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# First and second critical exponents for an inhomogeneous Schrödinger equation with combined nonlinearities

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**Abstract.** We study the large-time behavior of solutions for the inhomogeneous nonlinear Schrödinger equation

$$iu_t + \Delta u = \lambda |u|^p + \mu |\nabla u|^q + w(x), \quad t > 0,\ x \in \mathbb{R}^N,$$

where $N \geq 1$, $p, q > 1$, $\lambda, \mu \in \mathbb{C}$, $\lambda \neq 0$, and $u(0, \cdot), w \in L^1_{\text{loc}}(\mathbb{R}^N, \mathbb{C})$. We consider both the cases where $\mu = 0$ and $\mu \neq 0$, respectively. We establish existence/nonexistence of global weak solutions. In each studied case, we compute the critical exponents in the sense of Fujita, and Lee and Ni. When $\mu \neq 0$, we show that the nonlinearity $|\nabla u|^q$ induces an interesting phenomenon of discontinuity of the Fujita critical exponent.

**Mathematics Subject Classification.** 35B33, 35B44, 35Q55.

**Keywords.** Critical exponent, Global weak solution, Nonlinear Schrödinger equation.

## Contents



## 1. Introduction and main results

Let $N \geq 1$, $p, q > 1$, $\lambda, \mu \in \mathbb{C}$ with $\lambda \neq 0$. Our goal here is to investigate the existence and nonexistence of global weak solutions to the inhomogeneous nonlinear Schrödinger problem

$$\begin{cases} iu_t + \Delta u = \lambda |u|^p + \mu |\nabla u|^q + w(x) & \text{in } (0, \infty) \times \mathbb{R}^N, \\ u(0, x) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

subject to the assumptions $u_0, w \in L^1_{\text{loc}}(\mathbb{R}^N, \mathbb{C})$.

The nonlinearity in the right side of the main equation presents the combined effects of two potential terms, depending by powers of $|u|$ and of $|\nabla u|$, respectively. Thus, when $\mu \neq 0$ our problem is gradient dependent, and this requires some specific estimates in our arguments of proofs. However, we know that

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the dependence on gradient occurs in many physical models of (heat) transport and fluid mechanics of plasma. Thus, it is an important key feature to model "rich dynamics" in evolution processes. Here, the cases $\mu = 0$ and $\mu \neq 0$ will be discussed separately. Precisely, we will show the suitability of our approach for the gradient dependence ($\mu \neq 0$) in the nonlinearity. We will develop this case as a perturbation of the starting case $\mu = 0$. In both the situations, we will also point out the effects of the inhomogeneous term $w \in L^1_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{C})$ on the global behavior of the problem. A crucial research direction is to establish the appropriate framework space, where the boundary data belong.

Starting with the case where $\mu = 0$ and $w \equiv 0$, (1.1) reduces to the Cauchy problem

$$\begin{cases} iu_t + \Delta u = \lambda |u|^p & \text{in } (0,\infty) \times \mathbb{R}^N, \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{1.2}$$

Problem (1.2) has been studied by many authors, whose finding leads to the consideration of certain special exponents' values. For $1 < p < 1 + \frac{4}{N-2s}$, $0 \leq s < \frac{N}{2}$, it is well known that local well-posedness for (1.2) holds in Sobolev spaces $H^s(\mathbb{R}^N)$ (see, e.g., [1,17]). When $N = 1$ and $p = 2$, it was shown in [10] that (1.2) is locally well posed in $H^s(\mathbb{R}^N)$, for $s > -\frac{1}{4}$. For arbitrary $N$, if $u_0 \in L^2(\mathbb{R}^N) \cap L^{1+\frac{1}{p}}(\mathbb{R}^N)$ and $1 + \frac{2}{N} < p_s < p < 1 + \frac{4}{N}$, where $p_s = \frac{N+2+\sqrt{N^2+4N+12}}{2N}$ is the Strauss exponent, the global existence for (1.2) for small initial data holds (see [16]). In [6], when $1 < p \leq 1 + \frac{2}{N}$, it was shown that, if $u_0 \in L^2(\mathbb{R}^N)$ and

$$\operatorname{Re} \lambda \operatorname{Im} \int_{\mathbb{R}^N} u_0(x)\,\mathrm{d}x < 0 \quad \text{or} \quad \operatorname{Im} \lambda \operatorname{Re} \int_{\mathbb{R}^N} u_0(x)\,\mathrm{d}x > 0,$$

then the $L^2$-norm of the solution to (1.2) blows up at finite time. Clearly, here by $z \in \mathbb{C}$, Re $z$ and Im $z$ we mean the real part and the imaginary part of $z$, respectively. Later, in [4], when $1 < p < 1 + \frac{4}{N}$, a small initial data blow-up result of the $L^2$-solution to (1.2) was derived. In [5], when $p > 1 + \frac{4}{N}$, for suitable $L^2$-data, it was shown that (1.2) admits no local weak solution. In [2], the authors extended the obtained results in [6] to the fractional Schrödinger equation

$$\begin{cases} iu_t - (-\Delta)^{\frac{\alpha}{2}} u = \lambda |u|^p & \text{in } (0,\infty) \times \mathbb{R}^N, \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{1.3}$$

where $0 < \alpha < 2$, $u_0 \in H^{\frac{\alpha}{2}}(\mathbb{R}^N)$, and $(-\Delta)^{\frac{\alpha}{2}}$ is the fractional Laplacian operator of order $\frac{\alpha}{2}$. Namely, they investigated the local well-posedness of solutions to (1.3) in $H^{\frac{\alpha}{2}}(\mathbb{R}^N)$ and derived a finite-time blow-up result, under suitable conditions on the initial data. In [11], the authors investigated the nonlocal in time nonlinear Schrödinger equation

$$\begin{cases} iu_t + \Delta u = \lambda J^{\alpha}_{0|t} |u|^p & \text{in } (0,\infty) \times \mathbb{R}^N, \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{1.4}$$

where $0 < \alpha < 1$ and $J^{\alpha}_{0|t}$ is the Riemann–Liouville fractional integral operator of order $\alpha$. Namely, they derived a blow-up exponent and obtained an estimate of the life span of blowing-up solutions to (1.4). In [19], the authors studied the time-fractional Schrödinger equation

$$\begin{cases} i^{\alpha} \partial_t^{\alpha} u + \Delta u = \lambda |u|^p & \text{in } (0,\infty) \times \mathbb{R}^N, \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{1.5}$$

where $0 < \alpha < 1$ and $\partial_t^{\alpha}$ is the Caputo fractional derivative of order $\alpha$. Namely, it was shown that (1.5) admits no global weak solution with suitable initial data when $1 < p < 1 + \frac{2}{N}$. Moreover, the authors derived sufficient conditions for which (1.5) admits no global weak solution for every $p > 1$. For other works related to (1.2), see, e.g., [7–9,14] and the manuscripts cited therein.

Motivated by the above-mentioned contributions, (1.1) is investigated in this paper. As already mentioned before, we first assume that $\mu = 0$. Focusing on this case, we consider the problem

$$\begin{cases} iu_t + \Delta u = \lambda |u|^p + w(x) & \text{in } (0,\infty) \times \mathbb{R}^N, \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{1.6}$$

Before stating in which sense solutions to (1.6) are considered, we indicate some of the notations used throughout this paper. Let $Q = [0,\infty) \times \mathbb{R}^N$, we denote by $C_c^2(Q,\mathbb{R})$ the space of $C^2$ real-valued functions compactly supported in $Q$. For $z \in \mathbb{C}$, let $z_1 = \text{Re } z$ and $z_2 = \text{Im } z$. Moreover, the symbol $C$ will denote always a generic positive constant, which is independent of the scaling parameter $T$ and the solution $u$.

