A GENERAL NONEXISTENCE RESULT FOR INHOMOGENEOUS SEMILINEAR WAVE EQUATIONS WITH DOUBLE DAMPING AND POTENTIAL TERMS

MOHAEMD JLELI, BESSEM SAMET, CALOGERO VETRO

Abstract. We investigate the large-time behavior of solutions for a class of inhomogeneous semilinear wave equations involving double damping and potential terms. Namely, we first establish a general criterium for the absence of global weak solutions. Next, some special cases of potential and inhomogeneous terms are studied. In particular, when the inhomogeneous term depends only on the variable space, the Fujita critical exponent and the second critical exponent in the sense of Lee and Ni are derived.

1. Introduction

In this paper, we investigate the large-time behavior of solutions to the following Cauchy problem for the inhomogeneous semilinear wave equation

$$u_{tt} - \Delta u + u_t - \Delta u_t = V(x)|u|^p + W(t,x), \quad t>0,\, x\in\mathbb{R}^N, \tag{1.1}$$

where $N \geq 2$, $p>1$ and Δ is the Laplacian operator. Here, the left-hand side of (1.1) presents two types of damping terms. In the right-hand side, we have the combined effects of a potential term $V = V(x) > 0$ and an inhomogeneous term $W = W(t,x) \geq 0$ with $W \not\equiv 0$ (i.e. not identically zero).

Concerning the semilinear wave equation with frictional damping

$$u_{tt} - \Delta u + u_t = |u|^p, \quad t>0,\, x\in\mathbb{R}^N, \tag{1.2}$$

we know from the literature that such an equation admits a critical behavior. Precisely, there exists an exponent value $p_c > 1$ leading to a bifurcation in the following sense:

(i) if $p \leq p_c$, then problem (1.2) does not admit any nontrivial global solution for some initial data;
(ii) if $p > p_c$, then problem (1.2) has a small data global in time solution.

Notice that $p_c = 1 + \frac{2}{N}$, which is called Fujita exponent in the semilinear heat equation case. For more details on those mentioned results, we refer the reader to [3,7,8,12,13,15] and the references therein.

In the viscoelastic damping case, i.e.

$$u_{tt} - \Delta u - \Delta u_t = |u|^p, \quad t>0,\, x\in\mathbb{R}^N,$$

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the exact value of the critical exponent is still an open question. For some discussion about this problem, see, for example, [2, 10, 11] and the references therein.

In [4], the authors studied the special case of (1.1) when $V \equiv 1$ and $W \equiv 0$. Namely, it was shown that the number $1+\frac{2}{N}$ is critical when $N \in \{1,2\}$. Later, in [5], it was shown that $1+\frac{2}{N}$ is still critical in the case $N=3$. In [1], the question of the critical exponent for this problem has been solved in any space dimension, requiring suitable assumptions on the data, and L^1-regularities.

Motivated by the above mentioned contributions, the nonexistence of global weak solutions and the existence of stationary solutions to (1.1) are investigated in this paper. Namely, we first obtain a general criterium for the absence of global weak solutions to (1.1). Next, some special cases of V and W are studied. In particular, when W depends only on the variable space, the Fujita critical exponent and the second critical exponent in the sense of Lee and Ni are derived. To the best of our knowledge, problems of type (1.1) were not previously investigated.

Before stating our main results, we first fix some notations and define the notion of solutions to (1.1). Let $Q=(0,\infty)\times\mathbb{R}^N$. By $C_c^2(Q)$, we mean the space of C^2 real valued functions compactly supported in Q. It is supposed that the potential term $V>0$ is continuous and the inhomogeneous term $W \geq 0$ is a nontrivial $L^1_{loc}(Q)$-function.

Definition 1.1. *We say that $u \in L^p_{loc}(Q)$ is a global weak solution to (1.1), if*

$$\int_Q u(\varphi_{tt}-\Delta\varphi-\varphi_t+(\Delta\varphi)_t)\,dx\,dt=\int_Q V(x)|u|^p\varphi\,dx\,dt+\int_Q W(t,x)\varphi\,dx\,dt, \tag{1.3}$$

for every $\varphi \in C_c^2(Q)$.

Our first main result is the following.

Theorem 1.1. *Suppose that there exist $0<c_1<c_2<1$ such that*

$$\limsup_{T\to\infty} T^{\frac{1}{p-1}-\frac{N}{2}}\Big(\int_{0<|y|<\sqrt{2}} V(\sqrt{T}y)^{\frac{-1}{p-1}}\,dy\Big)^{-1}\int_{c_1T}^{c_2T}\int_{0<|y|<\sqrt{T}} W(t,y)\,dy\,dt=\infty. \tag{1.4}$$

Then (1.1) admits no global weak solution.

