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A quasidouble of the affine plane of order 4 and the solution of a problem on additive designs

Marco Pavone

Dipartimento di Ingegneria, Università degli Studi di Palermo, Palermo 90128, Italy

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A B S T R A C T

A 2-(v, k, λ) block design $(\mathcal{P}, \mathcal{B})$ is *additive* if, up to isomorphism, $\mathcal{P}$ can be represented as a subset of a commutative group $(G, +)$ in such a way that the k elements of each block in $\mathcal{B}$ sum up to zero in G. If, for some suitable G, the embedding of $\mathcal{P}$ in G is also such that, conversely, any zero-sum k-subset of $\mathcal{P}$ is a block in $\mathcal{B}$, then $(\mathcal{P}, \mathcal{B})$ is said to be *strongly additive*.

In this paper we exhibit the very first examples of additive 2-designs that are not strongly additive, thereby settling an open problem posed in 2019. Our main counterexample is a resolvable 2-$(16, 4, 2)$ design $(\mathbb{F}_4 \times \mathbb{F}_4, \mathcal{B}_2)$, which decomposes into two disjoint isomorphic copies of the affine plane of order four.

An essential part of our construction is a (cyclic) decomposition of the point-plane design of $AG(4, 2)$ into seven disjoint isomorphic copies of the affine plane of order four. This produces, in addition, a solution to Kirkman's schoolgirl problem.

E-mail address: marco.pavone@unipa.it.

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1. Introduction

In two recent papers [8,9], A. Caggegi, G. Falcone, and the present author developed a theory of *additive* block designs, in order to extend to a larger setting a property that is satisfied by some classic designs in finite geometry.

Essentially, a 2-(v, k, λ) block design $(\mathcal{P}, \mathcal{B})$ is *additive* if, up to isomorphism, $\mathcal{P}$ is a subset of a commutative group $(G, +)$, and the k elements of any block in $\mathcal{B}$ sum up to zero (see Definition 1 and Proposition 3 in Section 2).

Some geometric designs, such as the point-flat designs of either an affine geometry $\mathrm{AG}(d, q)$ over $\mathbb{F}_q$, or a projective geometry $\mathrm{PG}(d, 2)$ over $\mathbb{F}_2$, are basic examples of additive 2-designs. In this case, the group G is somehow intrinsic, as $\mathcal{P}$ can be seen as the set of elements of the additive group of the vector space $\mathbb{F}_q^d$ (respectively, as the set of nonzero elements of $\mathbb{F}_2^{d+1}$), and, for $k > 2$, the sum of the points in each block is zero.

For designs that are defined in a purely combinatorial way, it is not as evident that they should have an algebraic representation of this sort. Moreover, whenever such a group G exists, a natural question is whether the blocks of the design $(\mathcal{P}, \mathcal{B})$ can be characterized as the only k-subsets of $\mathcal{P}$ whose elements add up to zero in G.

Essentially, a 2-(v, k, λ) block design $(\mathcal{P}, \mathcal{B})$ is *strongly additive* if, up to isomorphism, $\mathcal{P}$ can be represented a subset of a commutative group $(G, +)$ in such a way that, for any k-set $\{X_1, \ldots, X_k\} \subseteq \mathcal{P}$,

$$\{X_1, \ldots, X_k\} \text{ is a block in } \mathcal{B} \Longleftrightarrow X_1 + \cdots + X_k = 0$$

(see Definition 4 and Proposition 6 in Section 2 below).

This strong version of additivity is satisfied by some of the main families of additive 2-designs, that is, symmetric designs [8], affine resolvable designs [9], geometric Steiner triple systems [8], Boolean Steiner quadruple systems [15,22], and the other classes of additive designs introduced in [15] and [26] (which are strongly additive by construction).

Until very recently, no single example was known of an additive 2-design that was not strongly additive. An open problem was posed in 2019 in [9, 3.10] as to whether any additive design is also strongly additive.

The main purpose of this paper is to exhibit a resolvable 2-$(16, 4, 2)$ design that is additive but not strongly additive. It is a simple design (that is, with no repeated blocks) and is a *quasidouble* of the affine plane of order four in the strongest possible sense, in that it decomposes into two disjoint isomorphic copies of $\mathrm{AG}(2, 4)$. Moreover, it is also the first example of a resolvable additive design that is not affine resolvable, nor the point-flat design of some affine geometry $\mathrm{AG}(d, q)$ or some projective geometry $\mathrm{PG}(d, 2)$. Our construction, originally inspired by the main result in [10], is based on the idea that $\mathbb{F}_4 \times \mathbb{F}_4$ can be both seen as a 2-dimensional vector space over $\mathbb{F}_4$, as well as a 4-dimensional vector space over $\mathbb{F}_2$.

It is appropriate to mention that the question of the existence of an additive but not strongly additive 2-(v, k, λ) design remains still open for the case $\lambda = 1$, that is, for Steiner systems.

In the final section of the paper, we extend the construction of the 2-$(16, 4, 2)$ design to show that the point-plane design $\mathrm{AG}_2(4, 2)$ of the affine geometry $\mathrm{AG}(4, 2)$ can be partitioned as the disjoint union of seven subdesigns, each isomorphic to the affine plane of order four, which are obtained as the orbit of $\mathrm{AG}(2, 4)$ under a $\mathbb{F}_2$-linear endomorphism of $\mathbb{F}_4 \times \mathbb{F}_4 \simeq \mathbb{F}_2^4$ of order seven. As an application, for each n in $\{3, 4, 5, 6\}$, we exhibit a resolvable 2-$(16, 4, n)$ design that is additive but not strongly additive, and which can be partitioned as the disjoint union of n copies of the affine plane of order four. Moreover, by looking at the derived design of $\mathrm{AG}_2(4, 2)$ at the origin, and by considering also a different order-7 endomorphism, we find two (cyclic) non-isomorphic resolutions of the 35 projective lines of $\mathrm{PG}(3, 2)$, which provide two solutions to Kirkman's schoolgirl problem. We also show that $\mathrm{AG}_2(4, 2)$ can be represented as the set of the 16 points of the affine hyperplane $x_1 + x_2 + x_3 + x_4 + x_5 = 1$ in the 5-dimensional vector space $\mathbb{F}_2^5$, whereas its blocks are the 4-subsets of the hyperplane whose elements sum up to zero.

2. Additive and strongly additive designs

In this section we give the definitions of additive and strongly additive design, and, together with a few known properties, we present a new characterization of strong additivity and a characterization of the automorphism group of any strongly additive design. For a general treatment of block designs, see for instance [1,11].

Definition 1. ([8, Definition 2.1]) For a t-(v, k, λ_t) design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$, one lets $(\mathfrak{G}_{\mathcal{D}}, +)$ denote the finitely presented commutative group whose generators are the points of $\mathcal{P}$ and whose relations are the equalities $X_1 + \cdots + X_k = 0$, as $\{X_1, \ldots, X_k\}$ ranges over the blocks in $\mathcal{B}$. The design $\mathcal{D}$ is said to be *additive* if distinct points in $\mathcal{P}$ are still distinct in the group $\mathfrak{G}_{\mathcal{D}}$.

If the design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is additive, then one may identify the points of $\mathcal{P}$ with the corresponding elements of the group $\mathfrak{G}_{\mathcal{D}}$, and, by construction, the k points in any block in $\mathcal{B}$ sum up to zero. The group $\mathfrak{G}_{\mathcal{D}}$ is finite for any 2-(v, k, λ) design with $k < v$ [8, Theorem 2.2], whereas it can be infinite otherwise, for instance for any 1-design having more points than blocks (for example, if $\mathcal{P} = \{a, b, c, d\}$ and $\mathcal{B} = \{\{a, b\}, \{c, d\}\}$, then $\mathfrak{G}_{\mathcal{D}}$ is isomorphic to $\mathbb{Z}^2$).

Equivalently, the notion of additive design can be given as follows. First, one may identify the free commutative group generated by the v points $P_1, \ldots, P_v$ of $\mathcal{P}$ with $\mathbb{Z}^v$, where $P_1 = (1, 0, \ldots, 0), \ldots, P_v = (0, 0, \ldots, 1)$. If $b = |\mathcal{B}|$, and if A is a $b \times v$ incidence matrix for $\mathcal{D}$, then each row of A is precisely the sum in $\mathbb{Z}^v$ of the k points in the corresponding block of $\mathcal{D}$. Hence the additivity of $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ can be described, alternatively, as in the following result.

