$(p,2)$-EQUATIONS WITH A CROSSING NONLINEARITY AND CONCAVE TERMS

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Abstract. We consider a parametric Dirichlet problem driven by the sum of a p-Laplacian ($p > 2$) and a Laplacian (a $(p,2)$-equation). The reaction consists of an asymmetric $(p-1)$-linear term which is resonant as $x \to -\infty$, plus a concave term. However, in this case the concave term enters with a negative sign. Using variational tools together with suitable truncation techniques and Morse theory (critical groups), we show that when the parameter is small the problem has at least three nontrivial smooth solutions.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we deal with the following nonlinear Dirichlet problem

$$(P_\mu) \qquad \begin{cases} -\Delta_p u(z) - \Delta u(z) = f(z,u(z)) - \mu |u(z)|^{q-2}u(z) & \text{in } \Omega, \\ u\Big|_{\partial\Omega} = 0. \end{cases}$$

In this problem $1 < q < 2 < p < +\infty$ and for $r \in (1,+\infty)$, Δ_r denotes the r-Laplace differential operator defined by

$$\Delta_r u = \operatorname{div}\,(|\nabla u|^{r-2}\nabla u) \quad \text{ for all } u \in W_0^{1,r}(\Omega).$$

When $r = 2$, then $\Delta_2 = \Delta$ is the usual Laplacian. In the reaction term the nonlinearity $f(z,x)$ is an $\mathbb{R}$-valued function defined on $\Omega \times \mathbb{R}$, which is jointly measurable and for a.a. $z \in \Omega$ $x \to f(z,x)$ is a C^1-function. We assume that $f(z,\cdot)$ exhibits $(p-1)$-linear growth near $\pm\infty$ but it has asymmetric behavior as $x \to \pm\infty$. Also, $\mu > 0$ is a parameter and since $1 < q < 2$, $-\mu|u|^{q-2}u$ is a concave contribution

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in the reaction. However note that in our problem the concave component of the reaction enters with a negative sign.

The starting point of our work here, is the paper of Papageorgiou-Winkert [15] where problem (P_μ) was studied under the assumption that the quotient $\frac{f(z,x)}{|x|^{p-2}x}$ has the same asymptotic behavior as $x \to \pm\infty$ and stays below $\widehat{\lambda}_1(p) > 0$ the principal eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$. This way in [25] the energy functional of the problem is coercive and so the direct method of the calculus of variations can be used. In contrast here $\frac{f(z,x)}{|x|^{p-2}x}$ has different asymptotic behavior as $x \to \pm\infty$. So, in the negative direction (that is, as $x \to -\infty$), the quotient stays below $\widehat{\lambda}_1(p) > 0$, while in the positive direction (that is, as $x \to +\infty$), it remains above $\widehat{\lambda}_1(p) > 0$ (crossing or jumping nonlinearity). Note that no use of the Fučik spectrum is made and in fact we do not assume that the limits $\lim_{x\to\pm\infty} \frac{f(z,x)}{|x|^{p-2}x}$ exist.

Elliptic problems with concave nonlinearities (problems with competition phenomena), were first investigated by Ambrosetti-Brezis-Cerami [3], who considered semilinear problems driven by the Dirichlet Laplacian and a parametric reaction of the form

$$f_\mu(x) = \mu|x|^{q-2}x + |x|^{r-2}x$$

with $1 < q < 2 < r \leq 2^* = \begin{cases} \frac{2N}{N-2} & \text{if } N \geq 3 \\ +\infty & \text{if } N = 1,2 \end{cases}$ and $\mu > 0$.

So, in the equation of [3], the concave term enters with a positive sign and in the reaction of the problem we have the competing effects of concave (sublinear) and convex (superlinear) terms. Extensions of the work of Ambrosetti-Brezis-Cerami [3] can be found in the papers of Garcia Azorero-Peral Alonso-Manfredi [7], Guo-Zhang [11], Marano-Papageorgiou [16], Papageorgiou-Rădulescu [22, 23]. Equations in which the concave term enters with a negative sign were investigated by de Paiva-Massa [18], de Paiva-Presoto [19], Perera [26].

We prove a multiplicity theorem producing three nontrivial smooth solutions when the parameter $\mu > 0$ is small. Our approach combines variational methods together with truncation and comparison techniques and Morse theory (critical groups). In the next section, for easy reference, we recall the main mathematical tools which we will use in this paper.

2. Mathematical Background - Hypotheses

Let X be a Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). Given $\varphi \in C^1(X, \mathbb{R})$, we say that φ satisfies the "Cerami condition" (the "C-condition" for short), if the following holds:

"Every sequence $\{u_n\}_{n\geq 1} \subseteq X$ such that

$$\{\varphi(u_n)\}_{n\geq 1} \subseteq \mathbb{R} \text{ is bounded,}$$
$$(1 + \|u_n\|)\varphi'(u_n) \to 0 \text{ in } X^* \text{ as } n \to +\infty,$$

admits a strongly convergent subsequence".

This compactness-type condition on the functional φ is crucial in developing the minimax theory for the critical values of φ. One of the main results in this theory is the so-called "mountain pass theorem" of Ambrosetti-Rabinowitz [4], which we recall here.

Theorem 1. *If X is a Banach space, $\varphi \in C^1(X,\mathbb{R})$ satisfies the C-condition, $u_0, u_1 \in X$, $\|u_1-u_0\|>\rho>0$,*

$$\max\{\varphi(u_0),\varphi(u_1)\}<\inf\{\varphi(u) : \|u-u_0\|=\rho\}=m_\rho,$$

$$\text{and } c=\inf_{\gamma\in\Gamma}\max_{0\le t\le 1}\varphi(\gamma(t)) \text{ with } \Gamma=\{\gamma\in C([0,1],X) : \gamma(0)=u_0, \gamma(1)=u_1\},$$

then $c\ge m_\rho$ and c is a critical value of φ, that is there exists $\widehat{u}\in X$ such that $\varphi'(\widehat{u})=0$ and $\varphi(\widehat{u})=c$.

Two are the main spaces in the study of problem (P_μ). The Sobolev space $W_0^{1,p}(\Omega)$ and the Banach space $C_0^1(\overline{\Omega})=\left\{u\in C^1(\overline{\Omega}) : u\Big|_{\partial\Omega}=0\right\}$. By $\|\cdot\|$ we denote the norm of the Sobolev space $W_0^{1,p}(\Omega)$. On account of the Poincaré inequality, we can have

$$\|u\|=\|\nabla u\|_p \text{ for all } u\in W_0^{1,p}(\Omega).$$

The Banach space $C_0^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone

$$C_+=\left\{u\in C_0^1(\overline{\Omega}) : u(z)\ge 0 \text{ for all } z\in\overline{\Omega}\right\}.$$

This cone has a nonempty interior given by

$$\text{int } C_+=\left\{u\in C_+ : u(z)>0 \text{ for all } z\in\Omega, \quad \frac{\partial u}{\partial n}\Big|_{\partial\Omega}<0\right\}.$$

Here $\frac{\partial u}{\partial n}$ denotes the usual normal derivative of $u(\cdot)$, with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. The space $C_0^1(\overline{\Omega})$ is dense in the Sobolev spaces $W_0^{1,p}(\Omega)$ and $H_0^1(\Omega)$.

Consider the following nonlinear eigenvalue problem

$$-\Delta_p u(z)=\widehat{\lambda}|u(z)|^{p-2}u(z) \quad \text{in } \Omega, \quad u\Big|_{\partial\Omega}=0 \quad (1<p<+\infty). \tag{1}$$

A number $\widehat{\lambda}\in\mathbb{R}$ is an "eigenvalue", if problem (1) admits a nontrivial solution $\widehat{u}\in W_0^{1,p}(\Omega)$, known as an "eigenfunction" corresponding to $\widehat{\lambda}$. There is a smallest eigenvalue $\widehat{\lambda}_1(p)$ which has the following properties:

- $\widehat{\lambda}_1(p)>0$ and is isolated (that is, if $\widehat{\sigma}(p)$ denotes the spectrum of (1), then there exists $\varepsilon>0$ such that $(\widehat{\lambda}_1(p),\widehat{\lambda}_1(p)+\varepsilon)\cap\widehat{\sigma}(p)=\emptyset$).
- $\widehat{\lambda}_1(p)$ is simple (that is, if $\widehat{u}$, $\widehat{v}$ are eigenfunctions corresponding to $\widehat{\lambda}_1(p)$, then $\widehat{u}=\xi\widehat{v}$ for some $\xi\neq 0$).
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$$\widehat{\lambda}_1(p)=\inf\left[\frac{\|\nabla u\|_p^p}{\|u\|_p^p} : u\in W_0^{1,p}(\Omega),\, u\neq 0\right]. \tag{2}$$

In (2) the infimum is realized on the one-dimensional eigenspace corresponding to $\widehat{\lambda}_1(p)$. From the above properties it is clear that the elements of this eigenspace do not change sign. By $\widehat{u}_1(p)$ we denote the positive, L^p-normalized (that is, $\|\widehat{u}_1(p)\|_p=1$) eigenfunction corresponding to $\widehat{\lambda}_1(p)$. From the nonlinear regularity theory and the nonlinear maximum principle (see, for example, Gasiński-Papageorgiou [8], pp. 737-738) we have $\widehat{u}_1(p)\in\text{int } C_+$. It is easily seen that the spectrum $\widehat{\sigma}(p) \subseteq W_0^{1,p}(\Omega)$ is closed. Then since $\widehat{\lambda}_1(p)$ is isolated, the second eigenvalue $\widehat{\lambda}_2(p) > \widehat{\lambda}_1(p)$ is well-defined by

$$\widehat{\lambda}_2(p) = \inf \left[\widehat{\lambda} \in \widehat{\sigma}(p) : \widehat{\lambda} > \widehat{\lambda}_1(p) \right].$$

We mention that every eigenfunction corresponding to an eigenvalue $\widehat{\lambda} \neq \widehat{\lambda}_1(p)$ is automatically nodal (that is, sign changing).

We can have a weighted version of problem (1). So, let $m \in L^\infty(\Omega)$, $m(z) \geq 0$ a.e. on Ω, $m \not\equiv 0$. We consider the following nonlinear eigenvalue problem:

$$(3) \qquad -\Delta_p u(z) = \widetilde{\lambda} m(z)|u(z)|^{p-2}u(z) \quad \text{in } \Omega, \quad u\Big|_{\partial\Omega} = 0 \quad (1 < p < +\infty).$$

Again we have a smallest eigenvalue $\widetilde{\lambda}_1(p, m) > 0$ which is isolated and simple. Moreover, it admits the following variational characterization

$$(4) \qquad \widetilde{\lambda}_1(p, m) = \inf \left[\frac{\|\nabla u\|_p^p}{\int_\Omega m(z)|u|^p dz} : u \in W_0^{1,p}(\Omega),\, u \neq 0 \right].$$

The infimum in (4) is realized on the corresponding one-dimensional eigenspace, the elements of which do not change sign. These facts lead to the following monotonicity property for the maps $m \to \widetilde{\lambda}_1(p, m)$, $m \to \widetilde{\lambda}_2(p, m)$ (see [17]).

Proposition 1. *If $m_1, m_2 \in L^\infty(\Omega)$, $0 \leq m_1(z) \leq m_2(z)$ for a.a. $z \in \Omega$, $m_1 \not\equiv 0$, $m_2 \not\equiv m_1$, then $\widetilde{\lambda}_1(p, m_2) < \widetilde{\lambda}_1(p, m_1)$; also if $m_1(z) < m_2(z)$ for a.a. $z \in \Omega$, then $\widehat{\lambda}_2(p, m_2) < \widehat{\lambda}_2(p, m_1)$.*

Another useful consequence of the properties of $\widehat{\lambda}_1(p) > 0$, is given in the next proposition (see Motreanu-Motreanu-Papageorgiou [17], Lemma 11.3, p. 305).

