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A geometric point of view on biderivations

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ABSTRACT

In this paper, we extend the notion of tangent vectors at a point of a smooth manifold (also known as derivations) by introducing *biderivations on manifolds*. This is a geometric counterpart of the algebraic notion of biderivation, which has been extensively studied in the beginning in rings, and subsequently in the context of non-associative algebra. Our construction leads to the definition of a bitangent space at a pair of points. When the construction is carried out on a single manifold and the two points coincide, this object reduces to a contravariant 2-tensor. We then introduce the *bidifferential* of a pair of smooth maps as a linear map between bitangent spaces, and we describe its associated matrix explicitly. Finally, we illustrate the theory through applications to some partial differential equations. Indeed, we discuss connections with the eikonal equation, a classical problem arising in the study of wave propagation, and elliptic operators.

1. Introduction

In the theory of smooth manifolds, one of the key tools is introducing tangent vectors at a point as a natural generalisation of tangent vectors of Euclidean spaces, which are given simply by directional derivatives. In fact, by the use of coordinates, one can define diffeomorphisms between open sets of the manifold and open sets of $\mathbb{R}^n$. The set T_pM of tangent vectors at a point p of a manifold M, called the tangent space of M at p, allows us to introduce the differential (or *pushforward*) of a smooth map, which, as well as in the classical calculus, plays the role of the best linear approximation of the function. The aim of this work is to follow this geometric approach to introduce *biderivations* for (a pair of) manifolds.

Biderivations may be seen as a generalization of derivations. Indeed, they are bilinear maps, acting as derivations in each argument. Derivations and biderivations have been studied in depth for algebraic structures. Indeed, derivations may be seen as an extension of the concept of differentiation from real-valued functions as linear operators satisfying the Leibniz rule. For instance, it is well-known that in Lie algebras, the set of derivations forms a Lie algebra via the commutator bracket. Furthermore, the study of derivations gives some information about the internal properties of the corresponding algebraic structure. Similarly, one can investigate the relationship between biderivations and the structure of the algebra on which they are defined. The earliest work where biderivations, although not called as such, appear is that of Maksa [21], but later they were formally introduced by Vukman in [27] in a purely algebraic context. However, the first example of biderivation for Lie algebras dates only to 2011 [28]. The study of biderivations has become a standard research topic in Lie algebras and in the related algebraic structures [3,4,10]. Indeed, biderivations have been subsequently introduced and studied for many

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other algebraic structures, as, for instance, for Lie superalgebras in [12], for complete Lie algebras in [13], and for complete Leibniz algebras in [11].

In differential geometry, the vector space $T_pM \otimes \cdots \otimes T_pM$ given by the tensor product of k copies of the tangent space T_pM is called the space of *contravariant tensors* of rank k. As will become clear in Section 3, the bitangent space $T^2_{p,p}(M,M) = T^2_{p,p}M$ consists of contravariant 2-tensors. It is worth pointing out that the bitangent space that we will introduce is constructed *a priori* starting from two points belonging to two distinct manifolds. On the other hand, in [15,22] tensors defined on pairs of points belonging to two manifolds connected by a transport map are considered. These are called *two-point tensors* and they play an important role in continuum mechanics. A natural question is whether the biderivations defined in this work are closely related to such two-point tensors. A further motivation for making this geometric construction explicit comes from partial differential equations. On the diagonal, namely when the two points coincide, a symmetric biderivation field is a contravariant symmetric 2-tensor. If it is positive (semi)definite, it may be interpreted as a cometric, that is, as a positive definite quadratic form on the cotangent space. This point of view naturally appears in first-order Hamilton–Jacobi equations: in the eikonal equation, a positive definite biderivation field prescribes the quadratic Hamiltonian, or equivalently the ellipsoidal level sets in the cotangent space. The same diagonal object also appears at second order, where it determines the principal part of divergence-form elliptic operators and the associated carré du champ or Dirichlet form.

The paper is organised in four sections. After this introduction section, in Section 2, we define biderivations on manifolds, and we study the vector space formed by all the biderivations at two points. In Section 3, we introduce the notion of bidifferential, a linear map defined between the bitangent spaces and investigate what the associated matrix of this map is. Lastly, Section 4 is devoted to the applications of the results obtained in the previous sections to some PDE problems. For instance, the eikonal equation and elliptic operators.

2. Tangent bivectors on manifolds

Before the introduction of the notion of biderivations within the framework of differential geometry, it is necessary to provide a clarification regarding the choice of the term *bivector*. In the relevant literature, this term is widely used in the context of geometric algebra. Let V be a vector space and let $\wedge$ denote the wedge product (or exterior product) on V. A decomposable element of the form $v = v_1 \wedge v_2 \in \bigwedge^2 V$ is called a *bivector*. The primary reason behind this terminology is that such elements are, informally speaking, neither scalars nor vectors (we refer the reader to [14] for further details, as well as to other standard references on the subject). However, the subsequent subsections will demonstrate that, in a similar manner to how a derivation can be regarded as a tangent vector at a point on a smooth manifold, biderivations can be interpreted as tangent bivectors. Nevertheless, this interpretation will be entirely unrelated to the notion of bivectors in geometric algebra, as previously defined.

2.1. First definitions and properties

Let M, N be two smooth manifolds and let $p \in M$, $q \in N$. Let $\mathcal{C}^\infty_p(M)$ and $\mathcal{C}^\infty_q(N)$ denote the algebras of real-valued $\mathcal{C}^\infty$ functions defined on some neighbourhood of p and q, respectively.

Definition 2.1. A *biderivation of M and N in p and q* is a bilinear form $B : \mathcal{C}^\infty_p(M) \times \mathcal{C}^\infty_q(N) \to \mathbb{R}$ such that

$$B(f_1 f_2, g) = f_1(p)B(f_2, g) + f_2(p)B(f_1, g), \tag{1}$$

$$B(f, g_1 g_2) = g_1(q)B(f, g_2) + g_2(q)B(f, g_1) \tag{2}$$

for every $f, f_1, f_2 \in \mathcal{C}^\infty_p(M)$ and $g, g_1, g_2 \in \mathcal{C}^\infty_q(N)$.

An equivalent definition of the latter is the following.

Definition 2.2. A map $B : \mathcal{C}^\infty_p(M) \times \mathcal{C}^\infty_q(N) \to \mathbb{R}$ is a *biderivation* if and only if $B(\cdot, g) \in T_pM$ for every $g \in \mathcal{C}^\infty_q(N)$, and $B(f, \cdot) \in T_qN$ for every $f \in \mathcal{C}^\infty_p(M)$.

The set of these maps will be denoted by $T^2_{p,q}(M,N)$. When $M = N$, and then $p, q \in M$, we will denote this vector space as $T^2_{p,q}M$. The following result is straightforward.

Theorem 2.3. *$T^2_{p,q}(M,N)$ is a real vector space.*

Let us suppose that $f \in \mathcal{C}^\infty_p(M)$ is a constant function on a neighbourhood U_p of p. Then $\delta = B(\cdot, g) \in T_pM$, for any $g \in \mathcal{C}^\infty_q(N)$. By a well-known result on derivations [20, Lemma 3.1 a], we have $\delta(f) = 0$, and hence $B(f, g) = 0$. Furthermore, if $f_1(p) = f_2(p) = 0$ with $f_1, f_2 \in \mathcal{C}^\infty_p(M)$, then, by [20, Lemma 3.1 b], $\delta(f_1 f_2) = 0$ and $B(f_1 f_2, g) = 0$, for every $g \in \mathcal{C}^\infty_q(N)$. These results are formally summarised below.

Proposition 2.4. *Let $B \in T^2_{p,q}(M,N)$, $f, f_1, f_2 \in \mathcal{C}^\infty_p(M)$, and $g, g_1, g_2 \in \mathcal{C}^\infty_q(N)$. Then*

a) If f is constant in a neighbourhood $U_p \subseteq M$ of p, then $B(f,g)=0$, for every $g \in \mathcal{C}^\infty_q(N)$.
b) If g is constant in a neighbourhood $W_q \subseteq N$ of q, then $B(f,g)=0$, for every $f \in \mathcal{C}^\infty_p(M)$.
c) If $f_1(p)=f_2(p)=0$, then $B(f_1 f_2, g)=0$, for every $g \in \mathcal{C}^\infty_q(N)$.
d) If $g_1(q)=g_2(q)=0$, then $B(f, g_1 g_2)=0$, for every $f \in \mathcal{C}^\infty_p(M)$.

Remark 2.1. We note that, as well as derivations, a biderivation of $T^2_{p,q}(M,N)$ applied to a pair of suitable functions f and g such that $f(p)=g(q)=0$ is not, in general, equal to zero. Hence, under mixed conditions where $f_1(p)=g(q)=0$ and $f_2(p)\neq 0$, the point c) in the last proposition shows that, in general, B does not vanish.

Just as the tangent space T_pM to M at a point p has a purely local nature, that is, independent of the global structure of the manifold, we will show that the same holds for $T^2_{p,q}(M,N)$.

