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Response to Reviewers:	AMPA-S-23-00154 Dear Editor-in-Chief, Prof. Graziano Gentili, and Referee, first of all we thank you very much for your work and suggestions to improve the quality of the article we submitted. Please find enclosed our revised version of the manuscript: "Convergence for varying measures in the topological case" by L. Di Piazza, V. Marraffa, K. Musiał and A. R. Sambucini, that we have submitted for publication in “Annali di Matematica Pura e Applicata”.

In the revised version we have:

- formatted the article with the journal's template;
- made all the typos changes suggested by the Referee.

In particular for what concerns:

- point 1) and 2) we have correct the parts, following the referee's suggestions;
- point 3) in the class M(Omega) the measures are finite. In any case we have highlighted this on page 3 of the manuscript
- point 4) the proof of (old) Proposition 3.1 holds for a general Radon measure m. In fact the measure m can be decomposed in a purely atomic summand and in a non atomic one. We have observed moreover that our proof can be simplified and we put this new version in the revised paper
- point 5) and point 6) we have rewritten the two parts following the referee's comments.

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We look forward to hearing from you at your earliest convenience.
Yours sincerely,

Anna Rita Sambucini

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Convergence for varying measures in the topological case

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Abstract

In this paper convergence theorems for sequences of scalar, vector and multivalued Pettis integrable functions on a topological measure space are proved for varying measures vaguely convergent.

Keywords: setwise convergence, vaguely convergence, weak convergence of measures, locally compact Hausdorff space, Vitali's Theorem.

MSC Classification: 28B20 , 26E25 , 26A39 , 28B05 , 46G10 , 54C60 , 54C65

1 Introduction

Conditions for the convergence of sequences of measures $(m_n)_n$ and of their integrals $(\int f_n dm_n)_n$ in a measurable space Ω are of interest in many areas of pure and applied mathematics such as statistics, transportation problems, interactive partial systems, neural networks and signal processing (see, for instance, [1–3, 9–12, 17]). In particular, for the image reconstruction, which is a branch of signal theory, in the last years, interval-valued functions have been considered since the process of discretization of an image is affected by quantization errors ([19]) and its numerical approximation can be interpreted as a suitable sequence of interval-valued functions (see for instance [22, 28]).

Obviously, suitable convergence notions are needed for the varying measures, see for example [15, 16, 18, 21, 23, 24, 30] and the references therein. In a previous paper [13] we have examined the problem when the varying measures converge setwisely in an arbitrary measurable space. This type of convergence is a powerful tool since it permits to obtain strong results, for example the Vitali-Hahn-Saks Theorem or a Dominated Convergence Theorem [18].
But sometime in the applications it is difficult, at least technically, to prove that the sequence $(m_n(A))_n$ converges to $m(A)$ for every measurable set A, unless e.g. the sequence $(m_n)_n$ is decreasing or increasing. So other types of convergence are studied, based on the structure of the topological space Ω, such as the vague and the weak convergence which are, in general, weaker than the setwise. These convergences are useful, for example, from the point of view of applications on non-interactive particle systems (see [9, 23]).

In the present paper we continue the research started in [13] and we provide sufficient conditions in order to obtain Vitali's type convergence results for a sequence of (multi)functions $(f_n)_n$ integrable with respect to a sequence $(m_n)_n$ of measures when $(m_n)_n$ converges vaguely or weakly to a finite measure m.
The known results, in literature, as far as we know, require that the topological space Ω, endowed with the Borel σ-algebra is a metric space ([15, 16]), or a locally compact space which is also: separable and metric ([18]), metrizable ([20]) or Hausdorff second countable ([30]). An interesting comparison among all these results is given in [23].