**Definition 1.1.** Let $\lambda \in \mathbb{C}$, $\lambda \neq 0$, $p > 1$, and $u_0, w \in L^1_{\text{loc}}(\mathbb{R}^N, \mathbb{C})$. We say that $u \in L^p_{\text{loc}}(Q, \mathbb{C})$ is a global weak solution to (1.6), if

$$-i \int_{\mathbb{R}^N} u_0(x)\varphi(0,x)\,\mathrm{d}x + \int_Q (-i\varphi_t + \Delta\varphi)u\,\mathrm{d}x\,\mathrm{d}t = \int_Q (\lambda|u|^p + w(x))\varphi\,\mathrm{d}x\,\mathrm{d}t, \tag{1.7}$$

for every $\varphi \in C_c^2(Q,\mathbb{R})$.

**Remark 1.1.** Observe that (1.7) is equivalent to the system of integral equations

$$\int_{\mathbb{R}^N} u_{02}(x)\varphi(0,x)\,\mathrm{d}x + \int_Q \left(\varphi_t u_2 + \Delta\varphi\, u_1\right)\,\mathrm{d}x\,\mathrm{d}t = \int_Q \left(\lambda_1|u|^p + w_1(x)\right)\varphi\,\mathrm{d}x\,\mathrm{d}t \tag{1.8}$$

and

$$-\int_{\mathbb{R}^N} u_{01}(x)\varphi(0,x)\,\mathrm{d}x + \int_Q \left(-\varphi_t u_1 + \Delta\varphi\, u_2\right)\,\mathrm{d}x\,\mathrm{d}t = \int_Q \left(\lambda_2|u|^p + w_2(x)\right)\varphi\,\mathrm{d}x\,\mathrm{d}t. \tag{1.9}$$

Now, we present our main results for (1.6). The following bifurcation-type theorem provides the critical exponent for (1.6) in the sense of Fujita [3].

**Theorem 1.1.** (First critical exponent for (1.6))

(i) *Let $\lambda \in \mathbb{C}$, $\lambda \neq 0$, and $u_0, w \in L^1_{\text{loc}}(\mathbb{R}^N, \mathbb{C})$. Let*

$$p^*(N) = \begin{cases} \infty & \text{if } N \in \{1,2\}, \\ \frac{N}{N-2} & \text{if } N \geq 3. \end{cases}$$

*Suppose that for some $i \in \{1,2\}$,*

$$w_i \in L^1(\mathbb{R}^N, \mathbb{R}) \quad \text{and} \quad \lambda_i \int_{\mathbb{R}^N} w_i(x)\,\mathrm{d}x > 0.$$

*Then, for all $1 < p < p^*(N)$, (1.6) admits no global weak solution.*

(ii) *Let $N \geq 3$. If $p > p^*(N)$, then (1.6) admits global solutions (stationary solutions) for some $\lambda < 0$, $w < 0$ and $u_0 > 0$.*

**Remark 1.2.** We underline the following two facts about the situations considered in Theorem 1.1:

(i) The only condition on the initial value $u_0$ is that $u_0 \in L^1_{\text{loc}}(\mathbb{R}^N, \mathbb{C})$. We just need this condition to guarantee that the integral term from the left side of (1.7) is well defined.

(ii) For $N \geq 3$, the critical exponent $p^*(N)$ is the same as the one obtained by Zhang [18] for the inhomogeneous semilinear heat equation

$$u_t - \Delta u = u^p + w(x),\ u > 0, \tag{1.10}$$

where $w = w(x) \geq 0$, $w \not\equiv 0$. Namely, the following results were established in [18]:

(a) If $1 < p \leq \frac{N}{N-2}$, then problem (1.10) possesses no global positive solution.

(b) If $p > \frac{N}{N-2}$, then problem (1.10) admits global positive solutions for some $w, u_0 > 0$.

Notice that the proof of the nonexistence result in the case $p = \frac{N}{N-2}$ makes use of Lemma 1 in [15], which holds only for $u \geq 0$, while in our work, the solutions to (1.6) are complex-valued functions. The critical case $p = \frac{N}{N-2}$ for problem (1.6) is left open.

Next, we study the influence of the inhomogeneous term $w$ on the critical behavior of (1.6) when $N \geq 3$ and $p > p^*(N)$ and determine the second critical exponent in the sense of Lee and Ni [12]. Just before, for $\sigma < N$, we introduce the following sets:

$$\Lambda_\sigma = \left\{ f \in C(\mathbb{R}^N, \mathbb{R}) : \, f(x) \geq 0, \, f(x) \geq C|x|^{-\sigma} \text{ for sufficiently large } |x| \right\},$$

and

$$\Sigma_\sigma = \left\{ f \in C(\mathbb{R}^N, \mathbb{R}) : \, f(x) > 0, \, f(x) \leq C|x|^{-\sigma} \text{ for sufficiently large } |x| \right\}.$$

We have the following bifurcation-type result.

**Theorem 1.2.** (Second critical exponent for (1.6))

(i) *Let $\lambda \in \mathbb{C}$, $\lambda \neq 0$, and $u_0, w \in L^1_{\text{loc}}(\mathbb{R}^N, \mathbb{C})$. Let $N \geq 3$, $p > \frac{N}{N-2}$, and*

$$\sigma^* = \frac{2p}{p-1}. \tag{1.11}$$

*If $\sigma < \sigma^*$ and*

$$\lambda_i w_i \in \Lambda_\sigma,$$

*for some $i \in \{1, 2\}$, then (1.6) admits no global weak solution.*

(ii) *If $\sigma^* \leq \sigma < N$, then (1.6) admits global solutions (stationary solutions) for some $u_0 > 0$ and $\lambda, w$ with $\lambda w \in \Sigma_\sigma$.*

**Remark 1.3.** We underline the following two facts on Theorem 1.2:

(i) Since $p > \frac{N}{N-2}$, we have that the set of $\sigma$ satisfying $\sigma^* \leq \sigma < N$ is nonempty.

(ii) The second critical exponent for (1.6) in the sense of Lee and Ni is given by (1.11).

Now, we consider (1.1) when $\mu \neq 0$. Namely, our aim is to study the influence of the nonlinear term $|\nabla u|^q$ on the previous obtained results. Let us mention in which sense solutions to (1.1) are considered this time.

**Definition 1.2.** Let $\lambda, \mu \in \mathbb{C}$, $\lambda, \mu \neq 0$, $p, q > 1$, and $u_0, w \in L^1_{\text{loc}}(\mathbb{R}^N, \mathbb{C})$. We say that $u$ is a global weak solution to (1.1), if the following conditions hold:

(i) $(u, \nabla u) \in L^p_{\text{loc}}(Q, \mathbb{C}) \times L^q_{\text{loc}}(Q, \mathbb{C}^N)$.

(ii) For all $\varphi \in C^2_c(Q)$,

$$-i \int_{\mathbb{R}^N} u_0(x)\varphi(0,x)\,\mathrm{d}x + \int_Q (-i\varphi_t + \Delta\varphi)u\,\mathrm{d}x\,\mathrm{d}t = \int_Q (\lambda|u|^p + \mu|\nabla u|^q + w(x))\varphi\,\mathrm{d}x\,\mathrm{d}t. \tag{1.12}$$

**Remark 1.4.** Observe that (1.12) is equivalent to the system of integral equations

$$\int_{\mathbb{R}^N} u_{02}(x)\varphi(0,x)\,\mathrm{d}x + \int_Q \left(\varphi_t u_2 + \Delta\varphi\, u_1\right)\,\mathrm{d}x\,\mathrm{d}t = \int_Q \left(\lambda_1|u|^p + \mu_1|\nabla u|^q + w_1(x)\right)\varphi\,\mathrm{d}x\,\mathrm{d}t \tag{1.13}$$

and

$$-\int_{\mathbb{R}^N} u_{01}(x)\varphi(0,x)\,\mathrm{d}x + \int_Q \left(-\varphi_t u_1 + \Delta\varphi\, u_2\right)\,\mathrm{d}x\,\mathrm{d}t = \int_Q \left(\lambda_2|u|^p + \mu_2|\nabla u|^q + w_2(x)\right)\varphi\,\mathrm{d}x\,\mathrm{d}t.$$

As previously, we first determine the critical exponent for (1.1) in the sense of Fujita.