Proposition 2.5. *Let $f \in \mathcal{C}^\infty_p(M)$ with domain U, and let $U_1 \subseteq U$ be a smaller neighbourhood of p. Similarly, let $g \in \mathcal{C}^\infty_q(N)$ with domain V, and $V_1 \subseteq V$ a smaller neighbourhood of q. Then, for every $B \in T^2_{p,q}(M,N)$, we have*

$$B\left(f|_{U_1}, g|_{V_1}\right) = B(f,g).$$

Proof. Let h (respectively, ℓ) be the function with domain $U_1 \subseteq U \subseteq M$ (respectively, $V_1 \subseteq V \subseteq N$) that takes the constant value 1; that is,

$$h\colon U_1 \to \{1\}, \qquad \ell\colon V_1 \to \{1\}.$$

Then $f|_{U_1} = fh$ and $g|_{V_1} = g\ell$,[1] and hence, by applying part a) of Proposition 2.4 to function h, we have

$$\begin{aligned} B\left(f|_{U_1}, \varphi\right) &= B(fh, \varphi) \\ &= f(p)B(h,\varphi) + h(p)B(f,\varphi) \\ &= B(f,\varphi), \end{aligned}$$

for any $\varphi \in \mathcal{C}^\infty_q(N)$. Similarly, one can prove that $B(\psi, g|_{V_1}) = B(\psi, g)$, for any $\psi \in \mathcal{C}^\infty_q(N)$. Putting together these two results, then we have $B\left(f|_{U_1}, g|_{V_1}\right) = B(f,g)$. □

Corollary 2.6. *Let $f_1, f_2 \in \mathcal{C}^\infty_p(M)$ such that f_1 and f_2 coincide in a neighbourhood U of p and let $g_1, g_2 \in \mathcal{C}^\infty_q(N)$ such that g_1 and g_2 coincide in a neighbourhood V of q. Then $B(f_1,g) = B(f_2,g)$, for every $g \in \mathcal{C}^\infty_q(N)$, and $B(f,g_1) = B(f,g_2)$, for every $f \in \mathcal{C}^\infty_p(M)$.*

Proof. If $f_1|_U = f_2|_U$, then by Proposition 2.5, for any $g \in \mathcal{C}^\infty_q(N)$ we have

$$B(f_1,g) = B\left(f_1|_U, g\right) = B\left(f_2|_U, g\right) = B(f_2,g).$$

Similarly, one can prove the other equality of the statement. □

2.2. The vector space $T^2_{p,q}(\mathbb{E}^m, \mathbb{E}^n)$

Let $M=\mathbb{E}^m$ and $N=\mathbb{E}^n$. Let π_i be the projection of a k-tuple on its i-th component, i.e. $\pi_i(x_1,x_2,\ldots,x_k)=x_i$. Let $f \in \mathcal{C}^\infty_p(\mathbb{E}^m)$ and $g \in \mathcal{C}^\infty_q(\mathbb{E}^n)$. Then, by Taylor's theorem, we obtain

$$f(x) = f(p) + \sum_{i=1}^m \frac{\partial f}{\partial x_i}(p)(x_i - p_i) + \sum_{i=1}^m f_i(x)(x_i - p_i)$$

and

$$g(y) = g(q) + \sum_{j=1}^n \frac{\partial g}{\partial y_j}(q)(y_j - q_j) + \sum_{j=1}^n g_j(y)(y_j - q_j),$$

[1] Recall that the pointwise product of two functions is defined on the intersection of their domains.

where $f_1, f_2, \dots, f_m \in \mathcal{C}^\infty_p(\mathbb{E}^m)$ and $g_1, g_2, \dots, g_n \in \mathcal{C}^\infty_q(\mathbb{E}^n)$, with $f_i(p) = g_j(q) = 0$ for every $1 \le i \le m$ and $1 \le j \le n$. More precisely, the following formulas hold:

$$f_i(x) = \sum_{k=1}^{m} (x_k - p_k) \int_0^1 (1-t)\, \frac{\partial^2 f}{\partial x_i \partial x_k}\big(p + t(x-p)\big)\, dt,$$

and

$$g_j(y) = \sum_{k=1}^{n} (y_k - q_k) \int_0^1 (1-t)\, \frac{\partial^2 g}{\partial y_j \partial y_k}\big(q + t(y-q)\big)\, dt.$$

Then, we have

$$\begin{aligned} B(f,g) &= B\left(f(p), g(q)\right) \\ &\quad + \sum_{i=1}^{m}\sum_{j=1}^{n}\Bigg[B\left(\frac{\partial f}{\partial x_i}(p)(x_i - p_i), \frac{\partial g}{\partial y_j}(q)(y_j - q_j)\right) \\ &\quad + B\left(\frac{\partial f}{\partial x_i}(p)(x_i - p_i), g_j(y)(y_j - q_j)\right) \\ &\quad + B\left(f_i(x)(x_i - p_i), \frac{\partial g}{\partial y_j}(q)(y_j - q_j)\right) \\ &\quad + B\left(f_i(x)(x_i - p_i), g_j(y)(y_j - q_j)\right)\Bigg]. \end{aligned}$$

Since $f(p)$ and $g(q)$ are constant, by parts a) and b) of Proposition 2.4, we obtain $B\left(f(p), g(q)\right) = 0$. Moreover, the third, fourth, and fifth addends in the expansion of $B(f,g)$ vanish as a consequence of parts c) and d) of Proposition 2.4. Lastly, we have

$$\begin{aligned} B(f,g) &= B\left(\sum_{i=1}^{m} \frac{\partial f}{\partial x_i}(p)(x_i - p_i), \sum_{j=1}^{n} \frac{\partial g}{\partial y_j}(q)(y_j - q_j)\right) \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{\partial f}{\partial x_i}(p)\frac{\partial g}{\partial y_j}(q) B(x_i - p_i, y_j - q_j) \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{\partial f}{\partial x_i}(p)\frac{\partial g}{\partial y_j}(q)\left(B(x_i, y_j) - B(x_i, q_j) - B(p_i, y_j) + B(p_i, q_j)\right) \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{\partial f}{\partial x_i}(p)\frac{\partial g}{\partial y_j}(q) B(\pi_i(x), \pi_j(y)), \end{aligned}$$

where $B(x_i, -q_j) = B(-p_i, y_j) = B(p_i, q_j) = 0$ because p_i, q_j are constant functions. The factor $B(\pi_i(x), \pi_j(y))$ can be thought as a matrix $B = (b_{ij}) \in M_{m\times n}(\mathbb{R})$ defined as $b_{ij} = B(\pi_i(x), \pi_j(y))$. Given that all the quantities involved are scalars and therefore commute, the scalar expression in the final equality can naturally be interpreted as a bilinear form. Indeed, let

$$\nabla f(p) = \left(\frac{\partial f}{\partial x_1}(p), \frac{\partial f}{\partial x_2}(p), \dots, \frac{\partial f}{\partial x_m}(p)\right) \quad \text{and} \quad \nabla g(q) = \left(\frac{\partial g}{\partial y_1}(q), \frac{\partial g}{\partial y_2}(q), \dots, \frac{\partial g}{\partial y_n}(q)\right)$$

denote the gradients of the functions f and g at the points $p \in \mathbb{E}^m$ and $q \in \mathbb{E}^n$, respectively. Hence, the equation

$$B(f,g) = \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{\partial f}{\partial x_i}(p) b_{ij} \frac{\partial g}{\partial y_j}(q)$$

can be written as

$$B(f,g) = \nabla f(p) B \left(\nabla g(q)\right)^t. \tag{3}$$

In other words, if we identify with B not only the biderivation, but also the matrix $B = (b_{ij})$ and the bilinear form that it defines, B can be thought as a bilinear form. Moreover, if one interprets

$$\frac{\partial}{\partial x_i}\bigg|_p \otimes \frac{\partial}{\partial y_j}\bigg|_q (f,g) = \frac{\partial f}{\partial x_i}(p)\frac{\partial g}{\partial y_j}(q),$$

every biderivation B can be written as

$$B(f,g) = \sum_{\substack{1\leq i\leq m\\ 1\leq j\leq n}} B_{ij}\left(\frac{\partial}{\partial x_i}\bigg|_p \otimes \frac{\partial}{\partial y_j}\bigg|_q\right)(f,g). \tag{4}$$

Proposition 2.7. *$T^2_{p,q}(\mathbb{E}^m,\mathbb{E}^n) \cong T_p\mathbb{E}^m \otimes T_q\mathbb{E}^n$ and a basis is given by*

$$\left\{\frac{\partial}{\partial x_i}\bigg|_p \otimes \frac{\partial}{\partial y_j}\bigg|_q : 1\leq i\leq m,\ 1\leq j\leq n\right\}.$$

Therefore, the dimension of $T^2_{p,q}(\mathbb{E}^m,\mathbb{E}^n)$ is $m\cdot n$.

We conclude this section with an explicit example of a biderivation in the framework of Euclidean spaces. The following falls in the setting of Remark 2.1.

Example 2.7.1. Let B be the biderivation of $M = N = \mathbb{E}^2$ given by the matrix $\begin{pmatrix}1 & 0\\ 1 & -1\end{pmatrix}$. If $p = (1,0)$, $q = (0,1)$ are two points of $\mathbb{E}^2$ and $f_1(x_1,x_2) = x_1 + x_2 - 1$, $f_2(x_1,x_2) = x_1 + x_2$, and $g(y_1,y_2) = y_1 + y_2 - 1$, three $\mathcal{C}^\infty$-functions, then $f_1(p) = g(q) = 0$ and $f_2(p) \neq 0$. On the other hand, by the Leibniz rule and the formula (3) we have that

$$B(f_1f_2,g) = f_1(p)B(f_2,g) + f_2(p)B(f_1,g) = \nabla f_1(p)B(\nabla g(q))^t = (1,1)\begin{pmatrix}1 & 0\\ 1 & -1\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} = 1,$$

which does not vanish.

3. Bidifferential

Let $F\colon M_1 \to M_2$ be a smooth map. We recall here that the classical differential of F in $p \in M_1$ is defined as $dF_p\colon T_pM_1 \to T_{F(p)}M_2$, with $dF_p(\delta)f = \delta(f\circ F)$, for every $\delta \in T_pM_1$ and $f \in \mathcal{C}^\infty_{F(p)}(M_2)$. Following the same guiding principle adopted so far, namely, extending differential-topological concepts dimensionally to provide geometric interpretations, we now introduce the following definition of a bidifferential, understood as a natural generalisation in this geometric framework.

Definition 3.1. Let M_1, M_2, N_1, N_2 be smooth manifolds and $F\colon M_1 \to M_2$, $G\colon N_1 \to N_2$ two smooth maps. The map $d^2(F,G)_{p,q}\colon T^2_{p,q}(M_1,N_1) \to T^2_{F(p),G(q)}(M_2,N_2)$ is the *bidifferential* of F and G in p and q, and it is defined as

$$d^2(F,G)_{p,q}(B)(f,g) = B(f\circ F, g\circ G), \tag{5}$$

for every $B \in T^2_{p,q}(M_1,N_1)$, $f \in \mathcal{C}^\infty_{F(p)}(M_2)$, and $g \in \mathcal{C}^\infty_{G(q)}(N_2)$.

We note that, since $f\circ F\colon M_1 \xrightarrow{F} M_2 \xrightarrow{f} \mathbb{R}$ and $g\circ G\colon N_1 \xrightarrow{G} N_2 \xrightarrow{g} \mathbb{R}$, then $f\circ F \in \mathcal{C}^\infty_p(M_1)$ and $g\circ G \in \mathcal{C}^\infty_q(N_1)$, the last definition is properly stated. We shall prove that the bidifferential of F and G in p and q yields a biderivation of M in p and q.