Remark 1.1. *We point out that Theorem 1.1 does not need any assumption about the initial values. This is due to the definition of global weak solutions to (1.1) that we considered, where (1.3) is satisfied for all $\varphi \in C_c^2((0,\infty)\times\mathbb{R}^N)$. Notice that for such test functions, one has $\varphi(0,x)=\varphi_t(0,x)=\Delta\varphi(0,x)=0$, for all $x\in\mathbb{R}^N$. On the other hand, it is possible to define a global weak solution to (1.1) subject to the initial conditions $(u(0,x),u_t(0,x))=(u_0(x),u_1(x))$, $x\in\mathbb{R}^N$, as a function $u\in L^p_{loc}(\overline{Q})$ satisfying*

$$\begin{aligned}&\int_Q u(\varphi_{tt}-\Delta\varphi-\varphi_t+(\Delta\varphi)_t)\,dx\,dt\\&=\mathcal{L}(u_0,u_1,\varphi)+\int_Q V(x)|u|^p\varphi\,dx\,dt+\int_Q W(t,x)\varphi\,dx\,dt,\end{aligned}$$

for all $\varphi \in C_c^2(\overline{Q})$*, where* $\overline{Q} = [0,\infty) \times \mathbb{R}^N$ *and*

$$\mathcal{L}(u_0,u_1,\varphi) = \int_{\mathbb{R}^N} u_0(x)\left(\varphi(0,x) - \varphi_t(0,x) - \Delta\varphi(0,x)\right)\,dx + \int_{\mathbb{R}^N} u_1(x)\varphi(0,x)\,dx.$$

In this case, following exactly the proof of Theorem 1.1, one can show that the result given by this theorem holds true under the additional assumption: $u_0, u_1 \in L^1_{loc}(\mathbb{R}^N)$*. Notice that this assumption is needed only to to guarantee that* $\mathcal{L}(u_0,u_1,\varphi) < \infty$*, for all* $\varphi \in C_c^2(\overline{Q})$*.*

Remark 1.2. *In this paper, we are concerned essentially with the existence and nonexistence of global weak solutions in the sense of Definition 1.1. For the local existence of solutions, see, for example [6].*

Consider now the functions V and W defined by

$$V(x) = (1+|x|^2)^{\frac{\alpha}{2}}, \quad x \in \mathbb{R}^N \tag{1.5}$$

and

$$W(t,x) = t^\sigma \mathcal{W}(x), \quad t>0,\, x\in\mathbb{R}^N, \tag{1.6}$$

where $\alpha > -2$, $-1 < \sigma \leq 0$ and $\mathcal{W}$ is a nontrivial $L^1_{loc}(\mathbb{R}^N)$-function. Such a kind of fuctions recall, and are inspired, to usual forms of perturbation terms (nonlinearity) exhibiting a polynomial-type growth condition. Also, the use of (locally) integrable functions is related to the decay properties as z goes to infinity.

We first discuss the case $\sigma \neq 0$.

Corollary 1.1. *Let* V *and* W *be the functions defined respectively by* (1.5) *and* (1.6)*, where* $\alpha > -2$ *and* $-1 < \sigma < 0$*. If*

$$1 < p < 1 + \frac{\alpha+2}{N-2\sigma-2}, \tag{1.7}$$

then (1.1) *admits no global weak solution.*

Remark 1.3. *In the case* $p \geq 1 + \frac{\alpha+2}{N-2\sigma-2}$*, we do not know whether global solutions exist or not. This question is open.*

Next, we consider the case $\sigma = 0$.

Corollary 1.2. *Let* V *and* W *be the functions defined respectively by* (1.5) *and* (1.6)*, where* $\alpha > -2$ *and* $\sigma = 0$*. If*

$$1 < p < p^*(\alpha, N), \tag{1.8}$$

where

$$p^*(\alpha,N) = \begin{cases} \infty & \text{if} \quad N = 2, \\ \frac{N+\alpha}{N-2} & \text{if} \quad N \geq 3, \end{cases}$$

then (1.1) *admits no global weak solution.*

Remark 1.4. *Note that* (1.8) *is optimal, i.e.* $p^*(\alpha,N)$ *is critical (in the sense of Fujita) for problem* (1.1)*. Namely, if* $N \geq 3$ *and* $p > p^*(\alpha,N)$*, then* (1.1) *admits stationary solutions for some* $W > 0$ *(see e.g. [14]), and hence global solutions. In the critical case* $p = p^*(\alpha,N)$ *and* $N \geq 3$*, we do not know whether global solutions exist or not. This question is open.*

For $N \geq 3$ and $a < N$, let

$$I_a^+ = \left\{\mathcal{W} \in C(\mathbb{R}^N) : \mathcal{W} \geq 0, \mathcal{W}(x) \geq C|x|^{-a} \text{ for } |x| \text{ large}\right\}$$

and

$$I_a^- = \left\{\mathcal{W} \in C(\mathbb{R}^N) : \mathcal{W} \geq 0, \mathcal{W}(x) \leq C|x|^{-a} \text{ for } |x| \text{ large}\right\},$$

where $C > 0$ is a constant (independent of x). The next result provides the second critical exponent for (1.1) in the sense of Lee and Ni [9]. Namely, for $N \geq 3$ and $p > p^*(\alpha, N)$, a second critical exponent is obtained when $\mathcal{W}$ is independent of t and it decays/grows as $|x|^{-a}$, for some $a < N$. Consequently, there exists a critical parameter value a^* leading to a bifurcation in the sense of Theorem 1.2 below. To this end, here we define the critical parameter

$$a^* = \frac{\alpha + 2p}{p-1}.$$

Theorem 1.2. *Let V and W be the functions defined respectively by (1.5) and (1.6), where $\alpha > -2$ and $\sigma = 0$. Let $N \geq 3$ and $p > p^*(\alpha, N)$.*