Proposition 2. ([8, §2]) *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a t-(v, k, λ_t) design with $b = |\mathcal{B}|$ blocks, and let A be a $b \times v$ incidence matrix for $\mathcal{D}$. If H is the subgroup of $\mathbb{Z}^v$ generated by the b rows of A, then $\mathfrak{G}_\mathcal{D} \simeq \mathbb{Z}^v/H$ and, moreover, $\mathcal{D}$ is additive if and only if the quotient map $\pi : \mathcal{P} \to \mathbb{Z}^v/H$, $\pi(P) = P + H$, is injective.*

Finally, the notion of additivity can be given in yet another equivalent way, which for practical purposes is often the most suitable one to establish whether a given design is additive.

Proposition 3. ([8, Proposition 2.7]) *A block design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is additive if and only if, up to isomorphism, $\mathcal{P}$ can be represented as a subset of a commutative group $(G, +)$ (not necessarily isomorphic to $\mathfrak{G}_\mathcal{D}$) in such a way that the sum of the points in any block in $\mathcal{B}$ is zero in G.*

In [8] and [9] it is shown that symmetric and affine resolvable 2-designs are additive, whereas the only additive Steiner triple systems are the geometric STSs, that is, the point-line designs of $\mathrm{AG}(d, 3)$ and $\mathrm{PG}(d, 2)$. In [22, Theorem 3, 2)] it is shown that the only additive Steiner quadruple systems are the Boolean ones, that is, the point-plane designs of the affine geometries $\mathrm{AG}(d, 2)$. Also, the so-called 2-(v, k, λ) designs over $\mathbb{F}_2$, when seen as 2-$(2^v - 1, 2^k - 1, \lambda)$ designs, form a notable class of additive 2-designs [2–4,28], which are, in turn, subdesigns of one of the classes of additive *Boolean* 2-designs considered in [15]. A similar class of additive 2-(p^n, mp, λ) designs, with p an odd prime, is described in [26], where the full automorphism group is found, on the basis of the results given in [16].

Additive designs have connections with several theories, such as additive combinatorics [16], representation theory of finite groups [9, Example 3.7], loop theory [13], combinatorial algebraic geometry [22], and coding theory [15].

It is worth noting that the quest for new additive designs sometimes produces new designs, which, in addition to being additive, happen to be also the first known examples of designs with a certain set of parameters. For instance, in [20] an additive 2-$(81, 6, 2)$ design is given, which is also the first known example of a simple 2-design (that is, with no repeated blocks) with these parameters.

Some infinite classes of new additive Steiner 2-designs, whose parameters are not those of the point-line designs of $\mathrm{AG}(d, q)$, $\mathrm{PG}(2, q)$, and $\mathrm{PG}(d, 2)$, are constructed in [5] by means of *difference methods* of various kinds, and new additive 2-designs have been obtained in [6] as a notable application of the method of *partial differences*. Recently, again by difference methods, several new examples of additive Steiner 2-designs have been obtained with a much smaller point-set [21], for instance by considering the design $\mathrm{PG}_1(d, q)$ for $(d, q) = (3, 3), (4, 3), (3, 4), (3, 5)$. More generally, it was shown in [7] that the point-line design of the projective geometry $\mathrm{PG}(d, q)$ is an additive Steiner 2-design for any choice of the dimension $d > 1$ and of the prime power q. Finally, the main result in [3], improving [28], could be the starting point for constructing new designs over $\mathbb{F}_2$, hence new additive designs of order a power of 2 minus 1.

The algebraic representation of an additive 2-(v, k, λ) design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ as a subset of the group $\mathfrak{G}_{\mathcal{D}}$ (see Definition 1) is even more significant in the case where the blocks in $\mathcal{B}$ can be characterized as the only k-subsets of $\mathcal{P}$ whose elements sum up to zero in $\mathfrak{G}_{\mathcal{D}}$. This leads to the following definition, which extends to arbitrary additive 2-designs a property that holds for symmetric designs [8, Theorem 4.1(ii)] and affine resolvable designs [9, Theorem 3.1(ii)].

Definition 4. An additive 2-(v, k, λ) design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is *strongly additive* if, for any k-set $\{X_1, \ldots, X_k\} \subseteq \mathcal{P} \subseteq \mathfrak{G}_{\mathcal{D}}$, $\{X_1, \ldots, X_k\}$ is a block in $\mathcal{B}$ if and only if $X_1 + \cdots + X_k = 0$ in the group $\mathfrak{G}_{\mathcal{D}}$.

Equivalently, in view of the isomorphism $\mathfrak{G}_{\mathcal{D}} \simeq \mathbb{Z}^v/H$ described above in Proposition 2, the notion of strong additivity can also be characterized in terms of the incidence matrix of $\mathcal{D}$, as follows.

Proposition 5. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be an additive 2-(v, k, λ) design with $b = |\mathcal{B}|$ blocks, let A be a $b \times v$ incidence matrix for $\mathcal{D}$, and let H be the subgroup of $\mathbb{Z}^v$ generated by the b rows of A. Then $\mathcal{D}$ is strongly additive if and only if the rows of A are the only elements $(x_1, \ldots, x_v)$ of H with the property that $x_i = 1$ for precisely k values of the index i and $x_i = 0$ for the remaining $v - k$ values.*

Finally, we recall an alternative characterization, which is the analogue of Proposition 3 above for the notion of strong additivity.

Proposition 6. ([9, Remark 2.2]) *A 2-(v, k, λ) design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is strongly additive if and only if, up to isomorphism, $\mathcal{P}$ can be represented as a subset of a suitable commutative group $(G, +)$ (not necessarily isomorphic to $\mathfrak{G}_{\mathcal{D}}$) in such a way that a k-subset of $\mathcal{P}$ is a block in $\mathcal{B}$ if and only if the sum of its elements is zero in G.*

This characterization provides a simpler and more effective way to prove that a given design is strongly additive, as in the case of the Steiner quadruple system $\mathrm{AG}_2(4, 2)$, that is, the point-plane design of the affine geometry $\mathrm{AG}(4, 2)$, whose blocks are precisely all the 4-subsets of $G = \mathbb{F}_2^4$ whose elements sum up to zero. For this reason, Proposition 6 was actually presented as the definition of strong additivity in [5, Definition 1.1] and [24], where, in the latter case, one also finds an account of additivity for Steiner triple systems. On the other hand, this characterization does not make it any easier to ascertain that a given design is not strongly additive, as we will further clarify in Remark 7.

The main examples of strongly additive designs are given by symmetric designs [8], affine resolvable designs and their complements [9], geometric Steiner triple systems [8], Boolean Steiner quadruple systems [15,22], and the 2-designs introduced in [15] and [26] (which are strongly additive by Proposition 6 above).

In the special case where $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a geometric STS, not only $\mathcal{D}$ is strongly additive, but it also satisfies the additivity condition in the strongest possible sense, in that, if $\mathcal{P}$ is embedded in a commutative group $(G, +)$ (not necessarily isomorphic to $\mathfrak{G}_{\mathcal{D}}$), in such a way that the sum of the three elements of each block is zero, then, conversely, $\{x, y, z\}$ is a block in $\mathcal{B}$ for any 3-subset $\{x, y, z\}$ of $\mathcal{P}$ such that $x + y + z = 0$ in G [8,24]. In this respect, for STSs, the group $\mathfrak{G}_{\mathcal{D}}$ does not appear to be any different from any other group G, in order to describe and understand the strong additivity of $\mathcal{D}$.

Remark 7. Unlike for Steiner triple systems, the distinguished group $\mathfrak{G}_{\mathcal{D}}$ associated with a general strongly additive design $\mathcal{D}$ plays a crucial role for the validity of the strong additivity. Indeed, there exist strongly additive designs $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ with the property that $\mathcal{P}$ can be embedded in a suitable commutative group $(G, +)$ (necessarily *not* isomorphic to $\mathfrak{G}_{\mathcal{D}}$) in such a way that the family of blocks $\mathcal{B}$ is strictly contained in the class of the k-subsets of $\mathcal{P}$ whose elements sum up to zero in G (such an embedding could be called a *non-strongly additive embedding* of $\mathcal{D}$). This happens, for instance, in the case where $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is the 2-$(11, 5, 2)$ (symmetric) Hadamard design $\mathcal{H}_{11}$ (equivalently, the 11-point biplane), whose associated group $\mathfrak{G}_{\mathcal{D}}$ is essentially isomorphic to (the additive group of) the 5-dimensional vector space $\mathbb{F}_3^5$ [8, Example 2.3], and which can be embedded in $\mathbb{F}_3^3$ and in $\mathbb{F}_3^4$ in such a way that each block is a zero-sum 5-subset of $\mathcal{P}$, but not conversely [8, Remark 4.4] (this is actually the case for any embedding in $\mathbb{F}_3^4$ [23]). Other examples are the affine plane $\mathrm{AG}(2, p)$, for any prime $p > 3$ [9, Remark 3.8], and the affine plane of order four $\mathrm{AG}(2, 4)$ (see Remarks 11 and 16 in Sections 3 and 4 of the present paper).