Proposition 2. *If $\theta_0 \in L^\infty(\Omega)$, $\theta_0(z) \leq \widehat{\lambda}_1(p)$ for a.a. $z \in \Omega$, $\theta_0 \not\equiv \widehat{\lambda}_1(p)$, then there exists $c_0 > 0$ such that*

$$\|\nabla u\|_p^p - \int_\Omega \theta_0(z)|u|^p dz \geq c_0 \|u\|^p \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

We mention that when $p \neq 2$ we do not have full knowledge of the spectrum $\widehat{\sigma}(p)$. When $p = 2$, the eigenvalue problem is linear and then the spectral theorem for compact, self-adjoint operators on a Hilbert space gives a full description of the spectrum $\widehat{\sigma}(2)$ which consists of a strictly increasing sequence $\{\widehat{\lambda}_k(2)\}_{k\geq 1}$ of distinct eigenvalues such that $\widehat{\lambda}_k(2) \to +\infty$ as $k \to +\infty$. Every eigenvalue $\widehat{\lambda}_k(2)$, $k \in \mathbb{N}$, has an eigenspace $E(\widehat{\lambda}_k(2))$, which is a finite dimensional subspace of $H_0^1(\Omega)$. In fact standard regularity theory implies that $E(\widehat{\lambda}_k(2)) \subseteq C_0^1(\overline{\Omega})$ for all $k \in \mathbb{N}$. Also, we have the following orthogonal direct sum decompositon

$$H_0^1(\Omega) = \overline{\oplus_{k\geq 1} E(\widehat{\lambda}_k(2))}.$$

Each eigenspace $E(\widehat{\lambda}_k(2))$ has the "Unique Continuation Property" (the "UCP" for short). This means that if $u \in E(\widehat{\lambda}_k(2))$ and vanishes on a set of positive measure, then $u \equiv 0$.

We have variational characterizations for all the eigenvalues.

$$\widehat{\lambda}_1(2) = \inf\left[\frac{\|\nabla u\|_2^2}{\|u\|_2^2} : u \in H_0^1(\Omega),\ u \neq 0\right], \tag{5}$$

$$\widehat{\lambda}_k(2) = \inf\left[\frac{\|\nabla u\|_2^2}{\|u\|_2^2} : u \in \widehat{H}_k = \overline{\oplus_{i\geq k} E(\widehat{\lambda}_i(2))}\right]$$

$$= \sup\left[\frac{\|\nabla u\|_2^2}{\|u\|_2^2} : u \in \overline{H}_k = \oplus_{i=1}^k E(\lambda_i(2))\right], \quad k \geq 2. \tag{6}$$

Again the infimum in (5) is realized on $E(\widehat{\lambda}_1(2))$. Also, both the infimum and the supremum in (6) are realized on $E(\widehat{\lambda}_k(2))$.

Using the UCP of the eigenspaces, we can have the following useful inequalities (see D'Aguì-Marano-Papageorgiou [6]).

Proposition 3. *We have:*

(a) *If $\eta \in L^\infty(\Omega)$ and $\eta(z) \leq \widehat{\lambda}_k(2)$ for a.a. $z \in \Omega$, $\eta \not\equiv \widehat{\lambda}_k(2)$, $k \in \mathbb{N}$, then there exists $c_1 > 0$ such that*
$$\|\nabla u\|_2^2 - \int_\Omega \eta(z) u^2 dz \geq c_1\|u\|^2 \quad \textit{for all } u \in \widehat{H}_k.$$

(b) *If $\eta \in L^\infty(\Omega)$ and $\eta(z) \geq \widehat{\lambda}_k(2)$ for a.a. $z \in \Omega$, $\eta \not\equiv \widehat{\lambda}_k(2)$, $k \in \mathbb{N}$, then there exists $c_2 > 0$ such that*
$$\|\nabla u\|_2^2 - \int_\Omega \eta(z) u^2 dz \leq -c_2\|u\|^2 \quad \textit{for all } u \in \overline{H}_k.$$

Consider the nonlinear operator $A_p : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)^* = W^{-1,p'}(\Omega)$, with $\frac{1}{p} + \frac{1}{p'} = 1$, defined by
$$\langle A_p(u), h\rangle = \int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla h)_{\mathbb{R}^N} dz \quad \text{for all } u, h \in W_0^{1,p}(\Omega).$$

This map has the following properties (see Motreanu-Motreanu-Papageorgiou [17], p. 40).

Proposition 4. *The map $A_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is bounded (that is, it maps bounded sets to bounded sets), continuous, strictly monotone (thus, maximal monotone too) and of type $(S)_+$ (that is, $u_n \xrightarrow{w} u$ in $W_0^{1,p}(\Omega)$ and $\limsup_{n\to+\infty}\langle A(u_n), u_n - u\rangle \leq 0 \Rightarrow u_n \to u$ in $W_0^{1,p}(\Omega)$).*

Also $A \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$ is the operator defined by
$$\langle A(u), h\rangle = \int_\Omega (\nabla u, \nabla h)_{\mathbb{R}^N} dz \quad \text{for all } u, h \in H_0^1(\Omega).$$

Consider a Carathéodory function $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ such that
$$|f_0(z,x)| \leq a_0(z)[1 + |x|^{r-1}] \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R},$$
with $1 < r \leq p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } p \geq N \end{cases}$ (the critical Sobolev exponent). We set $F_0(z,x) = \int_0^x f_0(z,s)ds$ and consider the C^1-functional $\varphi_0 : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by
$$\varphi_0(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega F_0(z,u)dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

The next result is an outgrowth of the nonlinear regularity theory of Lieberman [14] and can be found in Motreanu-Motreanu-Papageorgiou [17] (p. 409) and in Papageorgiou-Rădulescu [24].

Proposition 5. *If $u_0 \in W_0^{1,p}(\Omega)$ is a local $C_0^1(\overline{\Omega})$-minimizer of φ_0, that is, there exists $\rho_0 > 0$ such that*

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \quad \text{for all } h \in C_0^1(\overline{\Omega}),\ \|h\|_{C_0^1(\overline{\Omega})} \leq \rho_0,$$

then $u_0 \in C_0^{1,\eta}(\overline{\Omega})$ for some $\eta \in (0,1)$ and it is also a local $W_0^{1,p}(\Omega)$-minimizer of φ_0, that is, there exists $\rho_1 > 0$ such that

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \quad \text{for all } h \in W_0^{1,p}(\Omega),\ \|h\| \leq \rho_1.$$

Next let us recall some basic facts about critical groups. So, let X be a Banach space, $\varphi \in C^1(X, \mathbb{R})$ and $c \in \mathbb{R}$. We introduce the following sets:

$$\begin{aligned} &\varphi^c = \{u \in X : \varphi(u) \leq c\}, \\ &K_\varphi = \{u \in X : \varphi'(u) = 0\} \quad \text{(the critical set of } \varphi), \\ &K_\varphi^c = \{u \in K_\varphi : \varphi(u) = c\} \quad \text{(the critical set of } \varphi \text{ at the level } c). \end{aligned}$$

Let (Y_1, Y_2) be a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$ and let $k \in \mathbb{N}_0$. By $H_k(Y_1, Y_2)$ we denote the k^{th} relative singular homology group for (Y_1, Y_2) with integer coefficients. For any isolated $u \in K_\varphi^c$, the critical groups of φ at u are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}) \quad \text{for all } k \in \mathbb{N}_0,$$

where U is a neighborhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of singular homology guarantees that the above definition of critical groups is independent of the choice of the isolating neighborhood U.

Assume that $\varphi \in C^1(X, \mathbb{R})$ satisfies the C-condition and $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. Then the critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all } k \in \mathbb{N}_0.$$

The second deformation theorem (see Gasiński-Papageorgiou [8], p. 628) guarantees that this definition is independent of the choice of the level $c < \inf \varphi(K_\varphi)$. Indeed, if $c' < c < \inf \varphi(K_\varphi)$, then by the second deformation theorem $\varphi^{c'}$ is a strong deformation retract of φ^c, hence

$$H_k(X, \varphi^c) = H_k(X, \varphi^{c'}) \quad \text{for all } k \in \mathbb{N}_0$$

(see Motreanu-Motreanu-Papageorgiou [17], p. 145).

Consider $\varphi \in C^1(X, \mathbb{R})$ and assume that φ satisfies the C-condition and that K_φ is finite. We introduce the following items

$$\begin{aligned} M(t, u) &= \sum_{k \geq 0} \text{rank } C_k(\varphi, u) t^k \quad \text{for all } t \in \mathbb{R},\ u \in K_\varphi, \\ P(t, \infty) &= \sum_{k \geq 0} \text{rank } C_k(\varphi, \infty) t^k \quad \text{for all } t \in \mathbb{R}. \end{aligned}$$

Then the "Morse relation" says that

$$\sum_{u \in K_\varphi} M(t, u) = P(t, \infty) + (1 + t) Q(t) \quad \text{for all } t \in \mathbb{R}, \tag{7}$$

where $Q(t)=\sum_{k\geq 0}\beta_k t^k$ is a formal series with nonnegative integer coefficients. In what follows by $\delta_{k,l}$ we denote the Kronecker symbol defined by

$$\delta_{k,l}=\begin{cases}1 & \text{if } k=l,\\ 0 & \text{if } k\neq l.\end{cases}$$

If $\varphi\in C^1(X,\mathbb{R})$ and $u\in X$ is an isolated local minimizer of φ, then

$$C_k(\varphi,u)=\delta_{k,0}\mathbb{Z}.$$

If $K_\varphi=\{0\}$, then $C_k(\varphi,\infty)=C_k(\varphi,0)$ for all $k\in\mathbb{N}_0$.

Suppose that $X=H$ is a Hilbert space and $U\subseteq H$ an open set. Let $\varphi\in C^2(U,\mathbb{R})$. For each $u\in U$, $\varphi''(u)$ can be seen as a bilinear form on H and there is a unique $L_u\in\mathcal{L}(H,H)$ such that

$$(L_u(x),h)_H=\varphi''(u)(x,h)\quad\text{for all } x,h\in H.$$

Here by $(\cdot,\cdot)_H$ we denote the inner product of H. The operator L_u is self-adjoint and we identify L_u with $\varphi''(u)$. We say that $u\in K_\varphi$ is "nondegenerate", if $\varphi''(u)$ is invertible. The "Morse index" of $u\in K_\varphi$ is the supremum of the dimensions of the vector subspaces of H on which $\varphi''(u)$ is negative definite. Note that by the inverse function theorem, every nondegenerate critical point is isolated. If $\varphi\in C^2(U,\mathbb{R})$ and $u\in K_\varphi$ is nondegenerate and of Morse index m, then

$$C_k(\varphi,u)=\delta_{k,m}\mathbb{Z}\quad\text{for all } k\in\mathbb{N}_0.$$

For further details on critical groups, we refer to the book of Motreanu-Motreanu-Papageorgiou [17], Chapter 6.