**Theorem 1.3.** (First critical exponent for (1.1) in the case $\mu \neq 0$)

(i) *Let* $\lambda, \mu \in \mathbb{C}$, $\lambda, \mu \neq 0$, *and* $u_0, w \in L^1_{\text{loc}}(\mathbb{R}^N, \mathbb{C})$. *Let*

$$q^*(N) = \begin{cases} \infty & \text{if } N = 1, \\ \frac{N}{N-1} & \text{if } N \geq 2. \end{cases}$$

*If*

$$1 < p < p^*(N), \quad \mu_i \lambda_i \geq 0, \quad w_i \in L^1(\mathbb{R}^N, \mathbb{R}), \quad \lambda_i \int_{\mathbb{R}^N} w_i(x)\,\mathrm{d}x > 0$$

*or*

$$1 < q < q^*(N), \quad \lambda_i \mu_i > 0, \quad w_i \in L^1(\mathbb{R}^N, \mathbb{R}), \quad \mu_i \int_{\mathbb{R}^N} w_i(x)\,\mathrm{d}x > 0,$$

*for some* $i \in \{1, 2\}$, *then* (1.1) *admits no global weak solution.*

(ii) *Let* $N \geq 3$. *If* $p > p^*(N)$ *and* $q > q^*(N)$, *then* (1.1) *admits global solutions (stationary solutions) for some* $\lambda, \mu < 0$, $u_0 > 0$, *and* $w < 0$.

**Remark 1.5.** We point out the following two facts established in Theorem 1.3:

(i) Similarly to the case $\mu = 0$, the only condition on the initial value $u_0$ is that $u_0 \in L^1_{\text{loc}}(\mathbb{R}^N, \mathbb{C})$.

(ii) From Theorem 1.3, we deduce that for $N \geq 3$, the critical exponent in the sense of Fujita for (1.1) is given by

$$p^*(N, q) = \begin{cases} \infty & \text{if } 1 < q < \frac{N}{N-1}, \\ \frac{N}{N-2} & \text{if } q \geq \frac{N}{N-1}. \end{cases}$$

One observes that the gradient term induces an interesting phenomenon of discontinuity of the critical exponent, jumping from $p = \frac{N}{N-2}$ (the Fujita critical exponent for (1.6)) to $p = \infty$ as $q$ reaches the value $\frac{N}{N-1}$ from above.

Next, for $N \geq 3$, $p > p^*(N)$, and $q > q^*(N)$, we investigate the influence of the inhomogeneous term $w$ on the critical behavior of (1.1).

**Theorem 1.4.** (Second critical exponent for (1.1) in the case $\mu \neq 0$)

(i) *Let* $\lambda, \mu \in \mathbb{C}$, $\lambda, \mu \neq 0$, *and* $u_0, w \in L^1_{\text{loc}}(\mathbb{R}^N, \mathbb{C})$. *Let* $N \geq 3$, $p > p^*(N)$, $q > q^*(N)$, *and*

$$\sigma^{**} = \frac{q}{q-1}.$$

(a) *If* $\sigma < \sigma^*$, *where* $\sigma^*$ *is defined by* (1.11), *and*

$$\mu_i \lambda_i \geq 0, \quad \lambda_i w_i \in \Lambda_\sigma,$$

*for some* $i \in \{1, 2\}$, *then* (1.1) *admits no global weak solution.*

(b) *If* $\sigma < \sigma^{**}$ *and*

$$\lambda_i \mu_i > 0, \quad \mu_i w_i \in \Lambda_\sigma,$$

*for some* $i \in \{1, 2\}$, *then* (1.1) *admits no global weak solution.*

(ii) *If* $\max\{\sigma^*, \sigma^{**}\} \leq \sigma < N$, *then* (1.1) *admits global solutions (stationary solutions) for some* $u_0 > 0$ *and* $\lambda, \mu, w$ *with* $\lambda = \mu < 0$ *and* $\lambda w \in \Sigma_\sigma$.

**Remark 1.6.** It is important to note two aspects of Theorem 1.4:

(i) Since $p > p^*(N) = \frac{N}{N-2}$ and $q > q^*(N) = \frac{N}{N-1}$, we have that the set of $\sigma$ satisfying $\max\{\sigma^*, \sigma^{**}\} \leq \sigma < N$ is nonempty.

(ii) If $N \geq 3$, $p > p^*(N)$, and $q > q^*(N)$, then the second critical exponent for (1.1) in the sense of Lee and Ni is given by

$$\sigma^*(p,q) = \max\{\sigma^*, \sigma^{**}\} = \max\left\{\frac{2p}{p-1}, \frac{q}{q-1}\right\}.$$

The rest of the paper is organized as follows. In Sect. 2, we focus on the case $\mu = 0$ and establish the proofs of Theorems 1.1 and 1.2. In Sect. 3, we extend the study to the case where $\mu \neq 0$, and hence we establish the proofs of Theorems 1.3 and 1.4.

## 2. The case $\mu = 0$

In this section, we prove Theorems 1.1 and 1.2, studying problem (1.6) (that is, problem (1.1) when $\mu = 0$). We establish both the absence and the existence of global solutions in the sense of Definition 1.2. The proofs are based on a rescaled test function argument (see [13] for a general account of these methods).

### 2.1. Proof of Theorem 1.1

(i) Let $1 < p < p^*(N)$. Suppose that $u \in L^p_{\text{loc}}(Q, \mathbb{C})$ is a global weak solution to (1.6). We first consider the case

$$w_1 \in L^1(\mathbb{R}^N, \mathbb{R}) \quad \text{and} \quad \lambda_1 \int_{\mathbb{R}^N} w_1(x)\,\mathrm{d}x > 0. \tag{2.1}$$

Since the second assumption in (2.1) holds, it is obvious that $\lambda_1 \neq 0$. So, using the integral equation (1.8) (recall Remark 1.1), we obtain

$$\int_Q |u|^p \varphi\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\lambda_1}\int_Q w_1(x)\varphi\,\mathrm{d}x\,\mathrm{d}t \leq \frac{1}{|\lambda_1|}\int_Q |\varphi_t||u_2|\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{|\lambda_1|}\int_Q |\Delta\varphi||u_1|\,\mathrm{d}x\,\mathrm{d}t, \tag{2.2}$$

for all $\varphi \in C^2_c(Q, \mathbb{R})$ with $\varphi \geq 0$ and $\varphi(0,\cdot) \equiv 0$. This yields the following inequality in $u$

$$\int_Q |u|^p \varphi\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\lambda_1}\int_Q w_1(x)\varphi\,\mathrm{d}x\,\mathrm{d}t \leq \frac{1}{|\lambda_1|}\int_Q |\varphi_t||u|\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{|\lambda_1|}\int_Q |\Delta\varphi||u|\,\mathrm{d}x\,\mathrm{d}t. \tag{2.3}$$

With respect to the right side of (2.3), the $\varepsilon$-Young inequality for $0 < \varepsilon < \frac{1}{2}$ gives us two estimates:

$$\frac{1}{|\lambda_1|}\int_Q |\varphi_t||u|\,\mathrm{d}x\,\mathrm{d}t \leq \varepsilon\int_Q |u|^p\varphi\,\mathrm{d}x\,\mathrm{d}t + C\int_Q \varphi^{\frac{-1}{p-1}}|\varphi_t|^{\frac{p}{p-1}}\,\mathrm{d}x\,\mathrm{d}t, \tag{2.4}$$

and

$$\frac{1}{|\lambda_1|}\int_Q |\Delta\varphi||u|\,\mathrm{d}x\,\mathrm{d}t \leq \varepsilon\int_Q |u|^p\varphi\,\mathrm{d}x\,\mathrm{d}t + C\int_Q \varphi^{\frac{-1}{p-1}}|\Delta\varphi|^{\frac{p}{p-1}}\,\mathrm{d}x\,\mathrm{d}t. \tag{2.5}$$