If a 2-design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is strongly additive, then one can characterize its automorphism group in terms of the embedding of $\mathcal{D}$ into $\mathfrak{G}_{\mathcal{D}}$, thereby generalizing [8, Corollary 4.3] and [9, Corollary 3.4].

Proposition 8. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a strongly additive 2-design, and let $\mathcal{P}$ be represented as a subset of the group $\mathfrak{G}_{\mathcal{D}}$. Then $\mathrm{Aut}(\mathcal{D})$, the automorphism group of $\mathcal{D}$, is (isomorphic to) the group of automorphisms of $\mathfrak{G}_{\mathcal{D}}$ that leave $\mathcal{P}$ invariant.*

Proof. Let $(\mathfrak{G}, +)$ be the free commutative group generated by the points of $\mathcal{P}$, and let $\mathfrak{R}$ be the subgroup of $\mathfrak{G}$ generated by the $|\mathcal{B}|$ elements of the form $\sum_{X \in \mathfrak{b}} X$, as $\mathfrak{b}$ ranges over the blocks in $\mathcal{B}$. Then, by Definition 1, the group $\mathfrak{G}_{\mathcal{D}}$ is the quotient group $\mathfrak{G}/\mathfrak{R}$, and the quotient map $\chi : \mathcal{P} \longrightarrow \mathfrak{G}_{\mathcal{D}}$, defined by $\chi(X) = X + \mathfrak{R}$, is injective. By construction, $\sum_{X \in \mathfrak{b}} \chi(X) = 0$ for any block $\mathfrak{b}$ in $\mathcal{B}$.

Let f be an automorphism of $\mathcal{D}$, and let $\mathfrak{f}$ be the extension of f to an automorphism of $\mathfrak{G}$. Since the subgroup $\mathfrak{R}$ of $\mathfrak{G}$ is invariant under $\mathfrak{f}$, the automorphism $\mathfrak{f}$ induces in turn an automorphism F of $\mathfrak{G}_{\mathcal{D}} = \mathfrak{G}/\mathfrak{R}$ defined by $F(g + \mathfrak{R}) = \mathfrak{f}(g) + \mathfrak{R}$ for all g in $\mathfrak{G}$.

Hence $F(\chi(X)) = F(X + \mathfrak{R}) = \mathfrak{f}(X) + \mathfrak{R} = f(X) + \mathfrak{R} = \chi(f(X))$ for all X in $\mathcal{P}$, that is, $\chi(\mathcal{P})$ is invariant under F.

Conversely, if an automorphism $F \in \mathrm{Aut}(\mathfrak{G}_{\mathcal{D}})$ leaves the set $\chi(\mathcal{P})$ invariant, then it maps a block (in $\chi(\mathcal{B})$), which is a k-set of elements of $\chi(\mathcal{P})$ whose sum is zero, onto a k-set of elements of $\chi(\mathcal{P})$ whose sum is zero, which, by Definition 4, is again a block, since $\mathcal{D}$ is strongly additive by hypothesis. Hence the restriction of F to $\chi(\mathcal{P})$ induces an automorphism $f \in \mathrm{Aut}(\mathcal{D})$, such that $F(\chi(X)) = \chi(f(X))$ for all X in $\mathcal{P}$. $\square$

In some special cases, in addition to the property in Proposition 8, $\mathfrak{G}_{\mathcal{D}}$ is essentially the additive group of a finite vector space, and the points in $\mathcal{P}$ can be coordinatized as elements $(x_1, x_2, \ldots, x_d)$ of $\mathfrak{G}_{\mathcal{D}}$ in such a way that each block can be described as the set of all points in $\mathcal{P}$ satisfying a suitable equation $a_1x_1 + a_2x_2 + \cdots + a_dx_d = c$. This happens, for instance, in the case where $\mathcal{D}$ is the above mentioned 11-point biplane $\mathcal{H}_{11}$ [8, Example 4.11], or the 3-$(12, 6, 2)$ affine Hadamard design [9, Example 3.6], or the Steiner triple system of order 9 [24]. In the latter case, moreover, the design can be represented as the set of the nine points of the affine hyperplane $x_1 + x_2 + x_3 = 1$ in the 3-dimensional vector space $\mathbb{F}_3^3$ [8, Remark 3.8(b)]. For all these examples, the automorphisms of $\mathcal{D}$ can be represented as linear maps on $\mathfrak{G}_{\mathcal{D}}$, seen as a finite vector space. In particular, these examples show how strongly additive designs can be sometimes applied to the representation theory of finite groups.

After this overview of the properties of additive and strongly additive designs, the main question to ask, which had remained unanswered thus far, is whether the class of strongly additive 2-designs is strictly contained in the class of additive 2-designs. Until not too long ago, no single example was known of an additive 2-design that was not strongly additive. After repeated attempts to settle the question, an open problem was finally posed in [9, 3.10] as to whether any additive 2-design is also strongly additive.

The main goal of this paper is to provide explicit examples of additive 2-designs that are not strongly additive. The construction of the counterexamples will be the main content of the following part of the paper.

3. An example of an additive 2-design that is not strongly additive

Let

$$\mathbb{F}_4 = \{0, 1, \alpha, \alpha^2\}$$

be the (unique) field of order four. As it is well known, $\mathbb{F}_4$ is a field of characteristic 2, and necessarily $1 + \alpha = \alpha^2$ and $\alpha^3 = 1$.

The main result of this paper is the construction of a 2-design, with point-set $\mathbb{F}_4 \times \mathbb{F}_4$, that is additive but not strongly additive.

Theorem 9. *There exists a (resolvable)* 2-$(16, 4, 2)$ *design that is additive but not strongly additive.*

Let

$$\mathcal{D}_1 = (\mathbb{F}_4 \times \mathbb{F}_4, \mathcal{B}_1)$$

be the point-line design of the affine plane $\mathrm{AG}(2,4)$ of order four (sometimes denoted by $\mathrm{AG}_2(4)$), where $\mathcal{B}_1$ is the family of the 20 affine lines in the 2-dimensional vector space $\mathbb{F}_4 \times \mathbb{F}_4$ over $\mathbb{F}_4$. By definition, each block in $\mathcal{B}_1$ is a translate (or coset) of a 1-dimensional subspace, that is, a 4-subset of $\mathbb{F}_4 \times \mathbb{F}_4$ of the form

$$\{\boldsymbol{v} + t\boldsymbol{w} \mid t \in \mathbb{F}_4\},$$

with $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{F}_4 \times \mathbb{F}_4$, $\boldsymbol{w} \neq \mathbf{0}$. In particular, it immediately follows that

$$(\forall\, \mathfrak{b} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\} \in \mathcal{B}_1) \;\; \boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3 + \boldsymbol{v}_4 = \mathbf{0}, \tag{1}$$

since $(\boldsymbol{v} + 0\boldsymbol{w}) + (\boldsymbol{v} + 1\boldsymbol{w}) + (\boldsymbol{v} + \alpha\boldsymbol{w}) + (\boldsymbol{v} + \alpha^2\boldsymbol{w}) = (1 + \alpha + \alpha^2)\boldsymbol{w} = \mathbf{0}$ for all $\boldsymbol{v}, \boldsymbol{w}$ in $\mathbb{F}_4 \times \mathbb{F}_4$, $\boldsymbol{w} \neq \mathbf{0}$.