In the present paper, following the ideas of Bogachev ([4]), we assume that Ω is only an arbitrary locally compact Hausdorff space. The paper is organized as follows: in Section 2 the topological structure of the space Ω is introduced together with the convergence types considered and some of their properties. In Section 3 the scalar case is studied; the main result of this section is Theorem 3.4, where we obtain the convergence of the integrals $(\int f_n dm_n)_n$ over arbitrary Borel sets under suitable conditions. In Section 4 Theorem 3.4 is applied in order to obtain analogous results for the multivalued case, obtaining as a corollary also the vector case. In both cases the Pettis integrability of the integrands is considered. Finally, adding a condition as in [13, Theorem 3.2] we obtain a convergence result for (multi)functions in Proposition 4.4 on measurable spaces.

2 Topological case, preliminaries

Let Ω be a locally compact Hausdorff space and let $\mathcal{B}$ be its Borel σ-algebra. The symbol $\mathcal{F}(\Omega)$ indicates the class of all $\mathcal{B}$-measurable functions $f : \Omega \to \mathbb{R}$. We denote by $C(\Omega)$, $C_0(\Omega)$, $C_c(\Omega)$ and $C_b(\Omega)$ respectively the family of all continuous functions, and the subfamilies of all continuous functions that

vanish at infinity, have compact support, are bounded.
Throughout, we will use Urysohn's Lemma in the form ([29, Lemma 2.12]):

- If K is compact and $U \supset K$ is open in a locally compact Ω, then there exists $f : \Omega \to [0,1]$, $f \in C_c(\Omega)$, such that $\chi_K \leq f \leq \chi_U$.

All the measures we will consider on $(\Omega, \mathcal{B})$ are finite and by $\mathcal{M}(\Omega)$ we denote the family of *finite nonnegative measures*. As usual a measure $m \in \mathcal{M}(\Omega)$ is Radon if it is inner regular in the sense of approximation by compact sets.

We recall the following definitions of convergence for measures.

Definition 2.1 Let m and m_n be in $\mathcal{M}(\Omega)$. We say that

2.1.a) $(m_n)_n$ *converges vaguely* to m ($m_n \xrightarrow{v} m$) ([18, Section 2.3]) if

$$\int_\Omega g dm_n \to \int_\Omega g dm, \quad \text{for every } g \in C_0(\Omega).$$

2.1.b) $(m_n)_n$ *converges weakly* to m ($m_n \xrightarrow{w} m$) ([18, Section 2.1]) if

$$\int_\Omega g dm_n \to \int_\Omega g dm, \quad \text{for every } g \in C_b(\Omega).$$

2.1.c) $(m_n)_n$ *converges setwisely to* m *(*$m_n \xrightarrow{s} m$*) if* $\lim_n m_n(A) = m(A)$ *for every* $A \in \mathcal{B}$ ([18, Section 2.1], [16, Definition 2.3]) or, equivalently ([23]), if

$$\int_\Omega g dm_n \to \int_\Omega g dm, \quad \text{for every bounded } g \in \mathcal{F}(\Omega).$$

2.1.d) $(m_n)_n$ is *uniformly absolutely continuous with respect to* m if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$(E \in \mathcal{B} \quad \text{and} \quad m(E) < \delta) \implies \sup_n m_n(E) < \varepsilon. \tag{1}$$

We would like to note that the condition $m_n \leq m$, for every $n \in \mathbb{N}$, implies that $(m_n)_n$ is uniformly absolutely continuous with respect to m.

Remark 2.2 As observed in [18] the setwise convergence is stronger than the vague and the weak convergence. For the converse implications we know, by [21, Lemma 4.1 (ii)], that if $(m_n)_n$ is a sequence in $\mathcal{M}(\Omega)$ with $m_n \leq m$, where $m \in \mathcal{M}(\Omega)$ and $(m_n)_n$ converges vaguely to m, then $(m_n)_n$ converges setwisely to m. If m is $\overline{\mathbb{R}}$-valued this is not true in general, see for example [18, page 143]. The weak convergence is stronger than the vague convergence; as an example we can consider $m_n := \delta_n$ (the Dirac measure at the point $x = n$) and $m := 0$. The sequence $(m_n)_n$ converges vaguely to m, but since $m_n(\mathbb{R}) = 1 \not\to 0 = m(\mathbb{R})$ the convergence cannot be weak.
Moreover we note that if $(m_n)_n$ converges weakly to m, then $m_n(\Omega) \to m(\Omega)$ (it is enough to take $g = 1$ in the definition).