Turning to (2.3), and using the estimates (2.4) and (2.5) in the right side, we deduce that

$$(1-2\varepsilon)\int_Q |u|^p\varphi\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\lambda_1}\int_Q w_1(x)\varphi\,\mathrm{d}x\,\mathrm{d}t \leq C\left(\int_Q \varphi^{\frac{-1}{p-1}}|\varphi_t|^{\frac{p}{p-1}}\,\mathrm{d}x\,\mathrm{d}t + \int_Q \varphi^{\frac{-1}{p-1}}|\Delta\varphi|^{\frac{p}{p-1}}\,\mathrm{d}x\,\mathrm{d}t\right),$$

which yields

$$\frac{1}{\lambda_1}\int_Q w_1(x)\varphi\,\mathrm{d}x\,\mathrm{d}t \leq \left(\int_Q \varphi^{\frac{-1}{p-1}}|\varphi_t|^{\frac{p}{p-1}}\,\mathrm{d}x\,\mathrm{d}t + \int_Q \varphi^{\frac{-1}{p-1}}|\Delta\varphi|^{\frac{p}{p-1}}\,\mathrm{d}x\,\mathrm{d}t\right). \tag{2.6}$$

At this step, we consider two cutoff functions $\xi, \eta \in C^\infty([0,\infty))$ satisfying, respectively,

$$\xi \geq 0, \quad \xi \not\equiv 0, \quad \mathrm{supp}(\xi) \subset\subset (0,1)$$

and

$$0 \leq \eta \leq 1, \quad \eta(\sigma) = \begin{cases} 1 & \text{if } 0 \leq \sigma \leq 1, \\ 0 & \text{if } \sigma \geq 2. \end{cases}$$

For sufficiently large $T$, we define the following function:

$$\varphi_T(t,x) = \xi\left(\frac{t}{T}\right)^k \eta\left(\frac{|x|^2}{T^{2\rho}}\right)^k, \quad (t,x) \in Q, \tag{2.7}$$

where $k > \frac{2p}{p-1}$ and $\rho > 0$ are constants. We observe that $\varphi_T \in C_c^2(Q,\mathbb{R})$, $\varphi_T \geq 0$, and $\varphi_T(0,\cdot) \equiv 0$. So, for sufficiently large $T$, by (2.7) and using elementary calculations, we obtain the following:

$$|\varphi_T(t,x)|^{\frac{-1}{p-1}} |\Delta\varphi_T(t,x)|^{\frac{p}{p-1}} \leq C T^{\frac{-2\rho p}{p-1}} \xi\left(\frac{t}{T}\right)^k \eta\left(\frac{|x|^2}{T^{2\rho}}\right)^{k-\frac{2p}{p-1}}, \quad t>0,\ T^\rho < |x| < \sqrt{2}T^\rho \tag{2.8}$$

and

$$|\varphi_T(t,x)|^{\frac{-1}{p-1}} |(\varphi_T)_t(t,x)|^{\frac{p}{p-1}} \leq C T^{\frac{-p}{p-1}} \xi\left(\frac{t}{T}\right)^{k-\frac{p}{p-1}} \eta\left(\frac{|x|^2}{T^{2\rho}}\right)^k, \quad t>0,\ |x| < \sqrt{2}T^\rho. \tag{2.9}$$

Hence, by (2.7) and (2.8), for sufficiently large $T$ we obtain

$$\begin{aligned}
\int_Q |\varphi_T|^{\frac{-1}{p-1}} |\Delta\varphi_T|^{\frac{p}{p-1}} \,\mathrm{d}x\,\mathrm{d}t &= \int_0^T \int_{T^\rho<|x|<\sqrt{2}T^\rho} |\varphi_T|^{\frac{-1}{p-1}} |\Delta\varphi_T|^{\frac{p}{p-1}} \,\mathrm{d}x\,\mathrm{d}t \\
&\leq C T^{\frac{-2\rho p}{p-1}} \int_0^T \int_{T^\rho<|x|<\sqrt{2}T^\rho} \xi\left(\frac{t}{T}\right)^k \eta\left(\frac{|x|^2}{T^{2\rho}}\right)^{k-\frac{2p}{p-1}} \,\mathrm{d}x\,\mathrm{d}t \\
&= C T^{\frac{-2\rho p}{p-1}} \left(\int_0^T \xi\left(\frac{t}{T}\right)^k \,\mathrm{d}t\right) \left(\int_{T^\rho<|x|<\sqrt{2}T^\rho} \eta\left(\frac{|x|^2}{T^{2\rho}}\right)^{k-\frac{2p}{p-1}} \,\mathrm{d}x\right) \\
&\leq C T^{1-\frac{2\rho p}{p-1}} \int_{|x|<\sqrt{2}T^\rho} \mathrm{d}x \\
&= C T^{1-\frac{2\rho p}{p-1}+N\rho},
\end{aligned}$$

that is,

$$\int_Q |\varphi_T|^{\frac{-1}{p-1}} |\Delta\varphi_T|^{\frac{p}{p-1}} \,\mathrm{d}x\,\mathrm{d}t \leq C T^{1-\frac{2\rho p}{p-1}+N\rho}. \tag{2.10}$$

Similarly, using (2.7) and (2.9), for sufficiently large $T$ we obtain

$$\begin{aligned}
\int_Q |\varphi_T|^{\frac{-1}{p-1}} |(\varphi_T)_t|^{\frac{p}{p-1}} \,\mathrm{d}x\,\mathrm{d}t &= \int_0^T \int_{|x|<\sqrt{2}T^\rho} |\varphi_T|^{\frac{-1}{p-1}} |(\varphi_T)_t|^{\frac{p}{p-1}} \,\mathrm{d}x\,\mathrm{d}t \\
&\leq C T^{\frac{-p}{p-1}} \left(\int_0^T \xi\left(\frac{t}{T}\right)^{k-\frac{p}{p-1}} \,\mathrm{d}t\right) \left(\int_{|x|<\sqrt{2}T^\rho} \eta\left(\frac{|x|^2}{T^{2\rho}}\right)^k \,\mathrm{d}x\right)
\end{aligned}$$

$$\leq CT^{1-\frac{p}{p-1}} \int\limits_{|x|<\sqrt{2}T^{\rho}} \mathrm{d}x$$
$$= CT^{1-\frac{p}{p-1}+N\rho},$$

that is,

$$\int\limits_{Q} |\varphi_T|^{\frac{-1}{p-1}} |(\varphi_T)_t|^{\frac{p}{p-1}} \,\mathrm{d}x\,\mathrm{d}t \leq CT^{1-\frac{p}{p-1}+N\rho}. \tag{2.11}$$

On the other hand,

$$\frac{1}{\lambda_1}\int\limits_{Q} w_1(x)\varphi_T \,\mathrm{d}x\,\mathrm{d}t = \left(\int\limits_0^T \xi\left(\frac{t}{T}\right)^k \mathrm{d}t\right)\left(\frac{1}{\lambda_1}\int\limits_{\mathbb{R}^N} w_1(x)\eta\left(\frac{|x|^2}{T^{2\rho}}\right)^k \mathrm{d}x\right)$$
$$= CT\left(\frac{1}{\lambda_1}\int\limits_{\mathbb{R}^N} w_1(x)\eta\left(\frac{|x|^2}{T^{2\rho}}\right)^k \mathrm{d}x\right). \tag{2.12}$$

Since $w_1 \in L^1(\mathbb{R}^N, \mathbb{R})$ by (2.1), the dominated convergence theorem leads to

$$\lim_{T\to\infty} \frac{1}{\lambda_1}\int\limits_{\mathbb{R}^N} w_1(x)\eta\left(\frac{|x|^2}{T^{2\rho}}\right)^k \mathrm{d}x = \frac{1}{\lambda_1}\int\limits_{\mathbb{R}^N} w_1(x)\,\mathrm{d}x.$$