This makes $\mathcal{D}_1$ an additive, affine resolvable, 2-$(16, 4, 1)$ design (hence also strongly additive by [9, Theorem 3.1]), whose 20 blocks can be collected in five parallel classes as follows (where, for short, any element (x, y) of $\mathbb{F}_4 \times \mathbb{F}_4$ is denoted by xy).

$$\begin{array}{llll}
\{00, 10, \alpha 0, \alpha^2 0\} & \{01, 11, \alpha 1, \alpha^2 1\} & \{0\alpha, 1\alpha, \alpha\alpha, \alpha^2\alpha\} & \{0\alpha^2, 1\alpha^2, \alpha\alpha^2, \alpha^2\alpha^2\} \\
\{00, 01, 0\alpha, 0\alpha^2\} & \{10, 11, 1\alpha, 1\alpha^2\} & \{\alpha 0, \alpha 1, \alpha\alpha, \alpha\alpha^2\} & \{\alpha^2 0, \alpha^2 1, \alpha^2\alpha, \alpha^2\alpha^2\} \\
\{00, 11, \alpha\alpha, \alpha^2\alpha^2\} & \{01, 10, \alpha\alpha^2, \alpha^2\alpha\} & \{0\alpha, 1\alpha^2, \alpha 0, \alpha^2 1\} & \{0\alpha^2, 1\alpha, \alpha 1, \alpha^2 0\} \\
\{00, \alpha 1, \alpha^2\alpha, 1\alpha^2\} & \{01, \alpha 0, \alpha^2\alpha^2, 1\alpha\} & \{10, \alpha^2 1, \alpha\alpha, 0\alpha^2\} & \{11, \alpha^2 0, \alpha\alpha^2, 0\alpha\} \\
\{00, 1\alpha, \alpha\alpha^2, \alpha^2 1\} & \{10, 0\alpha, \alpha^2\alpha^2, \alpha 1\} & \{01, 1\alpha^2, \alpha\alpha, \alpha^2 0\} & \{11, 0\alpha^2, \alpha^2\alpha, \alpha 0\}.
\end{array} \tag{2}$$

We now look for a bijection $\varphi : \mathbb{F}_4 \times \mathbb{F}_4 \to \mathbb{F}_4 \times \mathbb{F}_4$ satisfying the two following conditions.

(I) $\varphi(\mathcal{B}_1) \cap \mathcal{B}_1 = \emptyset$,
(II) $(\forall\, \mathfrak{b} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\} \in \mathcal{B}_1)\; \varphi(\boldsymbol{v}_1) + \varphi(\boldsymbol{v}_2) + \varphi(\boldsymbol{v}_3) + \varphi(\boldsymbol{v}_4) = \mathbf{0}$,

where $\varphi(\mathcal{B}_1)$ denotes the family of all the 4-sets $\varphi(\mathfrak{b}) = \{\varphi(\boldsymbol{v}_1), \varphi(\boldsymbol{v}_2), \varphi(\boldsymbol{v}_3), \varphi(\boldsymbol{v}_4)\} \subseteq \mathbb{F}_4 \times \mathbb{F}_4$, as $\mathfrak{b} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ ranges in $\mathcal{B}_1$.

By property (1), condition (II) would be trivially satisfied if φ were linear over $\mathbb{F}_4$, but, if this were the case, then φ would induce a permutation of the 20 affine lines in $\mathcal{B}_1$, whence condition (I) would fail to hold. Hence we make a change of perspective, and regard $\mathbb{F}_4 \times \mathbb{F}_4$ as a 4-dimensional vector space over $\mathbb{F}_2$, with “canonical” basis $10, \alpha 0, 01, 0\alpha$. It now suffices to consider the invertible $\mathbb{F}_2$-linear map

$$\varphi : \mathbb{F}_4 \times \mathbb{F}_4 \to \mathbb{F}_4 \times \mathbb{F}_4$$

associated with the matrix

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix} \tag{3}$$

with respect to the canonical basis. By linearity, and by property (1), the condition (II) above is trivially satisfied. Also, it is easy to check that

$$\begin{array}{llllllll} \varphi(00) & = 00 & \varphi(01) & = \alpha 0 & \varphi(0\alpha) & = \alpha\alpha^2 & \varphi(0\alpha^2) & = 0\alpha^2 \\ \varphi(10) & = \alpha^2\alpha^2 & \varphi(11) & = 1\alpha^2 & \varphi(1\alpha) & = 10 & \varphi(1\alpha^2) & = \alpha^2 0 \\ \varphi(\alpha 0) & = 1\alpha & \varphi(\alpha 1) & = \alpha^2\alpha & \varphi(\alpha\alpha) & = \alpha^2 1 & \varphi(\alpha\alpha^2) & = 11 \\ \varphi(\alpha^2 0) & = \alpha 1 & \varphi(\alpha^2 1) & = 01 & \varphi(\alpha^2\alpha) & = 0\alpha & \varphi(\alpha^2\alpha^2) & = \alpha\alpha, \end{array} \tag{4}$$

whence one can verify, by inspection, that the condition (I) above is also satisfied. Alternatively, since each parallel class in (2) consists of the translates of some line through the origin, and since φ maps translates to translates by linearity, it actually suffices to verify that the five affine lines through the origin in (2), together with their images under φ (see the first column in (5)), form ten distinct 4-subsets of $\mathbb{F}_4 \times \mathbb{F}_4$. This also confirms, in passing, that φ is not linear over $\mathbb{F}_4$.

If we now define

$$\varphi(\mathcal{D}_1) = (\mathbb{F}_4 \times \mathbb{F}_4, \varphi(\mathcal{B}_1)),$$

then $\varphi(\mathcal{D}_1)$ is a 2-$(16, 4, 1)$ design and, by construction,

$$\varphi(\mathcal{D}_1) \simeq \mathcal{D}_1 = \mathrm{AG}(2, 4)$$

(on the other hand, $\mathrm{AG}(2, 4)$ is the only 2-$(16, 4, 1)$ design, up to isomorphism). In particular, $\varphi(\mathcal{D}_1)$ is an (additive) affine resolvable design, whose 20 blocks can be collected in five parallel classes as follows.

$$\begin{array}{llll} \{00, \alpha^2\alpha^2, 1\alpha, \alpha 1\} & \{\alpha 0, 1\alpha^2, \alpha^2\alpha, 01\} & \{\alpha\alpha^2, 10, \alpha^2 1, 0\alpha\} & \{0\alpha^2, \alpha^2 0, 11, \alpha\alpha\} \\ \{00, \alpha 0, \alpha\alpha^2, 0\alpha^2\} & \{\alpha^2\alpha^2, 1\alpha^2, 10, \alpha^2 0\} & \{1\alpha, \alpha^2\alpha, \alpha^2 1, 11\} & \{\alpha 1, 01, 0\alpha, \alpha\alpha\} \\ \{00, 1\alpha^2, \alpha^2 1, \alpha\alpha\} & \{\alpha 0, \alpha^2\alpha^2, 11, 0\alpha\} & \{\alpha\alpha^2, \alpha^2 0, 1\alpha, 01\} & \{0\alpha^2, 10, \alpha^2\alpha, \alpha 1\} \\ \{00, \alpha^2\alpha, 0\alpha, \alpha^2 0\} & \{\alpha 0, 1\alpha, \alpha\alpha, 10\} & \{\alpha^2\alpha^2, 01, \alpha^2 1, 0\alpha^2\} & \{1\alpha^2, \alpha 1, 11, \alpha\alpha^2\} \\ \{00, 10, 11, 01\} & \{\alpha^2\alpha^2, \alpha\alpha^2, \alpha\alpha, \alpha^2\alpha\} & \{\alpha 0, \alpha^2 0, \alpha^2 1, \alpha 1\} & \{1\alpha^2, 0\alpha^2, 0\alpha, 1\alpha\}. \end{array} \tag{5}$$

We can finally define

$$\mathcal{D}_2 = (\mathbb{F}_4 \times \mathbb{F}_4, \mathcal{B}_2),$$

where

$$\mathcal{B}_2 = \mathcal{B}_1 \cup \varphi(\mathcal{B}_1).$$

By construction, $\mathcal{B}_2$ contains precisely 40 distinct blocks, which make $\mathcal{D}_2$ a simple (that is, with no repeated blocks) resolvable 2-$(16, 4, 2)$ design. Also, because of properties (1) and (II) above, $\mathcal{D}_2$ is an additive design by [8, Proposition 2.7] (see Proposition 3 above). Note that $\mathcal{D}_2$, unlike $\mathcal{D}_1$ and $\varphi(\mathcal{D}_1)$, is resolvable but not affine resolvable, hence it is not necessarily strongly additive. According to the terminology in [17,18] (see also [1, Remark 2.24(a)]), $\mathcal{D}_2$ is a *quasidouble* (or 2-*fold quasimultiple*) of the 2-$(16, 4, 1)$ design, and, in fact, $\mathcal{D}_2$ decomposes into two disjoint isomorphic copies of the affine plane of order four. Following [19, 4.1], this is the case for the majority of the 325062 non-isomorphic resolvable 2-$(16, 4, 2)$ designs, as only 5001 *indecomposable* 2-$(16, 4, 2)$ designs are resolvable but do not contain $\mathrm{AG}(2, 4)$ as a subdesign.