Proposition 3.2. *$d^2(F,G)_{p,q}(B) \in T^2_{F(p),G(q)}(M_2,N_2)$, for every $B \in T^2_{p,q}(M_1,N_1)$.*

Proof. Let us start to prove that $d^2(F,G)_{p,q}(B)$ is bilinear. Indeed,

$$\begin{aligned}
d^2(F,G)_{p,q}(B)(\alpha_1f_1+\alpha_2f_2,g) &= B\left((\alpha_1f_1+\alpha_2f_2)\circ F, g\circ G\right)\\
&= B\left(\alpha_1f_1\circ F+\alpha_2f_2\circ F, g\circ G\right)\\
&= B\left(\alpha_1f_1\circ F, g\circ G\right) + B\left(\alpha_2f_2\circ F, g\circ G\right)\\
&= \alpha_1B(f_1\circ F, g\circ G) + \alpha_2B(f_2\circ F, g\circ G)\\
&= \alpha_1d^2(F,G)_{p,q}(B)(f_1,g) + \alpha_2d^2(F,G)_{p,q}(B)(f_2,g),
\end{aligned}$$

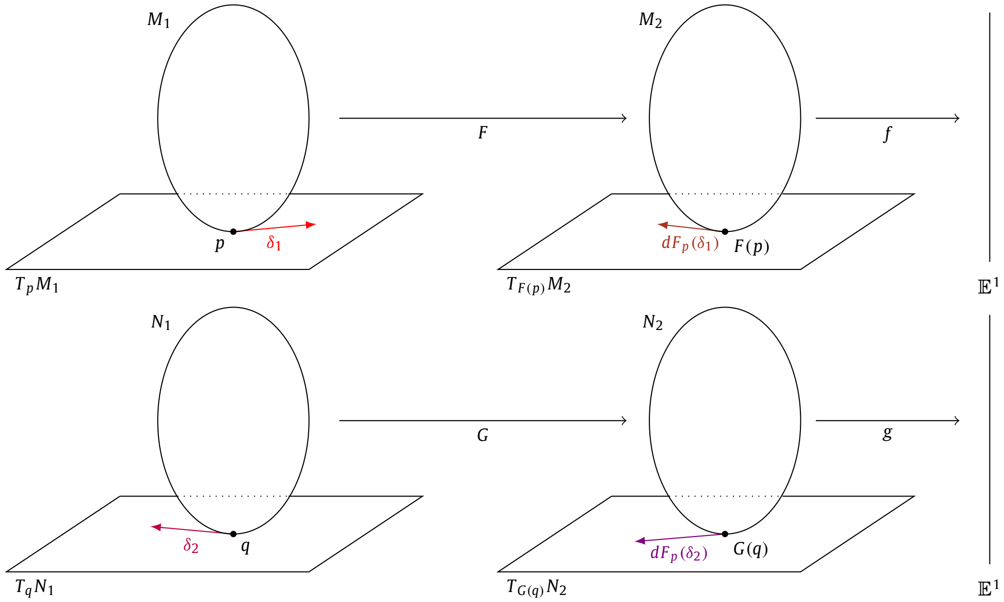

Fig. 1. Setting of Proposition 3.2.

for every $f_1, f_2 \in C^\infty_{F(p)}(M_2)$, $g \in \mathcal{C}^\infty_{G(q)}(N_2)$, and $\alpha_1, \alpha_2 \in \mathbb{R}$. Similarly, the reader can prove the linearity in the second argument. Now, we will prove that $d^2(F,G)_{p,q}(B)$ is a biderivation of M_2 and N_2 in $F(p)$ and $G(q)$ in the sense of Definition 2.1. Let $f_1, f_2 \in \mathcal{C}^\infty_{F(p)}(M_2)$ and $g \in \mathcal{C}^\infty_{G(q)}(N_2)$. Hence, we have

$$\begin{aligned} d^2(F,G)_{p,q}(B)(f_1f_2, g) &= B\left((f_1f_2)\circ F, g\circ G\right) \\ &= B\left((f_1\circ F)(f_2\circ F), g\circ G\right) \\ &= f_1\circ F(p)B\left(f_2\circ F, g\circ G\right) + f_2\circ F(p)B\left(f_1\circ F, g\circ G\right) \\ &= f_1(F(p))B\left(f_2\circ F, g\circ G\right) + f_2(F(p))B\left(f_1\circ F, g\circ G\right) \end{aligned}$$

and Equation (1) holds. To prove that Equation (2) holds as well, the computations are similar and therefore will be omitted. See Fig. 1. □

The next result can be regarded as a chain rule for bidifferentials, describing an identity that arises from the composition of two pairs of smooth maps.

Proposition 3.3. *Let M_1, M_2, M_3 and N_1, N_2, N_3 be smooth manifolds, and let $p_1 \in M_1$, and $q_1 \in N_1$. Consider smooth maps $F_i\colon M_i \to M_{i+1}$ and $G_i\colon N_i \to N_{i+1}$, for $i = 1, 2$. Then, the following chain rule for bidifferentials holds:*

$$d^2\big(F_2\circ F_1,\, G_2\circ G_1\big)_{p_1,q_1} = d^2\big(F_2, G_2\big)_{p_2,q_2}\circ d^2\big(F_1, G_1\big)_{p_1,q_1}, \tag{6}$$

where $p_2 = F_1(p_1)$ and $q_2 = G_1(q_1)$ are points in M_2 and N_2 respectively.

Proof. Let $f \in \mathcal{C}^\infty_{F_2(F_1(p_1))}(M_3)$ and $g \in \mathcal{C}^\infty_{G_2(G_1(q_1))}(N_3)$ and consider a biderivation $B \in T^2_{p_1,q_1}(M_1,N_1)$. Hence by Definition 3.1 we have

$$d^2\big(F_2\circ F_1,\, G_2\circ G_1\big)_{p_1,q_1}(B)(f,g) = B\left(f\circ F_2\circ F_1, g\circ G_2\circ G_1\right),$$

where we have omitted the parentheses in the composition of functions, since function composition is associative. For the same reason, we have $f\circ F_2\circ F_1 = (f\circ F_2)\circ F_1$ and $g\circ G_2\circ G_1 = (g\circ G_2)\circ G_1$. By observing that $f\circ F_2 \in \mathcal{C}^\infty_{F_1(p_1)}(M_2)$ and $g\circ G_2 \in \mathcal{C}^\infty_{G_1(q_1)}(N_2)$ (this is forced by the choices of $f \in \mathcal{C}^\infty_{F_2(F_1(p_1))}(M_3)$ and $g \in \mathcal{C}^\infty_{G_2(G_1(q_1))}(N_3)$), we obtain

$$B\left((f\circ F_2)\circ F_1, (g\circ G_2)\circ G_1\right) = d^2\left(F_1, G_1\right)_{p_1,q_1}(B)\left(f\circ F_2, g\circ G_2\right).$$

Now, by Proposition 3.2, the element $C = d^2\big(F_1, G_1\big)_{p_1,q_1}(B)$ is a biderivation on M_2 and N_2 at the points $p_2 = F(p_1)$ and $q_2 = G(q_1)$. Therefore, the right-hand side of the previous equation can be conveniently written as

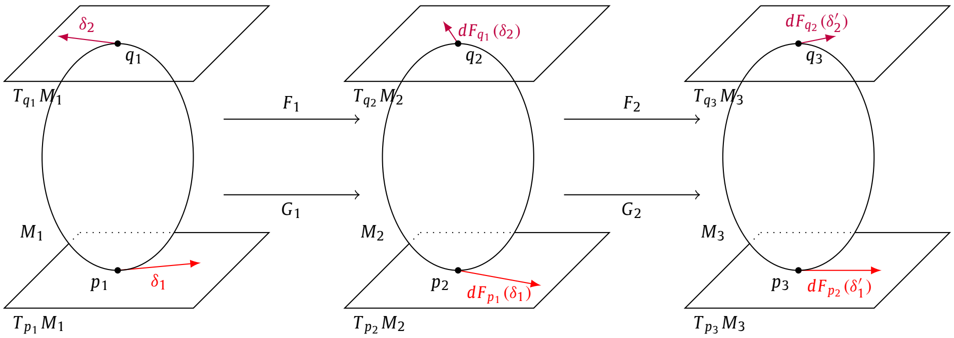

Fig. 2. Setting of Equation (7).

$$C\big(f \circ F_2, g \circ G_2\big).$$

By applying Definition 3.1 one final time, we obtain

$$d^2\big(F_2, G_2\big)_{F_1(p_1),G_1(q_1)}(C)(f,g) = \left(d^2\big(F_2, G_2\big)_{F_1(p_1),G_1(q_1)} \circ d^2\big(F_1, G_1\big)_{p_1,q_1}\right)(B)(f,g),$$

which concludes the proof. $\square$

The proof of the following result is straightforward.

Corollary 3.4. *With the same assumptions as in Proposition 3.3, the map*

$$d^2\big(F_2 \circ F_1,\, G_2 \circ G_1\big)_{p_1,q_1} : T^2_{p_1,q_1}(M_1, N_1) \longrightarrow T^2_{F_2\circ F_1(p_1),\, G_2\circ G_1(q_1)}(M_3, N_3)$$

is linear.

When $M_i = N_i$ for $i = 1, 2, 3$, hence $p_1, q_1 \in M_1$ and $F_i, G_i \colon M_i \to M_{i+1}$, the linear map is given by (see Fig. 2)

$$d^2\big(F_2 \circ F_1,\, G_2 \circ G_1\big)_{p_1,q_1} : T^2_{p_1,q_1}M_1 \longrightarrow T^2_{F_2\circ F_1(p_1),\, G_2\circ G_1(q_1)}M_3. \tag{7}$$

As well as the classical differential, if the smooth maps $F \colon M_1 \to M_2$, $G \colon N_1 \to N_2$ are such that each of them restricts to a diffeomorphism between two open sets, then the bidifferential $d^2(F,G)_{p,q}$ defines an isomorphism between the bitangent spaces $T^2_{p,q}(M_1, N_1)$ and $T^2_{F(p),G(q)}(M_2, N_2)$.

