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Multiple solutions for a class of anisotropic \vec{p} -Laplacian problems

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Abstract

In this paper we present some existence and multiplicity results for a class of anisotropic \vec{p} -Laplacian problems with Dirichlet boundary conditions. In particular, the existence of three solutions is pointed out. The approach is based on variational methods and our main tool is a three critical point theorem.

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1 Introduction

In the present work we deal with multiplication results of solutions for the following anisotropic problem

$$\begin{cases} -\Delta_{\vec{p}} u = \lambda f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (D_{\lambda}^{\vec{p}})$$

We suppose that Ω is a nonempty bounded open set of the real Euclidean space \mathbb{R}^N , with $N \geq 2$, whose boundary is of class C^1 , $f: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is an L^1 -Carathéodory function and $\vec{p} = (p_1, p_2, \dots, p_N)$, $\vec{p} \in \mathbb{R}^N$, with

$$p^- = \min\{p_1, p_2, \dots, p_N\} > N \quad \text{and} \quad p^+ = \max\{p_1, p_2, \dots, p_N\}, \quad (1.1)$$

respectively the minimum and the maximum value of the anisotropic configuration. Moreover, λ is a positive real parameter.

The anisotropic \vec{p} -Laplacian operator is defined as

$$\Delta_{\vec{p}} u = \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{\partial u}{\partial x_i} \right), \quad (1.2)$$

which is a generalization of the usual Laplacian operator in the case of $p_i = 2$, for all $i = 1, \dots, N$. We also observe that formula (1.2) becomes the pseudo- p -Laplacian operator if \vec{p} is constant (that is, $p_i = p$ for all $i = 1, \dots, N$) (see, for instance, [5, 11]).

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The theory of anisotropic Sobolev space appears for the first time in [22, 28, 30, 31, 35], where the authors introduce the anisotropic Sobolev space starting from a generic set of multi-indices. Recently, many authors have investigated anisotropic boundary value problems. For an overview of these subjects, we refer to [3, 4, 9, 13, 14, 18–20, 24] and the references therein. In [18] the authors study the existence of positive solutions for a class of anisotropic quasilinear systems. The approach is based on a suitable combination of the sub-supersolution method with the mountain pass theorem. In [24], the authors use an approximation approach to prove the existence and regularity of the solutions to an anisotropic problem involving a singular nonlinearity.

The theory of anisotropic problems is also extended for the case when the indexes of the operator are continuous functions. Then, the operator describing these problems becomes the following anisotropic $\vec{p}(x)$ -Laplacian operator

$$\Delta_{\vec{p}(x)}u = \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)-2} \frac{\partial u}{\partial x_i} \right). \tag{1.3}$$

It is easy to see that in the case of $p_i(x) = p(x)$ for all $i = 1, \dots, N$, the operator (1.3) becomes the pseudo- $p(x)$ -Laplacian operator (see, for instance, [10]). In this framework, the study of nonlinear elliptic problems involving operators of the type $p(x)$ -Laplacian is based on the theory of generalized Lebesgue-Sobolev spaces (see for instance [15, 17, 21, 25–27] and references therein). Nonlinear differential problems involving nonlocal operators as well as non standard and/or non uniform operators have been widely studied (see [12, 16, 29]). Anisotropic problems are of increasing interest, especially for their applications. Operators such as (1.2) model phenomena in which partial derivatives vary with direction. For instance, the study of an epidemic disease in heterogeneous habitat or many reaction–diffusion processes depending on different environments can be expressed by an anisotropic nonlinear operator. For more details about these arguments, we refer to [1, 6, 7, 23, 32, 33, 36, 37] and the references therein.

The aim of this paper is to obtain the existence of multiple solutions for the problem $(D_{\lambda}^{\vec{p}})$ using variational methods. In particular, our main tool is a critical points theorem (Theorem 2.1) established in [8]. Here a special case of our main result is presented.

