,0,y,1-y)\}$ be a subset S of all solutions of (A.9). We have that $M_1' > 0$, $M_2' = 0$ (as defined in (7)). Then, $I_0' = \{2\}$. We observe that the sub-assessment z on $\{(B|K)|_F(A|H)\}$ is coherent for every $z \in [0,1]$. Then, by Theorem 2, the assessment (x,y,z) on $\mathcal{F}$ is coherent $\forall (x,y,z) \in [0,1]^3$. □

A.4. Proof of Theorem 13

Theorem 13. *Let A, B, H, K be any logically independent events. The set Π of all the coherent assessments (x,y,z,μ) on the family $\mathcal{F} = \{A|H, B|K, (A|H) \wedge_L (B|K), (B|K)|_L(A|H)\}$ is $\Pi = \Pi' \cup \Pi''$, where $\Pi' = \{(x,y,z,\mu) : x \in (0,1], y \in [0,1], z \in [z', z''], \mu = \frac{z}{x}\}$ with $z' = 0$, $z'' = min\{x,y\}$, and $\Pi'' = \{(0,y,0,\mu) : (y,\mu) \in [0,1]^2\}$.*

Proof. It is well-known that the assessment (x,y) on $\{A|H, B|K\}$ is coherent for every $(x,y) \in [0,1]^2$. By Table 2, the assessment $z = P((A|H) \wedge_L (B|K))$ is a coherent extension of (x,y) if and only if $z \in [z', z'']$ where $z' = 0$ and $z'' = min\{x,y\}$. Assuming $x > 0$, from (44) it holds that $\mu = \frac{z}{x}$. Then, every $(x,y,z,\mu) \in \Pi'$ is coherent, i.e., $\Pi' \subseteq \Pi$. Of course, if $x > 0$ and $(x,y,z,\mu) \notin \Pi'$, then the assessment (x,y,z,μ) is not coherent, i.e. $(x,y,z,\mu) \notin \Pi$.

Let us consider now the case $x = 0$, so that $z' = 0$ and $z'' = 0$. We show that the assessment $(0,y,0,\mu)$ is coherent if and only if $(y,\mu) \in [0,1]^2$, that is $\Pi'' \subseteq \Pi$. As $x = 0$, from (49), it holds that

$$(B|K)|_L(A|H) = \begin{cases} 1, & \text{if } AHBK \text{ is true}, \\ 0, & \text{if } AH\bar{B}K \vee AH\bar{K} \text{ is true}, \\ \mu_L, & \text{if } \bar{A}H \vee \overline{H}BK \vee \overline{H}\,\bar{B}K \vee \overline{H}\,\bar{K} \text{ is true}. \end{cases} \tag{A.10}$$

that is $(B|K)|_L(A|H) = BK|AH$ (see proof of Theorem 11). Therefore $\mathcal{F} = \{A|H, B|K, (A|H) \wedge_L (B|K), (B|K)|_L(A|H)\} = \{A|H, B|K, AHBK|(AHBK \vee \overline{A}\overline{H} \vee \overline{B}\overline{K} \vee \overline{H}\,\overline{K}), BK|AH\}$ and $\mathcal{H}_4 = H \vee K \vee (AHBK \vee \overline{A}\overline{H} \vee \overline{B}\overline{K} \vee \overline{H}\,\overline{K}) \vee AH = \Omega$. The constituents $C_h's$ and the points $Q_h's$ associated with $(\mathcal{F}, \mathcal{P})$, where $\mathcal{P} = (0,y,0,\mu)$, are the following (see also Table A.16):

$$C_1 = AHBK, C_2 = AH\bar{B}K, C_3 = AH\bar{K}, C_4 = \bar{A}HBK, C_5 = \bar{A}H\bar{B}K,$$
$$C_6 = \bar{A}H\bar{K}, C_7 = \overline{H}BK, C_8 = \overline{H}\,\bar{B}K, C_9 = \overline{H}\,\bar{K},$$

and

$$Q_1 = (1,1,1,1), Q_2 = (1,0,0,0), Q_3 = (1,y,0,0), Q_4 = (0,1,0,\mu), Q_5 = (0,0,0,\mu),$$
$$Q_6 = (0,y,0,\mu), Q_7 = (0,1,0,\mu), Q_8 = (0,0,0,\mu), Q_9 = (0,y,0,\mu).$$