We now prove that $\mathcal{D}_2$ is not strongly additive. By Definition 4 in Section 2, it suffices to compute the group $\mathfrak{G}_{\mathcal{D}_2}$ and show that, when $\mathbb{F}_4 \times \mathbb{F}_4$ is identified with the corresponding subset of $\mathfrak{G}_{\mathcal{D}_2}$, there exists a 4-subset $\{X_1, X_2, X_3, X_4\}$ of $\mathbb{F}_4 \times \mathbb{F}_4$ that is not a block in $\mathcal{B}_2$ and is such that $X_1 + X_2 + X_3 + X_4 = 0$ in $\mathfrak{G}_{\mathcal{D}_2}$.

Lemma 10. *$\mathfrak{G}_{\mathcal{D}_2} \simeq (\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/4\mathbb{Z})^3$.*

Proof. By definition, $\mathfrak{G}_{\mathcal{D}_2}$ is the finitely presented commutative group whose generators are the sixteen elements of $\mathbb{F}_4 \times \mathbb{F}_4$ and whose relations are the equalities $X_1 + X_2 + X_3 + X_4 = 0$, as $\{X_1, X_2, X_3, X_4\}$ ranges over the 40 blocks in $\mathcal{B}_2$. As we will now show, it actually suffices to consider only 16 blocks in $\mathcal{B}_2$. Indeed, let us consider the 16 blocks

$$\begin{array}{l}
\{00, 10, 01, 11\} \\
\{00, 10, \alpha 0, \alpha^2 0\} \\
\{00, 01, 0\alpha, 0\alpha^2\} \\
\{11, \alpha 0, 0\alpha, \alpha^2\alpha^2\} \\
\{00, 11, \alpha\alpha, \alpha^2\alpha^2\} \\
\{01, \alpha\alpha, \alpha^2 0, 1\alpha^2\} \\
\{10, \alpha\alpha, 0\alpha^2, \alpha^2 1\} \\
\{00, \alpha 0, 0\alpha^2, \alpha\alpha^2\} \\
\{00, 0\alpha, \alpha^2 0, \alpha^2\alpha\} \\
\{11, 1\alpha, \alpha^2 1, \alpha^2\alpha\} \\
\{11, \alpha 1, 1\alpha^2, \alpha\alpha^2\} \\
\{10, 01, \alpha\alpha^2, \alpha^2\alpha\} \\
\{\alpha 0, 0\alpha, 1\alpha^2, \alpha^2 1\} \\
\{10, 11, 1\alpha, 1\alpha^2\} \\
\{00, \alpha 1, 1\alpha^2, \alpha^2\alpha\} \\
\{\alpha 0, \alpha\alpha, \alpha 1, \alpha\alpha^2\},
\end{array} \tag{6}$$

in this order. If we replace each block $\{X_1, X_2, X_3, X_4\}$ in (6) by the corresponding relation $X_1 + X_2 + X_3 + X_4 = 0$, then we get a system of 16 equations, whose unknowns are the 16 elements of $\mathbb{F}_4 \times \mathbb{F}_4$, which is easily seen to produce, in the same order as the corresponding blocks, the following equalities.

$$\begin{array}{lcl}
11 & = & -00 - 10 - 01 \\
\alpha^2 0 & = & -00 - 10 - \alpha 0 \\
0\alpha^2 & = & -00 - 01 - 0\alpha \\
\alpha^2\alpha^2 & = & 00 + 10 + 01 - \alpha 0 - 0\alpha \\
\alpha\alpha & = & -00 + \alpha 0 + 0\alpha \\
1\alpha^2 & = & 2 \cdot 00 + 10 - 01 - 0\alpha \\
\alpha^2 1 & = & 2 \cdot 00 - 10 + 01 - \alpha 0 \\
\alpha\alpha^2 & = & 01 - \alpha 0 + 0\alpha \\
\alpha^2\alpha & = & 10 + \alpha 0 - 0\alpha \\
1\alpha & = & -00 + 10 + 0\alpha \\
\alpha 1 & = & -00 + 01 + \alpha 0 \\
0 & = & 2 \cdot 10 + 2 \cdot 01 \\
0 & = & 4 \cdot 00 \\
0 & = & 4 \cdot 10 \\
0 & = & 2 \cdot 00 + 2 \cdot 10 + 2 \cdot \alpha 0 - 2 \cdot 0\alpha \\
0 & = & 4 \cdot \alpha 0 \ .
\end{array} \tag{7}$$

It can be easily verified that, because of the previous equalities, each of the relations corresponding to the remaining 24 blocks in $\mathcal{B}_2$ just reduces to the trivial relation $0 = 0$, thereby showing that $\mathfrak{G}_{\mathcal{D}_2}$ is only determined by the relations corresponding to the 16 blocks in (6). Also, each of the eleven elements $11, \alpha^2 0, 0\alpha^2, \alpha^2\alpha^2, \alpha\alpha, 1\alpha^2, \alpha^2 1, \alpha\alpha^2, \alpha^2\alpha, 1\alpha, \alpha 1$ is expressed in terms of the five elements $00, 10, 01, \alpha 0, 0\alpha$. This, together with the last five equalities in (7), finally shows that $\mathfrak{G}_{\mathcal{D}_2}$ is the finitely presented commutative group with generators $00, 10, 01, \alpha 0, 0\alpha$ and relations $4 \cdot 00 = 0$, $4 \cdot 10 = 0$, $4 \cdot \alpha 0 = 0$, $2 \cdot 10 + 2 \cdot 01 = 0$, $2 \cdot 0\alpha = 2 \cdot 00 + 2 \cdot 10 + 2 \cdot \alpha 0$. Hence we can conclude that, up to isomorphism, $\mathfrak{G}_{\mathcal{D}_2} = (\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/4\mathbb{Z})^3$, as claimed. $\square$

Note, in passing, that it can be shown that $\mathfrak{G}_{\mathcal{D}_1} \simeq (\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/4\mathbb{Z})^5$ by a similar argument.

Proof of Theorem 9. It suffices to show that the 2-$(16, 4, 2)$ design $\mathcal{D}_2 = (\mathbb{F}_4 \times \mathbb{F}_4, \mathcal{B}_2)$ is not strongly additive.

It follows from the proof of Lemma 10 that the generators $00, 10, 01, \alpha 0, 0\alpha$ of $\mathfrak{G}_{\mathcal{D}_2}$ can be represented as elements of $(\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/4\mathbb{Z})^3$ as follows.

$$\begin{array}{lcl}
00 & = & (0,0,0,0,1) \\
10 & = & (0,0,0,1,0) \\
01 & = & (0,1,0,1,0) \\
\alpha 0 & = & (0,0,1,0,0) \\
0\alpha & = & (1,0,1,1,1).
\end{array}$$

Also, because of the first eleven equalities in (7), one gets as well the following representation of the remaining eleven points of $\mathbb{F}_4 \times \mathbb{F}_4$.

$$\begin{array}{lcl} 11 & = & (0,1,0,2,3) \\ \alpha^2 0 & = & (0,0,3,3,3) \\ 0\alpha^2 & = & (1,1,3,2,2) \\ \alpha^2\alpha^2 & = & (1,1,2,1,0) \\ \alpha\alpha & = & (1,0,2,1,0) \\ 1\alpha^2 & = & (1,1,3,3,1) \\ \alpha^2 1 & = & (0,1,3,0,2) \\ \alpha\alpha^2 & = & (1,1,0,2,1) \\ \alpha^2\alpha & = & (1,0,0,0,3) \\ 1\alpha & = & (1,0,1,2,0) \\ \alpha 1 & = & (0,1,1,1,3). \end{array}$$