As we already said by $\|\cdot\|$ we denote norm of the Sobolev space $W_0^{1,p}(\Omega)$. Also, by $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair $((W^{-1,p'}(\Omega),W_0^{1,p}(\Omega))$. By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$. For $x\in\mathbb{R}$, we set $x^\pm=\max\{\pm x,0\}$ and then for $u\in W_0^{1,p}(\Omega)$, we define $u^\pm(\cdot)=u(\cdot)^\pm$. We have

$$u^\pm\in W_0^{1,p}(\Omega),\quad u=u^+-u^-,\quad |u|=u^++u^-.$$

Given a measurable function $g:\Omega\times\mathbb{R}\to\mathbb{R}$, we set

$$N_g(u)(\cdot)=g(\cdot,u(\cdot))\quad\text{for all } u\in W_0^{1,p}(\Omega).$$

Since $2<p$, we have $W_0^{1,p}(\Omega)\subseteq H_0^1(\Omega)$.

Now we are ready to introduce the hypotheses on the perturbation $f(z,x)$.

H(f): $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a measurable function such that for a.a. $z\in\Omega$ $f(z,0)=0$, $f(z,\cdot)\in C^1(\mathbb{R})$ and

- (i) $|f'_x(z,x)|\leq a(z)[1+|x|^{r-2}]$ for a.a. $z\in\Omega$, all $x\in\mathbb{R}$, with $p\leq r<p^*$;
- (ii) there exist a function $\eta\in L^\infty(\Omega)$ and constants $\widehat{\eta}<\widehat{\lambda}_2(p)$ and $\widehat{\theta}>0$ such that

$$\widehat{\lambda}_1(p)\leq\eta(z)\quad\text{for a.a. } z\in\Omega,\quad \widehat{\eta}\not\equiv\widehat{\lambda}_1(p),$$

$$\widehat{\theta}\leq\liminf_{x\to-\infty}\frac{f(z,x)}{|x|^{p-2}x}\leq\limsup_{x\to-\infty}\frac{f(z,x)}{|x|^{p-2}x}\leq\widehat{\lambda}_1(p),$$

$$\eta(z)\leq\liminf_{x\to+\infty}\frac{f(z,x)}{x^{p-1}}\leq\limsup_{x\to+\infty}\frac{f(z,x)}{x^{p-1}}\leq\widehat{\eta}\quad\text{uniformly for a.a. } z\in\Omega;$$

- (iii) if $F(z,x)=\int_0^x f(z,s)ds$, then $f(z,x)x-pF(z,x)\to+\infty$ as $x\to-\infty$ uniformly for a.a. $z\in\Omega$ and $f(z,x)x-pF(z,x)\geq 0$ for a.a. $z\in\Omega$, all $x\geq M_0>0$;

(iv) there exist $m \in \mathbb{N}$, $m \geq 2$, $\delta > 0$ and $\zeta_0 \in L^\infty(\Omega)$ such that

$$\widehat{\lambda}_m(2) \leq \zeta_0(z) \quad \text{for a.a. } z \in \Omega, \quad \zeta_0 \not\equiv \widehat{\lambda}_m(2),$$
$$\zeta_0(z)x^2 \leq f(z,x)x \leq \widehat{\lambda}_{m+1}(2)x^2 \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \leq |x| \leq \delta$$

and the second inequality is strict on a set of positive measure.

Remark 1. Clearly $f(z,\cdot)$ exhibits $(p-1)$-linear growth near $\pm\infty$. Hypothesis $H(f)\,(ii)$ implies that the perturbation term $f(z,x)$ is a crossing (jumping) nonlinearity. Indeed as we move from $-\infty$ to $+\infty$ the quotient $\dfrac{f(z,x)}{|x|^{p-2}x}$ crosses at least the principal eigenvalue $\widehat{\lambda}_1(p) > 0$. Note that in the negative direction (that is, as $x \to -\infty$) we can have resonance with respect to $\widehat{\lambda}_1(p)$. In the positive direction (that is, as $x \to +\infty$), we can only have nonuniform nonresonance with respect to $\widehat{\lambda}_1(p)$. Hypothesis $H(f)\,(iv)$ says that near zero the quotient $\dfrac{f(z,x)}{x}$ is within the spectral interval $[\widehat{\lambda}_m(2), \widehat{\lambda}_{m+1}(2)]$ with possible full interaction (resonance) with the right endpoint as $x \to 0$.

For every $\mu > 0$, the energy functional $\varphi_\mu : W_0^{1,p}(\Omega) \to \mathbb{R}$ of problem (P_μ) is defined by

$$\varphi_\mu(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 + \frac{\mu}{q}\|u\|_q^q - \int_\Omega F(z,u)dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Evidently $\varphi_\mu \in C^1(W_0^{1,p}(\Omega),\mathbb{R}) \cap C^2(W_0^{1,p}(\Omega)\setminus\{0\},\mathbb{R})$ (since $q < 2$). Also let $f_-(z,x) = f(z,-x^-)$ for all $(z,x) \in \Omega \times \mathbb{R}$ and $F_-(z,x) = \int_0^x f_-(z,s)ds$. We consider the C^1-functional $\varphi_\mu^- : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_\mu^-(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 + \frac{\mu}{q}\|u^-\|_q^q - \int_\Omega F_-(z,u)dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

3. Properties of the Functionals

In this section we show that the functional φ_μ satisfies the C-condition, while for φ_μ^- we show that it is coercive.

Proposition 6. *If hypotheses $H(f)$ hold and $\mu > 0$, then the functional φ_μ satisfies the C-condition.*

Proof. Consider a sequence $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ such that

$$|\varphi(u_n)| \leq M_1 \quad \text{for some } M_1 > 0, \text{ all } n \in \mathbb{N}, \tag{8}$$

$$(1 + \|u_n\|)\varphi'(u_n) \to 0 \quad \text{in } W^{-1,p'}(\Omega) \text{ as } n \to +\infty. \tag{9}$$

From (9) we have

$$\left|\langle A_p(u_n),h\rangle + \langle A(u_n),h\rangle + \mu\int_\Omega |u_n|^{q-2}u_n h dz - \int_\Omega f(z,u_n)hdz\right| \leq \frac{\varepsilon_n\|h\|}{1+\|u_n\|} \tag{10}$$

for all $h \in W_0^{1,p}(\Omega)$ with $\varepsilon_n \to 0^+$.

In (10) we choose $h = u_n \in W_0^{1,p}(\Omega)$. We obtain

$$-\|\nabla u_n\|_p^p - \|\nabla u_n\|_2^2 - \mu\|u_n\|_q^q + \int_\Omega f(z,u_n)u_n dz \leq \varepsilon_n \quad \text{for all } n \in \mathbb{N}. \tag{11}$$

From (8) we have

$$\|\nabla u_n\|_p^p + \frac{p}{2}\|\nabla u_n\|_2^2 + \frac{\mu p}{q}\|u_n\|_q^q - \int_\Omega pF(z, u_n)dz \leq pM_1 \quad \text{for all } n \in \mathbb{N}. \tag{12}$$

Adding (11) and (12) and using the fact that $q < 2 < p$, we obtain

$$\int_\Omega [f(z, u_n)u_n - pF(z, u_n)]dz \leq M_2 \text{ for some } M_2 > 0, \text{ all } n \in \mathbb{N},$$

$$\Rightarrow \int_\Omega [f(z, -u_n^-)(-u_n^-) - pF(z, -u_n^-)]dz \leq M_3 \text{ for some } M_3 > 0, \text{ all } n \in \mathbb{N} \tag{13}$$

(see hypotheses $H(f)\,(i), (iii)$).

Claim: $\{u_n^-\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ is bounded.

Arguing by contradiction, suppose that at least for a subsequence we have

$$\|u_n^-\| \to +\infty \quad \text{as } n \to +\infty. \tag{14}$$

Let $y_n = \frac{u_n^-}{\|u_n^-\|}$, $n \in \mathbb{N}$. Then $\|y_n\| = 1$, $y_n \geq 0$ for all $n \in \mathbb{N}$. We may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega) \text{ and } y_n \to y \text{ in } L^p(\Omega) \text{ with } y \geq 0. \tag{15}$$

In (10) we choose $h = -u_n^- \in W_0^{1,p}(\Omega)$. Then

$$\|\nabla u_n^-\|_p^p + \|\nabla u_n^-\|_2^2 + \mu\|u_n^-\|_q^q - \int_\Omega f(z, -u_n^-)(-u_n^-)dz \leq \varepsilon_n \quad \text{for all } n \in \mathbb{N}.$$

Multiplying the above equality with $\frac{1}{\|u_n^-\|^p}$, we obtain

$$\|\nabla y_n\|_p^p + \frac{1}{\|u_n^-\|^{p-2}}\|\nabla y_n\|_2^2 + \frac{\mu}{\|u_n^-\|^{p-q}}\|y_n\|_q^q - \int_\Omega \frac{N_f(-u_n^-)}{\|u_n^-\|^{p-1}}y_n dz \leq \frac{\varepsilon_n}{\|u_n^-\|^p} \tag{16}$$

for all $n \in \mathbb{N}$. Hypotheses $H(f)\,(i), (ii)$ imply that

$$|f(z, x)| \leq c_3(1 + |x|^{p-1}) \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ some } c_3 > 0, \tag{17}$$

$$\Rightarrow \left\{\frac{N_f(-u_n^-)}{\|u_n^-\|^p}\right\}_{n\geq 1} \subseteq L^{p'}(\Omega) \quad \text{is bounded.}$$

Then by passing to a suitable subsequence if necessary and using hypothesis $H(f)\,(ii)$, we obtain

$$\frac{N_f(-u_n^-)}{\|u_n^-\|^{p-1}} \xrightarrow{w} \theta_0(z)y^{p-1} \quad \text{in } L^{p'}(\Omega) \text{ as } n \to +\infty, \tag{18}$$

with $-\widehat{\theta} \leq \theta_0(z) \leq \widehat{\lambda}_1(p)$ for a.a. $z \in \Omega$ (see Aizicovici-Papageorgiou-Staicu [1], proof of Proposition 16). In (16) we pass to the limit as $n \to +\infty$ and use (14), (15), (18) and the fact that $q < 2 < p$. Then

$$\|\nabla y\|_p^p \leq \int_\Omega \theta_0(z)|y|^p dz. \tag{19}$$

First we assume that $\theta_0 \not\equiv \widehat{\lambda}_1(p)$ (see (18)). Then from (19) and Proposition 2 we have

$$c_0\|y\|_p^p \leq 0,$$

$$\Rightarrow \quad y = 0.$$

From (16) it follows that

$$\begin{aligned}&\|\nabla y_n\|_p \to 0 \quad \text{(see (14) and recall that } q<2<p),\\ \Rightarrow\ & y_n \to 0 \quad \text{in } W_0^{1,p}(\Omega).\end{aligned}$$

But this contradicts the fact that $\|y_n\| = 1$ for all $n \in \mathbb{N}$.

Next we assume that $\theta_0(z) = \widehat{\lambda}_1(p)$ for a.a. $z \in \Omega$. From (19) and (2) we have

$$\begin{aligned}&\|\nabla y\|_p^p = \widehat{\lambda}_1(p)\|y\|_p^p,\\ \Rightarrow\ & y = \xi \widehat{u}_1(p) \quad \text{for some } \xi \geq 0 \text{ (recall } y \geq 0).\end{aligned}$$

If $\xi = 0$, then $y \equiv 0$ and as above we reach a contradiction to the fact that $\|y_n\| = 1$ for all $n \in \mathbb{N}$.