The system (Σ) in (5) associated with the pair $(\mathcal{F}, \mathcal{P})$ is

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = 0, \\ \lambda_1 + y\lambda_3 + \lambda_4 + y\lambda_6 + \lambda_7 + y\lambda_9 = y, \\ \lambda_1 = 0, \\ \lambda_1 + \mu\lambda_4 + \mu\lambda_5 + \mu\lambda_6 + \mu\lambda_7 + \mu\lambda_8 + \mu\lambda_9 = \mu \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 + \lambda_8 + \lambda_9 = 1, \\ \lambda_i \geq 0,\ i = 1, \dots, 9. \end{cases} \tag{A.11}$$

We observe that $\mathcal{P}$ belongs to the segment with end points Q_4, Q_5; indeed $(0,y,0,\mu) = yQ_4 + (1-y)Q_5 = y(0,1,0,\mu) + (1-y)(0,0,0,\mu)$. Then, the vector $\Lambda = (0,0,0,y,1-y,0,0,0,0)$ is a solution of (A.11), with

Table A.16
Constituents and points Q_h's associated with $\mathcal{F} = \{A|H, B|K, (A|H) \wedge_L (B|K), (B|K)|_L(A|H)\}$ and $\mathcal{P} = (0, y, 0, \mu)$.

	C_h	$A\|H$	$B\|K$	$(A\|H) \wedge_L (B\|K)$	$(B\|K)\|_L(A\|H)$	Q_h
C_1	$AHBK$	1	1	1	1	Q_1
C_2	$AH\bar{B}K$	1	0	0	0	Q_2
C_3	$AH\bar{K}$	1	y	0	0	Q_3
C_4	$\bar{A}HBK$	0	1	0	μ	Q_4
C_5	$\bar{A}H\bar{B}K$	0	0	0	μ	Q_5
C_6	$\bar{A}H\bar{K}$	0	y	0	μ	Q_6
C_7	$\bar{H}BK$	0	1	0	μ	Q_7
C_8	$\bar{H}\bar{B}K$	0	0	0	μ	Q_8
C_9	$\bar{H}\bar{K}$	0	y	0	μ	Q_9

$$\phi_1(\Lambda) = \sum_{h:C_h\subseteq H} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 1 > 0,$$
$$\phi_2(\Lambda) = \sum_{h:C_h\subseteq K} \lambda_h = \lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 = 1 > 0,$$
$$\phi_3(\Lambda) = \sum_{h:C_h\subseteq(AHBK\vee\bar{A}H\vee\bar{B}K\vee\bar{H}\bar{K})} \lambda_h = \lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_8 + \lambda_9 = 1 > 0,$$
$$\phi_4(\Lambda) = \sum_{h:C_h\subseteq AH} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 = 0.$$

Let $\mathcal{S}' = \{(0,0,0,y,1-y,0,0,0,0)\}$ denote a subset of the set $\mathcal{S}$ of all solutions of (A.11). We have that $M_1' = M_2' = M_3' = 1$ and $M_4' = 0$ (as defined in (7)). It follows that $I_0' = \{4\}$. As the sub-assessment $\mathcal{P}_0' = \mu$ on $\mathcal{F}_0' = \{BK|AH\}$ is coherent $\forall\mu \in [0,1]$, by Theorem 1, it follows that the assessment $(0, y, 0, \mu)$ on $\mathcal{F} = \{A|H, B|K, (A|H) \wedge_L (B|K), (B|K)|_L(A|H)\}$ is coherent for every $(y,\mu) \in [0,1]^2$, that is $(0, y, 0, \mu) \in \Pi''$. Thus $\Pi'' \subseteq \Pi$. Of course, if $(0, y, z, \mu) \notin \Pi''$ the assessment $(0, y, z, \mu)$ is not coherent and hence $(0, y, z, \mu) \notin \Pi$. Therefore $\Pi = \Pi' \cup \Pi''$. □