(i) *If $a < a^*$ and $\mathcal{W} \in I_a^+$, then (1.1) admits no global weak solution.*
(ii) *If $a^* \leq a < N$, then (1.1) admits stationary solutions (then global solutions) for some $\mathcal{W} \in I_a^-$.*

The rest of the paper is organized as follows. In Section 2, after establishing some preliminary estimates, we give the proof of the general nonexistence result given by Theorem 1.1. In Section 3, we prove Corollaries 1.1 and 1.2. Finally, Section 4 is devoted to the proof of Theorem 1.2.

2. General nonexistence result

We first introduce two cut-off functions $\lambda, \mu \in C^\infty(\mathbb{R}_+)$ satisfying

$$\lambda \geq 0, \quad \operatorname{supp}(\lambda) \subset (0,1), \quad \lambda(t) = 1, \quad c_1 \leq t \leq c_2 \tag{2.1}$$

and

$$0 \leq \mu \leq 1, \quad \mu(\sigma) = \begin{cases} 1 & \text{if} \quad 0 \leq \sigma \leq 1, \\ 0 & \text{if} \quad \sigma \geq 2, \end{cases} \tag{2.2}$$

where $0 < c_1 < c_2 < 1$.

For $T > 0$, let

$$\xi(t,x) = \lambda\left(\frac{t}{T}\right)^k \mu\left(\frac{|x|^2}{T^{2\rho}}\right)^k := a_T(t) b_T(x), \tag{2.3}$$

where $k \geq 2$ and $\rho > 0$ are constants.

2.1. Some preliminaries.

Lemma 2.1. *Let $k \geq \frac{2p}{p-1}$. There exists a constant $C > 0$ such that*

$$\int_Q V(x)^{\frac{-1}{p-1}} \xi^{\frac{-1}{p-1}} |\xi_{tt}|^{\frac{p}{p-1}} \, dx \, dt \leq C T^{N\rho - \frac{p+1}{p-1}} \int_{0<|y|<\sqrt{2}} V(T^\rho y)^{\frac{-1}{p-1}} \, dy.$$

Proof. By (2.3), one has

$$\int_Q V(x)^{\frac{-1}{p-1}}\xi^{\frac{-1}{p-1}}|\xi_{tt}|^{\frac{p}{p-1}}\,dx\,dt=\left(\int_0^\infty a_T(t)^{\frac{-1}{p-1}}|a_T''(t)|^{\frac{p}{p-1}}\,dt\right)\left(\int_{\mathbb{R}^N}V(x)^{\frac{-1}{p-1}}b_T(x)\,dx\right).\tag{2.4}$$

By (2.1), one obtains

$$\int_0^\infty a_T(t)^{\frac{-1}{p-1}}|a_T''(t)|^{\frac{p}{p-1}}\,dt=T^{\frac{-2p}{p-1}}\int_0^T\lambda\Big(\frac{t}{T}\Big)^{k-\frac{2p}{p-1}}\left|\theta\left(\frac{t}{T}\right)\right|^{\frac{p}{p-1}}\,dt,$$

where

$$\theta(s)=k\left[(k-1)\lambda'(s)^2+\lambda(s)\lambda''(s)\right],\quad 0<s<1.$$

Hence, one deduces that

$$\begin{aligned}\int_0^\infty a_T(t)^{\frac{-1}{p-1}}|a_T''(t)|^{\frac{p}{p-1}}\,dt &\leq CT^{\frac{-2p}{p-1}}\int_0^T\lambda\Big(\frac{t}{T}\Big)^{k-\frac{2p}{p-1}}\,dt\\ &= CT^{-\left(\frac{p+1}{p-1}\right)}\int_0^1\lambda(s)^{k-\frac{2p}{p-1}}\,ds,\end{aligned}$$

which yields (since $k\geq\frac{2p}{p-1}$)

$$\int_0^\infty a_T(t)^{\frac{-1}{p-1}}|a_T''(t)|^{\frac{p}{p-1}}\,dt\leq CT^{-\left(\frac{p+1}{p-1}\right)}.\tag{2.5}$$

Here and below, $C>0$ is a generic suitable constant whose value may change from line to line. Next, using (2.2), one obtains

$$\int_{\mathbb{R}^N}V(x)^{\frac{-1}{p-1}}b_T(x)\,dx=\int_{0<|x|<\sqrt{2}T^\rho}V(x)^{\frac{-1}{p-1}}\mu\Big(\frac{|x|^2}{T^{2\rho}}\Big)^k\,dx.$$