If $\xi > 0$, then $y(z) > 0$ for all $z \in \Omega$ (recall $\widehat{u}_1(p) \in \operatorname{int} C_+$). Hence

$$\begin{aligned}&u_n^-(z) \to +\infty \quad \text{for a.a. } z \in \Omega,\\ \Rightarrow\ & f(z,-u_n^-(z))(-u_n^-(z)) - pF(z,-u_n^-(z)) \to +\infty \quad \text{for a.a. } z \in \Omega\\ &\qquad\qquad \text{(see hypothesis } H(f)\,(iii)),\\ \Rightarrow\ & \int_\Omega [f(z,-u_n^-)(-u_n^-) - pF(z,-u_n^-)]dz \to +\infty \quad \text{(by Fatou's lemma).} \qquad (20)\end{aligned}$$

Comparing (13) and (20), we have a contradiction. This proves the Claim.

Now we will use the Claim to show that $\{u_n^+\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ is bounded.

Again we proceed indirectly. So, suppose that at least for a subsequence we have

$$\|u_n^+\| \to +\infty \quad \text{as } n \to +\infty. \qquad (21)$$

We set $v_n = \dfrac{u_n^+}{\|u_n^+\|}$, $n \in \mathbb{N}$. Then $\|v_n\| = 1$, $v_n \geq 0$ for all $n \in \mathbb{N}$. We may assume that

$$v_n \xrightarrow{w} v \text{ in } W_0^{1,p}(\Omega) \text{ and } v_n \to v \text{ in } L^p(\Omega),\ v \geq 0. \qquad (22)$$

From (10) and the Claim we obtain

$$\left| \langle A_p(u_n^+), h\rangle + \langle A(u_n^+), h\rangle + \mu \int_\Omega (u_n^+)^{q-1} h dz - \int_\Omega f(z, u_n^+) h dz \right| \leq \varepsilon_n' \|h\| \qquad (23)$$

for all $h \in W_0^{1,p}(\Omega)$ with $\varepsilon_n' \to 0^+$.

We multiply (23) with $\dfrac{1}{\|u_n^+\|^{p-1}}$. Then

$$\begin{aligned}&\Big| \langle A_p(v_n), h\rangle + \frac{1}{\|u_n^+\|^{p-2}} \langle A(v_n), h\rangle + \frac{\mu}{\|u_n^+\|^{p-q}} \int_\Omega v_n^{q-1} h dz\\ &- \int_\Omega \frac{N_f(u_n^+)}{\|u_n^+\|^{p-1}} h dz \Big| \leq \frac{\varepsilon_n' \|h\|}{\|u_n^+\|^{p-1}} \quad \text{for all } n \in \mathbb{N}. \qquad (24)\end{aligned}$$

It is clear from (17) that

$$\left\{ \frac{N_f(u_n^+)}{\|u_n^+\|^{p-1}} \right\}_{n\geq 1} \subseteq L^{p'}(\Omega) \quad \text{is bounded.} \qquad (25)$$

In (24) we choose $h = v_n - v \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (21), (22), (25) and the fact that $q < 2 < p$. We obtain

$$\lim_{n\to+\infty} \langle A_p(v_n), v_n - v\rangle = 0,$$
$$\Rightarrow \quad v_n \to v \text{ in } W_0^{1,p}(\Omega),\ \|v\| = 1,\ v \geq 0 \text{ (see Proposition 4).} \tag{26}$$

From (25) and hypothesis $H(f)\,(ii)$ we see that at least for a subsequence we have

$$\frac{N_f(u_n^+)}{\|u_n^+\|^{p-1}} \xrightarrow{w} \eta_0(z)v^{p-1} \quad \text{in } L^{p'}(\Omega) \text{ as } n \to +\infty \tag{27}$$

with $\eta(z) \leq \eta_0(z) \leq \widehat{\eta}$ for a.a. $z \in \Omega$. So, if in (24) we pass to the limit as $n \to +\infty$ and use (21), (26), (27) and the fact that $q < 2 < p$, then

$$\langle A_p(v), h\rangle = \int_\Omega \eta_0(z)v^{p-1}h dz \quad \text{for all } h \in W_0^{1,p}(\Omega),$$
$$\Rightarrow \quad -\Delta_p v(z) = \eta_0(z)v(z)^{p-1} \quad \text{for a.a. } z \in \Omega, \quad v\big|_{\partial\Omega} = 0,\ v \neq 0. \tag{28}$$

Using Proposition 1, we have

$$\widetilde{\lambda}_1(p, \eta_0) < \widetilde{\lambda}_1(p, \widehat{\lambda}_1(p)) = 1$$

(see (27) and hypothesis $H(f)\,(ii)$. Then from (28) we infer that v must be nodal, which contradicts (26). This proves that

$$\{u_n^+\}_{n\geq1} \subseteq W_0^{1,p}(\Omega) \quad \text{is bounded,}$$
$$\Rightarrow \quad \{u_n\}_{n\geq1} \subseteq W_0^{1,p}(\Omega) \quad \text{is bounded (see the Claim).}$$

So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega) \text{ and } u_n \to u \text{ in } L^p(\Omega). \tag{29}$$

From (17) it follows that

$$\{N_f(u_n)\}_{n\geq1} \subseteq L^{p'}(\Omega) \quad \text{is bounded.} \tag{30}$$

If in (10) we choose $h = u_n - u \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (29), (30), then

$$\lim_{n\to+\infty} [\langle A_p(u_n), u_n - u\rangle + \langle A(u_n), u_n - u\rangle] = 0,$$
$$\Rightarrow \quad \limsup_{n\to+\infty} [\langle A_p(u_n), u_n - u\rangle + \langle A(u_n), u_n - u\rangle] \leq 0 \quad \text{(since } A \text{ is monotone),}$$
$$\Rightarrow \quad \limsup_{n\to+\infty} \langle A_p(u_n), u_n - u\rangle \leq 0,$$
$$\Rightarrow \quad u_n \to u \quad \text{in } W_0^{1,p}(\Omega) \text{ (see Proposition 4),}$$
$$\Rightarrow \quad \varphi_\mu \quad \text{satisfies the } C\text{-condition.}$$

□

For the functional φ_μ^- we show that it is coercive. We will need the next lemma which shows that the resonance with respect to $\widehat{\lambda}_1(p) > 0$ at $-\infty$, is from the left of the eigenvalue.

Lemma 1. *$\widehat{\lambda}_1(p)\,|v|^p - pF(z,v) \to +\infty$ as $v \to -\infty$ uniformly for a.a. $z \in \Omega$.*

Proof. Hypothesis $H(f)\,(iii)$ implies that given any $\beta > 0$, we can find $M_4 = M_4(\beta) > 0$ such that

$$0 < \beta \leq f(z,x)x - pF(z,x) \quad \text{for a.a. } z \in \Omega, \text{ all } x \leq -M_4. \tag{31}$$

We have

$$\begin{aligned}\frac{d}{dx}\left[\frac{F(z,x)}{|x|^p}\right] &= \frac{f(z,x)|x|^p - p|x|^{p-2}xF(z,x)}{|x|^{2p}} \\ &= \frac{f(z,x)x - pF(z,x)}{|x|^p x} \\ &\leq \frac{\beta}{|x|^p x} \quad \text{for a.a. } z \in \Omega, \text{ all } x \leq -M_4 \text{ (see (31))},\end{aligned}$$

$$\Rightarrow \frac{F(z,w)}{|w|^p} - \frac{F(z,v)}{|v|^p} \geq \frac{\beta}{p}\left[\frac{1}{|v|^p} - \frac{1}{|w|^p}\right] \text{for a.a. } z \in \Omega, \text{ all } w \leq v \leq -M_4. \tag{32}$$

Hypothesis $H(f)\,(ii)$ implies that

$$-\frac{\widehat{\theta}}{p} \leq \liminf_{x\to-\infty} \frac{F(z,x)}{|x|^p} \leq \limsup_{x\to-\infty} \frac{F(z,x)}{|x|^p} \leq \frac{\widehat{\lambda}_1(p)}{p} \quad \text{uniformly for a.a. } z \in \Omega. \tag{33}$$

So, if in (32) we let $w \to -\infty$ and use (33), then

$$\widehat{\lambda}_1(p)|v|^p - pF(z,v) \geq \beta \quad \text{for a.a. } z \in \Omega, \text{ all } v \leq -M_4.$$

Since $\beta > 0$ is arbitrary, we conclude that

$$\widehat{\lambda}_1(p)|v|^p - pF(z,v) \to +\infty \quad \text{as } v \to -\infty \text{ uniformly for a.a. } z \in \Omega.$$

□

Using this Lemma, we can prove the coercivity of φ_μ^-.

Proposition 7. *If hypotheses $H(f)$ hold and $\mu > 0$, then the functional φ_μ^- is coercive.*

Proof. Suppose that φ_μ^- is not coercive. We can find $\{u_n\}_{n\geq1} \subseteq W_0^{1,p}(\Omega)$ and $M_5 > 0$ such that

$$\|u_n\| \to +\infty \text{ and } \varphi_\mu^-(u_n) \leq M_5 \text{ for all } n \in \mathbb{N}. \tag{34}$$

Let $y_n = \dfrac{u_n}{\|u_n\|}$, $n \in \mathbb{N}$. Then $\|y_n\| = 1$ for all $n \in \mathbb{N}$ and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega) \text{ and } y_n \to y \text{ in } L^p(\Omega) \text{ as } n \to +\infty. \tag{35}$$

From (34) we have

$$\begin{aligned}&\frac{1}{p}\|\nabla u_n\|_p^p + \frac{1}{2}\|\nabla u_n\|_2^2 + \frac{\mu}{q}\|u_n^-\|_q^q - \int_\Omega F_-(z,u_n)dz \leq M_5 \quad \text{for all } n \in \mathbb{N}, \\ &\Rightarrow \frac{1}{p}\|\nabla y_n\|_p^p + \frac{1}{2\|u_n\|^{p-2}}\|\nabla y_n\|_2^2 + \frac{\mu}{q\|u_n\|^{p-q}}\|y_n^-\|_q^q - \int_\Omega \frac{N_F(-u_n^-)}{\|u_n\|^p}dz \leq \frac{M_5}{\|u_n\|^p}\end{aligned} \tag{36}$$

for all $n \in \mathbb{N}$. From (17) it follows that

$$\begin{aligned}&|F(z,x)| \leq c_4(1+|x|^p) \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ some } c_4 > 0, \\ \Rightarrow \ &\left\{\frac{N_F(-u_n^-)}{\|u_n\|^p}\right\}_{n\geq1} \subseteq L^1(\Omega) \quad \text{is uniformly integrable.}\end{aligned}$$

The Dunford-Pettis theorem and (33) imply that

$$\frac{N_F(-u_n^-)}{\|u_n\|^p} \xrightarrow{w} \frac{1}{p}\theta_0(z)(y^-)^p \text{ in } L^1(\Omega) \tag{37}$$

with $-\widehat{\theta} \leq \theta_0(z) \leq \widehat{\lambda}_1(p)$ for a.a. $z \in \Omega$. In (36) we pass to the limit as $n \to +\infty$ and use (34), (35), (37) and the fact that $q < 2 < p$. We obtain

$$\begin{aligned} & \|\nabla y\|_p^p \leq \int_\Omega \theta_0(z)(y^-)^p dz, \\ \Rightarrow \quad & \|\nabla y^-\|_p^p \leq \int_\Omega \theta_0(z)(y^-)^p dz. \end{aligned} \tag{38}$$

First we assume that $\theta_0 \not\equiv \widehat{\lambda}_1(p)$ (see (37)). Then from (38) and Proposition 2, we have

$$\begin{aligned} & c_0\|y^-\|_p^p \leq 0, \\ \Rightarrow \quad & y \geq 0. \end{aligned} \tag{39}$$

From (35) and (39) it follows that

$$y_n^- \xrightarrow{w} 0 \text{ in } W_0^{1,p}(\Omega) \text{ and } y_n^- \to 0 \text{ in } L^p(\Omega). \tag{40}$$

Then from (36), (37) and (40), we infer that

$$\begin{aligned} & \|\nabla y_n^+\|_p \to 0, \\ \Rightarrow \quad & y_n^+ \to 0 \quad \text{in } W_0^{1,p}(\Omega), \\ \Rightarrow \quad & y_n \to 0 \quad \text{in } W_0^{1,p}(\Omega) \text{ (see (36) and (40))}. \end{aligned}$$

But this contradicts the fact that $\|y_n\| = 1$ for all $n \in \mathbb{N}$.