Theorem 1.1 *Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function which is nonnegative and non zero in $[0, +\infty)$ and nonpositive and nonzero in $(-\infty, 0]$ and such that*

$$\lim_{|t| \rightarrow +\infty} \frac{g(t)}{|t|^{p^*-1}} = \lim_{t \rightarrow 0} \frac{g(t)}{|t|^{p^*-1}} = 0. \tag{1.4}$$

Then there exists $\lambda^ > 0$ such that for each $\lambda > \lambda^*$ the problem*

$$\begin{cases} -\Delta_{\vec{p}}u = \lambda g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{AD_{\lambda}^{\vec{p}}}$$

admits at least four distinct nontrivial weak solutions, two positive and two negative.

The paper is arranged as follows. In Sect. 2 we define some preliminaries and basic notations that we are going to use to define the anisotropic Sobolev space. Moreover, properties

of the functional associated to problem $(D_{\lambda}^{\vec{p}})$ are pointed out. In Sect. 3, our main result (Theorem 3.1) and consequences are presented (see Theorem 3.2). In particular, under suitable sign conditions for the nonlinearity, in Theorem 3.2 the existence of two positive solutions is established. Finally, an example (see Example 3.4) is presented.

2 Preliminaries and basic notations

The present section is devoted to defining the anisotropic Sobolev space and to recalling some properties and basic results that we will use in the sequel. Given a vector $\vec{p} = (p_0, p_1, \dots, p_N)$ with $p_i \geq 1$ for $i = 0, 1, \dots, N$, the anisotropic Sobolev space (see [30, Definition 7]) is defined as

$$W^{1,\vec{p}}(\Omega) = \left\{ u \in L^{p_0}(\Omega) : \frac{\partial u}{\partial x_i} \in L^{p_i}(\Omega), \text{ for } i = 1, \dots, N \right\}, \tag{2.1}$$

endowed with the norm

$$\|u\|_{W^{1,\vec{p}}(\Omega)} = \|u\|_{L^{p_0}(\Omega)} + \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)}. \tag{2.2}$$

We define as $W_0^{1,\vec{p}}(\Omega)$ the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.2) and denote by $(W_0^{1,\vec{p}}(\Omega))^*$ its dual space. Moreover, $(W^{1,\vec{p}}(\Omega), \|\cdot\|_{W^{1,\vec{p}}(\Omega)})$ and $(W_0^{1,\vec{p}}(\Omega), \|\cdot\|_{W_0^{1,\vec{p}}(\Omega)})$ are separable Banach spaces, which are reflexive if $p_i > 1$ for $i = 0, 1, \dots, N$, and, taking into account the smoothness of the boundary the Ω , the embedding theorems are verified, (see [22, 30, 31]).

Here and in the sequel assume $p^- = \min\{p_1, p_2, \dots, p_N\} > N$. Clearly, from the Hölder inequality, one has

$$\|u\|_{W^{1,p^-}(\Omega)} \leq c \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)}$$

for all $u \in W_0^{1,\vec{p}}(\Omega)$ (see [9, page 234]). Taking into account that $W^{1,p^-}(\Omega)$ is compactly embedded in $C(\bar{\Omega})$, one has $\|u\|_{C(\bar{\Omega})} \leq k \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)}$ for which $\|u\|_{L^{p_0}(\Omega)} \leq \tilde{k} \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)}$. Hence, on $W_0^{1,\vec{p}}(\Omega)$ we can also define the following norm

$$\|u\|_{W_0^{1,\vec{p}}(\Omega)} := \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)}, \tag{2.3}$$

which is equivalent to the usual one (2.2).

Now, recalling that the usual Sobolev space $W_0^{1,p^-}(\Omega)$ is compactly embedded in $C^0(\bar{\Omega})$, we explicit that one also has

$$\|u\|_{C^0(\bar{\Omega})} \leq m_{p^-} \|u\|_{W_0^{1,p^-}(\Omega)}, \tag{2.4}$$

for every $u \in W_0^{1,p^-}(\Omega)$, where

$$m_{p^-} = \frac{N^{-\frac{1}{p^-}}}{\sqrt{\pi}} \left[\Gamma\left(1 + \frac{N}{2}\right) \right]^{\frac{1}{N}} \left(\frac{p^- - 1}{p^- - N} \right)^{1 - \frac{1}{p^-}} |\Omega|^{\frac{1}{N} - \frac{1}{p^-}}, \tag{2.5}$$

with Γ the Gamma function and $|\Omega|$ the Lebesgue measure of Ω . In particular, if Ω is the N -dimensional ball, (2.5) is the best constant such that (2.4) is verified (see [34, Formula (6b)]).