Using the change of variable $x=T^\rho y$, it holds that

$$\int_{\mathbb{R}^N}V(x)^{\frac{-1}{p-1}}b_T(x)\,dx=T^{N\rho}\int_{0<|y|<\sqrt{2}}V(T^\rho y)^{\frac{-1}{p-1}}\mu(|y|^2)^k\,dy,$$

which yields

$$\int_{\mathbb{R}^N}V(x)^{\frac{-1}{p-1}}b_T(x)\,dx\leq T^{N\rho}\int_{0<|y|<\sqrt{2}}V(T^\rho y)^{\frac{-1}{p-1}}\,dy.\tag{2.6}$$

Therefore, using (2.4), (2.5) and (2.6), the desired estimate follows. □

Lemma 2.2. *Let $k\geq\frac{p}{p-1}$. There exists a constant $C>0$ such that*

$$\int_Q V(x)^{\frac{-1}{p-1}}\xi^{\frac{-1}{p-1}}|\xi_t|^{\frac{p}{p-1}}\,dx\,dt\leq CT^{N\rho-\frac{1}{p-1}}\int_{0<|y|<\sqrt{2}}V(T^\rho y)^{\frac{-1}{p-1}}\,dy.$$

Proof. By (2.3), one has

$$\int_Q V(x)^{\frac{-1}{p-1}}\xi^{\frac{-1}{p-1}}|\xi_t|^{\frac{p}{p-1}}\,dx\,dt=\left(\int_0^\infty a_T(t)^{\frac{-1}{p-1}}|a_T'(t)|^{\frac{p}{p-1}}\,dt\right)\left(\int_{\mathbb{R}^N}V(x)^{\frac{-1}{p-1}}b_T(x)\,dx\right).\tag{2.7}$$

On the other hand, by (2.1), one obtains

$$\int_0^\infty a_T(t)^{\frac{-1}{p-1}}|a_T'(t)|^{\frac{p}{p-1}}\,dt = k^{\frac{p}{p-1}}T^{\frac{-p}{p-1}}\int_0^T \lambda\Big(\frac{t}{T}\Big)^{k-\frac{p}{p-1}}\left|\lambda'\left(\frac{t}{T}\right)\right|^{\frac{p}{p-1}}\,dt$$
$$\le CT^{1-\frac{p}{p-1}}\int_0^1 \lambda(s)^{k-\frac{p}{p-1}}\,ds,$$

which yields

$$\int_0^\infty a_T(t)^{\frac{-1}{p-1}}|a_T'(t)|^{\frac{p}{p-1}}\,dt \le CT^{\frac{-1}{p-1}}. \tag{2.8}$$

Hence, using (2.6), (2.7) and (2.8), the desired estimate follows. □

Lemma 2.3. *Let $k \ge \frac{2p}{p-1}$. There exists a constant $C>0$ such that*

$$\int_Q V(x)^{\frac{-1}{p-1}}\xi^{\frac{-1}{p-1}}|\Delta\xi|^{\frac{p}{p-1}}\,dx\,dt \le CT^{\frac{-2\rho p}{p-1}+N\rho+1}\int_{1<|y|<\sqrt{2}} V(T^\rho y)^{\frac{-1}{p-1}}\,dy$$

Proof. By (2.3), one has

$$\int_Q V(x)^{\frac{-1}{p-1}}\xi^{\frac{-1}{p-1}}|\Delta\xi|^{\frac{p}{p-1}}\,dx\,dt = \left(\int_0^\infty a_T(t)\,dt\right)\left(\int_{\mathbb{R}^N} V(x)^{\frac{-1}{p-1}}b_T(x)^{\frac{-1}{p-1}}|\Delta b_T(x)|^{\frac{p}{p-1}}\,dx\right). \tag{2.9}$$

Using (2.1), one obtains

$$\int_0^\infty a_T(t)\,dt = \int_0^T \lambda\Big(\frac{t}{T}\Big)^k\,dt = T\int_0^1\lambda(s)^k\,ds = CT. \tag{2.10}$$

On the other hand, an elementary calculation shows that

$$\Delta b_T(x) = T^{-2\rho}\mu\Big(\frac{|x|^2}{T^{2\rho}}\Big)^{k-2}\phi\Big(\frac{|x|^2}{T^{2\rho}}\Big),\quad x\in\mathbb{R}^N,$$

where

$$\phi\Big(\frac{|x|^2}{T^{2\rho}}\Big)$$
$$= 2k\left[N\mu\Big(\frac{|x|^2}{T^{2\rho}}\Big)\mu'\Big(\frac{|x|^2}{T^{2\rho}}\Big) + 2(k-1)|x|^2T^{-2\rho}\mu'\Big(\frac{|x|^2}{T^{2\rho}}\Big)^2 + 2|x|^2T^{-2\rho}\mu\Big(\frac{|x|^2}{T^{2\rho}}\Big)\mu''\Big(\frac{|x|^2}{T^{2\rho}}\Big)\right].$$