We have

Proposition 2.3 *Let m_n, $n \in \mathbb{N}$, and m be in $\mathcal{M}(\Omega)$, with m Radon. If $(m_n)_n$ is uniformly absolutely continuous with respect to m and $(m_n)_n$ is vaguely convergent to m, then $(m_n)_n$ is weakly convergent to m.*

Proof We fix $\varepsilon > 0$ and let $f \in C_b(\Omega)$. We set $c := \max\{1, \sup_\Omega |f(\omega)|\}$, let $\delta \in]0, \varepsilon[$ be taken in such a way that if E is a Borel set with $m(E) < \delta$, then

$$\max\left\{\int_E |f|\, dm, \sup_n m_n(E)\right\} < \varepsilon.$$

Let K be a compact set such that $m(K^c) < \delta$. Then by Urysohn's Lemma let $h : \Omega \to [0,1]$ be a continuous function with compact support such that $h(\omega) = 1$ for $\omega \in K$. Let $g := f \cdot h$. Then $g \in C_0(\Omega)$. We have for sufficiently large $n \in \mathbb{N}$, depending on the vaguely convergence,

$$\begin{aligned}
&\Big|\int_\Omega f dm - \int_\Omega f dm_n\Big| \\
&\leq \int_\Omega |f-g| dm + \int_\Omega |f-g| dm_n + \Big|\int_\Omega g\ dm - \int_\Omega g\ dm_n\Big| \\
&= \int_{K^c} |f|\cdot|1-h|\, dm + \int_{K^c} |f|\cdot|1-h|\, dm_n + \Big|\int_\Omega g\ dm - \int_\Omega g\ dm_n\Big| \\
&\leq \varepsilon(c+2).
\end{aligned}$$

□

For other relations among weak or vague convergence and setwise convergence see [21, Lemma 4.1]. Moreover

Proposition 2.4 *Let m_n, $n \in \mathbb{N}$, and m be in $\mathcal{M}(\Omega)$ with m Radon. If $m_n \leq m$, for every $n \in \mathbb{N}$, and $(m_n)_n$ is vaguely convergent to m, then for every $f \in L^1(m)$ and $A \in \mathcal{B}$*

$$\lim_n \int_A f\, dm_n = \int_A f\, dm. \tag{2}$$

In particular $(m_n)_n$ converges to m setwisely.

Proof Let $f \in L^1(m)$ be fixed. Given $\varepsilon > 0$ there exists $g \in C_c(\Omega)$ such that

$$\int_\Omega |f-g| dm_n \leq \int_\Omega |f-g| dm < \frac{\varepsilon}{3}. \tag{3}$$

Moreover, since $(m_n)_n$ is vaguely convergent to m, let $N(\varepsilon/3)$ be such that

$$\Big|\int_\Omega g\ dm - \int_\Omega g\ dm_n\Big| < \frac{\varepsilon}{3} \tag{4}$$

for $n > N$. Therefore by (3) and (4) for $n > N$ we obtain

$$\begin{aligned}
&\Big|\int_\Omega f dm - \int_\Omega f dm_n\Big| \\
&\leq \int_\Omega |f-g| dm + \int_\Omega |f-g| dm_n + \Big|\int_\Omega g\ dm - \int_\Omega g\ dm_n\Big| < \varepsilon.
\end{aligned}$$

Now if $A \in \mathcal{B}$, also $f\chi_A \in L^1(m)$ and (2) follows. In particular $m_n \xrightarrow{s} m$. □

Results of the previous type are contained for example in [18, Proposition 2.3] for the setwise convergence when the measures m_n are equibounded by a measure ν for non negative $f \in L^1(\nu)$ or in [18, Proposition 2.4] under the additional hypothesis of separability of Ω for non negative and lower semicontinuous functions f.