Again by (2.1), we know that

$$\frac{1}{\lambda_1}\int\limits_{\mathbb{R}^N} w_1(x)\,\mathrm{d}x > 0,$$

and hence for sufficiently large $T$, we get

$$\frac{1}{\lambda_1}\int\limits_{\mathbb{R}^N} w_1(x)\eta\left(\frac{|x|^2}{T^{2\rho}}\right)^k \mathrm{d}x \geq \frac{1}{2\lambda_1}\int\limits_{\mathbb{R}^N} w_1(x)\,\mathrm{d}x. \tag{2.13}$$

Hence, it follows from (2.12) and (2.13) that

$$\frac{1}{\lambda_1}\int\limits_{Q} w_1(x)\varphi_T \,\mathrm{d}x\,\mathrm{d}t \geq CT\frac{1}{\lambda_1}\int\limits_{\mathbb{R}^N} w_1(x)\,\mathrm{d}x. \tag{2.14}$$

Therefore, for sufficiently large $T$, taking $\varphi = \varphi_T$ in (2.6) and using (2.10), (2.11), and (2.14), we obtain

$$\frac{1}{\lambda_1}\int\limits_{\mathbb{R}^N} w_1(x)\,\mathrm{d}x \leq CT^{-1}\left(T^{1-\frac{2\rho p}{p-1}+N\rho} + T^{1-\frac{p}{p-1}+N\rho}\right). \tag{2.15}$$

Observe that for $\rho = \frac{1}{2}$, we have

$$1 - \frac{2\rho p}{p-1} + N\rho = 1 - \frac{p}{p-1} + N\rho = \frac{N}{2} - \frac{1}{p-1}.$$

So, taking $\rho = \frac{1}{2}$ in (2.15), we obtain

$$\frac{1}{\lambda_1}\int\limits_{\mathbb{R}^N} w_1(x)\,\mathrm{d}x \leq CT^{\frac{N}{2}-\frac{p}{p-1}}. \tag{2.16}$$

Since $1 < p < p^*(N)$, passing to the limit as $T \to \infty$ in (2.16), we arrive to a contradiction to $\lambda_1 \int_{\mathbb{R}^N} w_1(x)\,\mathrm{d}x > 0$.

Consider now the case

$$w_2 \in L^1(\mathbb{R}^N,\mathbb{R}), \quad \lambda_2 \int_{\mathbb{R}^N} w_2(x)\,\mathrm{d}x > 0. \tag{2.17}$$

Taking $\widetilde{u_1} = u_2$, $\widetilde{u_2} = -u_1$, $\widetilde{u_{01}} = u_{02}$, $\widetilde{u_{02}} = -u_{01}$, $\widetilde{u} = \widetilde{u_1} + i\widetilde{u_2}$, $\widetilde{\lambda_1} = \lambda_2$, and $\widetilde{w_1} = w_2$, we deduce from (1.9) and (2.17) that

$$\widetilde{w_1} \in L^1(\mathbb{R}^N,\mathbb{R}), \quad \widetilde{\lambda_1} \int_{\mathbb{R}^N} \widetilde{w_1}(x)\,\mathrm{d}x > 0,$$

and

$$\int_{\mathbb{R}^N} \widetilde{u_{02}}(x)\varphi(0,x)\,\mathrm{d}x + \int_Q \left(\varphi_t \widetilde{u_2} + \Delta\varphi\,\widetilde{u_1}\right)\,\mathrm{d}x\,\mathrm{d}t = \int_Q \left(\widetilde{\lambda_1}|\widetilde{u}|^p + \widetilde{w_1}(x)\right)\varphi\,\mathrm{d}x\,\mathrm{d}t,$$

for every $\varphi \in C_c^2(Q,\mathbb{R})$. Hence, from the previous case, we obtain a contradiction with $\widetilde{\lambda_1} \int_{\mathbb{R}^N} \widetilde{w_1}(x)\,\mathrm{d}x > 0$.

Therefore, in both cases (case (2.1) and case (2.17)), (1.6) admits no global weak solution. This proves part (i) of Theorem 1.1.

(ii) To construct the second part of the proof, we focus on the situation where

$$N \geq 3 \quad \text{and} \quad p > p^*(N) = \frac{N}{N-2}. \tag{2.18}$$

We consider the class of functions

$$u_{\delta,\varepsilon}(x) = \varepsilon(1+r^2)^{-\delta}, \quad r = |x|, \quad x \in \mathbb{R}^N, \tag{2.19}$$

subject to

$$\frac{1}{p-1} \leq \delta < \frac{N-2}{2} \tag{2.20}$$

and

$$0 < \varepsilon < \left[2\delta(N-2\delta-2)\right]^{\frac{1}{p-1}}. \tag{2.21}$$

Notice that from (2.18), the set of values of $\delta$ satisfying (2.20) is nonempty. An elementary calculation shows that

$$\Delta u_{\delta,\varepsilon}(x) = -2\delta\varepsilon(1+r^2)^{-\delta-2}\left[(N-2\delta-2)r^2 + N\right].$$

Let

$$w(x) = \Delta u_{\delta,\varepsilon}(x) + u_{\delta,\varepsilon}^p(x),$$

that is,

$$w(x) = -2\delta\varepsilon(1+r^2)^{-\delta-2}\left[(N-2\delta-2)r^2 + N\right] + \varepsilon^p(1+r^2)^{-\delta p}. \tag{2.22}$$

Using the constraints (2.20) and (2.21), we obtain the following sign information for $w$:

$$\begin{aligned} w(x) &\leq \varepsilon\left[-2\delta(N-2\delta-2)(1+r^2)^{-\delta-1} + \varepsilon^{p-1}(1+r^2)^{-\delta p}\right] \\ &< \varepsilon\left[-2\delta(N-2\delta-2)(1+r^2)^{-\delta-1} + 2\delta(N-2\delta-2)(1+r^2)^{-\delta p}\right] \\ &= 2\varepsilon\delta(N-2\delta-2)\left[(1+r^2)^{-\delta p} - (1+r^2)^{-\delta-1}\right] \\ &\leq 0. \end{aligned}$$

Hence, for all $\delta$ and $\varepsilon$ satisfying, respectively, (2.20) and (2.21), the function $u_{\delta,\varepsilon}$ defined by (2.19) is a stationary positive solution to (1.6), where $\lambda = -1$ and $w = w(x)$ $(< 0)$ is defined by (2.22). This proves part (ii) of Theorem 1.1. □

### 2.2. Proof of Theorem 1.2

This proof is based on the previous one, but it is slightly modified according to the different assumptions. Since some a priori estimates used in the proof of Theorem 1.1 still work, we remain to consider certain additional factors as follows.

(i) Suppose that $u \in L^p_{\mathrm{loc}}(Q,\mathbb{C})$ is a global weak solution to (1.6). Without restriction of the generality, we suppose that

$$\lambda_1 w_1 \in \Lambda_\sigma. \tag{2.23}$$

The case $\lambda_2 w_2 \in \Lambda_\sigma$ can be treated in a similar way (see the case (2.17) in the proof of part (i) of Theorem 1.1).