Notice that by (2.2), one has

$$\phi\Big(\frac{|x|^2}{T^{2\rho}}\Big) = 0,\quad 0\le|x|<T^\rho.$$

Now, the continuity of the function ϕ and (2.2) lead to

$$\left|\phi\Big(\frac{|x|^2}{T^{2\rho}}\Big)\right| \le C,\quad x\in\mathbb{R}^N,$$

for some $C>0$, which yields

$$|\Delta b_T(x)| \le CT^{-2\rho}\mu\Big(\frac{|x|^2}{T^{2\rho}}\Big)^{k-2}.$$

Therefore, using (2.2), one obtains

$$\begin{aligned}
&\int_{\mathbb{R}^N} V(x)^{\frac{-1}{p-1}} b_T(x)^{\frac{-1}{p-1}} |\Delta b_T(x)|^{\frac{p}{p-1}}\, dx \\
&\leq CT^{\frac{-2\rho p}{p-1}} \int_{T^\rho<|x|<\sqrt{2}T^\rho} V(x)^{\frac{-1}{p-1}} \mu\Big(\frac{|x|^2}{T^{2\rho}}\Big)^{k-\frac{2p}{p-1}}\, dx \\
&\leq CT^{\frac{-2\rho p}{p-1}} \int_{T^\rho<|x|<\sqrt{2}T^\rho} V(x)^{\frac{-1}{p-1}}\, dx.
\end{aligned}$$

Using the change of variable $x = T^\rho y$, it holds that

$$\int_{\mathbb{R}^N} V(x)^{\frac{-1}{p-1}} b_T(x)^{\frac{-1}{p-1}} |\Delta b_T(x)|^{\frac{p}{p-1}}\, dx \leq CT^{\frac{-2\rho p}{p-1}+N\rho} \int_{1<|y|<\sqrt{2}} V(T^\rho y)\, dy. \tag{2.11}$$

Next, using (2.9), (2.10) and (2.11), the desired estimate follows. □

Lemma 2.4. *Let $k \geq \frac{2p}{p-1}$. There exists a constant $C > 0$ such that*

$$\int_Q V(x)^{\frac{-1}{p-1}} \xi^{\frac{-1}{p-1}} |(\Delta\xi)_t|^{\frac{p}{p-1}}\, dx\, dt \leq CT^{N\rho-\frac{2\rho p+1}{p-1}} \int_{1<|y|<\sqrt{2}} V(T^\rho y)\, dy.$$

Proof. By (2.3), one has

$$\begin{aligned}
&\int_Q V(x)^{\frac{-1}{p-1}} \xi^{\frac{-1}{p-1}} |(\Delta\xi)_t|^{\frac{p}{p-1}}\, dx\, dt \\
&= \left(\int_0^\infty a_T(t)^{\frac{-1}{p-1}} |a_T'(t)|^{\frac{p}{p-1}}\, dt\right) \left(\int_{\mathbb{R}^N} V(x)^{\frac{-1}{p-1}} b_T(x)^{\frac{-1}{p-1}} |\Delta b_T(x)|^{\frac{p}{p-1}}\, dx\right).
\end{aligned}$$

Hence, by (2.8) and (2.11), the desired estimate follows. □

2.2. **Proof of Theorem 1.1.** Suppose that $u \in L^p_{loc}(Q)$ is a global weak solution to (1.1). Using (1.3), one obtains

$$\begin{aligned}
&\int_Q V(x)|u|^p \varphi dx dt + \int_Q W(t,x)\varphi dx dt \\
&\leq \int_Q |u||\varphi_{tt}| dx dt + \int_Q |u||\Delta\varphi| dx dt + \int_Q |u||\varphi_t| dx dt + \int_Q |u||(\Delta\varphi)_t| dx dt,
\end{aligned} \tag{2.12}$$

for all $\varphi \in C_c^2(Q)$, $\varphi \geq 0$. Writing

$$\int_Q |u||\varphi_{tt}|\, dx\, dt = \int_Q V(x)^{\frac{1}{p}} \varphi^{\frac{1}{p}} |u| V(x)^{\frac{-1}{p}} \varphi^{\frac{-1}{p}} |\varphi_{tt}|\, dx\, dt$$

and using ε- Young inequality with $0 < \varepsilon < \frac{1}{4}$, one obtains

$$\int_Q |u||\varphi_{tt}|\, dx\, dt \leq \varepsilon \int_Q V(x)\varphi|u|^p\, dx\, dt + C\int_Q V(x)^{\frac{-1}{p-1}} \varphi^{\frac{-1}{p-1}} |\varphi_{tt}|^{\frac{p}{p-1}}\, dx\, dt \tag{2.13}$$