Now we recall the following two propositions (see [9, Proposition 2.1] and [9, Proposition 2.2]) that we need for our purposes.

Proposition 2.1 Suppose that $p^- > N$; one has

$$\|u\|_{C^0(\bar{\Omega})} \leq T_0 \|u\|_{W_0^{1,\bar{p}}(\Omega)}, \tag{2.6}$$

for each $u \in W_0^{1,\bar{p}}(\Omega)$, where

$$T_0 = 2^{\frac{(N-1)(p^- - 1)}{p^-}} m_{p^-} \max_{1 \leq i \leq N} \left\{ |\Omega|^{\frac{p_i - p^-}{p_i p^-}} \right\}. \tag{2.7}$$

Moreover, the embedding of $W_0^{1,\bar{p}}(\Omega)$ in $C^0(\bar{\Omega})$ is compact.

Proposition 2.2 Fix $r > 0$. Then for each $u \in W_0^{1,\bar{p}}(\Omega)$ such that

$$\sum_{i=1}^N \frac{1}{p_i} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)}^{p_i} < r,$$

one has

$$\|u\|_{C^0(\bar{\Omega})} < T \max \{ r^{1/p^-}; r^{1/p^+} \},$$

where $T = T_0 \sum_{i=1}^N p_i^{1/p_i}$ and T_0 is given in (2.7).

Throughout the sequel, we suppose that $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is an L^1 -Carathéodory function, i.e.:

- (1) $x \mapsto f(x, \xi)$ is measurable for every $\xi \in \mathbb{R}$;
- (2) $\xi \mapsto f(x, \xi)$ is continuous for almost every $x \in \Omega$;
- (3) for every $s > 0$ there is a function $l_s \in L^1(\Omega)$ such that

$$\sup_{|\xi| \leq s} |f(x, \xi)| \leq l_s(x)$$

for a.e. $x \in \Omega$.

We recall that $u : \Omega \rightarrow \mathbb{R}$ is a weak solution of problem $(D_\lambda^{\bar{p}})$ if $u \in W_0^{1,\bar{p}}(\Omega)$ satisfies the following condition

$$\sum_{i=1}^N \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^{p_i - 2} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx = \lambda \int_{\Omega} f(x, u(x)) v(x) dx,$$

for all $v \in W_0^{1,\bar{p}}(\Omega)$.

Finally, we define the functionals $\Phi, \Psi : W_0^{1,\bar{p}}(\Omega) \rightarrow \mathbb{R}$ by setting

$$\Phi(u) := \sum_{i=1}^N \frac{1}{p_i} \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx, \quad \Psi(u) := \int_{\Omega} F(x, u(x)) dx, \tag{2.8}$$

for every $u \in W_0^{1,\bar{p}}(\Omega)$, where

$$F(x, t) = \int_0^t f(x, \tau) \, d\tau, \quad \text{for all } t \in \mathbb{R}.$$

Clearly, Φ and Ψ are Gâteaux differentiable functionals whose Gâteaux derivatives at the point $u \in W_0^{1,\bar{p}}(\Omega)$ are respectively given by

$$\begin{aligned} \Phi'(u)(v) &= \sum_{i=1}^N \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \, dx, \\ \Psi'(u)(v) &= \int_{\Omega} f(x, u(x))v(x) \, dx, \end{aligned} \tag{2.9}$$

for every $v \in W_0^{1,\bar{p}}(\Omega)$. Moreover, we observe that the critical points in $W_0^{1,\bar{p}}(\Omega)$ of the functional $I_{\lambda} = \Phi - \lambda\Psi$ are precisely the weak solutions of problem $(D_{\lambda}^{\bar{p}})$.

Now we prove the following propositions useful in the sequel.