We now introduce the following definition

Definition 2.5 Let $(m_n)_n$ be a sequence in $\mathcal{M}(\Omega)$. We say that:

2.5.a) A sequence $(f_n)_n \subset \mathcal{F}(\Omega)$ has *uniformly absolutely continuous* (m_n)*-integrals on* Ω, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $n \in \mathbb{N}$

$$(A \in \mathcal{B} \quad \text{and} \quad m_n(A) < \delta) \quad \Longrightarrow \quad \int_A |f_n| \, dm_n < \varepsilon. \tag{5}$$

Analogously a function $f \in \mathcal{F}(\Omega)$ has *uniformly absolutely continuous* (m_n)*-integrals on* Ω if previous condition (5) holds for $f_n := f$ for every $n \in \mathbb{N}$.

2.5.b) A sequence $(f_n)_n \subset \mathcal{F}(\Omega)$ is *uniformly* (m_n)*-integrable on* Ω if

$$\lim_{\alpha \to +\infty} \sup_n \int_{|f_n| > \alpha} |f_n| \, dm_n \; = 0. \tag{6}$$

Remark 2.6 As we observed in [13, Proposition 2.6] if $(m_n)_n$ is a bounded sequence of measures and $(f_n)_n \subset \mathcal{F}$, then, $(f_n)_n$ is uniformly (m_n)-integrable on Ω if and only if it has uniformly absolutely continuous (m_n)-integrals and

$$\sup_n \int_\Omega |f_n| \, dm_n < +\infty \, . \tag{7}$$

3 The scalar case

Proposition 3.1 *Let* $(m_n)_n$ *be a sequence in* $\mathcal{M}(\Omega)$ *which is uniformly absolutely continuous with respect to a Radon measure* $m \in \mathcal{M}(\Omega)$ *and vaguely convergent to* m*. Let* $f \in C(\Omega)$ *be a function which has uniformly absolutely continuous* (m_n)*-integrals on* Ω*. Then,*

$$\sup_n \int_\Omega |f| \, dm_n < +\infty \, . \tag{8}$$

Proof Let $\varepsilon > 0$ be fixed and let $\sigma = \sigma(\varepsilon)$ be that of the uniform absolutely continuous (m_n)-integrability of f as in formula (5) (with $f_n = f$ for each $n \in \mathbb{N}$). Moreover let $\delta = \delta(\sigma) > 0$ be that of the uniform absolute continuity of $(m_n)_n$ with respect to m, as in formula (1).

Since m is Radon, there is a compact set K such that $m(\Omega \setminus K) < \delta$. By Urysohn's Lemma there exists a continuous function $h : \Omega \to [0, 1]$ with compact support such that $h(\omega) = 1$ for $\omega \in K$. Let $g := |f| \cdot h$. Then $g \in C_0(\Omega)$. Hence

$$\int_\Omega |f| \, dm_n \le \int_K |f| \, dm_n + \int_{\Omega \setminus K} |f| \, dm_n \le \int_\Omega g \, dm_n + \varepsilon.$$