With this identification of $\mathbb{F}_4 \times \mathbb{F}_4$ with a set of 16 distinct points in $\mathfrak{G}_{\mathcal{D}_2} = (\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/4\mathbb{Z})^3$ (which further confirms, by Definition 1 in Section 2, that the design $\mathcal{D}_2 = (\mathbb{F}_4 \times \mathbb{F}_4, \mathcal{B}_2)$ is additive), the four elements of each block in $\mathcal{B}_2$ sum up to zero by construction. This can also be readily verified by inspection. For instance, $\{00, \alpha 1, 1\alpha, \alpha^2\alpha^2\}$ is a block in $\varphi(\mathcal{B}_1) \subseteq \mathcal{B}_2$, and

$$\begin{aligned} 00 + \alpha 1 + 1\alpha + \alpha^2\alpha^2 &= (0,0,0,0,1) + (0,1,1,1,3) + (1,0,1,2,0) + (1,1,2,1,0) \\ &= (0,0,0,0,0). \end{aligned}$$

On the other hand, the 4-set $\{00, 11, \alpha\alpha^2, \alpha^2\alpha\}$ is not a block in $\mathcal{B}_2$, and

$$\begin{aligned} 00 + 11 + \alpha\alpha^2 + \alpha^2\alpha &= (0,0,0,0,1) + (0,1,0,2,3) + (1,1,0,2,1) + (1,0,0,0,3) \\ &= (0,0,0,0,0) \end{aligned}$$

in $\mathfrak{G}_{\mathcal{D}_2}$. This finally proves, by Definition 4 in Section 2, that the 2-$(16,4,2)$ design $\mathcal{D}_2 = (\mathbb{F}_4 \times \mathbb{F}_4, \mathcal{B}_2)$ is not strongly additive.

This completes the proof of the theorem. □

Remark 11. Since the 4-subset $\{00, 11, \alpha\alpha^2, \alpha^2\alpha\}$ of $\mathbb{F}_4 \times \mathbb{F}_4$ is not a block in $\mathcal{B}_1$, but its elements sum up to zero in $(\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/4\mathbb{Z})^3$, the embedding of $\mathcal{D}_1 = (\mathbb{F}_4 \times \mathbb{F}_4, \mathcal{B}_1)$ in $(\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/4\mathbb{Z})^3$ is a further example of a non-strongly additive embedding of a strongly additive design (see Remark 7 in Section 2).

In the following section, by iterating the previous construction, we will exhibit for each n in $\{3,4,5,6\}$ an additive resolvable 2-$(16,4,n)$ design $\mathcal{D}_n = (\mathbb{F}_4 \times \mathbb{F}_4, \mathcal{B}_n)$ that is not strongly additive, and which is defined, as in the case $n = 2$, as the disjoint union of n copies of the affine plane of order four.

4. A decomposition of $\mathbf{AG_2(4,2)}$ into seven disjoint copies of the affine plane of order four

In this final section of the paper, we consider again the map

$$\varphi : \mathbb{F}_4 \times \mathbb{F}_4 \to \mathbb{F}_4 \times \mathbb{F}_4$$

defined above in Section 3, in order to extend the construction given therein and provide further examples of additive 2-designs that are not strongly additive. By doing so, we also obtain a remarkable property of the classic geometric design $\mathrm{AG}_2(4,2)$.

We first show that, by considering the orbit under φ of the family $\mathcal{B}_1$ of the 20 affine lines of $\mathrm{AG}(2,4)$, the 3-(16, 4, 1) Steiner quadruple system $\mathrm{AG}_2(4,2)$, that is, the point-plane design of the affine geometry $\mathrm{AG}(4,2)$, which is also a 2-(16, 4, 7) design with 140 blocks, decomposes into seven disjoint isomorphic copies of the affine plane of order four. In particular, $\mathrm{AG}_2(4,2)$ is a 7-fold quasimultiple of $\mathrm{AG}(2,4)$ in the strongest possible sense.

Let us first recall that $\mathrm{AG}_2(4,2)$ is the block design whose point-set is the 4-dimensional vector space $\mathbb{F}_2^4$ over $\mathbb{F}_2$, and whose blocks are all the affine planes in $\mathbb{F}_2^4$, that is, equivalently, all the 4-subsets of $\mathbb{F}_2^4$ whose elements sum up to zero. In particular, $\mathrm{AG}_2(4,2)$ is strongly additive by Proposition 6 in Section 2. In passing, $\mathrm{AG}_2(4,2)$ coincides with the *Boolean design* $\mathcal{D}_4 = (\mathbb{F}_2^4, \mathcal{B}_4)$ considered in [15, Proposition 2.5 and Remark 2.1(ii)], where it is also independently shown that the full group of automorphisms is (isomorphic to) the group of invertible affine mappings on $\mathbb{F}_2^4$ over $\mathbb{F}_2$. If we consider the standard group isomorphism

$$\mathbb{F}_4 \simeq \mathbb{F}_2 \times \mathbb{F}_2$$

defined by means of the identifications

$$0 \equiv (0,0) \quad 1 \equiv (1,0) \quad \alpha \equiv (0,1) \quad \alpha^2 \equiv (1,1), \tag{8}$$

then $\mathrm{AG}_2(4,2)$ can be seen as the design whose point-set is $\mathbb{F}_4 \times \mathbb{F}_4$, and whose blocks are all the 4-subsets of $\mathbb{F}_4 \times \mathbb{F}_4$ whose elements sum up to 00. In particular, by property (1) in Section 3, the affine plane $\mathrm{AG}(2,4)$ of order four, which we denoted by $\mathcal{D}_1 = (\mathbb{F}_4 \times \mathbb{F}_4, \mathcal{B}_1)$, can be seen, up to isomorphism, as a 2-(16, 4, 1) subdesign of $\mathrm{AG}_2(4,2)$.

The crucial property of the matrix M defined in (3) is that, as it can be easily verified,

$$M^7 = I,$$

that is, equivalently, $\varphi^7(\boldsymbol{v}) = \boldsymbol{v}$ for all $\boldsymbol{v}$ in $\mathbb{F}_4 \times \mathbb{F}_4$. We claim that the orbit under φ of the 20 affine lines in $\mathcal{B}_1$ contains precisely 140 distinct 4-subsets of $\mathbb{F}_4 \times \mathbb{F}_4$ whose elements sum up to 00, which must necessarily coincide with the 140 blocks of $\mathrm{AG}_2(4, 2)$.

Lemma 12. *The* 140 *blocks of* $\mathrm{AG}_2(4, 2)$*, seen as* 4*-subsets of* $\mathbb{F}_4 \times \mathbb{F}_4$*, are the orbit under* φ *of the* 20 *blocks of the affine plane* $\mathcal{D}_1 = (\mathbb{F}_4 \times \mathbb{F}_4, \mathcal{B}_1) \simeq \mathrm{AG}(2, 4)$*.*

Proof. Since each plane through the origin in $\mathbb{F}_4 \times \mathbb{F}_4$ has precisely four translates (or cosets) and, conversely, each affine plane in $\mathbb{F}_4 \times \mathbb{F}_4$ is the translate of a unique plane through the origin, and since, by linearity, each power of φ maps translates to translates, it suffices to prove that the orbit under φ of the 5 affine lines through 00 in $\mathcal{B}_1$ contains all the 35 distinct planes through the origin in $\mathrm{AG}_2(4, 2)$, that is, all the 35 distinct 4-subsets of $\mathbb{F}_4 \times \mathbb{F}_4$ containing 00 and whose elements sum up to 00. Since the orbit under φ of 00 only contains 00, it finally suffices to show that the orbit under φ of the 5 affine lines through 00 in $\mathcal{B}_1$, with 00 removed, contains all the 35 distinct 3-subsets of $\mathbb{F}_4 \times \mathbb{F}_4 \setminus \{00\}$ whose elements sum up to 00 (that is, the 35 blocks of the derived design of $\mathrm{AG}_2(4, 2)$ at the origin). In passing, any such φ is necessarily $\mathbb{F}_2$-linear on $\mathbb{F}_4 \times \mathbb{F}_4$ by [15, Theorem 3.1].