For sufficiently large $T$, we have

$$\begin{aligned}
\frac{1}{\lambda_1}\int_Q w_1(x)\varphi_T \,\mathrm{d}x\,\mathrm{d}t &= \left(\int_0^T \xi\left(\frac{t}{T}\right)^k \mathrm{d}t\right)\left(\frac{1}{\lambda_1}\int_{\mathbb{R}^N} w_1(x)\eta\left(\frac{|x|^2}{T^{2\rho}}\right)^k \mathrm{d}x\right)\\
&= CT\left(\frac{1}{\lambda_1}\int_{\mathbb{R}^N} w_1(x)\eta\left(\frac{|x|^2}{T^{2\rho}}\right)^k \mathrm{d}x\right)\\
&= CT\left(\frac{1}{\lambda_1}\int_{|x|<\sqrt{2}T^\rho} w_1(x)\eta\left(\frac{|x|^2}{T^{2\rho}}\right)^k \mathrm{d}x\right),
\end{aligned}$$

where $\varphi_T$ is again defined by (2.7). Hence, by the definition of the function $\eta$, and by (2.23), for sufficiently large $T$, we obtain

$$\begin{aligned}
\frac{1}{\lambda_1}\int_Q w_1(x)\varphi_T \,\mathrm{d}x\,\mathrm{d}t &\geq CT\left(\frac{1}{\lambda_1}\int_{|x|<T^\rho} w_1(x)\eta\left(\frac{|x|^2}{T^{2\rho}}\right)^k \mathrm{d}x\right)\\
&= CT\left(\frac{1}{\lambda_1}\int_{|x|<T^\rho} w_1(x)\,\mathrm{d}x\right)\\
&\geq CT\left(\frac{1}{\lambda_1}\int_{\frac{T^\rho}{2}<|x|<T^\rho} w_1(x)\,\mathrm{d}x\right)\\
&\geq CT\left(\frac{1}{\lambda_1}\int_{\frac{T^\rho}{2}<|x|<T^\rho} |x|^{-\sigma}\,\mathrm{d}x\right)\\
&= CT^{\rho(N-\sigma)+1}.
\end{aligned} \tag{2.24}$$

Hence, taking $\varphi = \varphi_T$ in (2.6) and using (2.10), (2.11), and (2.24), we obtain

$$CT^{\rho(N-\sigma)+1} \leq \left(T^{1-\frac{2\rho p}{p-1}+N\rho} + T^{1-\frac{p}{p-1}+N\rho}\right),$$

that is,

$$0 < C \leq T^{-\rho(N-\sigma)-1}\left(T^{1-\frac{2\rho p}{p-1}+N\rho} + T^{1-\frac{p}{p-1}+N\rho}\right).$$

Taking $\rho = \frac{1}{2}$ in the above inequality, we get

$$0 < C \leq T^{\frac{\sigma}{2}-\frac{p}{p-1}}. \tag{2.25}$$

Since $\sigma < \sigma^* = \frac{2p}{p-1}$, passing to the limit as $T \to \infty$ in (2.25), we obtain a contradiction with $C > 0$. This proves part (i) of Theorem 1.2.

(ii) In the second part of the proof, we focus on the case where

$$\sigma^* \leq \sigma < N.$$

We consider the class of functions (2.19), with

$$\frac{\sigma-2}{2} \leq \delta < \frac{N-2}{2} \tag{2.26}$$

and $\varepsilon$ satisfying (2.21). Notice that since $\sigma < N$, the set of values of $\delta$ satisfying (2.26) is nonempty. Moreover, since $\sigma^* \leq \sigma$, we have

$$\frac{\sigma-2}{2} \geq \frac{1}{p-1}.$$

Hence, by (2.26), we deduce that $\delta$ satisfies (2.20). Therefore, from the proof of part (ii) of Theorem 1.1, for all $\delta$ and $\varepsilon$ satisfying, respectively, (2.26) and (2.21), the function $u_{\delta,\varepsilon}$ defined by (2.19) is a stationary positive solution to (1.6), where $\lambda = -1$ and $w = w(x)$ $(< 0)$ is defined by (2.22). Now, we have just to show that $\lambda w \in \Sigma_\sigma$, that is,

$$-w \in \Sigma_\sigma. \tag{2.27}$$

By (2.22), we have

$$-w(x) = 2\delta\varepsilon(1+r^2)^{-\delta-2}\left[(N-2\delta-2)r^2+N\right] - \varepsilon^p(1+r^2)^{-\delta p}, \quad r = |x|.$$

Hence, using (2.26), for sufficiently large $r$, we obtain

$$\begin{aligned} -w(x) &\leq 2\delta\varepsilon(1+r^2)^{-\delta-2}\left[(N-2\delta-2)r^2+N\right] \\ &\leq 2\delta\varepsilon N(1+r^2)^{-\delta-1} \\ &\leq Cr^{-2(\delta+1)} \\ &\leq Cr^{-\sigma}, \end{aligned}$$

and hence (2.27) is established. This fact proves part (ii) of Theorem 1.2. □

## 3. The case $\mu \neq 0$

In this section, we prove Theorems 1.3 and 1.4, studying problem (1.1) in the case $\mu \neq 0$. Precisely, we consider the effects of the power term $\mu|\nabla u|^q$ on the behavior of solutions to (1.1).

### 3.1. Proof of Theorem 1.3

(i) Suppose that $u$ is a global weak solution to (1.1). We first consider the case

$$1 < p < p^*(N), \quad \mu_i\lambda_i \geq 0, \quad w_i \in L^1(\mathbb{R}^N, \mathbb{R}), \quad \lambda_i \int_{\mathbb{R}^N} w_i(x)\,\mathrm{d}x > 0,$$

for some $i \in \{1,2\}$. Without restriction of the generality, we may take $i = 1$, that is,

$$1 < p < p^*(N), \quad \mu_1\lambda_1 \geq 0, \quad w_1 \in L^1(\mathbb{R}^N, \mathbb{R}), \quad \lambda_1 \int_{\mathbb{R}^N} w_1(x)\,\mathrm{d}x > 0. \tag{3.1}$$

The last inequality in (3.1) implies that $\lambda_1 \neq 0$. So, it follows from the integral equation (1.13) that

$$\frac{1}{\lambda_1}\int_{\mathbb{R}^N} u_{02}(x)\varphi(0,x)\,\mathrm{d}x + \frac{1}{\lambda_1}\int_Q (\varphi_t u_2 + \Delta\varphi\, u_1)\,\mathrm{d}x\,\mathrm{d}t = \int_Q \left(|u|^p + \frac{\mu_1}{\lambda_1}|\nabla u|^q + \frac{1}{\lambda_1}w_1(x)\right)\varphi\,\mathrm{d}x\,\mathrm{d}t,$$

for all $\varphi \in C_c^2(Q)$, $\varphi \geq 0$. Since $\frac{\mu_1}{\lambda_1} \geq 0$ by (3.1), then (2.2) holds for every $\varphi \in C_c^2(Q, \mathbb{R})$ with $\varphi \geq 0$ and $\varphi(0,\cdot) \equiv 0$. Therefore, for sufficiently large $T$, taking $\varphi = \varphi_T$, where $\varphi_T$ is defined by (2.7), we obtain (2.16). Then, passing to the limit as $T \to \infty$ in (2.16), we obtain a contradiction with $\lambda_1 \int_{\mathbb{R}^N} w_1(x)\,\mathrm{d}x > 0$.