Since $(m_n)_n$ converges vaguely to m, then

$$\int_\Omega g\,dm_n \longrightarrow \int_\Omega g\,dm < +\infty.$$

Hence

$$\sup_n \int_K |f|\,dm_n \le \sup_n \int_\Omega g\,dm_n < +\infty.$$

$\square$

Proposition 3.2 *Let $(m_n)_n$ be a sequence in $\mathcal{M}(\Omega)$ which is uniformly absolutely continuous with respect to a Radon measure $m \in \mathcal{M}(\Omega)$ and vaguely convergent to m. Moreover let $f \in C(\Omega)$ be a function which has uniformly absolutely continuous (m_n)-integrals on Ω. Then $f \in L^1(m)$ and*

$$\lim_n \int_\Omega f\,dm_n = \int_\Omega f\,dm. \tag{9}$$

Proof By Proposition 3.1 $\sup_n \int_\Omega |f|dm_n < +\infty$. We denote by $(g_k)_k$ an increasing sequence of functions in $C_b(\Omega)$ such that $0 \le g_k \uparrow |f|$, m a.e.
By Proposition 2.3 $(m_n)_n$ is also weakly convergent to m. Now fix $k \in \mathbb{N}$. Let $N_1(k,1)$ be such that if $n > N_1$

$$\int_\Omega g_k\,dm - 1 < \int_\Omega g_k\,dm_n. \tag{10}$$

By Proposition 3.1 we infer

$$\int_\Omega g_k dm - 1 < \int_\Omega g_k dm_n \le \sup_n \int_\Omega |f| dm_n < \infty\,. \tag{11}$$

So, by the Monotone Convergence Theorem applied to the sequence $(g_k)_k$ we obtain $f \in L^1(m)$.
We are showing now that (9) holds. We fix $\sigma > 0$. Since $f \in L^1(m)$ there exists a positive δ_0 such that for every $A \in \mathcal{B}$ with $m(A) < \delta_0$ then

$$\int_A |f| dm < \sigma. \tag{12}$$

Moreover let $\varepsilon(\sigma) > 0$ be that of the uniform absolutely continuous (m_n)-integrability of f in Ω (with $f_n = f$ for each $n \in \mathbb{N}$) and $\delta = \delta(\varepsilon) \in]0, \min\{\varepsilon, \delta_0\}[$ be that of the absolute continuity of $(m_n)_n$ with respect to m.
So, if $m(A) < \delta$ then $\sup_n m_n(A) < \varepsilon$ and

$$\sup_n \int_A |f| dm_n < \sigma. \tag{13}$$

By Urysohn's Lemma one can find a compact set K with $m(K^c) < \delta$ and a function $h : \Omega \to [0,1]$ in $C_c(\Omega)$ and equal to 1 on K. So $g := f \cdot h \in C_c(\Omega)$.
Since the sequence $(m_n)_n$ is vaguely convergent to m, there is $N_2(\sigma) > N_1$ such that for $n > N_2$

$$\left| \int_\Omega g\,dm_n - \int_\Omega g\,dm \right| < \sigma \tag{14}$$

Then by (13), (14) and (12), for $n > N_2$, we have

$$\left| \int_\Omega f\,dm_n - \int_\Omega f\,dm \right| \le$$

$$\left|\int_\Omega (f-g)\,dm_n\right| + \left|\int_\Omega g\,dm_n - \int_\Omega g\,dm\right| + \left|\int_\Omega (g-f)\,dm\right| \leq$$
$$\left|\int_\Omega f(1-h)\,dm_n\right| + \left|\int_\Omega g\,dm_n - \int_\Omega g\,dm\right| + \left|\int_\Omega f(1-h)\,dm\right| \leq$$
$$\int_{K^c} |f|\,dm_n + \sigma + \int_{K^c} |f|\,dm < 3\sigma$$

and the thesis follows. □

Now our aim is to obtain a limit result

$$\lim_n \int_A f_n dm_n = \int_A f dm, \qquad \text{for every} A \in \mathcal{B}. \tag{15}$$

For the scalar case, using a Portmanteau's characterization of the vague convergence in metric spaces (see for example [20]), sufficient conditions when $A = \Omega$, are given