A.5. Proof of Theorem 15

Theorem 15. *Let A, B, H, K be any logically independent events. The set Π of all the coherent assessments (x, y, z, μ) on the family $\mathcal{F} = \{A|H, B|K, (A|H) \wedge_B (B|K), (B|K)|_B(A|H)\}$ is $\Pi = \Pi' \cup \Pi''$, where $\Pi' = \{(x, y, z, \mu) : x \in (0,1], y \in [0,1], z \in [z', z''], \mu = \frac{z}{x}\}$ with $z' = 0$, $z'' = 1$, and $\Pi'' = \{(0, y, 0, \mu) : (y,\mu) \in [0,1]^2\}$.*

Proof. Of course the assessment (x, y) on $\{A|H, B|K\}$ is coherent for every $(x,y) \in [0,1]^2$. By Table 2, the assessment $z = P((A|H) \wedge_B (B|K))$ is a coherent extension of (x, y) if and only if $z \in [z', z'']$ where $z' = 0$ and $z'' = 1$. Assuming $x > 0$, from (44) it holds that $\mu = \frac{z}{x}$. Then, every $(x, y, z, \mu) \in \Pi'$ is coherent, i.e., $\Pi' \subseteq \Pi$. Of course, if $x > 0$ and $(x, y, z, \mu) \notin \Pi'$, then the assessment (x, y, z, μ) is not coherent, i.e. $(x, y, z, \mu) \notin \Pi$.

Let us consider now the case $x = 0$. We recall that it is coherent to assess $(0, y, z)$, with $z > 0$, on $\{A|H, B|K, (A|H) \wedge_B (B|K)\}$ (see Table 2). However, for the further object $(B|K)|_B(A|H)$ coherence also requires that $z = \mu x$ (see (44)). Then, coherence requires that $z = 0$ when we consider the assessment $(0, y, z, \mu)$ on $\{A|H, B|K, (A|H) \wedge_B (B|K), (B|K)|_B(A|H)\}$. We show that the assessment $(0, y, 0, \mu)$ is coherent if and only if $(y,\mu) \in [0,1]^2$, that is $\Pi'' \subseteq \Pi$. As $x = 0$, from (51), it holds that

$$(B|K)|_B(A|H) = \begin{cases} 1, & \text{if } AHBK \text{ is true}, \\ 0, & \text{if } AH\bar{B}K \vee AH\bar{K} \text{ is true}, \\ \mu_B, & \text{if } \bar{H} \vee \bar{A}HBK \vee \bar{A}H\bar{B}K \vee \bar{A}H\bar{K} \text{ is true}, \end{cases} \tag{A.12}$$

that is $(B|K)|_B(A|H) = BK|AH$ (see proof Theorem 11). We show that the assessment $\mathcal{P} = (0, y, 0, \mu)$ on $\mathcal{F} = \{A|H, B|K, (A|H) \wedge_B (B|K), (B|K)|_B(A|H)\} = \{A|H, B|K, AHBK|HK, BK|AH\}$ is coherent if and only if $(y,\mu) \in [0,1]^2$, that is $\Pi'' \subseteq \Pi$. We observe that $\mathcal{H}_4 = H \vee K \vee HK \vee AH = H \vee K$. The constituents C_h's and the points Q_h's associated with $(\mathcal{F}, \mathcal{P})$, where $\mathcal{P} = (0, y, 0, \mu)$, are the following:

$$C_1 = AHBK, C_2 = AH\bar{B}K, C_3 = AH\bar{K}, C_4 = \bar{A}HBK, C_5 = \bar{A}H\bar{B}K,$$
$$C_6 = \bar{A}H\bar{K}, C_7 = \bar{H}BK, C_8 = \bar{H}\bar{B}K, C_0 = \bar{H}\bar{K},$$

and

$$Q_1 = (1,1,1,1), Q_2 = (1,0,0,0), Q_3 = (1,y,0,0), Q_4 = (0,1,0,\mu), Q_5 = (0,0,0,\mu),$$
$$Q_6 = (0,y,0,\mu), Q_7 = (0,1,0,\mu), Q_8 = (0,0,0,\mu), \mathcal{P} = Q_0 = (0,y,0,\mu).$$