Similarly, one has

$$\int_Q |u||\Delta\varphi|\, dx\, dt \leq \varepsilon \int_Q V(x)\varphi|u|^p\, dx\, dt + C\int_Q V(x)^{\frac{-1}{p-1}} \varphi^{\frac{-1}{p-1}} |\Delta\varphi|^{\frac{p}{p-1}}\, dx\, dt, \tag{2.14}$$

$$\int_Q |u||\varphi_t|\,dx\,dt \le \varepsilon \int_Q V(x)\varphi|u|^p\,dx\,dt + C\int_Q V(x)^{\frac{-1}{p-1}}\varphi^{\frac{-1}{p-1}}|\varphi_t|^{\frac{p}{p-1}}\,dx\,dt \tag{2.15}$$

and

$$\int_Q |u||(\Delta\varphi)_t|\,dx\,dt \le \varepsilon \int_Q V(x)\varphi|u|^p\,dx\,dt + C\int_Q V(x)^{\frac{-1}{p-1}}\varphi^{\frac{-1}{p-1}}|(\Delta\varphi)_t|^{\frac{p}{p-1}}\,dx\,dt. \tag{2.16}$$

Hence, using (2.12), (2.13), (2.14), (2.15) and (2.16), one deduces that

$$\begin{aligned}&\int_Q W(t,x)\varphi\,dx\,dt\\ &\le C\int_Q V(x)^{\frac{-1}{p-1}}\left(\varphi^{\frac{-1}{p-1}}|\varphi_{tt}|^{\frac{p}{p-1}} + \varphi^{\frac{-1}{p-1}}|\Delta\varphi|^{\frac{p}{p-1}} + \varphi^{\frac{-1}{p-1}}|\varphi_t|^{\frac{p}{p-1}} + \varphi^{\frac{-1}{p-1}}|(\Delta\varphi)_t|^{\frac{p}{p-1}}\right)dx\,dt,\end{aligned}$$

for all $\varphi\in C_c^2(Q)$, $\varphi\ge 0$. On the other hand, one can check easily that $\xi\in C_c^2(Q)$ and $\xi\ge 0$, where ξ is the function defined by (2.3) for $T>0$, $k\ge\frac{2p}{p-1}$ and $0<c_1<c_2<1$. Then, it holds that

$$\begin{aligned}&\int_Q W(t,x)\xi\,dx\,dt\\ &\le C\int_Q V(x)^{\frac{-1}{p-1}}\left(\xi^{\frac{-1}{p-1}}|\xi_{tt}|^{\frac{p}{p-1}} + \varphi^{\frac{-1}{p-1}}|\Delta\xi|^{\frac{p}{p-1}} + \xi^{\frac{-1}{p-1}}|\xi_t|^{\frac{p}{p-1}} + \xi^{\frac{-1}{p-1}}|(\Delta\xi)_t|^{\frac{p}{p-1}}\right)dx\,dt.\end{aligned} \tag{2.17}$$

On the other hand, by (2.1), (2.2) and (2.3), one has (since $W\ge 0$)

$$\begin{aligned}\int_Q W(t,x)\xi\,dx\,dt &= \int_0^\infty\int_{\mathbb{R}^N} W(t,x)\lambda\left(\frac{t}{T}\right)^k\mu\left(\frac{|x|^2}{T^{2\rho}}\right)^k dx\,dt\\ &\ge \int_{c_1T}^{c_2T}\int_{0<|y|<T^\rho} W(t,y)\,dy\,dt.\end{aligned} \tag{2.18}$$

Therefore, using (2.17), (2.18), together with Lemmas 2.1, 2.2, 2.3 and 2.4, one deduces that

$$\begin{aligned}&\int_{c_1T}^{c_2T}\int_{0<|y|<T^\rho} W(t,y)\,dy\,dt\\ &\le C\left(T^{N\rho-\frac{p+1}{p-1}} + T^{N\rho-\frac{1}{p-1}} + T^{\frac{-2\rho p}{p-1}+N\rho+1} + T^{N\rho-\frac{2\rho p+1}{p-1}}\right)\int_{0<|y|<\sqrt{2}} V(T^\rho y)^{\frac{-1}{p-1}}\,dy.\end{aligned}$$

Taking $\rho=\frac{1}{2}$ in the above inequality, for T sufficiently large, it holds that

$$\int_{c_1T}^{c_2T}\int_{0<|y|<\sqrt{T}} W(t,y)\,dy\,dt \le CT^{\frac{N}{2}-\frac{1}{p-1}}\int_{0<|y|<\sqrt{2}} V(\sqrt{T}y)^{\frac{-1}{p-1}}\,dy$$

i.e.

$$T^{\frac{1}{p-1}-\frac{N}{2}}\left(\int_{0<|y|<\sqrt{2}} V(\sqrt{T}y)^{\frac{-1}{p-1}}\,dy\right)^{-1}\int_{c_1T}^{c_2T}\int_{0<|y|<\sqrt{T}} W(t,y)\,dy\,dt \le C,$$

which contradicts (1.4). $\square$

3. Some special cases

In this section, we consider problem (1.1), where V and W are defined respectively by (1.5) and (1.6). Namely, we prove Corollaries 1.1 and 1.2.