As shown in (2) in Section 3, the five affine lines through 00 in $\mathcal{B}_1$, with 00 removed, are the five following 3-subsets of $\mathbb{F}_4 \times \mathbb{F}_4$:

$$\{10, \alpha 0, \alpha^2 0\} \quad \{01, 0\alpha, 0\alpha^2\} \quad \{11, \alpha\alpha, \alpha^2\alpha^2\} \quad \{\alpha 1, \alpha^2\alpha, 1\alpha^2\} \quad \{1\alpha, \alpha\alpha^2, \alpha^2 1\}.$$

In accordance with table (4) in Section 3, the orbit under φ of the previous five triples consists of the 35 following 3-subsets of $\mathbb{F}_4 \times \mathbb{F}_4$, where each row is obtained by applying φ to the preceding row.

$$\begin{array}{lllll}
\{10, \alpha 0, \alpha^2 0\} & \{01, 0\alpha, 0\alpha^2\} & \{11, \alpha\alpha, \alpha^2\alpha^2\} & \{\alpha 1, \alpha^2\alpha, 1\alpha^2\} & \{1\alpha, \alpha\alpha^2, \alpha^2 1\} \\
\{\alpha^2\alpha^2, 1\alpha, \alpha 1\} & \{\alpha 0, \alpha\alpha^2, 0\alpha^2\} & \{1\alpha^2, \alpha^2 1, \alpha\alpha\} & \{\alpha^2\alpha, 0\alpha, \alpha^2 0\} & \{10, 11, 01\} \\
\{\alpha\alpha, 10, \alpha^2\alpha\} & \{1\alpha, 11, 0\alpha^2\} & \{\alpha^2 0, 01, \alpha^2 1\} & \{0\alpha, \alpha\alpha^2, \alpha 1\} & \{\alpha^2\alpha^2, 1\alpha^2, \alpha 0\} \\
\{\alpha^2 1, \alpha^2\alpha^2, 0\alpha\} & \{10, 1\alpha^2, 0\alpha^2\} & \{\alpha 1, \alpha 0, 01\} & \{\alpha\alpha^2, 11, \alpha^2\alpha\} & \{\alpha\alpha, \alpha^2 0, 1\alpha\} \\
\{01, \alpha\alpha, \alpha\alpha^2\} & \{\alpha^2\alpha^2, \alpha^2 0, 0\alpha^2\} & \{\alpha^2\alpha, 1\alpha, \alpha 0\} & \{11, 1\alpha^2, 0\alpha\} & \{\alpha^2 1, \alpha 1, 10\} \\
\{\alpha 0, \alpha^2 1, 11\} & \{\alpha\alpha, \alpha 1, 0\alpha^2\} & \{0\alpha, 10, 1\alpha\} & \{1\alpha^2, \alpha^2 0, \alpha\alpha^2\} & \{01, \alpha^2\alpha, \alpha^2\alpha^2\} \\
\{1\alpha, 01, 1\alpha^2\} & \{\alpha^2 1, \alpha^2\alpha, 0\alpha^2\} & \{\alpha\alpha^2, \alpha^2\alpha^2, 10\} & \{\alpha^2 0, \alpha 1, 11\} & \{\alpha 0, 0\alpha, \alpha\alpha\}.
\end{array} \tag{9}$$

These 35 triples are all distinct, and our claim is proved. □

Corollary 13. $\mathrm{AG}_2(4, 2)$ *decomposes into the seven disjoint subdesigns* $(\mathbb{F}_4 \times \mathbb{F}_4, \mathcal{B}_1)$*,* $(\mathbb{F}_4 \times \mathbb{F}_4, \varphi(\mathcal{B}_1))$*,* $(\mathbb{F}_4 \times \mathbb{F}_4, \varphi^2(\mathcal{B}_1))$ *, . . . ,* $(\mathbb{F}_4 \times \mathbb{F}_4, \varphi^6(\mathcal{B}_1))$*, each of which is an isomorphic copy of the affine plane of order four.*

It is natural to ask whether in general, for every prime power q, $\mathrm{AG}_2(4, q)$ can be partitioned as the disjoint union of $q^2 + q + 1$ subdesigns, each isomorphic to the affine plane of order q^2, and whether such isomorphic copies form one orbit under a cyclic group of order q^2+q+1. By applying either of the two inequivalent cyclic regular packings of $\mathrm{PG}(3,q)$ constructed in [27], we expect that both properties are satisfied whenever $q \equiv 2 \pmod 3$.

We now apply the decomposition of $\mathrm{AG}_2(4,2)$ in Corollary 13 to the construction of further examples of additive 2-designs that are not strongly additive. By iterating the same construction as in Section 3, for each $n = 3,4,5,6,7$ we define

$$\mathcal{D}_n = (\mathbb{F}_4 \times \mathbb{F}_4, \mathcal{B}_n),$$

where

$$\mathcal{B}_n = \mathcal{B}_1 \cup \varphi(\mathcal{B}_1) \cup \varphi^2(\mathcal{B}_1) \cup \ldots \cup \varphi^{n-1}(\mathcal{B}_1).$$

By construction, $\mathcal{D}_n$ is an additive resolvable 2-$(16,4,n)$ design for each n, which can be partitioned as the disjoint union of n subdesigns, each isomorphic to the affine plane of order four. We claim that $\mathcal{D}_n$ is not strongly additive for all $n \in \{3,4,5,6\}$.

Lemma 14. $\mathfrak{G}_{\mathcal{D}_n} \simeq (\mathbb{Z}/2\mathbb{Z})^4 \times \mathbb{Z}/4\mathbb{Z}$ *for all* $n = 3,\ldots,7$.

Proof. By arguing as in Section 3, one can show that, for each $n = 3,\ldots,7$, $\mathfrak{G}_{\mathcal{D}_n}$ is the finitely presented commutative group with generators $00, 10, 01, \alpha 0, 0\alpha$ and relations $4 \cdot 00 = 0$, $2 \cdot 00 = 2 \cdot 10 = 2 \cdot 01 = 2 \cdot \alpha 0 = 2 \cdot 0\alpha$, whence, up to isomomorphism, $\mathfrak{G}_{\mathcal{D}_n} = (\mathbb{Z}/2\mathbb{Z})^4 \times \mathbb{Z}/4\mathbb{Z}$. □

Theorem 15. *The additive 2-design* $\mathcal{D}_n = (\mathbb{F}_4 \times \mathbb{F}_4, \mathcal{B}_n)$ *is not strongly additive for all* $n \in \{3,4,5,6\}$.

Proof. By the argument in the proof of Lemma 14, the generators $00, 10, 01, \alpha 0, 0\alpha$ of $\mathfrak{G}_{\mathcal{D}_n}$ can be represented as elements of $(\mathbb{Z}/2\mathbb{Z})^4 \times \mathbb{Z}/4\mathbb{Z}$ as follows.

$$\begin{array}{rcl} 00 & = & (0,0,0,0\,;1) \\ 10 & = & (1,0,0,0\,;1) \\ \alpha 0 & = & (0,1,0,0\,;1) \\ 01 & = & (0,0,1,0\,;1) \\ 0\alpha & = & (0,0,0,1\,;1). \end{array} \tag{10}$$

Also, because of the first eleven equalities in (7), one gets as well the following representation of the remaining eleven points of $\mathbb{F}_4 \times \mathbb{F}_4$.

$$\begin{array}{lcl} 11 & = & (1,0,1,0\,;1) \\ \alpha^2 0 & = & (1,1,0,0\,;1) \\ 0\alpha^2 & = & (0,0,1,1\,;1) \\ \alpha^2\alpha^2 & = & (1,1,1,1\,;1) \\ \alpha\alpha & = & (0,1,0,1\,;1) \\ 1\alpha^2 & = & (1,0,1,1\,;1) \\ \alpha^2 1 & = & (1,1,1,0\,;1) \\ \alpha\alpha^2 & = & (0,1,1,1\,;1) \\ \alpha^2\alpha & = & (1,1,0,1\,;1) \\ 1\alpha & = & (1,0,0,1\,;1) \\ \alpha 1 & = & (0,1,1,0\,;1). \end{array} \tag{11}$$

Now the 4-set $\{1\alpha, \alpha\alpha, \alpha^2\alpha^2, 0\alpha^2\} = \varphi^6\{10, \alpha^2 1, \alpha\alpha, 0\alpha^2\}$ is in $\mathcal{B}_7 \setminus \mathcal{B}_n$ for all $n = 3, 4, 5, 6$, and

$$\begin{aligned} 1\alpha + \alpha\alpha + \alpha^2\alpha^2 + 0\alpha^2 &= (1,0,0,1\,;1) + (0,1,0,1\,;1) + (1,1,1,1\,;1) + (0,0,1,1\,;1) \\ &= (0,0,0,0\,;0) \end{aligned}$$

in $\mathfrak{G}_{\mathcal{D}_n} \simeq (\mathbb{Z}/2\mathbb{Z})^4 \times \mathbb{Z}/4\mathbb{Z}$. This proves, by Definition 4 in Section 2, that the additive 2-$(16, 4, n)$ design $\mathcal{D}_n$ is not strongly additive for $n = 3, 4, 5, 6$.