Proposition 2.3 The functional Φ defined in (2.8) is coercive and sequentially weakly lower semicontinuous. Moreover, its Gâteaux derivative admits a continuous inverse on $(W_0^{1,\bar{p}}(\Omega))^*$.

Proof Let Φ the functional defined in (2.8). Put p_j such that

$$\left(\int_{\Omega} \left| \frac{\partial u}{\partial x_j} \right|^{p_j} \, dx \right)^{\frac{1}{p_j}} := \max_{1 \leq i \leq N} \left(\int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} \, dx \right)^{\frac{1}{p_i}};$$

simple computations show

$$\Phi(u) \geq \frac{1}{p^+ N^{p_j}} \|u\|_{W_0^{1,\bar{p}}(\Omega)}^{p_j}, \tag{2.10}$$

and so Φ is coercive.

Now, we observe that Φ is sequentially weakly lower semicontinuous. Indeed, it is convex and continuous and our claim follows from [38, Proposition 25.20]).

Finally, we prove that Φ' admits a continuous inverse on $(W_0^{1,\bar{p}}(\Omega))^*$. Since $p_i > 2$ for $i = 1, \dots, N$, arguing as [2, Proposition 2.4] one has that Φ' is uniformly monotone. Hence, the Browder-Minty Theorem (see [38, Theorem 26.A (d)]) ensures that there exists the inverse $(\Phi^{-1})'$, which is continuous. \square

Proposition 2.4 The functional Ψ is continuously Gâteaux differentiable and its derivative is compact.

Proof Our aim is to apply [38, Proposition 26.2]. To this end let $\{u_n\} \subset W_0^{1,\bar{p}}(\Omega)$ be bounded. Since $W_0^{1,\bar{p}}(\Omega)$ is reflexive (see [31, Theorem 1]), up to a subsequence, we have that

$$u_n \xrightarrow{w} u \quad \text{in } W_0^{1,\bar{p}}(\Omega). \tag{2.11}$$

Clearly, since the embedding of $W_0^{1,\bar{p}}(\Omega)$ in $C^0(\bar{\Omega})$ is compact (see Proposition 2.1) we have that

$$u_n \xrightarrow{s} u \text{ in } C^0(\bar{\Omega}).$$

that is

$$u_n(x) \rightarrow u(x) \text{ uniformly in } \Omega,$$

then there exists $K > 0$ such that $|u_n(x)| \leq K$ for all $x \in \Omega$ and then, since f is an L^1 -Carathéodory function, up to a subsequence we obtain that

$$f(x, u_n(x)) \rightarrow f(x, u(x)) \text{ in } L^1(\Omega).$$

Let $v \in W_0^{1,\bar{p}}(\Omega)$ be such that $\|v\|_{W_0^{1,\bar{p}}(\Omega)} \leq 1$; we observe that

$$\begin{aligned} |\Psi'(u_n)(v) - \Psi'(u)(v)| &= \left| \int_{\Omega} (f(x, u_n(x)) - f(x, u(x)))v(x) \, dx \right| \\ &\leq \int_{\Omega} |f(x, u_n(x)) - f(x, u(x))| |v(x)| \, dx \\ &\leq T_0 \int_{\Omega} |f(x, u_n(x)) - f(x, u(x))| \, dx, \end{aligned} \tag{2.12}$$

where in the last inequality we use formula (2.6). Finally, from (2.12), we obtain

$$\sup_{\|v\|_{W_0^{1,\bar{p}}(\Omega)} \leq 1} |\Psi'(u_n)(v) - \Psi'(u)(v)| \leq T_0 \|f(x, u_n(x)) - f(x, u(x))\|_{L^1(\Omega)}, \tag{2.13}$$

and from (2.13)

$$\begin{aligned} \lim_{n \rightarrow +\infty} \|\Psi'(u_n)(v) - \Psi'(u)(v)\|_{(W_0^{1,\bar{p}}(\Omega))^*} &= \lim_{n \rightarrow +\infty} \left[\sup_{\|v\|_{W_0^{1,\bar{p}}(\Omega)} \leq 1} |\Psi'(u_n)(v) - \Psi'(u)(v)| \right] \\ &\leq T_0 \lim_{n \rightarrow +\infty} \|f(x, u_n(x)) - f(x, u(x))\|_{L^1(\Omega)} = 0, \end{aligned}$$

that is Ψ' is strongly continuous and then, from comma (a) of [38, Proposition 26.2], Ψ' is compact. □

Our main tool is a three critical point theorem, that we recall here for reader convenience.