Different from the above, we now reason on the exponents $q$, $q^*$ (instead of $p$, $p^*$). Precisely, we deal with the assumptions

$$1 < q < q^*(N), \quad \lambda_i\mu_i > 0, \quad w_i \in L^1(\mathbb{R}^N, \mathbb{R}), \quad \mu_i \int_{\mathbb{R}^N} w_i(x)\,\mathrm{d}x > 0,$$

for some $i \in \{1,2\}$. Without restriction of the generality, we may suppose that $i = 1$, that is,

$$1 < q < q^*(N), \quad \lambda_1\mu_1 > 0, \quad w_1 \in L^1(\mathbb{R}^N, \mathbb{R}), \quad \mu_1 \int_{\mathbb{R}^N} w_1(x)\,\mathrm{d}x > 0. \tag{3.2}$$

Since $\mu_1 \neq 0$ by (3.2), using (1.13) (recall Remark 1.4), we obtain

$$\frac{1}{\mu_1}\int_Q (\varphi_t u_2 + \Delta\varphi\, u_1)\,\mathrm{d}x\,\mathrm{d}t = \int_Q \left(\frac{\lambda_1}{\mu_1}|u|^p + |\nabla u|^q + \frac{1}{\mu_1}w_1(x)\right)\varphi\,\mathrm{d}x\,\mathrm{d}t,$$

for all $\varphi \in C_c^2(Q)$ with $\varphi \geq 0$ and $\varphi(0,\cdot) \equiv 0$. Using an integration by parts, it holds that

$$\frac{1}{\mu_1}\int_Q (\varphi_t u_2 - \nabla\varphi \cdot \nabla u_1)\,\mathrm{d}x\,\mathrm{d}t = \int_Q \left(\frac{\lambda_1}{\mu_1}|u|^p + |\nabla u|^q + \frac{1}{\mu_1}w_1(x)\right)\varphi\,\mathrm{d}x\,\mathrm{d}t,$$

where "$\cdot$" denotes the inner product in $\mathbb{R}^N$. Hence, using Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} &\frac{\lambda_1}{\mu_1}\int_Q |u|^p\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_Q |\nabla u|^q\varphi\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\mu_1}\int_Q w_1(x)\varphi\,\mathrm{d}x\,\mathrm{d}t \\ &\leq \frac{1}{|\mu_1|}\int_Q |\varphi_t||u|\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{|\mu_1|}\int_Q |\nabla\varphi||\nabla u|\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.3}$$

On the other hand, by $\varepsilon$-Young inequality with $0<\varepsilon<\min\left\{1,\frac{\lambda_1}{\mu_1}\right\}$ (notice that $\frac{\lambda_1}{\mu_1}>0$ by (3.2)), we have

$$\frac{1}{|\mu_1|}\int_Q |\varphi_t||u|\,\mathrm{d}x\,\mathrm{d}t \le \varepsilon\int_Q |u|^p\varphi\,\mathrm{d}x\,\mathrm{d}t + C\int_Q \varphi^{\frac{-1}{p-1}}|\varphi_t|^{\frac{p}{p-1}}\,\mathrm{d}x\,\mathrm{d}t, \tag{3.4}$$

and

$$\frac{1}{|\mu_1|}\int_Q |\nabla\varphi||\nabla u|\,\mathrm{d}x\,\mathrm{d}t \le \varepsilon\int_Q |\nabla u|^q\varphi\,\mathrm{d}x\,\mathrm{d}t + C\int_Q \varphi^{\frac{-1}{q-1}}|\nabla\varphi|^{\frac{q}{q-1}}\,\mathrm{d}x\,\mathrm{d}t. \tag{3.5}$$

Then, combining (3.3), (3.4), and (3.5), it follows that

$$\begin{aligned}&\left(\frac{\lambda_1}{\mu_1}-\varepsilon\right)\int_Q |u|^p\varphi\,\mathrm{d}x\,\mathrm{d}t + (1-\varepsilon)\int_Q |\nabla u|^q\varphi\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\mu_1}\int_Q w_1(x)\varphi\,\mathrm{d}x\,\mathrm{d}t\\ &\le C\left(\int_Q \varphi^{\frac{-1}{q-1}}|\nabla\varphi|^{\frac{q}{q-1}}\,\mathrm{d}x\,\mathrm{d}t + \int_Q \varphi^{\frac{-1}{p-1}}|\varphi_t|^{\frac{p}{p-1}}\,\mathrm{d}x\,\mathrm{d}t\right),\end{aligned}$$

which implies that

$$\frac{1}{\mu_1}\int_Q w_1(x)\varphi\,\mathrm{d}x\,\mathrm{d}t \le C\left(\int_Q \varphi^{\frac{-1}{q-1}}|\nabla\varphi|^{\frac{q}{q-1}}\,\mathrm{d}x\,\mathrm{d}t + \int_Q \varphi^{\frac{-1}{p-1}}|\varphi_t|^{\frac{p}{p-1}}\,\mathrm{d}x\,\mathrm{d}t\right).$$

In particular, for sufficiently large $T$, taking $\varphi=\varphi_T$ with $k>\max\left\{\frac{p}{p-1},\frac{q}{q-1}\right\}$, where $\varphi_T$ is defined by (2.7), we obtain

$$\frac{1}{\mu_1}\int_Q w_1(x)\varphi_T\,\mathrm{d}x\,\mathrm{d}t \le C\left(\int_Q \varphi_T^{\frac{-1}{q-1}}|\nabla\varphi_T|^{\frac{q}{q-1}}\,\mathrm{d}x\,\mathrm{d}t + \int_Q \varphi_T^{\frac{-1}{p-1}}|(\varphi_T)_t|^{\frac{p}{p-1}}\,\mathrm{d}x\,\mathrm{d}t\right). \tag{3.6}$$

An elementary calculation shows that, for $t>0$ and $T^\rho<|x|<\sqrt{2}T^\rho$, we have

$$|\varphi_T(t,x)|^{\frac{-1}{q-1}}|\nabla\varphi_T(t,x)|^{\frac{q}{q-1}} \le CT^{\frac{-\rho q}{q-1}}\xi\left(\frac{t}{T}\right)^k\eta\left(\frac{|x|^2}{T^{2\rho}}\right)^{k-\frac{q}{q-1}}\eta'\left(\frac{|x|^2}{T^{2\rho}}\right)^{\frac{q}{q-1}}.$$

Hence, for sufficiently large $T$, we deduce that

$$\int_Q \varphi_T^{\frac{-1}{q-1}}|\nabla\varphi_T|^{\frac{q}{q-1}}\,\mathrm{d}x\,\mathrm{d}t \le CT^{1-\frac{\rho q}{q-1}+N\rho}. \tag{3.7}$$

Therefore, using (2.11), (3.6), and (3.7), for sufficiently large $T$, it holds that

$$\frac{1}{\mu_1}\int_Q w_1(x)\varphi_T\,\mathrm{d}x\,\mathrm{d}t \le C\left(T^{1-\frac{\rho q}{q-1}+N\rho}+T^{1-\frac{p}{p-1}+N\rho}\right). \tag{3.8}$$

Observe that for $\rho=\frac{p(q-1)}{(p-1)q}$, we have

$$1-\frac{\rho q}{q-1}+N\rho = 1-\frac{p}{p-1}+N\rho = \frac{1}{p-1}\left(\frac{N(q-1)p}{q}-1\right).$$

Therefore, taking $\rho=\frac{p(q-1)}{(p-1)q}$ in (3.8), for sufficiently large $T$, it holds that

$$\frac{1}{\mu_1}\int_Q w_1(x)\varphi_T\,\mathrm{d}x\,\mathrm{d}t \le CT^{\frac{1}{p-1}\left(\frac{N(q-1)p}{q}-1\right)}. \tag{3.9}$$

Recalling from (3.2) that $w_1 \in L^1(\mathbb{R}^N, \mathbb{R})$ and $\mu_1 \int\limits_{\mathbb{R}^N} w_1(x)\,\mathrm{d}x > 0$, and proceeding as in the proof of part (i) of Theorem 1.1 (see (2.14)), we deduce that for sufficiently large $T$,

$$\frac{1}{\mu_1} \int\limits_Q w_1(x)\varphi_T \,\mathrm{d}x\,\mathrm{d}t \geq CT \frac{1}{\mu_1} \int\limits_{\mathbb{R}^N} w_1(x)\,\mathrm{d}x. \tag{3.10}$$