The system (Σ) in (5) associated with the pair $(\mathcal{F}, \mathcal{P})$ is

$$\begin{cases} \lambda_1+\lambda_2+\lambda_3=0, \\ \lambda_1+y\lambda_3+\lambda_4+y\lambda_6+\lambda_7=y, \\ \lambda_1=0, \\ \lambda_1+\mu\lambda_4+\mu\lambda_5+\mu\lambda_6+\mu\lambda_7+\mu\lambda_8=\mu \\ \lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5+\lambda_6+\lambda_7+\lambda_8=1, \\ \lambda_i\geq 0,\ i=1,\dots,8. \end{cases} \tag{A.13}$$

We observe that $\mathcal{P}$ belongs to the segment with end points Q_4, Q_5; indeed $(0,y,0,\mu)=yQ_4+(1-y)Q_5=y(0,1,0,\mu)+(1-y)(0,0,0,\mu)$.
Then the vector $\Lambda=(0,0,0,y,1-y,0,0,0)$ is a solution of (A.13), with

$$\begin{aligned} &\phi_1(\Lambda)=\textstyle\sum_{h:C_h\subseteq H}\lambda_h=\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5+\lambda_6=1>0, \\ &\phi_2(\Lambda)=\textstyle\sum_{h:C_h\subseteq K}\lambda_h=\lambda_1+\lambda_2+\lambda_4+\lambda_5+\lambda_7+\lambda_8=1>0, \\ &\phi_3(\Lambda)=\textstyle\sum_{h:C_h\subseteq HK}\lambda_h=\lambda_1+\lambda_2+\lambda_4+\lambda_5=1>0, \\ &\phi_4(\Lambda)=\textstyle\sum_{h:C_h\subseteq AH}\lambda_h=\lambda_1+\lambda_2+\lambda_3=0. \end{aligned}$$

Let $\mathcal{S}'=\{(0,0,0,y,1-y,0,0,0)\}$ denote a subset of the set $\mathcal{S}$ of all solutions of (A.13). We have that $M_1'=M_2'=M_3'=1$ and $M_4'=0$ (as defined in (7)). It follows that $I_0'=\{4\}$. As the sub-assessment $\mathcal{P}_0'=\mu$ on $\mathcal{F}_0'=\{BK|AH\}$ is coherent $\forall\mu\in[0,1]$, by Theorem 1, it follows that the assessment $(0,y,0,\mu)$ on $\mathcal{F}=\{A|H, B|K, (A|H)\wedge_B(B|K), (B|K)|_B(A|H)\}$ is coherent for every $(y,\mu)\in[0,1]^2$, that is $(0,y,0,\mu)\in\Pi''$. Thus $\Pi''\subseteq\Pi$. Of course, if $(0,y,z,\mu)\notin\Pi''$ the assessment $(0,y,z,\mu)$ is not coherent and hence $(0,y,z,\mu)\notin\Pi$. Therefore $\Pi=\Pi'\cup\Pi''$. □

A.6. Proof of Theorem 16

Theorem 16. *Let A,B,H,K be any logically independent events. Given a coherent assessment (x,y) on $\{A|H,B|K\}$, for the iterated conditional $(B|K)|_B(A|H)$ the extension $\mu_B=\mathbb{P}[(B|K)|_B(A|H)]$ is coherent if and only if $\mu_B\in[\mu_B',\mu_B'']$, where*

$$\mu_B'=0,\ \ \mu_B''=\begin{cases}\frac{1}{x}, & \text{if } 0<x<1,\\ 1, & \text{if } x=0\vee x=1.\end{cases}$$