Theorem 2.1 (see [8, Theorem 7.1]) *Let X be a real Banach space and $\Phi, \Psi : X \rightarrow \mathbb{R}$ be two continuously Gâteaux differentiable functionals with Φ bounded from below. Assume that $\Phi(0) = \Psi(0) = 0$ and that there exist $r > 0$ and $\bar{u} \in X$, with $r < \Phi(\bar{u})$, such that:*

$$\frac{\sup_{u \in \Phi^{-1}([-\infty, r])} \Psi(u)}{r} < \frac{\Psi(\bar{u})}{\Phi(\bar{u})} \tag{2.14}$$

Moreover, for each $\lambda \in \Lambda_r =]\frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \frac{r}{\sup_{u \in \Phi^{-1}([-\infty, r])} \Psi(u)}[$, the functional $I_\lambda = \Phi - \lambda\Psi$ is bounded from below and satisfies the (PS)-condition.

Then, for each $\lambda \in \Lambda_r$, the functional $I_\lambda = \Phi - \lambda\Psi$ has at least three distinct critical points in X .

3 Main result and consequences

This section presents our main result and some consequences. Our goal is to apply the critical point theorem recalled in Sect. 2 (Theorem 2.1). More precisely, we point out an existence result of at least three weak solutions (see Theorem 3.1) and some consequences (Theorem 3.2, Example 3.15). Put

$$R := \sup_{x \in \Omega} \text{dist}(x, \partial\Omega);$$

simple calculations show that there is $x_0 \in \Omega$ such that $B(x_0, R) \subseteq \Omega$ and we denote by

$$\omega_R := |B(x_0, R)| = \frac{\pi^{\frac{N}{2}}}{\Gamma(1 + \frac{N}{2})} R^N,$$

the measure of the N -dimensional ball of radius R .

Let $x = (x_1, \dots, x_N)$, $x_0 = (x_{01}, \dots, x_{0N})$ be in \mathbb{R}^N and put

$$\delta_i = \int_{B(x_0, R) \setminus B(x_0, \frac{R}{2})} \frac{|x_i - x_{0i}|^{p_i}}{|x - x_0|_N^{p_i}} dx, \tag{3.1}$$

and

$$\mathcal{H} = \sum_{i=1}^N \left[\frac{1}{p_i} \left(\frac{2}{R} \right)^{p_i} \delta_i \right], \tag{3.2}$$

where $|\cdot|_N$ is the usual norm in \mathbb{R}^N .

Theorem 3.1 *Let $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be an L^1 -Carathéodory function. Assume that there exist two positive constants c and d , with*

$$\min\{c^{p^-}; c^{p^+}\} < \mathcal{H} \min\{T^{p^-}; T^{p^+}\} \min\{d^{p^-}; d^{p^+}\}, \tag{3.3}$$

such that

$$F(x, t) \geq 0, \quad \text{for all } (x, t) \in \Omega \times [0, d], \tag{3.4}$$

and

$$\frac{\int_{\Omega} \max_{|\xi| \leq c} F(x, \xi) dx}{\min\{c^{p^-}; c^{p^+}\}} < \frac{1}{\mathcal{H} \max\{T^{p^-}; T^{p^+}\}} \frac{\int_{B(x_0, \frac{R}{2})} F(x, d) dx}{\max\{d^{p^-}; d^{p^+}\}} \tag{3.5}$$

where T is given in Proposition 2.2. Moreover, suppose that

$$\limsup_{|t| \rightarrow +\infty} \frac{F(x, t)}{|t|^{p^-}} = 0, \quad \text{uniformly a.e. in } \Omega. \tag{3.6}$$

Then, for each $\lambda \in \tilde{\Lambda} :=]\mathcal{H} \frac{\max\{d^{p^-}, d^{p^+}\}}{\int_{B(x_0, \frac{R}{2})} F(x, d) dx}, \frac{1}{\max\{T^{p^-}, T^{p^+}\}} \frac{\min\{c^{p^-}, c^{p^+}\}}{\int_{\Omega} \max_{|\xi| \leq c} F(x, \xi) dx} [$, problem $(D_{\lambda}^{\bar{p}})$ has at least three weak solutions.