This completes the proof of the theorem. $\square$

Remark 16. Note that, since the 4-subset $\{1\alpha, \alpha\alpha, \alpha^2\alpha^2, 0\alpha^2\}$ of $\mathbb{F}_4 \times \mathbb{F}_4$ is not a block in $\mathcal{B}_1$, but its elements sum up to zero in $(\mathbb{Z}/2\mathbb{Z})^4 \times \mathbb{Z}/4\mathbb{Z}$, the embedding of $\mathcal{D}_1 = (\mathbb{F}_4 \times \mathbb{F}_4, \mathcal{B}_1)$ in $(\mathbb{Z}/2\mathbb{Z})^4 \times \mathbb{Z}/4\mathbb{Z}$ is a further example of a non-strongly additive embedding of a strongly additive design (see Remark 7 in Section 2).

Remark 17. For $n = 7$, we showed above that

$$\mathcal{D}_7 = (\mathbb{F}_4 \times \mathbb{F}_4, \mathcal{B}_7) \simeq \mathrm{AG}_2(4, 2),$$

and that $\mathbb{F}_4 \times \mathbb{F}_4$ can be represented as the set of the sixteen points in (10) and (11) in the group $\mathfrak{G}_{\mathcal{D}_7} = (\mathbb{Z}/2\mathbb{Z})^4 \times \mathbb{Z}/4\mathbb{Z}$. Note that, interestingly enough, the embedding of $\mathbb{F}_4 \times \mathbb{F}_4$ in $(\mathbb{Z}/2\mathbb{Z})^4 \times \mathbb{Z}/4\mathbb{Z}$ is essentially the identity (up to the irrelevant fifth coordinate, always equal to 1, in (10) and (11)). Indeed, if for each x in $\mathbb{F}_4 = \{0, 1, \alpha, \alpha^2\}$ we denote by $f(x)$ the corresponding element in $(\mathbb{Z}/2\mathbb{Z})^2$ as in (8), then it can be easily checked by inspection that the embedding of $\mathbb{F}_4 \times \mathbb{F}_4$ in $(\mathbb{Z}/2\mathbb{Z})^4 \times \mathbb{Z}/4\mathbb{Z}$ is given precisely by

$$xy \mapsto (f(x), f(y)\,; 1)$$

for each xy in $\mathbb{F}_4 \times \mathbb{F}_4$. Being $f : \mathbb{F}_4 \to (\mathbb{Z}/2\mathbb{Z})^2$ a group homomorphism, this implies, in particular, that, for each 4-subset $\{X_1, X_2, X_3, X_4\}$ of $\mathbb{F}_4 \times \mathbb{F}_4$, $X_1 + X_2 + X_3 + X_4 = 0$ in $\mathbb{F}_4 \times \mathbb{F}_4$ if and only if $X_1 + X_2 + X_3 + X_4 = 0$ in $\mathfrak{G}_{\mathcal{D}_7} = (\mathbb{Z}/2\mathbb{Z})^4 \times \mathbb{Z}/4\mathbb{Z}$. Since, in turn, the former condition is equivalent to saying that $\{X_1, X_2, X_3, X_4\}$ is a block in $\mathcal{B}_7$, we conclude that $\mathcal{D}_7$ is strongly additive by Definition 4 in Section 2, thereby obtaining an alternative, direct, proof of the strong additivity of $\mathrm{AG}_2(4,2)$.

Remark 18. It is worth noting that $\mathrm{AG}_2(4,2)$ has also an interesting representation in $\mathbb{F}_2^5$. Indeed, if we recall again that $\mathrm{AG}_2(4,2)$ is the 2-$(16,4,7)$ design whose point-set is $\mathbb{F}_2^4$, and whose blocks are all the 4-subsets of $\mathbb{F}_2^4$ whose elements sum up to zero, we may define an embedding of $\mathrm{AG}_2(4,2)$ in $\mathbb{F}_2^5$ by means of the map

$$(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3, x_4, 1 + x_1 + x_2 + x_3 + x_4).$$

Under this identification, the points of $\mathrm{AG}_2(4,2)$ are precisely the 16 points of the affine hyperplane of equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 1$$

in the 5-dimensional vector space $\mathbb{F}_2^5$, and the blocks are the 4-subsets of the hyperplane whose elements sum up to zero. Similarly, the affine plane $\mathrm{AG}(2,3)$ can be represented as the set of the nine points of the affine hyperplane $x_1 + x_2 + x_3 = 1$ in the 3-dimensional vector space $\mathbb{F}_3^3$ (see [8, Remark 3.8(b)] and [24]). More generally, any 2-(p^n, k, λ) design $(\mathbb{F}_p^n, \mathcal{B})$, with p a prime dividing k, can be represented as the set of the p^n points of the affine hyperplane $x_1 + x_2 + \cdots + x_{n+1} = 1$ in the vector space $\mathbb{F}_p^{n+1}$, in such a way that, for any k-subset of the point-set $\mathbb{F}_p^n$, the sum of the k elements is zero in $\mathbb{F}_p^n$ if and only if the sum of the corresponding points of the hyperplane is zero in $\mathbb{F}_p^{n+1}$. It suffices to consider the embedding of $\mathbb{F}_p^n$ in $\mathbb{F}_p^{n+1}$ defined by $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 1 + (p-1)(x_1 + \cdots + x_n))$.

Remark 19. As a final application of the above decomposition of $\mathrm{AG}_2(4,2)$ (see Corollary 13), we can regard the arrangement of the distinct 35 triples in (9) as a Kirkman triple system (KTS) of order 15, that is, as a solution to Kirkman's schoolgirl problem, where the fifteen schoolgirls are represented as the fifteen nonzero elements of $\mathbb{F}_4 \times \mathbb{F}_4$.

Note that φ induces an order-7 automorphism of the KTS, with one fixed point (that is, $0\alpha^2$) and two orbits of length 7 (a KTS with this property is called 2-*rotational*). A natural question to ask is which of the seven non-isomorphic solutions of Kirkman's problem is isomorphic to the solution presented here in (9).

Proposition 20. *The solution of Kirkman's schoolgirl problem given in (9) is isomorphic to the KTS(*15*) denoted by 1b.*

Proof. As we already mentioned earlier, each triple in the arrangement (9) is a 3-subset of $\mathbb{F}_4 \times \mathbb{F}_4 \setminus \{00\}$ whose elements sum up to 00. By identifying $\mathbb{F}_4$ and $\mathbb{F}_2 \times \mathbb{F}_2$ as in (8), we immediately conclude that the distinct 35 triples in (9) are the 35 lines of the projective geometry PG(3, 2). Now it is well known that the point-line design of PG(3, 2) is the underlying Steiner triple system of two non-isomorphic KTS(15)s, commonly denoted by 1a and 1b (see, for instance, [11, p. 67] and [12, Table 19.6]), where the latter (respectively, former) system is the first (resp., second) published solution of Kirkman's problem, due to Cayley (resp., Kirkman). By means of an algorithm introduced in [14] (see also Theorem 2.4 in [25]), one can easily show that the KTS(15) in (9) is isomorphic to system 1b. The details are given in [25, §4, 3)], where, as we did above, we regard the point-line design of PG(3, 2) as the derived design of $\mathrm{AG}_2(4, 2)$ at the origin. □

Similarly, one can show that, if $\psi : \mathbb{F}_4 \times \mathbb{F}_4 \to \mathbb{F}_4 \times \mathbb{F}_4$ is the order-7 invertible $\mathbb{F}_2$-linear map defined on the canonical basis by $\psi(10) = 0\alpha$, $\psi(\alpha 0) = 10$, $\psi(01) = \alpha 0$, $\psi(0\alpha) = 1\alpha^2$, then the orbit under ψ of the five affine lines through 00 in $\mathcal{B}_1$, with 00 removed, is a KTS(15) isomorphic to system 1a (see again [25, §4, 3)]).

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