- in locally compact second countable and Hausdorff spaces, ([30, Theorems 3.3 and 3.5]), by Serfozo, for the vague and weak convergence respectively, when the sequence $(f_n)_n$ converges continuously to f. Under a domination condition in the first result while, in the second, the uniform (m_n)-integrability of the sequence $(f_n)_n$, with $f_n \geq 0$ for every $n \in \mathbb{N}$, is required;
- in locally compact separable metric spaces ([18]) by Hernandez-Lerma and Lasserre, obtaining a Fatou result and asking for the convergence of the sequence of measures an inequality of the lim inf of the m_n on each Borelian set;
- in metric spaces ([15, 16]), where the authors obtained a dominated convergence result for sequences of equicontinuous functions $(f_n)_n$ satisfying the uniform (m_n)-integrability.

In Theorem 3.4, taking into account Remarks 2.2 and 2.6, we extend [30, Theorem 3.5], obtaining a sufficient condition when the convergence is vague, the functions f_n are real valued and using the uniformly absolutely continuous (m_n)-integrability of the sequence $(f_n)_n$. Later, in Section 4, we will also extend it to the vector and multivalued cases making use of the Pettis integrability.

We assume only that Ω is a locally compact Hausdorff space and then, in our setting, Ω is a Tychonoff space, i.e. a completely regular Hausdorff space ([14, Theorem 3.3.1]). So we are able to use the following Portmanteau's characterization of the vague convergence for positive measures given in [4].

Theorem 3.3 ([4, Corollary 8.1.8 and Remark 8.1.11]) *Let Ω be an arbitrary completely regular space and let m and m_n, $n \in \mathbb{N}$, be measures in $\mathcal{M}(\Omega)$ with m Radon and assume that $\lim_n m_n(\Omega) = m(\Omega)$. Then the following are equivalent:*

3.3.i) $(m_n)_n$ is vaguely convergent to m;

3.3.ii) for any closed set $F \subset \Omega$, $\limsup_n m_n(F) \leq m(F)$.

So we have

Theorem 3.4 *Let m and m_n, $n \in \mathbb{N}$, be measures in $\mathcal{M}(\Omega)$, with m Radon. Let $f, f_n \in \mathcal{F}(\Omega)$. Suppose that*

(3.4.i) $f_n(t) \to f(t)$, m-a.e.;

(3.4.ii) $f \in C(\Omega)$;

(3.4.iii) $(f_n)_n$ and f have uniformly absolutely continuous (m_n)-integrals on Ω;

(3.4.iv) $(m_n)_n$ is vaguely convergent to m and uniformly absolutely continuous with respect to m.

Then $f \in L^1(m)$ and

$$\lim_n \int_A f_n dm_n = \int_A f dm \quad \text{for every } A \in \mathcal{B}. \tag{16}$$

Proof By Proposition 3.2 the function $f \in L^1(m)$. We proceed by steps.

Step 1 We prove (16) for $A = \Omega$.
Fix $\varepsilon > 0$ and let $\delta := \min\left\{\frac{\varepsilon}{6}, \delta(\frac{\varepsilon}{6}), \delta_f(\frac{\varepsilon}{6})\right\} > 0$ where $\delta_f(\frac{\varepsilon}{6})$ is that of the absolute continuity of $\int_\cdot |f| dm$, and by (3.4.iii) $\delta(\frac{\varepsilon}{6})$ is that of (5) for both $(f_n)_n$ and f with respect to $(m_n)_n$.
By the hypothesis (3.4.iv) let $0 < \delta_0 < \delta$ be such that

$$(E \in \mathcal{B} \quad \text{and} \quad m(E) < \delta_0) \implies \sup_n m_n(E) < \delta. \tag{17}$$