Proof Put Φ and Ψ as in (2.8). It is well known that Φ and Ψ satisfy all regularity assumptions requested in Theorem 2.1, and furthermore one has $\inf_{u \in W_0^{1, \bar{p}}(\Omega)} \Phi(u) = \Phi(0) = \Psi(0) = 0$. Our aim is to verify condition (2.14). To this end, put $r = \min\{(\frac{c}{T})^{p^-}; (\frac{c}{T})^{p^+}\}$, where T is given in Proposition 2.2. Fix

$$\bar{u}(x) = \begin{cases} 0 & \text{if } x \in \Omega \setminus B(x_0, R), \\ \frac{2d}{R}(R - |x - x_0|) & \text{if } x \in B(x_0, R) \setminus B(x_0, \frac{R}{2}), \\ d & \text{if } x \in B(x_0, \frac{R}{2}). \end{cases} \tag{3.7}$$

Clearly, $\bar{u} \in W_0^{1, \bar{p}}(\Omega)$. From (3.3), we obtain that $\Phi(\bar{u}) > r$. Indeed,

$$\begin{aligned} \Phi(\bar{u}) &= \sum_{i=1}^N \frac{1}{p_i} \int_{\Omega} \left| \frac{\partial \bar{u}}{\partial x_i} \right|^{p_i} dx = \sum_{i=1}^N \frac{1}{p_i} \left(\frac{2d}{R}\right)^{p_i} \int_{B(x_0, R) \setminus B(x_0, \frac{R}{2})} \frac{|x_i - x_{0i}|^{p_i}}{|x - x_0|_N^{p_i}} dx \\ &= \sum_{i=1}^N \left[\frac{1}{p_i} \left(\frac{2d}{R}\right)^{p_i} \delta_i \right] \\ &\geq \sum_{i=1}^N \left[\frac{1}{p_i} \left(\frac{2}{R}\right)^{p_i} \delta_i \right] \min\{d^{p^-}; d^{p^+}\} = \mathcal{H} \min\{d^{p^-}; d^{p^+}\} > \frac{\min\{c^{p^-}; c^{p^+}\}}{\min\{T^{p^-}; T^{p^+}\}} \\ &= \min\{c^{p^-}; c^{p^+}\} \max\left\{ \left(\frac{1}{T}\right)^{p^-}; \left(\frac{1}{T}\right)^{p^+} \right\} \\ &\geq \min\left\{ \left(\frac{c}{T}\right)^{p^-}; \left(\frac{c}{T}\right)^{p^+} \right\} = r \end{aligned}$$

Moreover, for all $u \in W_0^{1, \bar{p}}(\Omega)$ such that $u \in \Phi^{-1}(]-\infty, r])$, from Proposition 2.2 one has

$$|u(x)| < T \max\{r^{1/p^-}; r^{1/p^+}\} = c \quad \text{for all } x \in \Omega. \tag{3.8}$$

So,

$$\Psi(u) = \int_{\Omega} F(x, u(x)) dx \leq \int_{\Omega} \max_{|\xi| \leq c} F(x, \xi) dx,$$

for all $u \in X$ such that $u \in \Phi^{-1}(]-\infty, r])$. Hence,

$$\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u) \leq \int_{\Omega} \max_{|\xi| \leq c} F(x, \xi) dx.$$

Therefore, one has

$$\begin{aligned} \frac{\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)}{r} &\leq \frac{\int_{\Omega} \max_{|\xi| \leq c} F(x, \xi) dx}{\min\{(\frac{c}{T})^{p^-}; (\frac{c}{T})^{p^+}\}} \\ &\leq \max\{T^{p^-}; T^{p^+}\} \frac{\int_{\Omega} \max_{|\xi| \leq c} F(x, \xi) dx}{\min\{c^{p^-}; c^{p^+}\}}. \end{aligned} \tag{3.9}$$