Now suppose that $\theta_0(z) = \widehat{\lambda}_1(p)$ for a.a. $z \in \Omega$. From (38) and (2), we have

$$\begin{aligned} & \|\nabla y^-\|_p^p = \widehat{\lambda}_1(p)\|y^-\|_p^p, \\ \Rightarrow \quad & y^- = \xi\widehat{u}_1(p) \quad \text{for some } \xi \geq 0. \end{aligned}$$

If $\xi = 0$, then $y = 0$ and reasoning as above we have

$$y_n \to 0 \text{ in } W_0^{1,p}(\Omega) \quad \text{as } n \to +\infty,$$

which contradicts the fact that $\|y_n\| = 1$ for all $n \in \mathbb{N}$.

So, assume that $\xi > 0$. Then $y^- \in \operatorname{int} C_+$ and so $y(z) < 0$ for all $z \in \Omega$. This means that

$$\begin{aligned} & u_n(z) \to -\infty \quad \text{for a.a. } z \in \Omega \text{ as } n \to +\infty, \\ \Rightarrow \quad & \widehat{\lambda}_1(p)|-u_n^-(z)|^p - pF(z,-u_n^-(z)) \to +\infty \quad \text{for a.a. } z \in \Omega \\ & \qquad \text{(see Lemma 1)}, \\ \Rightarrow \quad & \int_\Omega \left[\widehat{\lambda}_1(p)|-u_n^-(z)|^p - pF(z,-u_n^-(z))\right] dz \to +\infty \\ & \qquad \text{(by Fatou's lemma)}. \end{aligned}$$

Since $\widehat{\lambda}_1(p)\|u_n^-(z)\|_p^p \leq \|\nabla u_n^-\|_p^p$ for all $n \in \mathbb{N}$ (see (2)), it follows that

$$p\varphi_\mu^-(u_n) \to +\infty \quad \text{as } n \to +\infty,$$

which contradicts (34). This proves the coercivity of φ_μ^-. □

Remark 2. In particular Proposition 7 implies that φ_μ^- satisfies the C-condition (see Marano-Papageorgiou [15]).

4. Critical Groups

To prove our multiplicity theorem (three solutions theorem, see Section 5), we will use also Morse theoretic techniques (critical groups). For this reason in this section we compute the critical groups of φ_μ ($\mu > 0$) at infinity and at zero.

Proposition 8. *If hypotheses $H(f)$ hold and $\mu > 0$, then $C_k(\varphi_\mu, \infty) = 0$ for all $k \in \mathbb{N}_0$.*

Proof. Let $\lambda \in (\widehat{\lambda}_1(p), \widehat{\lambda}_2(p))$ and consider the C^2-functional $\psi : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi(u) = \frac{1}{p}\|\nabla u\|_p^p - \frac{\lambda}{p}\|u^+\|_p^p \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

We consider the homotopy $h(t, u)$ defined by

$$h(t, u) = (1 - t)\varphi_\mu(u) + t\psi(u) \quad \text{for all } t \in [0, 1], \text{ all } u \in W_0^{1,p}(\Omega).$$

Claim 1: There exist $\alpha \in \mathbb{R}$ and $\widehat{\delta} > 0$ such that

$$h(t, u) \le \alpha \Rightarrow (1 + \|u\|)\|h'_u(t, u)\|_* \ge \widehat{\delta} \quad \text{for all } t \in [0, 1].$$

We argue indirectly. So, suppose that the Claim is not true. Since $h(\cdot, \cdot)$ maps bounded sets to bounded sets, we see that we can find two sequences $\{t_n\}_{n\ge1} \subseteq [0, 1]$ and $\{u_n\}_{n\ge1} \subseteq W_0^{1,p}(\Omega)$ such that

$$t_n \to t, \|u_n\| \to +\infty, h(t_n, u_n) \to -\infty \text{ and } (1 + \|u_n\|)h'_u(t_n, u_n) \to 0 \text{ in } W^{-1,p'}(\Omega). \tag{41}$$

From the last convergence in (41), we have

$$\begin{aligned}\Big|\langle A_p(u_n), h\rangle + (1 - t_n)\langle A(u_n), h\rangle + (1 - t_n)\mu \int_\Omega |u_n|^{q-2}u_n h dz - \lambda t_n \int_\Omega (u_n^+)^{p-1} h dz \\ - (1 - t_n) \int_\Omega f(z, u_n)h dz\Big| \le \frac{\varepsilon_n\|h\|}{1 + \|u_n\|}, \text{ for all } h \in W_0^{1,p}(\Omega), \text{ with } \varepsilon_n \to 0^+.\end{aligned} \tag{42}$$

Also, from the third convergence in (41), we see that we can find $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}&\|\nabla u_n\|_p^p + \frac{(1 - t_n)p}{2}\|\nabla u_n\|_2^2 + \frac{\mu(1 - t_n)p}{q}\|u_n\|_q^q \\ &\quad - \lambda t_n\|u_n^+\|_p^p - (1 - t_n)\int_\Omega pF(z, u_n)dz \le 0 \quad \text{for all } n \ge n_0.\end{aligned} \tag{43}$$

In (42) we choose $h = u_n \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned}&- \|\nabla u_n\|_p^p - (1 - t_n)\|\nabla u_n\|_2^2 - (1 - t_n)\mu\|u_n\|_q^q \\ &\quad + \lambda t_n\|u_n^+\|_p^p + (1 - t_n)\int_\Omega f(z, u_n)u_n dz \le \varepsilon_n \quad \text{for all } n \in \mathbb{N}.\end{aligned} \tag{44}$$

Adding (43) and (44) and recalling that $q < 2 < p$, we obtain

$$(1 - t_n)\int_\Omega [f(z, u_n)u_n - pF(z, u_n)]dz \le M_6 \quad \text{for some } M_6 > 0, \text{ all } n \ge n_0. \tag{45}$$

Using hypothesis $H(f)\,(iii)$ and (45), we have

(46)
$$(1-t_n)\int_\Omega [f(z,-u_n^-)(-u_n^-)-pF(z,-u_n^-)]dz \le M_7 \text{ for some } M_7>0, \text{ all } n\ge n_0.$$

We claim that $t<1$. Indeed if $t=1$, then $t_n\to 1$. We set $y_n=\dfrac{u_n}{\|u_n\|}$, $n\in\mathbb{N}$. From (42) we have

(47)
$$\Big|\langle A_p(y_n),h\rangle+\frac{1-t_n}{\|u_n\|^{p-2}}\langle A(y_n),h\rangle+\frac{(1-t_n)}{\|u_n\|^{p-q}}\int_\Omega |y_n|^{q-2}y_nhdz-\lambda t_n\int_\Omega (y_n^+)^{p-1}hdz$$
$$-(1-t_n)\int_\Omega \frac{N_f(u_n)}{\|u_n\|^{p-1}}h\,dz\Big| \le \frac{\varepsilon_n\|h\|}{(1+\|u_n\|)\|u_n\|^{p-1}}, \text{ for all } n\in\mathbb{N}.$$

Since $\|y_n\|=1$ for all $n\in\mathbb{N}$, we may assume that

(48)
$$y_n\xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega) \text{ and } y_n\to y \text{ in } L^p(\Omega).$$

In (47), we choose $h=y_n-y\in W_0^{1,p}(\Omega)$, pass to the limit as $n\to+\infty$ and use (48), (17) and the facts that $q<2<p$ and that $t_n\to 1$. Then

$$\lim_{n\to+\infty}\langle A_p(y_n),y_n-y\rangle=0$$

(49)
$$\Rightarrow\quad y_n\to y \;\text{ in }\; W_0^{1,p}(\Omega) \;\text{ (see Proposition 4) and so } \|y\|=1.$$

So, if in (47) we pass to the limit as $n\to+\infty$ and use (49), then

(50)
$$\langle A_p(y),h\rangle=\lambda\int_\Omega (y^+)^{p-1}hdz \quad \text{for all } h\in W_0^{1,p}(\Omega) \quad (\text{recall } t_n\to 1).$$

In (50) we choose $h=-y^-\in W_0^{1,p}(\Omega)$ and infer that $y\ge 0$. Then from (50), we have

(51)
$$-\Delta_p y(z)=\lambda y(z)^{p-1} \text{ for a.a. } z\in\Omega,\quad y\Big|_{\partial\Omega}=0.$$

But recall that $\lambda\in(\widehat{\lambda}_1(p),\widehat{\lambda}_2(p))$. So, from (51) it follows that $y=0$, which contradicts (49). Hence $t<1$ and so from (46) we have

(52)
$$\int_\Omega [f(z,-u_n^-)(-u_n^-)-pF(z,-u_n^-)]dz\le M_8 \text{ for some } M_8>0, \text{ all } n\ge n_1\ge n_0.$$

Using (52) and reasoning as in the proof of Proposition 6 (see the claim in that proof), we establish that

(53)
$$\{u_n^-\}_{n\ge1}\subseteq W_0^{1,p}(\Omega) \text{ is bounded.}$$

Next we show that $\{u_n^+\}_{n\ge1}\subseteq W_0^{1,p}(\Omega)$ is bounded. We follow the argument in the proof of Proposition 6. So, we argue by contradiction and assume that

$$\|u_n^+\|\to+\infty \quad \text{as } n\to+\infty.$$

We set $v_n=\dfrac{u_n^+}{\|u_n^+\|}$, $n\in\mathbb{N}$. Then $\|v_n\|=1$, $v_n\ge 0$ for all $n\in\mathbb{N}$. So, we may assume that

(54)
$$v_n\xrightarrow{w} v \text{ in } W_0^{1,p}(\Omega) \text{ and } v_n\to v \text{ in } L^p(\Omega),\ v\ge 0.$$

From (42) and (53), we have

$$\begin{aligned}&\Big|\langle A_p(v_n),h\rangle+\frac{1-t_n}{\|u_n^+\|^{p-2}}\langle A(v_n),h\rangle+\frac{(1-t_n)\mu}{\|u_n^+\|^{p-q}}\int_\Omega v_n^{q-1}hdz-\lambda t_n\int_\Omega y_n^{p-1}hdz\\ &-(1-t_n)\int_\Omega\frac{N_f(u_n^+)}{\|u_n^+\|^{p-1}}\,h\,dz\Big|\le\frac{\varepsilon_n'\|h\|}{\|u_n^+\|^{p-1}},\ \text{for all } h\in W_0^{1,p}(\Omega),\text{ with } \varepsilon_n'\to0^+.\end{aligned}\tag{55}$$

As in the proof of Proposition 6, passing to the limit as $n\to+\infty$ in (55) and using (54), we obtain

$$\begin{aligned}&\langle A_p(v),h\rangle=\int_\Omega[\lambda t+(1-t)\eta_0(z)]v^{p-1}hdz\quad\text{for all } h\in W_0^{1,p}(\Omega)\text{ (see (27))},\\ \Rightarrow\quad &-\Delta_p v(z)=[\lambda t+(1-t)\eta_0(z)]v(z)^{p-1}\text{ for a.a. } z\in\Omega,\ v\big|_{\partial\Omega}=0,\ \|v\|=1.\end{aligned}\tag{56}$$