Proposition 3.5. *Let $F \colon M_1 \to M_2$, $G \colon N_1 \to N_2$ be two smooth maps such that*

(1) the function F restricts to a diffeomorphism between an open set V_1 containing p and an open set V_2 containing $F(p)$,
(2) the function G restricts to a diffeomorphism between an open set W_1 containing q and an open set W_2 containing $G(q)$.

Then, the bidifferential $d^2(F,G)_{p,q} \colon T^2_{p,q}(M_1, N_1) \to T^2_{F(p),G(q)}(M_2, N_2)$ is an isomorphism of vector spaces.

Proof. Since $F^{-1}_{|V_1} \circ F_{|V_1} = \mathrm{id}_{V_1}$ and $G^{-1}_{|W_1} \circ G_{|W_1} = \mathrm{id}_{W_1}$, by the formula (6),

$$d^2(\mathrm{id}_{V_1}, \mathrm{id}_{W_1})_{p,q} = d^2\left(F^{-1}_{|V_1}, G^{-1}_{|W_1}\right)_{F(p),G(q)} \circ d^2\left(F_{|V_1}, G_{|W_1}\right)_{p,q}.$$

On the other hand, for $f \in \mathcal{C}^\infty_p(M_1)$, $g \in \mathcal{C}^\infty_q(N_1)$ and $B \in T^2_{p,q}(M_1, N_1)$, we have that

$$d^2(\mathrm{id}_{V_1}, \mathrm{id}_{W_1})_{p,q}(B)(f,g) = B(f \circ \mathrm{id}_{V_1}, g \circ \mathrm{id}_{W_1}) = B(f,g),$$

which implies that the map $d^2(\mathrm{id}_{V_1}, \mathrm{id}_{W_1})_{p,q}$ is $\mathrm{id}_{T^2_{p,q}(M_1,N_1)}$. We conclude that

$$\mathrm{id}_{T^2_{p,q}(M_1,N_1)} = d^2\left(F^{-1}_{|V_1}, G^{-1}_{|W_1}\right)_{F(p),G(q)} \circ d^2\left(F_{|V_1}, G_{|W_1}\right)_{p,q}. \tag{8}$$

Similarly, the reader can prove the other equality

$$\mathrm{id}_{T^2_{F(p),G(q)}(M_2,N_2)} = d^2\left(F_{|V_1}, G_{|W_1}\right)_{p,q} \circ d^2\left(F^{-1}_{|V_1}, G^{-1}_{|W_1}\right)_{F(p),G(q)}. \tag{9}$$

Equations (8) and (9) together imply the thesis. $\square$

Corollary 3.6. *Under the hypotheses of Proposition 3.5, the followings hold.*

a) $d^2(\mathrm{id}_{V_1}, \mathrm{id}_{W_1})_{p,q} = \mathrm{id}_{T^2_{p,q}(M_1,N_1)}$ *and* $d^2(\mathrm{id}_{V_2}, \mathrm{id}_{W_2})_{F(p),G(q)} = \mathrm{id}_{T^2_{F(p),G(q)}(M_2,N_2)}$.
b) $d^2(F^{-1}, G^{-1})_{F(p),G(q)} = d^2(F,G)^{-1}_{p,q}$.

Remark 3.1. In Section 2.1 we showed that the construction of $T^2_{p,q}(M,N)$ is purely local and it does not depend on the whole varieties M and N. Formally, the function $\iota = (\iota_U, \iota_V)$, where ι_U and ι_V are the inclusions $U \hookrightarrow M$ and $V \hookrightarrow N$ respectively, gives a diffeomorphism between $U \times V \longleftrightarrow U \times V \subseteq M \times N$, hence, by Proposition 3.5, $d^2\iota_{p,q}$ is an isomorphism $T^2_{p,q}(U,V) \longleftrightarrow T^2_{p,q}(M,N)$.

3.1. Bidifferential of Euclidean spaces

Let V_1, W_1 be two open sets of $M_1 = N_1 = \mathbb{E}^m$ and V_2, W_2 two open sets of $M_2 = N_2 = \mathbb{E}^n$. The aim of this section is to compute the matrix associated, as linear map, with the bidifferential

$$d^2(F,G)_{p_1,q_1} \colon T^2_{p_1,q_1}(V_1,W_1) \to T^2_{p_2,q_2}(V_2,W_2),$$

for smooth maps $F \colon V_1 \to V_2$, $G \colon W_1 \to W_2$ and points $p_1 \in V_1$, $q_1 \in W_1$ and $p_2 = F(p_1) \in V_2$, $q_2 = G(q_1) \in W_2$. Recall that $T^2_{p_1,q_1}(V_1,W_1) \cong T^2_{p_1,q_1}(\mathbb{E}^m,\mathbb{E}^m) = T^2_{p_1,q_1}(\mathbb{E}^m)$. Let $(x_1,\dots,x_m)$ be the coordinates of the chart V_1 and $(y_1,\dots,y_m)$ the coordinates of W_1, so

$$F(p) = F(x_1,\dots,x_m) = (z_1,\dots,z_n) \quad \text{and} \quad G(q) = G(y_1,\dots,y_m) = (t_1,\dots,t_n),$$

for $p \in V_1$ and $q \in W_1$, where $(z_1,\dots,z_n)$ and $(t_1,\dots,t_n)$ are the coordinates of the charts V_2 and W_2, respectively.

Let us first restrict ourselves to the case $m = n = 2$, hence V_1, W_1, V_2, W_2 are open sets of $\mathbb{E}^2$, $F(x) = F(x_1,x_2) = (z_1,z_2)$, and $G(y) = G(y_1,y_2) = (t_1,t_2)$. By Proposition 2.7, a basis of $T^2_{p_1,q_1}(V_1,W_1) \cong T^2_{p_1,q_1}(\mathbb{E}^2)$ and $T^2_{p_2,q_2}(V_2,W_2) \cong T^2_{p_2,q_2}(\mathbb{E}^2)$ are given by

$$\left\{ \frac{\partial}{\partial x_1}\Big|_{p_1} \otimes \frac{\partial}{\partial y_1}\Big|_{q_1}, \frac{\partial}{\partial x_1}\Big|_{p_1} \otimes \frac{\partial}{\partial y_2}\Big|_{q_1}, \frac{\partial}{\partial x_2}\Big|_{p_1} \otimes \frac{\partial}{\partial y_1}\Big|_{q_1}, \frac{\partial}{\partial x_2}\Big|_{p_1} \otimes \frac{\partial}{\partial y_2}\Big|_{q_1} \right\}$$

and

$$\left\{ \frac{\partial}{\partial z_1}\Big|_{p_2} \otimes \frac{\partial}{\partial t_1}\Big|_{q_2}, \frac{\partial}{\partial z_1}\Big|_{p_2} \otimes \frac{\partial}{\partial t_2}\Big|_{q_2}, \frac{\partial}{\partial z_2}\Big|_{p_2} \otimes \frac{\partial}{\partial t_1}\Big|_{q_2}, \frac{\partial}{\partial z_2}\Big|_{p_2} \otimes \frac{\partial}{\partial t_2}\Big|_{q_2} \right\},$$

respectively. For $f \in \mathcal{C}^\infty_{p_1}(\mathbb{E}^2)$, $g \in \mathcal{C}^\infty_{q_1}(\mathbb{E}^2)$, let us calculate the element

$$d^2(F,G)\left(\frac{\partial}{\partial x_i}\Big|_{p_1} \otimes \frac{\partial}{\partial y_j}\Big|_{q_1} \right)(f,g) = \frac{\partial}{\partial x_i}\Big|_{p_1} \otimes \frac{\partial}{\partial y_j}\Big|_{q_1} \Big(f(F(x)), g(G(y)) \Big) \tag{10}$$

for $i, j \in \{1,2\}$. Since

$$\frac{\partial}{\partial x_i}\Big|_{p_1} (f \circ F)(x_1,x_2) = \frac{\partial}{\partial x_i}\Big|_{p_1} f(F(x_1,x_2)) = \sum_{i_0=1}^{2} \frac{\partial z_{i_0}}{\partial x_i}(p_1) \frac{\partial f}{\partial z_{i_0}}(F(p_1))$$

and

$$\frac{\partial}{\partial y_j}\Big|_{q_1} (g \circ G)(y_1,y_2) = \frac{\partial}{\partial y_j}\Big|_{q_1} g(G(y_1,y_2)) = \sum_{j_0=1}^{2} \frac{\partial t_{j_0}}{\partial y_j}(q_1) \frac{\partial g}{\partial t_{j_0}}(G(q_1)),$$

the left hand-side of Equation (10) is equal to

$$\begin{aligned}
&\left(\sum_{i_0=1}^{2} \frac{\partial z_{i_0}}{\partial x_i}(p_1) \frac{\partial f}{\partial z_{i_0}}(F(p_1)) \right) \left(\sum_{j_0=1}^{2} \frac{\partial t_{j_0}}{\partial y_j}(q_1) \frac{\partial g}{\partial t_{j_0}}(G(q_1)) \right) \\
&\qquad = \sum_{i_0,j_0=1}^{2} \frac{\partial z_{i_0}}{\partial x_i}(p_1) \frac{\partial t_{j_0}}{\partial y_j}(q_1) \frac{\partial f}{\partial y_{i_0}}(p_2) \frac{\partial g}{\partial z_{j_0}}(q_2) \\
&\qquad = \sum_{i_0,j_0=1}^{2} \frac{\partial z_{i_0}}{\partial x_i}(p_1) \frac{\partial t_{j_0}}{\partial y_j}(q_1) \left(\frac{\partial}{\partial y_{i_0}}\Big|_{p_2} \otimes \frac{\partial}{\partial z_{j_0}}\Big|_{q_2} \right)(f,g).
\end{aligned}$$