On the other hand, from (3.1) and (3.2), we have

$$\begin{aligned} \Phi(\bar{u}) &= \sum_{i=1}^N \frac{1}{p_i} \int_{\Omega} \left| \frac{\partial \bar{u}}{\partial x_i} \right|^{p_i} dx = \sum_{i=1}^N \frac{1}{p_i} \left(\frac{2d}{R} \right)^{p_i} \int_{B(x_0, R) \setminus B(x_0, \frac{R}{2})} \frac{|x_i - x_{0i}|^{p_i}}{|x - x_0|_N^{p_i}} dx \\ &= \sum_{i=1}^N \left[\frac{1}{p_i} \left(\frac{2d}{R} \right)^{p_i} \delta_i \right] \\ &\leq \sum_{i=1}^N \left[\frac{1}{p_i} \left(\frac{2}{R} \right)^{p_i} \delta_i \right] \max\{d^{p^-}; d^{p^+}\} = \mathcal{H} \max\{d^{p^-}; d^{p^+}\} \end{aligned}$$

and, taking (3.4) into account, one has

$$\Psi(\bar{u}) \geq \int_{B(x_0, \frac{R}{2})} F(x, d) dx.$$

Hence, we obtain

$$\frac{\Psi(\bar{u})}{\Phi(\bar{u})} \geq \frac{1}{\mathcal{H}} \frac{\int_{B(x_0, \frac{R}{2})} F(x, d) dx}{\max\{d^{p^-}; d^{p^+}\}}. \tag{3.10}$$

Hence, from (3.9), (3.10) and assumption (3.5), we obtain

$$\frac{\sup_{u \in \Phi^{-1}([-\infty, r])} \Psi(u)}{r} < \frac{\Psi(\bar{u})}{\Phi(\bar{u})}.$$

Now, we prove that the functional $I_\lambda = \Phi - \lambda\Psi$ is coercive.

Fix $0 < \epsilon < \frac{1}{2^{(p^- - 1)(N - 1)} p^+ \lambda T_0 |\Omega|}$, from (3.6) and since f is L^1 -Carathéodory, there exists a function $h_\epsilon \in L^1(\Omega)$ such that

$$F(x, t) \leq \epsilon |t|^{p^-} + h_\epsilon(x),$$

for each $(x, t) \in \Omega \times \mathbb{R}$. Then, for each $u \in W_0^{1, \bar{p}}(\Omega)$, from formula (2.6), taking (2.10) into account, we have that

$$I_\lambda(u) \geq \frac{1}{p^+ N^{p_j}} \|u\|_{W_0^{1, \bar{p}}(\Omega)}^{p_j} - \lambda \epsilon T_0^{p^-} |\Omega| \|u\|_{W_0^{1, \bar{p}}(\Omega)}^{p^-} - \lambda \|h_\epsilon\|_{L^1(\Omega)},$$

that is

$$\frac{I_\lambda(u)}{\|u\|_{W_0^{1, \bar{p}}(\Omega)}} \geq \frac{1}{p^+ N^{p_j}} \|u\|_{W_0^{1, \bar{p}}(\Omega)}^{p_j - 1} - \lambda \epsilon T_0^{p^-} |\Omega| \|u\|_{W_0^{1, \bar{p}}(\Omega)}^{p^- - 1} - \lambda \frac{\|h_\epsilon\|_{L^1(\Omega)}}{\|u\|_{W_0^{1, \bar{p}}(\Omega)}},$$

and therefore it is coercive. Indeed, let $u \in W_0^{1, \bar{p}}(\Omega)$ be such that $\|u\|_{W_0^{1, \bar{p}}(\Omega)} \rightarrow +\infty$. If $p_j \neq p^-$, the coercivity is trivial. Otherwise, it follows since $0 < \epsilon < \frac{1}{\lambda p^+ N^{p_j} T_0^{p^-} |\Omega|}$.

Clearly, I_λ is bounded from below, owing to the fact that it is coercive and sequentially weakly lower semicontinuous. Moreover, it satisfies (PS)-condition owing to Propositions 2.3 and 2.4 (see [39, Example 38.25]).