Proof. Assume that $x>0$. We simply write μ instead of μ_B. From Theorem 15 it follows that the set of all coherent assessments (x,y,z,μ) on $\mathcal{F}$ is $\Pi'=\{(x,y,z,\mu): 0<x\leq 1, 0\leq y\leq 1, z'\leq z\leq z'', \mu=\frac{z}{x}\}$, where $z'=0$ and $z''=1$ (see Table 2). Then, μ is a coherent extension of (x,y) if and only if $\mu\in[\mu',\mu'']$, where $\mu'=0$ and $\mu''=\frac{z''}{x}=\frac{1}{x}$. Assume that $x=0$. From Theorem 15 it follows that the set of all coherent assessments (x,y,z,μ) on $\mathcal{F}=\{A|H, B|K, (A|H)\wedge_B(B|K), (B|K)|_B(A|H)\}$ is $\Pi''=\{(0,y,0,\mu):(y,\mu)\in[0,1]^2\}$. Then, μ is a coherent extension to $(B|K)|_B(A|H)$ of the assessment $(0,y)$ on $\{A|H,B|K\}$ if and only if $\mu\in[\mu',\mu'']$, where $\mu'=0$ and $\mu''=1$. □

A.7. Proof of Theorem 17

Theorem 17. *Let A, B, H, K be any logically independent events. The set Π of all the coherent assessments (x,y,z,μ) on the family $\mathcal{F}=\{A|H, B|K, (A|H)\wedge_S(B|K), (B|K)|_S(A|H)\}$ is $\Pi=\Pi'\cup\Pi''$, where*

$$\Pi'=\{(x,y,z,\mu): x\in(0,1], y\in[0,1], z\in[z',z''], \mu=\tfrac{z}{x}\},$$
$$\text{with } z'=\max\{x+y-1,0\},\ z''=\begin{cases}\frac{x+y-2xy}{1-xy}, & \text{if } (x,y)\neq(1,1),\\ 1, & \text{if } (x,y)=(1,1),\end{cases}$$

and

$$\Pi''=\{(0,y,0,\mu): y\in[0,1], \mu\geq 0\}.$$

Proof. We know that the assessment (x,y) on $\{A|H,B|K\}$ is coherent for every $(x,y)\in[0,1]^2$. We also recall that the assessment $z=P((A|H)\wedge_S(B|K))$ is a coherent extension of (x,y) if and only if $z\in[z',z'']$, where (see Table 2)

$$z'=\max\{x+y-1,0\}\text{ and } z''=\begin{cases}\frac{x+y-2xy}{1-xy}, & \text{if } (x,y)\neq(1,1),\\ 1, & \text{if } (x,y)=(1,1).\end{cases}$$

Assuming $x>0$, then it follows from (44) that $\mu=\frac{z}{x}$. Thus, as every $(x,y,z,\mu)\in\Pi'$ is coherent, it holds that $\Pi'\subseteq\Pi$. Of course, if $x>0$ and $(x,y,z,\mu)\notin\Pi'$, then (x,y,z,μ) is not coherent, i.e. $(x,y,z,\mu)\notin\Pi$.

Let us consider now the case $x=0$. From Remark 19, coherence requires that $z=0$. We show that the assessment $\mathcal{M}=(0,y,0,\mu)$ is coherent if and only if $y\in[0,1]$ and $\mu\geq 0$, and hence $\Pi''\subseteq\Pi$. As $x=0$, by (53) it holds that

$$(B|K)|_S(A|H) = \begin{cases} 1, & \text{if } AHBK \vee AH\bar{K} \text{ is true,} \\ 0, & \text{if } AH\bar{B}K \text{ is true,} \\ \mu, & \text{if } \bar{A}H \vee \bar{H}\bar{K} \vee \bar{H}\bar{B}K \text{ is true,} \\ 1+\mu, & \text{if } \bar{H}BK \text{ is true,} \end{cases} \tag{A.14}$$