Therefore, the associated matrix is given by

$$\begin{pmatrix} \frac{\partial z_1}{\partial x_1}(p_1)\frac{\partial t_1}{\partial y_1}(q_1) & \frac{\partial z_1}{\partial x_1}(p_1)\frac{\partial t_1}{\partial y_2}(q_1) & \frac{\partial z_1}{\partial x_2}(p_1)\frac{\partial t_1}{\partial y_1}(q_1) & \frac{\partial z_1}{\partial x_2}(p_1)\frac{\partial t_1}{\partial y_2}(q_1) \\ \frac{\partial z_1}{\partial x_1}(p_1)\frac{\partial t_2}{\partial y_1}(q_1) & \frac{\partial z_1}{\partial x_1}(p_1)\frac{\partial t_2}{\partial y_2}(q_1) & \frac{\partial z_1}{\partial x_2}(p_1)\frac{\partial t_2}{\partial y_1}(q_1) & \frac{\partial z_1}{\partial x_2}(p_1)\frac{\partial t_2}{\partial y_2}(q_1) \\ \frac{\partial z_2}{\partial x_1}(p_1)\frac{\partial t_1}{\partial y_1}(q_1) & \frac{\partial z_2}{\partial x_1}(p_1)\frac{\partial t_1}{\partial y_2}(q_1) & \frac{\partial z_2}{\partial x_2}(p_1)\frac{\partial t_1}{\partial y_1}(q_1) & \frac{\partial z_2}{\partial x_2}(p_1)\frac{\partial t_1}{\partial y_2}(q_1) \\ \frac{\partial z_2}{\partial x_1}(p_1)\frac{\partial t_2}{\partial y_1}(q_1) & \frac{\partial z_2}{\partial x_1}(p_1)\frac{\partial t_2}{\partial y_2}(q_1) & \frac{\partial z_2}{\partial x_2}(p_1)\frac{\partial t_2}{\partial y_1}(q_1) & \frac{\partial z_2}{\partial x_2}(p_1)\frac{\partial t_2}{\partial y_2}(q_1) \end{pmatrix}, \tag{11}$$

which can be written as

$$\begin{pmatrix} \frac{\partial z_1}{\partial x_1}(p_1)\begin{pmatrix} \frac{\partial t_1}{\partial y_1}(q_1) & \frac{\partial t_1}{\partial y_2}(q_1) \\ \frac{\partial t_2}{\partial y_1}(q_1) & \frac{\partial t_2}{\partial y_2}(q_1)\end{pmatrix} & \frac{\partial z_1}{\partial x_2}(p_1)\begin{pmatrix} \frac{\partial t_1}{\partial y_1}(q_1) & \frac{\partial t_1}{\partial y_2}(q_1) \\ \frac{\partial t_2}{\partial y_1}(q_1) & \frac{\partial t_2}{\partial y_2}(q_1)\end{pmatrix} \\ \frac{\partial z_2}{\partial x_1}(p_1)\begin{pmatrix} \frac{\partial t_1}{\partial y_1}(q_1) & \frac{\partial t_1}{\partial y_2}(q_1) \\ \frac{\partial t_2}{\partial y_1}(q_1) & \frac{\partial t_2}{\partial y_2}(q_1)\end{pmatrix} & \frac{\partial z_2}{\partial x_2}(p_1)\begin{pmatrix} \frac{\partial t_1}{\partial y_1}(q_1) & \frac{\partial t_1}{\partial y_2}(q_1) \\ \frac{\partial t_2}{\partial y_1}(q_1) & \frac{\partial t_2}{\partial y_2}(q_1)\end{pmatrix} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{\partial z_1}{\partial x_1}(p_1) & \frac{\partial z_1}{\partial x_2}(p_1) \\ \frac{\partial z_2}{\partial x_1}(p_1) & \frac{\partial z_2}{\partial x_2}(p_1)\end{pmatrix} \otimes \begin{pmatrix} \frac{\partial t_1}{\partial y_1}(q_1) & \frac{\partial t_1}{\partial y_2}(q_1) \\ \frac{\partial t_2}{\partial y_1}(q_1) & \frac{\partial t_2}{\partial y_2}(q_1)\end{pmatrix},$$

that is, the tensor product (Kronecker product) of the matrices associated with dF_{p_1} and dG_{q_1}, respectively.

Let now V_i be a open set of $M_i = \mathbb{E}^{m_i}$ and W_i of $N_i = \mathbb{E}^{n_i}$, for $i \in \{1, 2\}$. A basis of $T^2_{p_1,q_1}(V_1, W_1)$ and $T^2_{p_2,q_2}(V_2, W_2)$ are given by

$$\left\{ \frac{\partial}{\partial x_i}\Big|_{p_1} \otimes \frac{\partial}{\partial y_j}\Big|_{q_1} \;:\; i = 1, \dots, m_1,\; j = 1, \dots, n_1 \right\}$$

and

$$\left\{ \frac{\partial}{\partial z_h}\Big|_{p_2} \otimes \frac{\partial}{\partial t_k}\Big|_{q_2} \;:\; h = 1, \dots, m_2,\; k = 1, \dots, n_2 \right\},$$

respectively. The arguments of case $m = n = 2$ can be easily generalized, hence the associated matrix with the linear map $d^2(F, G)_{p_1,q_1}$ with respect to the above bases is $(dF_{p_1}) \otimes (dG_{q_1})$, i.e. the $(m_2 n_2 \times m_1 n_1)$-real matrix

$$\begin{pmatrix} \frac{\partial z_1}{\partial x_1}(p_1)\frac{\partial t_1}{\partial y_1}(q_1) & \dots & \frac{\partial z_1}{\partial x_1}(p_1)\frac{\partial t_1}{\partial y_{n_1}}(q_1) & \dots & \frac{\partial z_1}{\partial x_{m_1}}(p_1)\frac{\partial t_1}{\partial y_1}(q_1) & \dots & \frac{\partial z_1}{\partial x_{m_1}}(p_1)\frac{\partial t_1}{\partial y_{n_1}}(q_1) \\ \vdots & & \vdots & & \vdots & & \vdots \\ \frac{\partial z_1}{\partial x_1}(p_1)\frac{\partial t_{n_2}}{\partial y_1}(q_1) & \dots & \frac{\partial z_1}{\partial x_1}(p_1)\frac{\partial t_{n_2}}{\partial y_{n_1}}(q_1) & \dots & \frac{\partial z_1}{\partial x_{m_1}}(p_1)\frac{\partial t_{n_2}}{\partial y_1}(q_1) & \dots & \frac{\partial z_1}{\partial x_{m_1}}(p_1)\frac{\partial t_{n_2}}{\partial y_{n_1}}(q_1) \\ \vdots & & \vdots & & \vdots & & \vdots \\ \frac{\partial z_{m_2}}{\partial x_1}(p_1)\frac{\partial t_1}{\partial y_1}(q_1) & \dots & \frac{\partial z_{m_2}}{\partial x_1}(p_1)\frac{\partial t_1}{\partial y_{n_1}}(q_1) & \dots & \frac{\partial z_{m_2}}{\partial x_{m_1}}(p_1)\frac{\partial t_1}{\partial y_1}(q_1) & \dots & \frac{\partial z_{m_2}}{\partial x_{m_1}}(p_1)\frac{\partial t_1}{\partial y_{n_1}}(q_1) \\ \vdots & & \vdots & & \vdots & & \vdots \\ \frac{\partial z_{m_2}}{\partial x_1}(p_1)\frac{\partial t_{n_2}}{\partial y_1}(q_1) & \dots & \frac{\partial z_{m_2}}{\partial x_1}(p_1)\frac{\partial t_{n_2}}{\partial y_{n_1}}(q_1) & \dots & \frac{\partial z_{m_2}}{\partial x_{m_1}}(p_1)\frac{\partial t_{n_2}}{\partial y_1}(q_1) & \dots & \frac{\partial z_{m_2}}{\partial x_{m_1}}(p_1)\frac{\partial t_{n_2}}{\partial y_{n_1}}(q_1) \end{pmatrix}. \tag{12}$$

3.2. Bidifferential of manifolds

Let M_1, N_1, M_2, N_2 be four manifolds and $F\colon M_1 \to M_2$, $G\colon N_1 \to N_2$ two smooth maps. The matrix associated with the linear map

$$d^2(F, G)_{p_1,q_1} : T^2_{p_1,q_1}(M_1, N_1) \to T^2_{p_2,q_2}(M_2, N_2),$$

where $p_2 = F(p_1)$ and $q_2 = G(q_1)$, can be found using coordinates for the manifolds. In particular, let (V_i, φ_i) be a chart of M_i containing p_i and (W_i, ψ_i) a chart of N_i containing q_i, for $i \in \{1, 2\}$. The functions φ_i and ψ_i define diffeomorphisms $V_i \longleftrightarrow \varphi_i(V_i) \subseteq \mathbb{E}^{m_i}$ and $W_i \longleftrightarrow \psi_i(W_i) \subseteq \mathbb{E}^{n_i}$, then by Proposition 3.5 $T^2_{p_i,q_i}(V_i, W_i) \cong T^2_{\varphi_i(p_i),\psi_i(q_i)}(\mathbb{E}^{m_i}, \mathbb{E}^{n_i})$ through the bidifferential $d^2(\varphi_i, \psi_i)_{p_i,q_i}$. These isomorphisms of vector spaces provide a basis for the bitangent space $T^2_{p_i,q_i}(V_i, W_i)$. We summarise in the following theorem.

Theorem 3.7. *Let M_1, N_1, M_2, N_2 be four manifolds, $F\colon M_1 \to M_2$, $G\colon N_1 \to N_2$ two smooth maps, (V_i, φ_i) a chart of M_i containing p_i and (W_i, ψ_i) a chart of N_i containing q_i, for $i \in \{1, 2\}$. Then*

$$\left\{ d^2(\varphi_1, \psi_1)^{-1}_{p_1,q_1}\left(\frac{\partial}{\partial x_i}\Big|_{\varphi_1(p_1)} \otimes \frac{\partial}{\partial z_h}\Big|_{\psi_1(q_1)} \right) \;:\; i = 1, \dots, m_1,\; h = 1, \dots, n_1 \right\}$$

is a basis of $T^2_{p_1,q_1}(M_1,N_1)$ *and*

$$\left\{ d^2(\varphi_2,\psi_2)^{-1}_{p_2,q_2}\left(\frac{\partial}{\partial y_j}\Big|_{\varphi_2(p_2)} \otimes \frac{\partial}{\partial t_k}\Big|_{\psi_2(q_2)}\right) \;:\; j=1,\dots,m_2,\; k=1,\dots,n_2 \right\}$$

is a basis of $T^2_{p_2,q_2}(M_2,N_2)$. *With respect to the above bases, the bidifferential is represented by the matrix in* (12).

In other words, the matrix associated with $d^2(F,G)_{p_1,q_1}$ is the one associated with the bidifferential of the functions defined between euclidean spaces

$$\varphi_2\circ F\circ\varphi_1^{-1}\colon \varphi_1(V_1)\to\varphi_2(V_2), \quad \text{and} \quad \psi_2\circ G\circ\psi_1^{-1}\colon \psi_1(W_1)\to\psi_2(W_2),$$

which are the functions F and G expressed in coordinates, respectively.