By the Egoroff's Theorem, we can find a compact set K such that $f_n \to f$ uniformly on K and $m(K^c) < \delta_0$.
We observe that by condition 3.4.iv) and by Proposition 2.3 $(m_n)_n$ weakly converges to m and then $\lim_n m_n(\Omega) = m(\Omega)$. So by Theorem 3.3, let $N_0 \in \mathbb{N}$ be such that

$$m_n(K) < m(K) + 1, \tag{18}$$

for every $n > N_0$. Moreover, since the convergence is uniform on K, let $N_1 > N_0 \in \mathbb{N}$ be such that

$$|f_n(t) - f(t)| < \frac{\varepsilon}{6(m(K)+1)}, \tag{19}$$

for every $t \in K$ and $n > N_1$. Then, for all $n > N_1$,

$$\int_K |f_n - f| dm < \frac{\varepsilon}{6}. \tag{20}$$

Therefore by (18) and (19) we obtain, for every for $n > N_1$,

$$\int_K |f_n - f| dm_n \le \frac{\varepsilon}{6(m(K)+1)} \cdot m_n(K) < \frac{\varepsilon}{6}. \tag{21}$$

Since $m(K^c) < \delta_0$ by (17) it follows that $m_n(K^c) < \delta$ for every $n \in \mathbb{N}$. Moreover, by hypothesis 3.4.iii) and by the choice of δ, we have that

$$\max\left\{\int_{K^c} |f| dm, \int_{K^c} |f_n| dm_n, \int_{K^c} |f| dm_n\right\} < \frac{\varepsilon}{6}. \tag{22}$$

By Urysohn's Lemma let $h : \Omega \to [0,1]$ be a continuous function with compact support equal to 1 on K. Then $g := f \cdot h \in C_c(\Omega)$ and by (22) we have

$$\max\left\{\int_{K^c} |f-g| dm_n, \int_{K^c} |f-g| dm\right\} < \frac{\varepsilon}{6}. \tag{23}$$

Moreover, since $(m_n)_n$ is vaguely convergent to m, let $n > N_2 \ge N_1$ be such that

$$\left|\int_\Omega g\ dm - \int_\Omega g\ dm_n\right| < \frac{\varepsilon}{6}. \tag{24}$$

Therefore by (21)–(24) for $n > N_2$ we obtain

$$\begin{aligned}
&\left|\int_\Omega f dm - \int_\Omega f_n dm_n\right| \le \left|\int_\Omega (f_n - f) dm_n\right| + \left|\int_\Omega f dm - \int_\Omega f dm_n\right| \\
&\le \left|\int_K (f_n - f) dm_n\right| + \int_{K^c} |f_n| dm_n + \int_{K^c} |f| dm_n + \left|\int_\Omega f dm - \int_\Omega f dm_n\right| \\
&\le \frac{\varepsilon}{2} + \left|\int_\Omega f dm - \int_\Omega f dm_n\right| \qquad (25)\\
&\le \frac{\varepsilon}{2} + \int_{K^c} |f-g| dm + \int_{K^c} |f-g| dm_n + \left|\int_\Omega g\ dm - \int_\Omega g\ dm_n\right| < \varepsilon
\end{aligned}$$

so (16) follows for $A = \Omega$.

Step 2 Now we are proving that (16) is valid for an arbitrary compact set K.
Let once again, $\varepsilon > 0$ be fixed. By (3.4.iii) and (3.4.iv) there exist $\delta_1, \delta_2 > 0$ such that:

j_1) if $m_n(E) < \delta_2$, then $\int_E |f_n|\, dm_n < \varepsilon$ for every $n \in \mathbb{N}$;

j_2) if $m(E) < \delta_1$, then $m_n(E) < \delta_2$ for every $n \in \mathbb{N}$;

j_3) if $m(E) < \delta_1$, then $\int_E |f|\, dm < \varepsilon$.