3.1. **Proof of Corollary 1.1.** By (1.5), one has

$$\begin{aligned}\int_{0<|y|<\sqrt{2}} V(\sqrt{T}y)^{\frac{-1}{p-1}}\,dy &= \int_{0<|y|<\sqrt{2}} \left(1+T|y|^2\right)^{\frac{-\alpha}{2(p-1)}}\,dy\\ &=C\int_0^{\sqrt{2}} (1+Tr^2)^{\frac{-\alpha}{2(p-1)}} r^{N-1}\,dr\\ &=C\int_0^{\sqrt{2}} 2r(1+Tr^2)^{\frac{-\alpha}{2(p-1)}} r^{N-2}\,dr\\ &\leq CT^{-1}\int_0^{\sqrt{2}} 2Tr(1+Tr^2)^{\frac{-\alpha}{2(p-1)}}\,dr.\end{aligned} \tag{3.1}$$

We have three possible cases.
Case 1: $\alpha=2(p-1)$. In this case, one has

$$\int_0^{\sqrt{2}} 2Tr(1+Tr^2)^{\frac{-\alpha}{2(p-1)}}\,dr = \ln(1+2T).$$

Hence, using (3.1), for sufficiently large T, one obtains

$$\int_{0<|y|<\sqrt{2}} V(\sqrt{T}y)^{\frac{-1}{p-1}}\,dy \leq CT^{-1}\ln T,$$

which yields

$$T^{\frac{1}{p-1}-\frac{N}{2}}\left(\int_{0<|y|<\sqrt{2}} V(\sqrt{T}y)^{\frac{-1}{p-1}}\,dy\right)^{-1} \geq CT^{\frac{p}{p-1}-\frac{N}{2}}(\ln T)^{-1}. \tag{3.2}$$

Case 2: $\alpha<2(p-1)$. In this case, one has

$$\int_0^{\sqrt{2}} 2Tr(1+Tr^2)^{\frac{-\alpha}{2(p-1)}}\,dr = C\left[(1+2T)^{1-\frac{\alpha}{2(p-1)}}-1\right].$$

Hence, using (3.1), for sufficiently large T, one obtains

$$\int_{0<|y|<\sqrt{2}} V(\sqrt{T}y)^{\frac{-1}{p-1}}\,dy \leq CT^{\frac{-\alpha}{2(p-1)}},$$

which yields

$$T^{\frac{1}{p-1}-\frac{N}{2}}\left(\int_{0<|y|<\sqrt{2}} V(\sqrt{T}y)^{\frac{-1}{p-1}}\,dy\right)^{-1} \geq CT^{\frac{\alpha+2}{2(p-1)}-\frac{N}{2}}. \tag{3.3}$$

Case 3: $\alpha>2(p-1)$. In this case, one has

$$\int_0^{\sqrt{2}} 2Tr(1+Tr^2)^{\frac{-\alpha}{2(p-1)}}\,dr = C\left[1-(1+2T)^{1-\frac{\alpha}{2(p-1)}}\right].$$

Hence, using (3.1), for sufficiently large T, one obtains

$$\int_{0<|y|<\sqrt{2}} V(\sqrt{T}y)^{\frac{-1}{p-1}}\,dy \leq CT^{-1},$$

which yields

$$T^{\frac{1}{p-1}-\frac{N}{2}}\left(\int_{0<|y|<\sqrt{2}} V(\sqrt{T}y)^{\frac{-1}{p-1}}\,dy\right)^{-1} \geq CT^{\frac{p}{p-1}-\frac{N}{2}}. \tag{3.4}$$

Hence, combining (3.2), (3.3) and (3.4), for sufficiently large T, it holds that

$$T^{\frac{1}{p-1}-\frac{N}{2}}\left(\int_{0<|y|<\sqrt{2}} V(\sqrt{T}y)^{\frac{-1}{p-1}}\,dy\right)^{-1} \geq CT^{\frac{\alpha+2}{2(p-1)}-\frac{N}{2}}(\ln T)^{-1}. \tag{3.5}$$

Moreover, for any $0 < c_1 < c_2 < 1$, and sufficiently large T, one has

$$\begin{aligned}\int_{c_1T}^{c_2T}\int_{0<|y|<\sqrt{T}} W(t,y)\,dy\,dt &= \left(\int_{c_1T}^{c_2T} t^\sigma\,dt\right)\left(\int_{0<|y|<\sqrt{T}} \mathcal{W}(y)\,dy\right)\\ &\geq CT^{\sigma+1}\int_{0<|y|<1} \mathcal{W}(y)\,dy\\ &= CT^{\sigma+1}.\end{aligned} \tag{3.6}$$

Therefore, using (3.5) and (3.6), one deduces that for sufficiently large T,

$$\begin{aligned}&T^{\frac{1}{p-1}-\frac{N}{2}}\Big(\int_{0<|y|<\sqrt{2}} V(\sqrt{T}y)^{\frac{-1}{p-1}}\,dy\Big)^{-1}\int_{c_1T}^{c_2T}\int_{0<|y|<\sqrt{T}} W(t,y)\,dy\,dt\\ &\geq CT^{\frac{\alpha+2}{2(p-1)}-\frac{N}{2}+\sigma+1}(\ln T)^{-1}.\end{aligned}$$

Next, using (1.7), the above estimate yields (1.4). Hence, by Theorem 1.1, one deduces that (1.1) admits no global weak solution. □