that is, when $x = 0$,

$$(B|K)|_S(A|H) = [AHBK + AH\bar{K} + (1+\mu)\bar{H}BK]|(AH \vee \bar{H}BK).$$

Then, we have $\mathcal{F} = \{A|H, B|K, (AH \vee \bar{H}) \wedge (BK \vee \bar{K})|(H \vee K), (AHBK + AH\bar{K} + (1+\mu)\bar{H}BK)|(AH \vee \bar{H}BK)\}$, with $\mathcal{H}_4 = H \vee K \vee (H \vee K) \vee (AH \vee \bar{H}BK) = H \vee K$. Therefore, the constituents and the points Q_h's associated with $(\mathcal{F}, \mathcal{M})$ are

$$C_1 = AHBK, C_2 = AH\bar{B}K, C_3 = AH\bar{K}, C_4 = \bar{A}HBK, C_5 = \bar{A}H\bar{B}K,$$
$$C_6 = \bar{A}H\bar{K}, C_7 = \bar{H}BK, C_8 = \bar{H}\bar{B}K, C_0 = \bar{H}\bar{K},$$

and

$$Q_1 = (1,1,1,1), Q_2 = (1,0,0,0), Q_3 = (1,y,1,1), Q_4 = (0,1,0,\mu), Q_5 = (0,0,0,\mu),$$
$$Q_6 = (0,y,0,\mu),\ Q_7 = (0,1,1,1+\mu), Q_8 = (0,0,0,\mu), \mathcal{M} = Q_0 = (0,y,0,\mu).$$

The system (Σ) in (5) associated with the pair $(\mathcal{F}, \mathcal{M})$ is

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = 0, \\ \lambda_1 + y\lambda_3 + \lambda_4 + y\lambda_6 + \lambda_7 = y, \\ \lambda_1 + \lambda_3 + \lambda_7 = 0, \\ \lambda_1 + \lambda_3 + \mu\lambda_4 + \mu\lambda_5 + \mu\lambda_6 + (1+\mu)\lambda_7 + \mu\lambda_8 = \mu, \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 + \lambda_8 = 1, \\ \lambda_i \geq 0,\ i = 1, \ldots, 8. \end{cases} \tag{A.15}$$

We observe that $\mathcal{M}$ belongs to the segment with end points Q_4, Q_5; indeed $(0,y,0,\mu) = yQ_4 + (1-y)Q_5 = y(0,1,0,\mu) + (1-y)(0,0,0,\mu)$.

Then, the vector $\Lambda = (0,0,0,y,1-y,0,0,0)$ is a solution of (A.15), with

$$\phi_1(\Lambda) = \sum_{h:C_h \subseteq H} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = 1 > 0,$$
$$\phi_2(\Lambda) = \sum_{h:C_h \subseteq K} \lambda_h = \lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 + \lambda_7 + \lambda_8 = 1 > 0,$$
$$\phi_3(\Lambda) = \sum_{h:C_h \subseteq H \vee K} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 + \lambda_8 = 1 > 0,$$
$$\phi_4(\Lambda) = \sum_{h:C_h \subseteq (AH \vee \bar{H}BK)} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_7 = 0.$$

Let $\mathcal{S}' = \{(0,0,0,y,1-y,0,0,0)\}$ denote a subset of the set $\mathcal{S}$ of all solutions of (A.15). We have that $M'_1 = M'_2 = M'_3 = 1$ and $M'_4 = 0$ (as defined in (7)). It follows that $I'_0 = \{4\}$. We set $\mathcal{F}'_0 = \{(AHBK + AH\bar{K} + (1+\mu)\bar{H}BK)|(AH \vee \bar{H}BK)\}$ and $\mathcal{M}'_0 = \mu$. By exploiting the betting scheme, we show that the assessment μ on the conditional random quantity $(AHBK + AH\bar{K} + (1+\mu)\bar{H}BK)|(AH \vee \bar{H}BK)$ is coherent for every $\mu \geq 0$. By recalling (2), the random gain for the assessment μ is