3.3. Change of coordinates

In this section, we show how two bases of $T^2_{p,q}(M,N)$ are related if one chooses more systems of coordinates for the manifolds.

Let (V,φ) and $(\widetilde{V},\widetilde{\varphi})$ be two charts of M of coordinates $(x_1,\dots,x_n)$ and $(\widetilde{x}_1,\dots,\widetilde{x}_n)$, respectively, and (W,ψ) and $(\widetilde{W},\widetilde{\psi})$ two charts of N of coordinates $(y_1,\dots,y_n)$ and $(\widetilde{y}_1,\dots,\widetilde{y}_n)$, respectively. In each manifold, in order to change from the first system of coordinates to the second one, one can consider the transition maps

$$\widetilde{\varphi}\circ\varphi^{-1}\colon \varphi(V\cap\widetilde{V})\longrightarrow\widetilde{\varphi}(V\cap\widetilde{V}) \quad \text{and} \quad \widetilde{\psi}\circ\psi^{-1}\colon \psi(W\cap\widetilde{W})\longrightarrow\widetilde{\psi}(W\cap\widetilde{W}).$$

Let now $p\in V\cap\widetilde{V}$ and $q\in W\cap\widetilde{W}$. The matrix associated with the bidifferential of functions defined between euclidean spaces has been computed in Section 3.1. Hence, it follows that

$$\begin{aligned} d^2\left(\widetilde{\varphi}\circ\varphi^{-1},\widetilde{\psi}\circ\psi^{-1}\right)_{\varphi(p),\psi(q)}&\left(\frac{\partial}{\partial x_i}\Big|_{\varphi(p)}\otimes\frac{\partial}{\partial y_j}\Big|_{\psi(q)}\right)\\ &=\sum_{i_0=1}^{m}\sum_{j_0=1}^{n}\frac{\partial\widetilde{x}_{i_0}}{\partial x_i}(\varphi(p))\frac{\partial\widetilde{y}_{j_0}}{\partial y_j}(\psi(q))\left(\frac{\partial}{\partial\widetilde{x}_i}\Big|_{\widetilde{\varphi}(p)}\otimes\frac{\partial}{\partial\widetilde{y}_j}\Big|_{\widetilde{\psi}(q)}\right). \end{aligned}$$

Combining this relation together with the composition property for bidifferentials in Equation (6) and with the inverse property in Corollary 3.6.b, we obtain that

$$\begin{aligned} d^2(\varphi,\psi)^{-1}_{p,q}&\left(\frac{\partial}{\partial x_i}\Big|_{\varphi(p)}\otimes\frac{\partial}{\partial y_j}\Big|_{\psi(q)}\right)=\\ &=d^2(\varphi^{-1},\psi^{-1})_{\varphi(p),\psi(q)}\left(\frac{\partial}{\partial x_i}\Big|_{\varphi(p)}\otimes\frac{\partial}{\partial y_j}\Big|_{\psi(q)}\right)\\ &=d^2\left(\widetilde{\varphi}^{-1},\widetilde{\psi}^{-1}\right)_{\widetilde{\varphi}(p),\widetilde{\psi}(q)}\left(d^2(\widetilde{\varphi}\circ\varphi^{-1},\widetilde{\psi}\circ\psi^{-1})_{\varphi(p),\psi(q)}\left(\frac{\partial}{\partial x_i}\Big|_{\varphi(p)}\otimes\frac{\partial}{\partial y_j}\Big|_{\psi(q)}\right)\right)\\ &=\sum_{i_0=1}^{m}\sum_{j_0=1}^{n}\frac{\partial\widetilde{x}_{i_0}}{\partial x_i}(\varphi(p))\frac{\partial\widetilde{y}_{j_0}}{\partial y_j}(\psi(q))\,d^2(\widetilde{\varphi},\widetilde{\psi})^{-1}_{p,q}\left(\frac{\partial}{\partial\widetilde{x}_i}\Big|_{\widetilde{\varphi}(p)}\otimes\frac{\partial}{\partial\widetilde{y}_j}\Big|_{\widetilde{\psi}(q)}\right). \end{aligned}$$

3.4. Bidifferential of curves

Let I and J be two open intervals of $\mathbb{R}$, and let $\gamma\colon I\to M$ and $\eta\colon J\to N$ be two smooth curves on M and N, respectively. For $a\in I$ and $b\in J$, the tangent lines to the curves $T_aI=\langle\frac{d}{dt}\big|_a\rangle$ and $T_bJ=\langle\frac{d}{ds}\big|_b\rangle$ at a and b, respectively, are well-defined. Then we can define the bilinear map

$$\frac{d}{dt}\Big|_a\otimes\frac{d}{ds}\Big|_b : \mathcal{C}^\infty_a(I)\times\mathcal{C}^\infty_b(J)\to\mathbb{R}$$

that belongs to $T^2_{a,b}\mathbb{R}$. Recall that $\gamma'(a)$ and $\eta'(b)$ denote the tangent vectors $d\gamma_a\left(\frac{d}{dt}\big|_a\right)\in T_{\gamma(a)}M$ and $d\eta_b\left(\frac{d}{ds}\big|_b\right)\in T_{\eta(b)}N$ at the curves γ and η at the points $\gamma(a)$ and $\eta(b)$, respectively. The following result provides the explicit computation of the bidifferential of γ and η at the points $a\in I$ and $b\in J$.

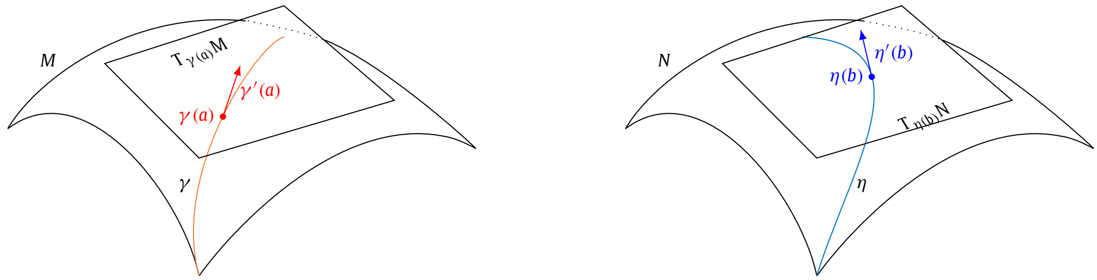

Fig. 3. Setting of Proposition 3.8.

Proposition 3.8. *The bilinear map $d^2(\gamma,\eta)_{a,b}$ associates to the pair $(f,g) \in \mathcal{C}^\infty_{\gamma(a)}(M) \times \mathcal{C}^\infty_{\eta(b)}(N)$ the product of the derivatives of f at a and g at b along the curves γ and η, respectively.*

Proof. If $f \in \mathcal{C}^\infty_{\gamma(a)}(M)$ and $g \in \mathcal{C}^\infty_{\eta(b)}(N)$, then the aim of the proof is to show that

$$d^2_{a,b}(\gamma,\eta)\left(\frac{d}{dt}\bigg|_a \otimes \frac{d}{ds}\bigg|_b\right) = (\gamma'(a)f)\,(\eta'(b)g). \tag{13}$$

By definition of bidifferential,

$$\begin{aligned} d^2_{a,b}(\gamma,\eta)\left(\frac{d}{dt}\bigg|_a \otimes \frac{d}{ds}\bigg|_b\right)(f,g) &= \left(\frac{d}{dt}\bigg|_a \otimes \frac{d}{ds}\bigg|_b\right)(f\circ\gamma, g\circ\eta) \\ &= \left(\frac{d}{dt}\bigg|_a f(\gamma(t))\right)\left(\frac{d}{ds}\bigg|_b g(\eta(s))\right). \quad \square \end{aligned}$$

If $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$ denotes the coordinates of charts of M and N containing $\gamma(a) \in M$ and $\eta(b) \in N$, respectively, then by the chain's rule

$$\gamma'(a)f = \sum_{i=1}^m \frac{\partial}{\partial x_i}\bigg|_{\gamma(a)} f(x)\,\frac{d}{dt}\bigg|_a x_i(t) \quad \text{and} \quad \eta'(b)g = \sum_{j=1}^n \frac{\partial}{\partial y_j}\bigg|_{\eta(b)} g(y)\,\frac{d}{ds}\bigg|_b y_j(s).$$

Finally, we obtain the following formula for bidifferential of curves (see Fig. 3)

$$d^2_{a,b}(\gamma,\eta)\left(\frac{d}{dt}\bigg|_a \otimes \frac{d}{ds}\bigg|_b\right) = \sum_{i,j}\left(\frac{d}{dt}\bigg|_a x_i(t)\,\frac{d}{ds}\bigg|_b y_j(s)\right)\left(\frac{\partial}{\partial x_i}\bigg|_{\gamma(a)} \otimes \frac{\partial}{\partial y_j}\bigg|_{\eta(b)}\right). \tag{14}$$

Example 3.8.1. The curve $\eta(t) = \exp(2\pi i t)$, for $t \in I =]0,1[$, defines the unit circle $\mathbb{S}^1$ without the point i on the complex plane. We set $M = \mathbb{S}^1 \setminus \{i\}$. If $\gamma = \eta$ and $a, b \in I$, then the bidifferential $d^2(\gamma,\eta)_{a,b} \colon T^2_{a,b}(I) \to T^2_{e^{2\pi i a}, e^{2\pi i b}}(M)$ satisfies

$$d^2(\gamma,\eta)_{a,b}(f,g) = \frac{d}{dt}\bigg|_a f(\gamma(t))\,\frac{d}{ds}\bigg|_b g(\eta(s)),$$

for every $f, g \in \mathcal{C}^\infty(M)$. For instance, if $f(z) = \mathrm{Re}(z)$ and $g(z) = \mathrm{Im}(z)$ which, in coordinates, are expressed by $f(t) = \cos(2\pi t)$ and $g(t) = \sin(2\pi t)$, respectively, we have

$$d^2(\gamma,\eta)_{a,b}(f,g) = (-2\pi\sin(2\pi a))(2\pi\cos(2\pi b)) = -4\pi^2\sin(2\pi a)\cos(2\pi b).$$

4. Applications

In this final section, we discuss some applications of our setting in the framework of partial differential equations. In particular, we first analyse the eikonal equation and, in general, differential problems in which the distance from the boundary is involved. In addition, we will take the opportunity to introduce the definition of biderivation field.