Hence, all assumptions of Theorem 2.1 are verified and the proof is complete. □

Now, we point out a consequence of our main result for the autonomous case. To be precise, let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function and consider the anisotropic Dirichlet problem $(AD_{\lambda}^{\vec{p}})$ in the Introduction.

Moreover, put

$$G(t) = \int_0^t g(\tau) d\tau \quad \text{for all } t \in \mathbb{R}.$$

Theorem 3.2 *Let $g : [0, +\infty) \rightarrow [0, +\infty)$ be a nonzero continuous function such that*

$$\lim_{t \rightarrow +\infty} \frac{g(t)}{t^{p^- - 1}} = 0, \tag{3.11}$$

and

$$\lim_{t \rightarrow 0^+} \frac{g(t)}{t^{p^+ - 1}} = 0. \tag{3.12}$$

Put $\lambda^* = \frac{\mathcal{H}}{|B(x_0, \frac{R}{2})|} \min\{\inf_{0 < d < 1} \frac{d^{p^-}}{G(d)}; \inf_{d \geq 1} \frac{d^{p^+}}{G(d)}\}.$

Then, for every $\lambda > \lambda^*$, problem $(AD_{\lambda}^{\vec{p}})$ admits at least two positive distinct weak solutions.

Proof Put

$$g^+(t) = \begin{cases} g(0), & \text{if } t < 0, \\ g(t), & \text{if } t \geq 0, \end{cases} \tag{3.13}$$

and consider the following problem

$$\begin{cases} -\Delta_{\vec{p}} u = \lambda g^+(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.14}$$

Since $g \not\equiv 0$ there is $d > 0$ such $G(d) = G^+(d) = \int_0^d g^+(t) dt > 0$, so that $\min\{\inf_{0 < d < 1} \frac{d^{p^-}}{G(d)}; \inf_{d \geq 1} \frac{d^{p^+}}{G(d)}\} < +\infty$. To fix ideas, assume $\min\{\inf_{0 < d < 1} \frac{d^{p^-}}{G(d)}; \inf_{d \geq 1} \frac{d^{p^+}}{G(d)}\} = \inf_{0 < d < 1} \frac{d^{p^-}}{G(d)}$. Therefore, fixed $\lambda > \lambda^*$, there is d , with $0 < d < 1$ such that $\lambda > \frac{\mathcal{H}}{|B(x_0, \frac{R}{2})|} \frac{d^{p^-}}{G(d)}$. On the other hand, from (3.12) one has $\lim_{t \rightarrow 0^+} \frac{t^{p^-}}{G(t)} = +\infty$ for which there is $c > 0$ small enough such that (3.3) holds and $\frac{1}{\max\{T^{p^-}, T^{p^+}\}} \frac{c^{p^-}}{G(c)|\Omega|} > \lambda$. Hence, assumption (3.5) is satisfied. Moreover, from (3.11), also the assumption (3.6) is verified and we can apply Theorem 3.1 for which problem (3.14) admits three weak solutions. Arguing in a standard way, such solutions are nonnegative and are also solutions of problem $(AD_{\lambda}^{\vec{p}})$ (see, for instance, [9, Lemma 2.2]). Finally, applying the strong maximum principle (see [9, Lemma 2.3]), at least two of them are positive and the proof is achieved. \square

Remark 3.3 Theorem 1.1 in the introduction is a consequence of Theorem 3.2.

As an application, we give the following example.

Example 3.4 Fix $N = 2$ and $\Omega = B(0, 2)$, put $p_1 = 3$, $p_2 = 4$, and consider the following problem

$$\begin{cases} -\sum_{i=1}^2 \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{\partial u}{\partial x_i} \right) = \frac{(t+1)t^4}{e^t} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.15)$$

Since $\mathcal{H} = \frac{3^2}{2^5}\pi$, owing to simple computations one has

$$\lambda^* = \frac{3^2 e}{2^5(144e - 391)},$$

for which Theorem 3.2 ensures that for each $\lambda > \lambda^*$ problem (3.15) admits at least two positive weak solutions.

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