Let $\eta_t(z)=\lambda t+(1-t)\eta_0(z)$. Then

$$\begin{aligned}&\widehat{\lambda}_1(p)\le\eta_t(z)\quad\text{for a.a. } z\in\Omega,\ \eta_t\not\equiv\widehat{\lambda}_1(p)\\ &\eta_t(z)<\widehat{\lambda}_2(p)\quad\text{for a.a. } z\in\Omega.\end{aligned}$$

So, by Proposition 1, we have

$$\widetilde{\lambda}_1(p,\eta_t)<\widetilde{\lambda}_1(p,\widehat{\lambda}_1(p))=1\quad\text{and}\quad 1=\widetilde{\lambda}_2(p,\widehat{\lambda}_2(p))<\widetilde{\lambda}_2(p,\eta_t).$$

From these inequalities and (56), we infer that $v=0$, which contradicts (56). Therefore

$$\begin{aligned}&\{u_n^+\}_{n\ge1}\subseteq W_0^{1,p}(\Omega)\text{ is bounded,}\\ &\{u_n\}_{n\ge1}\subseteq W_0^{1,p}(\Omega)\text{ is bounded (see (53)).}\end{aligned}$$

But this contradicts (41). So, we have proved Claim 1. On account of Claim 1 and Theorem 5.1.21, p. 334 of Chang [5] (see also Liang-Su [13], Proposition 3.2 and Gasiński-Papageorgiou [9], Proposition 2.5), we have

$$\begin{aligned}&C_k(h(0,\cdot),\infty)=C_k(h(1,\cdot),\infty)\quad\text{for all } k\in\mathbb{N}_0,\\ \Rightarrow\quad &C_k(\varphi_\mu,\infty)=C_k(\psi,\infty)\quad\text{for all } k\in\mathbb{N}_0.\end{aligned}\tag{57}$$

So, we need to compute $C_k(\psi,\infty)$. To this end we consider homotopy $\widehat{h}:[0,1]\times W_0^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\widehat{h}(t,u)=\psi(u)-t\int_\Omega udz\quad\text{for all } t\in[0,1],\text{ all } u\in W_0^{1,p}(\Omega).$$

Claim 2: $\widehat{h}'_u(t,u)\neq0$ for all $t\in[0,1]$, all $u\in W_0^{1,p}(\Omega)\setminus\{0\}$.

Arguing indirectly, suppose that we can find $t\in[0,1]$ and $u\in W_0^{1,p}(\Omega)$, $u\neq0$ such that

$$\begin{aligned}&\widehat{h}'_u(t,u)=0,\\ \Rightarrow\quad &\langle A_p(u),h\rangle-\lambda\int_\Omega(u^+)^{p-1}hdz-t\int_\Omega hdz=0\quad\text{for all } h\in W_0^{1,p}(\Omega).\end{aligned}\tag{58}$$

In (58) we choose $h=-u^-\in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned}&\|\nabla u^-\|_p^p\le0,\\ \Rightarrow\quad &u\ge0,\ u\neq0.\end{aligned}$$

So, from (58) we have

$$-\Delta_p u(z) = \lambda u(z)^{p-1} + t \quad \text{for a.a. } z \in \Omega, \quad u\big|_{\partial\Omega} = 0,\ u \geq 0,\ u \neq 0. \tag{59}$$

The nonlinear regularity theory and the nonlinear maximum principle (see, for example, Gasiński-Papageorgiou [8], pp. 737-738), imply that $u \in \operatorname{int} C_+$.

Let $v \in \operatorname{int} C_+$ and consider the function

$$R(v,u)(z) = |\nabla v(z)|^p - |\nabla u(z)|^{p-2}\left(\nabla u(z), \nabla\left(\frac{v^p}{u^{p-1}}\right)(z)\right)_{\mathbb{R}^N}.$$

From the nonlinear Picone's identity of Allegretto-Huang [2] (see also Motreanu-Motreanu-Papageorgiou [17], Proposition 9.61, p. 255), we have

$$\begin{aligned}
0 &\leq \int_\Omega R(v,u)dz \\
&= \|\nabla v\|_p^p - \int_\Omega (-\Delta_p u)\frac{v^p}{u^{p-1}}dz \text{ (by the nonlinear Green's identity, see [8], p. 211)} \\
&= \|\nabla v\|_p^p - \int_\Omega [\lambda u^{p-1} + t]\frac{v^p}{u^{p-1}}dz \quad \text{(see (59))} \\
&\leq \|\nabla v\|_p^p - \int_\Omega \lambda v^p dz \quad \left(\text{since } \frac{v^p}{u^{p-1}} > 0\right).
\end{aligned}$$

Let $v = \widehat{u}_1(p) \in \operatorname{int} C_+$. Then

$$0 \leq \int_\Omega [\widehat{\lambda}_1 - \lambda]\widehat{u}_1(p)^p dz = [\widehat{\lambda}_1 - \lambda] < 0 \text{ (recall } \|\widehat{u}_1(p)\|_p = 1 \text{ and } \lambda \in (\widehat{\lambda}_1(p), \widehat{\lambda}_2(p)),$$

a contradiction. This proves Claim 2.

From the homotopy invariance of singular homology, for $r > 0$ small we have

$$H_k(\widehat{h}(0,\cdot)^\circ \cap B_r, \widehat{h}(0,\cdot)^\circ \cap B_r \setminus \{0\}) = H_k(\widehat{h}(1,\cdot)^\circ \cap B_r, \widehat{h}(1,\cdot)^\circ \cap B_r \setminus \{0\}) \tag{60}$$

for all $k \in \mathbb{N}_0$ (recall $B_r = \{y \in W_0^{1,p}(\Omega) : \|y\| < r\}$). From Claim 2 and the Noncritical Interval Theorem (see Chang [5], Theorem 5.1.6, p. 320, a consequence of the Second Deformation Theorem, see [5] p. 320 and [8] p. 628), we have

$$H_k(\widehat{h}(1,\cdot) \cap B_r, \widehat{h}(1,\cdot) \cap B_r \setminus \{0\}) = 0 \quad \text{for all } k \in \mathbb{N}_0.$$

Also, from the definition of critical groups, we have

$$H_k(\widehat{h}(0,\cdot) \cap B_r, \widehat{h}(0,\cdot) \cap B_r \setminus \{0\}) = C_k(\psi, 0) \quad \text{for all } k \in \mathbb{N}_0.$$

So, from (60), we have

$$C_k(\psi, 0) = 0 \quad \text{for all } k \in \mathbb{N}_0.$$

But since $\lambda \in (\widehat{\lambda}_1(p), \widehat{\lambda}_2(p))$, we have $K_\psi = \{0\}$ and so

$$\begin{aligned}
& C_k(\psi, 0) = C_k(\psi, \infty) \quad \text{for all } k \in \mathbb{N}_0, \\
\Rightarrow\quad & C_k(\varphi_\mu, \infty) = 0 \quad \text{for all } k \in \mathbb{N}_0 \text{ (see (57))}.
\end{aligned}$$

□

Next we compute the critical groups of φ_μ at $u = 0$.

Proposition 9. *If hypotheses $H(f)$ hold and $\mu > 0$, then*

$$C_k(\varphi_\mu, 0) = \delta_{k,d_m}\mathbb{Z} \quad \textit{for all } k \in \mathbb{N}_0 \textit{ with } d_m = \dim \oplus_{k=1}^m E(\widehat{\lambda}_k(2)) \geq 2.$$

Proof. Consider the C^2-functional $\chi^* : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\chi^*(u) = \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega F(z,u)dz \quad \text{for all } u \in H_0^1(\Omega).$$

Let $\chi = \chi^*\big|_{W_0^{1,p}(\Omega)}$ (recall $p > 2$ and so $W_0^{1,p}(\Omega) \subseteq H_0^1(\Omega)$).

Claim: $C_k(\chi, 0) = \delta_{k,d_m}\mathbb{Z}$ for all $k \in \mathbb{N}_0$.

To this end let $\tau \in (\widehat{\lambda}_m(2), \widehat{\lambda}_{m+1}(2))$ and consider the C^2-functional $\gamma : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\gamma(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{\tau}{2}\|u\|_2^2 \quad \text{for all } u \in H_0^1(\Omega).$$

We consider the homotopy $h^*(t,u)$ defined by

$$h^*(t,u) = (1-t)\chi^*(u) + t\gamma(u) \quad \text{for all } t \in [0,1], \text{ all } u \in H_0^1(\Omega).$$

First suppose that $t \in (0,1]$. Let $u \in C_0^1(\overline{\Omega})$ with $\|u\|_{C_0^1(\overline{\Omega})} \leq \delta$ where $\delta > 0$ is as postulated by hypothesis $H(f)(iv)$. In what follows, let $\langle \cdot, \cdot \rangle_H$ denote the duality brackets for the pair $(H^{-1}(\Omega), H_0^1(\Omega))$. We have

$$\langle (h^*)'_u(t,u), v\rangle_H = (1-t)\langle (\chi^*)'(u), v\rangle_H + t\langle \gamma'(u), v\rangle_H \quad \text{for all } v \in H_0^1(\Omega). \tag{61}$$

Let $\overline{H}_m = \oplus_{k=1}^m E(\widehat{\lambda}_k(2))$ and $\widehat{H}_m = \overline{H}_m^\perp = \overline{\oplus_{k\geq m+1} E(\widehat{\lambda}_k(2))}$. We have the orthogonal direct sum decomposition

$$H_0^1(\Omega) = \overline{H}_m \oplus \widehat{H}_m. \tag{62}$$

So, if $u \in H_0^1(\Omega)$, then we can write u in a unique way as

$$u = \overline{u} + \widehat{u} \quad \text{with } \overline{u} \in \overline{H}_m,\ \widehat{u} \in \widehat{H}_m.$$

In (61) we choose $v = \widehat{u} - \overline{u} \in H_0^1(\Omega)$. Exploiting the orthogonality of the component spaces in (62), we obtain

$$\langle (\chi^*)'(u), \widehat{u} - \overline{u}\rangle = \|\nabla \widehat{u}\|_2^2 - \|\nabla \overline{u}\|_2^2 - \int_\Omega f(z,u)(\widehat{u} - \overline{u})dz. \tag{63}$$

From hypothesis $H(f)(iv)$ we have

$$\widehat{\lambda}_m(2) \leq \frac{f(z,x)}{x} \leq \widehat{\lambda}_{m+1}(2) \quad \text{for a.a. } z \in \Omega, \text{ all } 0 < |x| \leq \delta.$$

Therefore since $\|u\|_{C_0^1(\overline{\Omega})} \leq \delta$, we have

$$f(z,u(z))(\widehat{u} - \overline{u})(z) \leq \widehat{\lambda}_{m+1}(2)\widehat{u}(z)^2 - \widehat{\lambda}_m(2)\overline{u}(z)^2 \quad \text{for a.a. } z \in \Omega.$$

Using this inequality in (63), we obtain

(64)
$$\langle (\chi^*)'(u), \widehat{u} - \overline{u}\rangle \geq \|\nabla \widehat{u}\|_2^2 - \widehat{\lambda}_{m+1}(2)\|\widehat{u}\|_2^2 - \Big[\|\nabla \overline{u}\|_2^2 - \widehat{\lambda}_m(2)\|\overline{u}\|_2^2\Big] \geq 0 \text{ (see (5), (6))}.$$

Similarly we have

$$\langle \gamma'(u), \widehat{u} - \overline{u}\rangle = \|\nabla \widehat{u}\|_2^2 - \tau\|\widehat{u}\|_2^2 - \big[\|\nabla \overline{u}\|_2^2 - \tau\|\overline{u}\|_2^2\big] \geq c_5\|u\|^2 \tag{65}$$

for some $c_5 > 0$ (recall $\lambda \in (\widehat{\lambda}_m(2), \widehat{\lambda}_{m+1}(2))$ and see Proposition 3).