Combining (3.9) with (3.10), for sufficiently large $T$, it holds that

$$0 < C \leq T^{\frac{p}{p-1}\left[\frac{N(q-1)}{q}-1\right]}. \tag{3.11}$$

Since $1 < q < q^*(N)$ by (3.2), passing to the limit as $T \to \infty$ in (3.11), we obtain a contradiction with $C > 0$. Hence, part (i) of Theorem 1.3 is proved.
(ii) To construct the second part of the proof, we focus on the situation where

$$N \geq 3, \quad p > \frac{N}{N-2}, \quad q > \frac{N}{N-1}. \tag{3.12}$$

Consider the class of functions $u_{\delta,\varepsilon}$ defined by (2.19), where

$$\max\left\{\frac{1}{p-1}, \frac{2-q}{2(q-1)}\right\} \leq \delta < \frac{N-2}{2}, \tag{3.13}$$

and

$$0 < \varepsilon < \min\left\{[\delta(N-2\delta-2)]^{\frac{1}{p-1}}, \delta^{-1}2^{\frac{-q}{q-1}}(N-2\delta-2)^{\frac{1}{q-1}}\right\}. \tag{3.14}$$

Notice that due to (3.12), the set of $\delta$ satisfying (3.13) is nonempty. Let

$$w(x) = \Delta u_{\delta,\varepsilon}(x) + u^p_{\delta,\varepsilon}(x) + |\nabla u_{\delta,\varepsilon}(x)|^q.$$

An elementary calculation shows that

$$w(x) = -2\delta\varepsilon(1+r^2)^{-(\delta+2)}\left[(N-2\delta-2)r^2 + N\right] + \varepsilon^p(1+r^2)^{-\delta p} + 2^q\delta^q\varepsilon^q r^q(1+r^2)^{-(\delta+1)q}, \quad r = |x|. \tag{3.15}$$

Hence, we obtain

$$w(x) \leq -2\delta\varepsilon(N-2\delta-2)(1+r^2)^{-(\delta+1)} + \varepsilon^p(1+r^2)^{-\delta p} + 2^q\varepsilon^q\delta^q(1+r^2)^{-\left(\delta+\frac{1}{2}\right)q}.$$

Next, using (3.13) and (3.14), we deduce that

$$w(x) \leq \varepsilon\left[-2\delta(N-2\delta-2) + \varepsilon^{p-1} + 2^q\varepsilon^{q-1}\delta^q\right](1+r^2)^{-\delta-1} < 0.$$

For all $\delta$ and $\varepsilon$ satisfying, respectively, (3.13) and (3.14), the function $u_{\delta,\varepsilon}$ defined by (2.19) is a stationary positive solution to (1.1), where $\lambda = \mu = -1$ and $w = w(x)$ $(< 0)$ is defined by (3.15). This proves part (ii) of Theorem 1.3. □

### 3.2. Proof of Theorem 1.4

(i) Let $N \geq 3$, $p > \frac{N}{N-2}$, and $q > \frac{N}{N-1}$. Suppose that $u$ is a global weak solution to (1.1). We first consider the case

$$\sigma < \sigma^*, \quad \mu_i\lambda_i \geq 0, \quad \lambda_i w_i \in \Lambda_\sigma,$$

for some $i \in \{1, 2\}$. Without restriction of the generality, we may suppose that $i = 1$, that is,

$$\sigma < \sigma^*, \quad \mu_1\lambda_1 \geq 0, \quad \lambda_1 w_1 \in \Lambda_\sigma.$$

From (2.24), for sufficiently large $T$, we have

$$\frac{1}{\lambda_1}\int_Q w_1(x)\varphi_T \,\mathrm{d}x\,\mathrm{d}t \geq CT^{\rho(N-\sigma)+1}.$$

Next, following the same arguments used in the proof of the statement (i) of Theorem 1.2, we obtain (2.25), which leads to a contradiction with $C>0$. This proves part (i)-(a) of Theorem 1.4.

Consider now the case

$$\sigma<\sigma^{**},\quad \lambda_i\mu_i>0,\quad \mu_i w_i\in\Lambda_\sigma,$$

for some $i\in\{1,2\}$. Without restriction of the generality, we may suppose that $i=1$, that is,

$$\sigma<\sigma^{**},\quad \lambda_1\mu_1>0,\quad \mu_1 w_1\in\Lambda_\sigma.$$

Similar to the previous case, for sufficiently large $T$, we obtain

$$\frac{1}{\mu_1}\int_Q w_1(x)\varphi_T \,\mathrm{d}x\,\mathrm{d}t \geq CT^{\rho(N-\sigma)+1}.$$

Hence, taking $\rho=\frac{p(q-1)}{(p-1)q}$ and using (3.9), for sufficiently large $T$, it holds that

$$0<C\leq T^{\frac{p}{q(p-1)}[\sigma(q-1)-q]}. \tag{3.16}$$

Since $\sigma<\sigma^{**}$, passing to the limit as $T\to\infty$ in (3.16), we arrive to contradiction to $C>0$. This proves part (i)-(b) of Theorem 1.4.

(ii) To carry out the second part of the proof, we assume

$$\max\left\{\frac{2p}{p-1},\frac{q}{q-1}\right\}\leq\sigma<N, \tag{3.17}$$

and consider the class of functions $u_{\delta,\varepsilon}$ defined by (2.19), where

$$\frac{\sigma-2}{2}\leq\delta<\frac{N-2}{2} \tag{3.18}$$

and $\varepsilon$ satisfies (3.14). Notice that since $\sigma<N$, the set of values of $\delta$ satisfying (3.18) is nonempty. Moreover, due to (3.17) and (3.18), $\delta$ satisfies also (3.13). Hence, from the proof of part (ii) of Theorem 1.3, we deduce that

$$w(x)=\Delta u_{\delta,\varepsilon}(x)+u^p_{\delta,\varepsilon}(x)+|\nabla u_{\delta,\varepsilon}(x)|^q<0,\quad x\in\mathbb{R}^N. \tag{3.19}$$

For all $\delta$ and $\varepsilon$ satisfying, respectively, (3.18) and (3.14), $u_{\delta,\varepsilon}$ is a stationary positive solution to (1.1), where $\lambda=\mu=-1$ and $w$ is defined by (3.19). On the other hand, by (3.15) and (3.19), we have $\lambda w=-w>0$ and

$$\lambda w(x)=2\delta\varepsilon(1+r^2)^{-(\delta+2)}\left[(N-2\delta-2)r^2+N\right]-\varepsilon^p(1+r^2)^{-\delta p}-2^q\delta^q\varepsilon^q r^q(1+r^2)^{-(\delta+1)q},\ r=|x|.$$

From (3.18), for sufficiently large $|x|$, we obtain

$$\begin{aligned}\lambda w(x)&\leq 2\delta\varepsilon(1+r^2)^{-(\delta+2)}\left[(N-2\delta-2)r^2+N\right]\\&\leq 2\delta\varepsilon N(1+r^2)^{-(\delta+1)}\\&\leq Cr^{-2(\delta+1)}\\&\leq Cr^{-\sigma}.\end{aligned}$$

This shows that $\lambda w\in\Sigma_\sigma$, and so also part (ii) of Theorem 1.4 is established. □

## 4. Conclusions

This paper aimed to enlarge the discussion about the existence and nonexistence of solutions to certain Schrödinger equations, with suitable nonlinearities. We centered on global weak solutions in the sense of Definitions 1.1 and 1.2. So, we studied first the situation where we drop the $|\nabla u|^q$-dependence of the nonlinearity (case $\mu = 0$), and then we established the similar results in the situation where the nonlinearity depends by $|\nabla u|^q$ (case $\mu \neq 0$). The second case can be seen as a way to perform a perturbation analysis of the response of the Schrödinger equation to changes in nonlinearity properties. In particular, we observed that the gradient term induced a phenomenon of discontinuity of the critical exponent, jumping from the Fujita critical exponent for (1.6) ($p = \frac{N}{N-2}$) to the value $p = \infty$ as $q$ reaches the value $\frac{N}{N-1}$ from the above (see Theorem 1.3). The use of some classical rescaled test function arguments linked to the Mitidieri–Pokhozhaev method [13] was the crucial key in establishing the proofs, in a clear way.

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### Declarations

**Conflict of interest** The authors declare that there is no conflict of interest. The authors have no competing interests.



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