Secondly, we move to nonlinear second-order problems considering elliptic PDEs. More precisely, we will link some suitable biderivation fields to the notion of carrè du champ of an operator, and to the one of cometric in sub-Riemannian geometry.

Note that biderivations act on smooth functions, whereas solutions to PDE problems are generally assumed in suitable Sobolev spaces. In the following, however, we restrict our attention to smooth solutions. Indeed, every function in $H^1_0(U)$ can be approximated by smooth compactly supported functions, using the density of $\mathcal{C}^\infty_c(U)$ in $H^1_0(U)$.

4.1. Eikonal equation

The eikonal equation is a non-linear first-order partial differential equation of the form

$$|\nabla u(x)| = n(x), \tag{15}$$

where $x \in \Omega \subseteq \mathbb{R}^n$, being Ω open, $n(x)$ is a positive function, and $|\cdot|$ is the Euclidean norm.

Let Ω be an open domain with a smooth boundary $\partial\Omega$. From a physical point of view, the problem

$$\begin{cases} |\nabla u(x)| = \dfrac{1}{f(x)} & x \in \Omega, \\ u(x) = q(x) & x \in \partial\Omega, \end{cases} \tag{16}$$

being $f : \overline{\Omega} \to (0, +\infty)$ and $q : \partial\Omega \to [0, +\infty)$, allows us to describe the minimal amount of time required to travel from a point $x \in \Omega$ to the boundary of the domain. In this setting, f is the speed of travel, and q is an exit-time penalty. When $f \equiv 1$, the solution of (16) is the distance of the point x from the boundary of Ω, that is, a function $d : \overline{\Omega} \to [0, +\infty)$ defined as

$$d(x) = \inf_{y \in \partial\Omega} |x - y|.$$

This distance is widely used in the study of differential singular problems (blow-up at zero), throughout the Hardy-Sobolev inequality [25, Theorem 6.8.33]. We refer the interested reader to [8,9,14,16,18].

More precisely, the distance interpretation above corresponds to the case $q \equiv 0$: for a general boundary datum q, the value of u also includes the exit-time penalty at the point where the trajectory leaves Ω.

The eikonal equation admits several anisotropic generalisations. In the control-theoretic formulation, the speed may depend both on position and direction of motion, leading to static Hamilton–Jacobi equations of the form $H(x, \nabla u) = 1$ (see, for instance, [26]). A closely related viewpoint considers anisotropic eikonal equations tied to a Riemannian metric, as in [6,23]. Here we do not aim to treat arbitrary Finsler-type anisotropies. Instead, we show that symmetric positive definite biderivation fields naturally encode the quadratic, or Riemannian, class of anisotropic eikonal equations.

Since in this work we have regarded gradients as row vectors, for a symmetric positive definite matrix $C(x)$ we write

$$\|\nabla u(x)\|^2_{C(x)} := \nabla u(x) C(x) \nabla u(x)^t, \tag{17}$$

which is the row-vector version of the usual convention $\|w\|^2_C = \langle w, Cw \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the usual dot product in $\mathbb{R}^n$.

The classical eikonal equation is already encoded by a biderivation. For every $x \in \Omega$, let

$$B^E_x = \sum_{i=1}^n \left.\frac{\partial}{\partial x_i}\right|_x \otimes \left.\frac{\partial}{\partial x_i}\right|_x \in T^2_{x,x}\Omega. \tag{18}$$

Then $B^E_x(u, u) = \sum_{i=1}^n u_{x_i}(x)^2 = |\nabla u(x)|^2$, so the classical equation $|\nabla u(x)| = \frac{1}{f(x)}$ is equivalently written as $B^E_x(u, u) = \frac{1}{f(x)^2}$. This suggests replacing the Euclidean biderivation field (18) by a general positive definite one. For the purposes of this section, we restrict the following definition to a particular case.

Definition 4.1. Let $\Omega \subseteq \mathbb{R}^n$ be open. A *biderivation field* on Ω is a smooth assignment $x \in \Omega \longmapsto B_x \in T^2_{x,x}\Omega$. In coordinates, it can be written as

$$B_x(\varphi, \psi) = \sum_{i,j=1}^n b^{ij}(x) \frac{\partial \varphi}{\partial x_i}(x) \frac{\partial \psi}{\partial x_j}(x), \tag{19}$$

or, equivalently, $B_x(\varphi, \psi) = \nabla\varphi(x) B(x) \nabla\psi(x)^t$, where $B(x) = (b^{ij}(x))$. We say that B is symmetric positive definite if $B(x)$ is symmetric positive definite for every $x \in \Omega$.

Given a symmetric positive definite biderivation field B and a positive function f, we call

$$B_x(u, u) = \frac{1}{f(x)^2} \tag{20}$$

the *biderivation eikonal equation* associated with B and f; in coordinates, this reads

$$\nabla u(x) B(x) \nabla u(x)^t = \frac{1}{f(x)^2}.$$

Proposition 4.2. *Let $\Omega \subseteq \mathbb{R}^n$ be open, let $f : \Omega \to (0, +\infty)$ be smooth, and let B be a smooth field of symmetric positive definite biderivations on Ω, written as $B_x(\varphi, \psi) = \nabla\varphi(x)B(x)\nabla\psi(x)^t$. Set $N(x) := f(x)^2 B(x)$. Then the biderivation eikonal equation $B_x(u, u) = 1/f(x)^2$ is equivalent to*

$$\|\nabla u(x)\|_{N(x)} = 1. \tag{21}$$

Proof. For every $u \in \mathcal{C}^\infty(\Omega)$, we have

$$B_x(u, u) = \nabla u(x)B(x)\nabla u(x)^t,$$

so $B_x(u, u) = \frac{1}{f(x)^2}$ is equivalent to

$$f(x)^2\nabla u(x)B(x)\nabla u(x)^t = 1,$$

that is, $\nabla u(x)N(x)\nabla u(x)^t = 1$ by the definition of $N(x)$. Using Equation (17), this is exactly Equation (21). □

Equivalently, following the notation of [23], setting $M(x) := N(x)^{-1} = f(x)^{-2}B(x)^{-1}$, Equation (21) can be written as $\|\nabla u(x)\|_{M(x)^{-1}} = \|\nabla u(x)\|_{N(x)} = 1$.

Thus, a symmetric positive definite biderivation field plays the role of the cometric tensor of a Riemannian anisotropic eikonal equation.

Remark 4.1. An anisotropic Hamiltonian need not be quadratic in the covector variable. Symmetric positive definite biderivation fields single out precisely the case where it is: writing

$$\mathcal{H}_{\mathcal{B}}(x, p) := f(x)\sqrt{pB(x)p^t},$$

we have

$$\mathcal{H}_{\mathcal{B}}(x, p)^2 = f(x)^2\, pB(x)p^t,$$

which is a positive definite quadratic form in p, so its level sets

$$\left\{ p \in T_x^*\Omega : \ pB(x)p^t = \frac{1}{f(x)^2} \right\}$$

are ellipsoids in the cotangent space. Positive biderivation fields therefore correspond to the Riemannian, or quadratic, subclass of anisotropic eikonal equations.

The bidifferential introduced in Section 3 gives, finally, the natural transformation rule for the biderivation eikonal equation under changes of variables.

Proposition 4.3. *Let $U, V \subseteq \mathbb{R}^n$ be open sets and let $F : U \to V$ be a diffeomorphism. Let $x \in U \longmapsto B_x \in T^2_{x,x}U$ be a biderivation field on U, and define a biderivation field $\widetilde{B}$ on V by*

$$\widetilde{B}_{F(x)} = d^2(F, F)_{x,x}(B_x). \tag{22}$$

Let $u \in \mathcal{C}^\infty(U)$, let $f : U \to (0, +\infty)$, and set $\widetilde{u} = u \circ F^{-1}$, $\widetilde{f} = f \circ F^{-1}$. Then $B_x(u, u) = \dfrac{1}{f(x)^2}$ for every $x \in U$ if and only if $\widetilde{B}_y(\widetilde{u}, \widetilde{u}) = \dfrac{1}{\widetilde{f}(x)^2}$ for every $y \in V$.

Proof. Let $y = F(x)$. By the definition of $\widetilde{B}$ and of the bidifferential,

$$\widetilde{B}_y(\widetilde{u}, \widetilde{u}) = d^2(F, F)_{x,x}(B_x)(\widetilde{u}, \widetilde{u}) = B_x(\widetilde{u} \circ F, \widetilde{u} \circ F).$$

Since $\widetilde{u} \circ F = u \circ F^{-1} \circ F = u$, we get $\widetilde{B}_y(\widetilde{u}, \widetilde{u}) = B_x(u, u)$. Moreover $\widetilde{f}(y) = f(F^{-1}(y)) = f(x)$, so the equation for u at x is equivalent to the transformed equation for $\widetilde{u}$ at $y = F(x)$. □

4.2. Potential applications of biderivations to elliptic PDE problems

In this subsection, we briefly outline some future applications of the framework of biderivations to nonlinear second-order problems considering elliptic PDE, leaving a detailed analysis for future works.

Nonlinear second-order elliptic PDEs generalise Laplace's and Poisson's equations and arise in the calculus of variations and in differential geometry, concerning curvature issues. For the foundational theory of elliptic PDEs, we refer the reader to [15]. Let U be an open and bounded subset of $\mathbb{R}^n$, $f: U \to \mathbb{R}$ be a known function, and consider the boundary-value problem

$$\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U, \end{cases} \tag{23}$$

where L is a second-order partial differential operator given in divergence form by

$$Lu = -\sum_{i,j=1}^{n} (a^{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u,$$

and $u = u(x)$ is the unknown function defined on $\bar{U}$. For our purpose, we are not interested in the nondivergence form. However, if the functions a^{ij} are $\mathcal{C}^1$ one can rewrite the operator in divergence form into a nondivergence one, and vice versa. Assume that the matrix (a^{ij}) is symmetric, positive definite and with smallest eigenvalue greater than or equal to a positive constant θ. In particular, the last two conditions express the *ellipticity* of the operator L. We remark here that if (a^{ij}) is the identity matrix and $b^i = c = 0$ one obtains that L is the Laplace operator $-\Delta$. The standard technique for obtaining weak solutions is to multiply the equation $Lu = f$ by a test function $v \in \mathcal{C}_c^\infty(U)$ and integrate over U to find

$$\int_U \sum_{i,j=1}^{n} a^{ij}u_{x_i}v_{x_j} + \sum_{i=1}^{n} b^i u_{x_i} v + cuv\, dx = \int_U f v\, dx. \tag{24}$$

Note again that, using the density of smooth compactly supported functions in the Sobolev space $H_0^1(U)$, the identity (24) also holds replacing the smooth v by any $v \in H_0^1(U)$. Accordingly, we also seek u in $H_0^1(U)$.