Using (64) and (65) in (61) and recalling that $0 < t \leq 1$, we obtain

$$\langle (h^*)'_u(t,u), \widehat{u} - \overline{u}\rangle \geq tc_5\|u\|^2 > 0. \tag{66}$$

Standard regularity theory implies that

$$K_{h^*(t,\cdot)} \subseteq C_0^1(\overline{\Omega}) \quad \text{for all } t \in [0,1].$$

Therefore from (66), we conclude that $u=0$ is an isolated critical point of $h^*(t,\cdot)$ uniformily in $t\in(0,1]$.

We consider also the case $t=0$. Then we have $h^*(0,\cdot)=\chi^*(\cdot)$.

We will show that $u=0$ is isolated in K_{χ^*}. Arguing by contradiction, suppose we could find $\{u_n\}_{n\geq 1}\subseteq H_0^1(\Omega)$ such that

$$u_n\to 0 \text{ in } H_0^1(\Omega) \quad \text{and} \quad (\chi^*)'(u_n)=0 \quad \text{for all } n\in\mathbb{N}. \tag{67}$$

From the equation in (67), we have

$$-\Delta u_n(z)=f(z,u_n(z)) \quad \text{for a.a. } z\in\Omega, \quad u_n\big|_{\partial\Omega}=0 \text{ for all } n\in\mathbb{N}.$$

The regularity theory (see [8]) implies that there exist $\alpha\in(0,1)$ and $c_6>0$ such that

$$u_n\in C_0^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|u_n\|_{C_0^{1,\alpha}(\overline{\Omega})}\leq c_6 \text{ for all } n\in\mathbb{N}. \tag{68}$$

Taking into account the compact embedding of $C_0^{1,\alpha}(\overline{\Omega})$ into $C_0^1(\overline{\Omega})$ and (67), from (68) we infer that

$$\begin{aligned}
&u_n\to 0 \quad \text{in } C_0^1(\overline{\Omega}) \text{ as } n\to+\infty,\\
\Rightarrow\ &|u_n(z)|\leq\delta \quad \text{for all } z\in\overline{\Omega}, \text{ all } n\geq n_0,\\
\Rightarrow\ &f(z,u_n(z))(\widehat{u}_n-\overline{u}_n)(z)\leq\widehat{\lambda}_{m+1}(2)\widehat{u}_n(z)^2-\zeta_0(z)\overline{u}_n(z)^2\\
&\qquad\qquad \text{for a.a. } z\in\Omega, \text{ all } n\geq n_0 \text{ (see hypothesis } H(f)(iv)).
\end{aligned} \tag{69}$$

We know that

$$\langle A(u_n),v\rangle=\int_\Omega f(z,u_n)v\,dz \quad \text{for all } v\in H_0^1(\Omega), \text{ all } n\in\mathbb{N}. \tag{70}$$

As before, in (70) we choose $h=\widehat{u}_n-\overline{u}_n\in H_0^1(\Omega)$. From (69) we have

$$\begin{aligned}
&\|\nabla\widehat{u}_n\|_2^2-\|\nabla\overline{u}_n\|_2^2\leq\widehat{\lambda}_{m+1}(2)\|\widehat{u}_n\|_2^2-\int_\Omega\zeta_0(z)\overline{u}_n^2dz \quad \text{for all } n\geq n_0,\\
\Rightarrow\ &0\leq\|\nabla\widehat{u}_n\|_2^2-\widehat{\lambda}_{m+1}(2)\|\widehat{u}_n\|_2^2\leq\|\nabla\overline{u}_n\|_2^2-\int_\Omega\zeta_0(z)\overline{u}_n^2dz\leq 0\\
&\qquad\qquad \text{for all } n\geq n_0 \text{ (see (5), (6))}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\|\nabla\widehat{u}_n\|_2^2=\widehat{\lambda}_{m+1}(2)\|\widehat{u}_n\|_2^2 \quad \text{and} \quad \|\nabla\overline{u}_n\|_2^2=\int_\Omega\zeta_0(z)\overline{u}_n^2dz \quad \text{for all } n\geq n_0,\\
\Rightarrow\ &\widehat{u}_n\in E(\widehat{\lambda}_{m+1}(2)) \quad \text{and} \quad \overline{u}_n=0 \quad \text{for all } n\geq n_0 \text{ (see Proposition 3)},\\
\Rightarrow\ &u_n=\widehat{u}_n\in E(\widehat{\lambda}_{m+1}(2)) \quad \text{for all } n\geq n_0,\\
\Rightarrow\ &u_n(z)\neq 0 \quad \text{for a.a. } z\in\Omega, \text{ all } n\geq n_0 \text{ (by the UCP)}.
\end{aligned} \tag{71}$$

On account of the hypothesis $H(f)(iv)$ and since $u_n=\widehat{u}_n$, $n\geq n_0$, we have

$$\widehat{\lambda}_{m+1}(2)\|u_n\|_2^2=\|\nabla u_n\|_2^2=\int_\Omega f(z,u_n)u_n\,dz<\widehat{\lambda}_{m+1}(2)\|u_n\|_2^2$$
$$\text{for all } n\geq n_0 \text{ (see (71))}.$$

This proves that $u = 0$ is isolated in K_{χ^*}. Therefore we have proved that $u = 0$ is isolated in $K_{h^*(t,\cdot)}$ uniformly in $t \in [0, 1]$. Then the homotopy invariance of critical groups (see Gasiński-Papageorgiou [10], Theorem 5.1.125, p. 836), implies that

$$\begin{aligned} & C_k(h^*(0,\cdot),0) = C_k(h^*(1,\cdot),0) \quad \text{for all } k \in \mathbb{N}_0, \\ \Rightarrow \quad & C_k(\chi^*,0) = C_k(\gamma,0) \quad \text{for all } k \in \mathbb{N}_0. \end{aligned} \tag{72}$$

Recall that $\tau \in (\widehat{\lambda}_m(2), \widehat{\lambda}_{m+1}(2))$. Hence $K_\gamma = \{0\}$ and $u = 0$ is a nondegenerate critical point of Morse index $d_m \geq 2$. So, we have

$$\begin{aligned} & C_k(\gamma,0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0, \\ \Rightarrow \quad & C_k(\chi^*,0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0 \text{ (see (72))}. \end{aligned} \tag{73}$$

The space $W_0^{1,p}(\Omega)$ is dense in $H_0^1(\Omega)$. So, from Palais [20], we have

$$C_k(\chi,0) = C_k(\chi^*,0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0 \text{ (see (73))}. \tag{74}$$

Now note that

$$\begin{aligned} |\varphi_\mu(u) - \chi(u)| &= \frac{1}{p}\|\nabla u\|_p^p + \frac{\mu}{q}\|u\|_q^q, \\ |\langle \varphi'(u) - \chi'(u), v\rangle| &= \left|\langle A_p(u), v\rangle + \mu \int_\Omega |u|^{q-2}uv dz\right| \\ &\leq \|\nabla u\|_p^{p-1}\|v\| + c_7\mu\|u\|_q^{q-1}\|v\| \\ &\qquad \text{for some } c_7 > 0, \text{ all } v \in W_0^{1,p}(\Omega). \end{aligned}$$

Then the C^1-continuity of critical groups (see Gasiński-Papageorgiou [10], Theorem 5.1,125, p. 836), implies that

$$\begin{aligned} & C_k(\varphi_\mu,0) = C_k(\chi,0) \quad \text{for all } k \in \mathbb{N}_0, \\ \Rightarrow \quad & C_k(\varphi_\mu,0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0 \text{ (see (74))}. \end{aligned}$$

□

5. Multiplicity Theorem

In this section we state and prove our multiplicity theorem. We produce three nontrivial smooth solutions when the parameter $\mu > 0$ is small.

Theorem 2. *If hypotheses $H(f)$ hold, then there exists $\mu_0 > 0$ such that for all $\mu \in (0, \mu_0)$ problem (P_μ) has at least three nontrivial smooth solutions*

$$\begin{aligned} & u_o \in -C_+ \quad \textit{with } u_0(z) < 0 \textit{ for all } z \in \Omega, \\ & \widehat{u}, \widetilde{u} \in C_0^1(\overline{\Omega}). \end{aligned}$$

Proof. From Proposition 7 we know that for all $\mu > 0$, the functional φ_μ^- is coercive. Also, using the Sobolev embedding theorem, we see that φ_μ^- is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\varphi_\mu^-(u_0) = \inf\left[\varphi_\mu^-(u) : u \in W_0^{1,p}(\Omega)\right]. \tag{75}$$

Hypotheses $H(f)(i), (ii), (iv)$ imply that given $\varepsilon > 0$ and $r > p$, we can find

$$c_8 > \frac{\|\nabla \widehat{u}_1(2)\|_p^p}{\|\widehat{u}_1(2)\|_p^p} > 0 \quad \text{and} \quad c_9 > 0$$

such that

$$F(z,x) \geq \frac{1}{2}[\zeta_0(z)-\varepsilon]x^2 + \frac{c_8}{p}|x|^p - c_9|x|^r \quad \text{for a.a. } z\in\Omega, \text{ all } x\in\mathbb{R}. \tag{76}$$

Then for $t>0$ we have

$$\begin{aligned}\varphi_\mu^-(-t\widehat{u}_1(2)) \leq & \frac{t^p}{p}\|\nabla\widehat{u}_1(2)\|_p^p + \frac{t^2}{2}\left(\int_\Omega[\widehat{\lambda}_1(2)-\zeta_0(z)]\,\widehat{u}_1(2)^2dz+\varepsilon\right)\\ & + \frac{\mu t^q}{q}\|\widehat{u}_1(2)\|_q^q - \frac{t^p}{p}c_8\|\widehat{u}_1(2)\|_p^p + c_9t^r\|\widehat{u}_1(2)\|_r^r \quad \text{(see (76))},\\ \leq & \frac{t^2}{2}\left(\int_\Omega[\widehat{\lambda}_1(2)-\zeta_0(z)]\,\widehat{u}_1(2)^2dz+\varepsilon\right) + \frac{\mu t^q}{q}\|\widehat{u}_1(2)\|_q^q\\ & + c_9t^r\|\widehat{u}_1(2)\|_r^r \quad \left(\text{recall that } c_8 > \frac{\|\nabla\widehat{u}_1(2)\|_p^p}{\|\widehat{u}_1(2)\|_p^p}\right).\end{aligned} \tag{77}$$

Since $\widehat{u}_1(2)\in \operatorname{int} C_+$, we see that

$$d^* = \int_\Omega[\zeta_0(z)-\widehat{\lambda}_1(2)]\,\widehat{u}_1(2)^2dz > 0 \quad \text{(see hypothesis } H(f)(iv)).$$

Choosing $\varepsilon\in(0,d^*)$, from (77) we have

$$\begin{aligned} & \varphi_\mu^-(-t\widehat{u}_1(2)) \leq -c_{10}t^2 + \mu c_{11}t^q + c_{12}t^r \quad \text{for some } c_{10},c_{11},c_{12}>0,\\ \Rightarrow\quad & \varphi_\mu^-(-t\widehat{u}_1(2)) \leq [-c_{10}+\mu c_{11}t^{q-2}+c_{12}t^{r-2}]t^2 \quad \text{for all } t>0.\end{aligned} \tag{78}$$