$$G = s(AH + \bar{H}BK)(AHBK + AH\bar{K} + (1+\mu)\bar{H}BK - \mu).$$

Without loss of generality we can assume $s = 1$. The constituents in $AH \vee \bar{H}BK$ are $C_1 = AHBK, C_2 = AH\bar{B}K, C_3 = AH\bar{K}, C_4 = \bar{H}BK$ and the corresponding values for the random gain G are $g_1 = 1 - \mu$, $g_2 = -\mu$, $g_3 = 1 - \mu = g_1, g_4 = 1$. Hence the set of values of G restricted to $AH \vee \bar{H}BK$ is $\mathcal{G}_{AH \vee \bar{H}BK} = \{g_1, g_2, g_4\} = \{1-\mu, -\mu, 1\}$. We distinguish two cases: (i) $\mu < 0$; (ii) $\mu \geq 0$.
Case (i). We observe that, $1 - \mu > 0$, $-\mu > 0$, and hence $\min \mathcal{G}_{AH \vee \bar{H}BK} > 0$. Then the assessment $\mathcal{M}_0$ on $\mathcal{F}_0$ is incoherent.
Case (ii). We observe that, $\min \mathcal{G}_{AH \vee \bar{H}BK} = -\mu \leq 0$ and $\max \mathcal{G}_{AH \vee \bar{H}BK} = 1 > 0$. Then, as

$$\min \mathcal{G}_{AH \vee \bar{H}BK} \cdot \max \mathcal{G}_{AH \vee \bar{H}BK} \leq 0,$$

the assessment $\mathcal{M}_0$ on $\mathcal{F}_0$ is coherent. Then, as System (A.15) is solvable and $\mathcal{M}_0$ on $\mathcal{F}_0$ is coherent, by Theorem 2 it follows that every assessment $(0,y,0,\mu)$ on $\mathcal{F}$ is coherent if and only if $(0,y,0,\mu) \in \Pi'' = \{(0,y,0,\mu) : y \in [0,1], \mu \geq 0\}$. Thus, $\Pi'' \subseteq \Pi$. Of course, if $(0,y,z,\mu) \notin \Pi''$ the assessment $(0,y,z,\mu)$ is not coherent and hence $(0,y,z,\mu) \notin \Pi$. Therefore $\Pi = \Pi' \cup \Pi''$. □

A.8. Proof of Theorem 18

Theorem 18. *Let A, B, H, K be any logically independent events. Given a coherent assessment (x,y) on $\{A|H, B|K\}$, with $x \neq 0$, for the iterated conditional $(B|K)|_S(A|H)$ the extension $\mu_S = \mathbb{P}[(B|K)|_S(A|H)]$ is coherent if and only if $\mu_S \in [\mu'_S, \mu''_S]$, where*

$$\mu'_S = \max\{\frac{x+y-1}{x}, 0\} \text{ and } \mu''_S = \begin{cases} \frac{x+y-2xy}{x(1-xy)}, & \text{if } (x,y) \neq (1,1); \\ 1, & \text{if } (x,y) = (1,1). \end{cases}$$

Proof. We simply write μ instead of μ_S. As $x > 0$, from Theorem 17 it follows that the set of all coherent assessments (x, y, z, μ) on $\mathcal{F} = \{A|H, B|K, (A|H) \wedge_S (B|K), (B|K)|_S(A|H)\}$ is the set Π' given in Theorem 17. Then, μ is a coherent extension of (x, y) if and only if $\mu \in [\mu', \mu'']$, where μ' and μ'' are

$$\mu' = \frac{z'}{x} = \max\{\frac{x+y-1}{x}, 0\} \text{ and } \mu'' = \frac{z''}{x} = \begin{cases} \frac{x+y-2xy}{x(1-xy)}, & \text{if } (x,y) \neq (1,1); \\ 1, & \text{if } (x,y) = (1,1). \end{cases} \qquad \square$$

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