With the divergence elliptic operator L, one can associate the bilinear form

$$\overline{B}[u,v] = \int_U \sum_{i,j=1}^{n} a^{ij}(x)u_{x_i}v_{x_j} + \sum_{i=1}^{n} b^i u_{x_i} v + cuv\, dx \tag{25}$$

for smooth functions u, v. Let us restrict ourselves to the case $b^i = c = 0$, that is L consists only of the diffusion term. The bilinear form (25) can be written as

$$\overline{B}[u,v] = \int_U B_x(u,v)\, dx$$

where $B_x(u,v) = \nabla u(x) B_x (\nabla v(x))^t$ is a biderivation of $U \subseteq \mathbb{R}^n$ in x, as obtained in (3), where B_x denotes the matrix $(a^{ij}(x))$. In other words, we are integrating the biderivation field B_x, as given in Definition 4.1, on the open set U. The identity $\overline{B}[u,v] = \int_U B_x(u,v)\, dx$ is more than a notational remark: the biderivation field is the intrinsic object underlying the divergence structure of L, in two aspects that we now describe.

First, biderivation fields admit an exact operator-theoretic characterisation. Recall that with a linear operator $L: \mathcal{C}^\infty(U) \to \mathcal{C}^\infty(U)$ one associates the bilinear operator

$$\Gamma_L(u,v) = \tfrac{1}{2}\big(L(uv) - u\, Lv - v\, Lu\big), \tag{26}$$

called the *carré du champ* of L, which measures the failure of L to satisfy the Leibniz rule. This notion lies at the heart of the theory of Dirichlet forms and of the Bakry–Émery calculus, see [2,7,17]. The next result shows that, under the appropriate conditions, our notion of biderivation is exactly the pointwise object produced by this construction when L is a second-order operator in divergence form.

Proposition 4.4. *Let $U \subseteq \mathbb{R}^n$ be open, let $A(x) = (a^{ij}(x))$ be a smooth matrix field, $b(x) = (b^i(x))$ a smooth vector field, and c a smooth function on U, and consider the second-order operator*

$$Lu = \operatorname{div}\big(A(x)\nabla u(x)^t\big) + b(x)\cdot\nabla u(x) + c(x)u(x), \qquad u \in \mathcal{C}^\infty(U).$$

Then, for every $u, v \in \mathcal{C}^\infty(U)$ and $x \in U$,

$$\Gamma_L(u,v)(x) = \frac{1}{2}\,\nabla u(x)\big(A(x)+A(x)^t\big)\nabla v(x)^t - \frac{1}{2}\,c(x)u(x)v(x). \tag{27}$$

In particular:

a) The assignment $x \longmapsto \Gamma_L(\cdot,\cdot)(x)$ is a symmetric biderivation field on U, in the sense of Definition 4.1, if and only if $c \equiv 0$. In this case its matrix is the symmetric part of $A(x)$.

b) Conversely, every symmetric biderivation field B on U, with matrix $B(x)$, satisfies $B_x = \Gamma_{L_B}(\cdot,\cdot)(x)$ for the divergence-form operator $L_B u = \operatorname{div}\big(B(x)\nabla u(x)^t\big)$.

Proof. Since $L \mapsto \Gamma_L$ is linear, we may treat the three terms of L separately. For the second-order term $L_2 u = \operatorname{div}(A\nabla u^t)$, by the Leibniz rule for the gradient, $\nabla(uv) = u\nabla v + v\nabla u$, and for the divergence, $\operatorname{div}(\varphi V) = \varphi \operatorname{div} V + \nabla\varphi \cdot V$, we obtain

$$\begin{aligned} L_2(uv) &= \operatorname{div}\big(u\,A\nabla v^t\big) + \operatorname{div}\big(v\,A\nabla u^t\big) \\ &= u\,L_2 v + \nabla u\,A\,\nabla v^t + v\,L_2 u + \nabla v\,A\,\nabla u^t, \end{aligned}$$

whence $\Gamma_{L_2}(u,v) = \frac{1}{2}\big(\nabla u\,A\,\nabla v^t + \nabla v\,A\,\nabla u^t\big) = \frac{1}{2}\nabla u\,(A+A^t)\,\nabla v^t$. The first-order term $L_1 u = b\cdot\nabla u$ is a derivation, hence satisfies the Leibniz rule exactly and $\Gamma_{L_1} = 0$. For the zeroth-order term $L_0 u = cu$, we get $\Gamma_{L_0}(u,v) = \frac{1}{2}(cuv - u\,cv - v\,cu) = -\frac{1}{2}cuv$. Summing the three contributions yields (27).

For part a), if $c \equiv 0$ then Equation (27) is the biderivation field with matrix $\frac{1}{2}(A(x)+A(x)^t)$, as in Definition 4.1 and the Leibniz conditions (1)–(2) follow from the product rule for the gradient. Conversely, suppose that Γ_L is a biderivation field. Taking $u = v \equiv 1$, the constant function equal to 1, formula (26) gives $\Gamma_L(1,1) = -\frac{1}{2}L(1) = -\frac{1}{2}c$. On the other hand, by part a) of Proposition 2.4, every biderivation vanishes on constant functions, whence $c \equiv 0$.

Finally, part b) follows from Equation (27) applied with $A = B$ symmetric and $b = c = 0$. □

Remark 4.2. We would like to point out that Equation (27) does not depend on the vector field b. Moreover, the presence of the factor $A + A^t$ implies that only the symmetric part of the matrix field A plays a role.

Hence symmetric (diagonal) biderivation fields coincide with the carré du champ operators of second-order operators in divergence form without zeroth-order term, and the bilinear form (25) (with $b^i = c = 0$) is exactly the local Dirichlet form generated by the biderivation field.

Remark 4.3. The Leibniz conditions (1)–(2) are not an accident of the divergence structure: they characterise operators of order at most two. Indeed, let L be any linear operator on $\mathcal{C}^\infty(U)$ with $L(1) = 0$. A direct expansion of Equation (26) gives, for every $u, w, v \in \mathcal{C}^\infty(U)$,

$$\begin{aligned} &2\big[\,\Gamma_L(uw,v) - u\,\Gamma_L(w,v) - w\,\Gamma_L(u,v)\,\big] \\ &\qquad = L(uwv) - u\,L(wv) - w\,L(uv) - v\,L(uw) + uv\,Lw + uw\,Lv + vw\,Lu, \end{aligned}$$

and the right-hand side is the iterated commutator $[[[L, m_u], m_w], m_v]$ applied to the constant function equal to 1, where m_u denotes the multiplication operator by u. By the algebraic characterisation of differential operators, this expression vanishes for all u, w, v if and only if L is a differential operator of order at most two. Therefore, among operators annihilating constants, $\Gamma_L \equiv 0$ if and only if L has order at most one (i.e. L is a derivation), and Γ_L satisfies the Leibniz conditions of Definition 2.1 if and only if L has order at most two: derivations are the pointwise counterparts of first-order operators, and biderivations those of second-order ones. The correspondence fails already at order three: for $Lu = u'''$ on $\mathbb{R}$ one computes $\Gamma_L(u,v) = \frac{3}{2}(u''v' + u'v'')$ and $\Gamma_L(u^2,v) - 2u\,\Gamma_L(u,v) = 3(u')^2v' \not\equiv 0$. In the probabilistic literature, the failure of Γ_L when L involves derivatives of order at least three is known as the diffusion property of Markov generators (cf. [2, H10] and [7]).

Secondly, the same formalism points to a direction in which the ellipticity assumption may be relaxed. If the biderivation field B_x of Definition 4.1 is only positive *semi*definite, its matrix $B(x)$ may have a non-trivial kernel and the ellipticity condition on the constant θ fails. By Proposition 2.7 a symmetric biderivation field is a contravariant symmetric 2-tensor, that is, precisely the type of object (*cometric*) by which a sub-Riemannian structure is encoded in cotangent form [1,24]. The positive definite case corresponds to Riemannian metrics and elliptic operators, while the degenerate case corresponds to genuinely sub-Riemannian structures. A model example is the field $B_x = X|_x \otimes X|_x + Y|_x \otimes Y|_x$ on $\mathbb{R}^3$ generated by the Heisenberg vector fields $X = \partial_{x_1} - \frac{x_2}{2}\partial_{x_3}$ and $Y = \partial_{x_2} + \frac{x_1}{2}\partial_{x_3}$: here $B(x)$ has constant rank two, and the associated form $\overline{B}[u,v] = \int_U (Xu)(Xv) + (Yu)(Yv)\,dx$ is the Dirichlet form of the sub-Laplacian $X^2 + Y^2$, a hypoelliptic but non-elliptic operator [5,19]. We do not pursue these connections here; they do suggest, however, that a systematic study of degenerate biderivation fields, and of the sub-elliptic operators associated with them, could be a fruitful direction for future work.

Another approach to generalising elliptic PDE operators through biderivations could be given starting from the p-Laplacian, that is the operator

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$$

for a real number $1 < p < \infty$. We recall that for $p = 2$ we have the usual Laplace operator. A weak solution u of the Dirichlet problem

$$\begin{cases} -\Delta_p u = f, & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

satisfies the integral equation

$$\int_U |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx = \int_U f v \, dx$$

for any test function v. For a biderivation field B_x, one could consider the operator

$$\Delta_p^B u = \operatorname{div}\left(|B_x(u,u)|^{\frac{p-2}{2}}\nabla u\right).$$

Alternatively, using a method similar to that introduced above, one could generalise by considering weak solutions of a suitable PDE problem in the following way

$$\int_U |\nabla u|^{p-2} B_x(u,v) \, dx = \int_U f v \, dx.$$

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Data availability

No data was used for the research described in the article.

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