We consider the function

$$\sigma_\mu(t) = \mu c_{11}t^{q-2} + c_{12}t^{r-2} \quad \text{for all } t>0.$$

Evidently $\sigma_\mu\in C^1(0,+\infty)$ and since $q<2<r$, we have

$$\sigma_\mu(t)\to+\infty \quad \text{as } \ t\to0^+ \text{ and as } \ t\to+\infty.$$

Therefore we can find $t_0\in(0,+\infty)$ such that

$$\begin{aligned} & \sigma_\mu(t_0) = \inf[\sigma_\mu(t): t\geq 0]\\ \Rightarrow\quad & \sigma_\mu'(t_0) = \mu c_{11}(q-2)t_0^{q-3} + c_{12}(r-2)t_0^{r-3} = 0,\\ \Rightarrow\quad & t_0 = \left[\frac{\mu c_{11}(2-q)}{c_{12}(r-2)}\right]^{\frac{1}{r-q}}.\end{aligned}$$

Then we have

$$\sigma_\mu(t_0) = \mu c_{11}\left[\frac{c_{12}(r-2)}{\mu c_{11}(2-q)}\right]^{\frac{2-q}{r-q}} + c_{12}\left[\frac{\mu c_{11}(2-q)}{c_{12}(r-2)}\right]^{\frac{r-2}{r-q}}.$$

Recalling that $q<2<r$, we see that

$$\sigma_\mu(t_0)\to 0 \quad \text{as} \quad \mu\to0^+.$$

So, we can find $\mu_0>0$ such that

$$\begin{aligned} & \sigma_\mu(t_0) < c_{10} \quad \text{for all } \mu\in(0,\mu_0),\\ \Rightarrow\quad & \varphi_\mu^-(-t_0\,\widehat{u}_1(2)) < 0 \quad \text{for all } \mu\in(0,\mu_0) \text{ (see (78))},\\ \Rightarrow\quad & \varphi_\mu^-(u_0) < 0 = \varphi_\mu^-(0) \quad \text{for all } \mu\in(0,\mu_0) \text{ (see (75))},\\ \Rightarrow\quad & u_0\neq 0.\end{aligned}$$

From (75) we have

$$(\varphi_\mu^-)'(u_0) = 0,$$

(79)
$$\Rightarrow \quad \langle A_p(u_0), h\rangle + \langle A(u_0), h\rangle - \mu \int_\Omega (u_0^-)^{q-1} h dz - \int_\Omega f_-(z, u_0) h dz = 0$$
$$\text{for all } h \in W_0^{1,p}(\Omega).$$

In (79) we choose $h = u_0^+ \in W_0^{1,p}(\Omega)$. Then

$$\|\nabla u_0^+\|_p^p + \|\nabla u_0^+\|_2^2 = 0,$$
$$\Rightarrow \quad u_0^+ = 0 \quad \text{and so } u_0 \leq 0,\ u_0 \neq 0.$$

Then from (79) we have

$$-\Delta_p u_0(z) - \Delta u_0(z) = f(z, u_0(z)) - \mu |u_0(z)|^{q-2} u_0(z) \text{ for a.a. } z \in \Omega, \quad u_0\big|_{\partial\Omega} = 0.$$

From Theorem 7.1 of Ladyzhenskaya-Ural′tseva [12], we have $u_0 \in L^\infty(\Omega)$. Then Theorem 1 of Lieberman [14] implies that $u_0 \in (-C_+) \setminus \{0\}$.

Let $a(y) = |y|^{p-2} y + y$ for all $y \in \mathbb{R}^N$. Then $a \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ (recall that $p > 2$) and we have

$$\nabla a(y) = |y|^{p-2} \left[id + (p-2) \frac{y \otimes y}{|y|^2} \right],$$
$$\Rightarrow \quad (\nabla a(y) \xi, \xi)_{\mathbb{R}^N} \geq |\xi|^2 \quad \text{for all } y \in \mathbb{R}^N \setminus \{0\},\ \text{all } \xi \in \mathbb{R}^N.$$

Note that

$$\operatorname{div} a(\nabla u) = \Delta_p u + \Delta u \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

So, by the tangency principle of Pucci-Serrin [27], Theorem 2.5.2, p. 35, we have

$$u_0(z) < 0 \quad \text{for all } z \in \Omega. \tag{80}$$

<u>Claim</u>: u_0 is a local minimizer of φ_μ ($\mu \in (0, \mu_0)$).

According to Proposition 5, it suffices to show that u_0 is a local $C_0^1(\overline{\Omega})$-minimizer of φ_μ. To this end, let $u \in C_0^1(\overline{\Omega})$. We have

$$\begin{aligned} \varphi_\mu(u) - \varphi_\mu(u_0) &= \varphi_\mu(u) - \varphi_\mu^-(u_0) \quad \text{(see (80))} \\ &\geq \varphi_\mu(u) - \varphi_\mu^-(u) \quad \text{(see (75))} \\ &\geq \frac{\mu}{q} \int_\Omega (u^+)^q dz - \int_\Omega F(z, u^+) dz. \end{aligned} \tag{81}$$

From hypothesis $H(f)(iv)$ we have

$$F(z, x) \leq c_{13} x^2 \quad \text{for a.a. } z \in \Omega,\ \text{all } |x| \leq \delta,\ \text{some } c_{13} > 0.$$

Therefore, if $\|u\|_{C_0^1(\overline{\Omega})} \leq \delta_0$ with $\delta_0 \in (0, \delta]$, then

$$\begin{aligned} \varphi_\mu(u) - \varphi_\mu(u_0) &\geq \frac{\mu}{q} \int_\Omega (u^+)^q dz - c_{13} \int_\Omega (u^+)^2 dz \quad \text{(see (81))} \\ &\geq \left[\frac{\mu}{q} - c_{13} \|u\|_\infty^{2-q} \right] \int_\Omega (u^+)^q dz. \end{aligned} \tag{82}$$

If we choose $\delta_0 < \left(\frac{\mu}{qc_{13}}\right)^{\frac{1}{2-q}}$, then from (82) we have

$$\begin{aligned} & \varphi_\mu(u) - \varphi_\mu(u_0) > 0 \quad \text{for all } u \in C_0^1(\overline{\Omega}),\ 0 < \|u\|_{C_0^1(\overline{\Omega})} \leq \delta_0, \\ \Rightarrow\quad & u_0 \quad \text{is a local } C_0^1(\overline{\Omega})\text{-minimizer of } \varphi_\mu, \\ \Rightarrow\quad & u_0 \quad \text{is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \varphi_\mu \quad \text{(see Proposition 5)}. \end{aligned}$$

This proves the Claim. The Claim implies that

$$C_k(\varphi_\mu, u_0) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0,\ \mu \in (0, \mu_0). \tag{83}$$

We assume that K_φ is finite. Otherwise, we already have an infinity of nontrivial solutions in $C_0^1(\overline{\Omega})$ (by the nonlinear regularity theory) and so we are done. On account of the Claim, we can find $\rho \in (0,1)$ small such that

$$\varphi_\mu(u_0) < \inf\left[\varphi_\mu(u) : \|u - u_0\| = \rho\right] = m_\mu^\rho \tag{84}$$

(see Aizicovici-Papageorgiou-Staicu [1], proof of Proposition 29).

Hypotheses $H(f)(i),(ii)$ imply that given $\varepsilon > 0$, we can find $c_{14} = c_{14}(\varepsilon) > 0$ such that

$$F(z,x) \geq \frac{1}{p}[\eta(z) - \varepsilon]x^p - c_{14} \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq 0. \tag{85}$$

Since $\widehat{u}_1(p) \in \operatorname{int} C_+$, for $t > 0$ we have

$$\begin{aligned} \varphi_\mu(t\widehat{u}_1(p)) \leq & \frac{t^p}{p}\widehat{\lambda}_1(p) + \frac{t^2}{2}\|\nabla \widehat{u}_1(p)\|_2^2 + \frac{\mu t^q}{q}\|\nabla \widehat{u}_1(p)\|_q^q \\ & - \frac{t^p}{p}\int_\Omega \eta(z)\widehat{u}_1(p)^p dz + \frac{t^p \varepsilon}{p} + c_{14}|\Omega|_N \\ & \qquad\qquad \text{(see (85) and recall that } \|\widehat{u}_1(p)\|_p = 1) \\ = & \frac{t^p}{p}\left(\int_\Omega [\widehat{\lambda}_1(p) - \eta(z)]\widehat{u}_1(p)^p dz + \varepsilon\right) + \frac{t^2}{2}\|\nabla \widehat{u}_1(p)\|_2^2 \\ & + \frac{\mu t^q}{q}\|\nabla \widehat{u}_1(p)\|_q^q + c_{14}|\Omega|_N. \end{aligned} \tag{86}$$

Note that

$$\beta_* = \int_\Omega [\eta(z) - \widehat{\lambda}_1(p)]\widehat{u}_1(p)^p dz > 0 \quad \text{(see hypothesis } H(f)(ii)).$$

So, choosing $\varepsilon \in (0, \beta_*)$ and since $q < 2 < p$, from (86) we infer that

$$\varphi_\mu(t\widehat{u}_1(p)) \to -\infty \quad \text{as } t \to +\infty. \tag{87}$$

Finally from Proposition 6, we have that

$$\varphi_\mu \quad \text{satisfies the } C\text{-condition}. \tag{88}$$

Then (84), (87) and (88) permit the use of Theorem 1 (the mountain pass theorem). So, we can find $\widehat{u} \in K_\varphi \subseteq C_0^1(\overline{\Omega})$ (by the nonlinear regularity theory) $\mu \in (0, \mu_0)$, such that

$$\begin{aligned} & m_\mu^\rho \leq \varphi_\mu(\widehat{u}), \\ \Rightarrow\quad & \widehat{u} \neq u_0 \quad \text{(see (84))}. \end{aligned}$$

From Corollary 6.81, p. 168, of Motreanu-Motreanu-Papageorgiou [17], we have

$$C_1(\varphi_\mu, \widehat{u}) \neq \emptyset. \tag{89}$$

From Proposition 9 and (89), we have $\widehat{u} \neq 0$. Note that

$$\varphi_\mu \in C^2(W_0^{1,p}(\Omega) \setminus \{0\}).$$

Therefore from (89) and Papageorgiou-Rădulescu [21], we have

$$C_k(\varphi_\mu, \widehat{u}) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{90}$$

From Propositions 8 and 9, we have

$$C_k(\varphi_\mu, \infty) = 0 \quad \text{and} \quad C_k(\varphi_\mu, 0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{91}$$

Suppose $K_{\varphi_\mu} = \{0, u_0, \widehat{u}\}$. From (83), (90), (91) and the Morse relation with $t = -1$ (see (7)), we have

$$\begin{aligned} & (-1)^{d_m} + (-1)^0 + (-1)^1 = 0 \quad \text{(see Proposition 8)}, \\ \Rightarrow \quad & (-1)^{d_m} = 0, \quad \text{a contradiction.} \end{aligned}$$

So, there exists $\widetilde{u} \in K_{\varphi_\mu} \setminus \{0, u_0, \widehat{u}\}$. Then $\widetilde{u} \in C_0^1(\overline{\Omega})$ is the third nontrivial smooth solution of problem (P_μ) with $0 < \mu < \mu_0$.

Remark 3. In contrast to the case where the concave term enters with a positive sign, we can not guarantee that $u_0 \in -\operatorname{int} C_+$. The nonlinear maximum principle does not apply.

□

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