3.2. **Proof of Corollary 1.2.** Using (3.5) and (3.6) with $\sigma = 0$, for sufficiently large T, one obtains

$$\begin{aligned}&T^{\frac{1}{p-1}-\frac{N}{2}}\Big(\int_{0<|y|<\sqrt{2}} V(\sqrt{T}y)^{\frac{-1}{p-1}}\,dy\Big)^{-1}\int_{c_1T}^{c_2T}\int_{0<|y|<\sqrt{T}} W(t,y)\,dy\,dt\\ &\geq CT^{\frac{\alpha+2}{2(p-1)}-\frac{N}{2}+1}(\ln T)^{-1}.\end{aligned}$$

Next, using (1.8), the above estimate yields (1.4). Hence, by Theorem 1.1, one deduces that (1.1) admits no global weak solution. □

4. Second critical exponent

In this section, we prove Theorem 1.2 which provides the second critical exponent for (1.1) in the sense of Lee and Ni [9].

4.1. **Proof of Theorem 1.2 (i).** Let $\mathcal{W} \in I_a^+$. Then, for $0 < c_1 < c_2 < 1$ and sufficiently large T, one has

$$\begin{aligned}\int_{c_1T}^{c_2T}\int_{0<|y|<\sqrt{T}} W(t,y)\,dy\,dt &\geq CT\int_{\frac{\sqrt{T}}{2}<|y|<\sqrt{T}}\mathcal{W}(y)\,dy\\ &\geq CT\int_{\frac{\sqrt{T}}{2}<|y|<\sqrt{T}}|y|^{-a}\,dy\\ &= CT^{1+\frac{N-a}{2}}.\end{aligned}$$

Using (3.5) and the above estimate, one obtains

(4.1)

$$T^{\frac{1}{p-1}-\frac{N}{2}}\left(\int_{0<|y|<\sqrt{2}} V(\sqrt{T}y)^{\frac{-1}{p-1}}dy\right)^{-1}\int_{c_1T}^{c_2T}\int_{0<|x|<\sqrt{T}} W(x)dxdt \geq CT^{\frac{\alpha+2}{2(p-1)}+1-\frac{a}{2}}(\ln T)^{-1}.$$

Observe that $a < a^*$ is equivalent to

$$\frac{\alpha+2}{2(p-1)}+1-\frac{a}{2} > 0. \tag{4.2}$$

Hence, using (4.1) and (4.2), one obtains (1.4). Then, by Theorem 1.1, one deduces that (1.1) admits no global weak solution. □

4.2. **Proof of Theorem 1.2 (ii).** For

$$a-2 \leq \delta < N-2 \quad \text{and} \quad 0 < \epsilon < [\delta(N-\delta-2)]^{\frac{1}{p-1}}, \tag{4.3}$$

let

$$u_{\delta,\epsilon}(x) = \epsilon(1+|x|^2)^{-\frac{\delta}{2}}, \quad x \in \mathbb{R}^N.$$

Using (4.3), one can check easily that

$$\begin{aligned}\mathcal{W}(x) &:= -\Delta u_{\delta,\epsilon} - Vu_{\delta,\epsilon}^p\\ &= \delta\epsilon\Big[N(1+|x|^2)^{-\frac{\delta}{2}-1} - (\delta+2)|x|^2(1+|x|^2)^{-\frac{\delta}{2}-2}\Big] - \epsilon^p(1+|x|^2)^{\frac{\alpha-\delta p}{2}}\\ &> \delta\epsilon(N-\delta-2)(1+|x|^2)^{-\frac{\delta}{2}-1} - \epsilon^p(1+|x|^2)^{\frac{\alpha-\delta p}{2}}\\ &> 0.\end{aligned}$$

Moreover, for $|x|$ sufficiently large, one has

$$\begin{aligned}\mathcal{W}(x) &\leq \delta\epsilon\Big[N(1+|x|^2)^{-\frac{\delta}{2}-1} - (\delta+2)|x|^2(1+|x|^2)^{-\frac{\delta}{2}-2}\Big]\\ &= \delta\epsilon(1+|x|^2)^{-\frac{\delta}{2}-2}\Big[(N-\delta-2)|x|^2+N\Big]\\ &\leq \delta\epsilon N(1+|x|^2)^{-\frac{\delta}{2}-1}\\ &\leq \delta\epsilon N(1+|x|^2)^{-\frac{a}{2}}\\ &\leq C|x|^{-a}.\end{aligned}$$

Hence, for all δ and ϵ satisfying (4.3), the function $u_{\delta,\epsilon}$ is a stationary solution to (1.1) with $\mathcal{W} \in I_a^-$ □

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Mohamed Jleli
Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh, 11451, Saudi Arabia
E-mail: jleli@ksu.edu.sa

Bessem Samet
Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh, 11451, Saudi Arabia

E-mail: bsamet@ksu.edu.sa

Calogero Vetro
Department of Mathematics and Computer Science, University of Palermo, Via Archirafi 34, 90123, Palermo, Italy
E-mail: calogero.vetro@unipa.it