Let now $U \supset K$ be an open set such that $m(U \setminus K) < \delta_1$. Then let $g : \Omega \to [0,1]$ be continuous and such that $g = 1$ on K and zero on U^c.
Observe that the sequence $(f_n g)_n$ and the function fg satisfy all the hypotheses of Theorem 3.4 so, for the Step 1, we have

$$\lim_n \int_U f_n g\, dm_n = \lim_n \int_\Omega f_n g\, dm_n = \int_\Omega fg\, dm = \int_U fg\, dm.$$

Then, by the previous inequalities and for n sufficiently large, we have

$$\left|\int_K f_n\, dm_n - \int_K f\, dm\right| = \left|\int_K f_n g\, dm_n - \int_K fg\, dm + \Big(\int_{K^c} f_n g\, dm_n - \int_{K^c} fg\, dm\Big) + \right.$$

$$
\begin{aligned}
&-\Big(\int_{K^c} f_n g\, dm_n - \int_{K^c} fg\, dm\Big)\Big| \\
&\le \left|\int_{\Omega} f_n g\, dm_n - \int_{\Omega} fg\, dm\right| + \left|\int_{K^c} f_n g\, dm_n - \int_{K^c} fg\, dm\right| \\
&= \left|\int_{\Omega} f_n g\, dm_n - \int_{\Omega} fg\, dm\right| + \left|\int_{U\setminus K} f_n g\, dm_n - \int_{U\setminus K} fg\, dm\right| \\
&< \left|\int_{\Omega} f_n g\, dm_n - \int_{\Omega} fg\, dm\right| + \int_{U\setminus K} |f_n|\, dm_n + \int_{U\setminus K} |f|\, dm < 3\varepsilon.
\end{aligned}
$$

Step 3 Let now B a Borelian set and let $\varepsilon > 0$, $\delta_1, \delta_2 > 0$ as in Step 2. Let C_1 be a compact set with $C_1 \subset B$ such that $m(B \setminus C_1) < \delta_1$. So

$$
\begin{aligned}
&\left|\int_B f_n\, dm_n - \int_B f\, dm\right| \le \left|\int_{C_1} f_n\, dm_n - \int_{C_1} f\, dm\right| + \\
&+ \left|\int_{B\setminus C_1} f_n\, dm_n - \int_{B\setminus C_1} f\, dm\right| \\
&\le \left|\int_{C_1} f_n\, dm_n - \int_{C_1} f\, dm\right| + \int_{B\setminus C_1} |f_n|\, dm_n + \int_{B\setminus C_1} |f|\, dm.
\end{aligned}
$$

So the assertion follows from $j_1) - j_3)$ and the compact case in Step 2 and this ends the proof.

□

Corollary 3.5 Let m and m_n, $n \in \mathbb{N}$, be measures in $\mathcal{M}(\Omega)$, with m Radon. If $(m_n)_n$ is vaguely convergent to m and uniformly absolutely continuous with respect to m, then $(m_n)_n$ converges setwisely to m.

Proof It is a consequence of Theorem 3.4 if we assume $f_n = f \equiv 1$ for every $n \in \mathbb{N}$.

□

Remark 3.6

3.6.a) We observe that under the hypotheses of Theorem 3.4, if $f \in C(\Omega)$, then also $f^{\pm}$ are in $C(\Omega)$ and $f_n^{\pm}(t) \to f^{\pm}(t)$ m-a.e. as $n \to \infty$. In fact

$$
\begin{aligned}
&\Big||f_n| - |f|\Big| \le |f_n - f| \\
&2f_n^+ = f_n + |f_n| \to f + |f| = 2f^+; \\
&2f_n^- = |f_n| - f_n \to |f| - f = 2f^-.
\end{aligned}
$$

Moreover also $(f_n^{\pm})_n$ and $f^{\pm}$ satisfy condition 3.4.iii) since

$$f_n^{\pm} \le |f_n| \quad \text{and} \quad f^{\pm} \le |f|.$$

Therefore in the hypotheses of Theorem 3.4 we get also

$$\lim_n \int_A f_n^{\pm} dm_n = \int_A f^{\pm} dm, \quad \text{for every } A \in \mathcal{B}.$$

3.6.b) Theorem 3.4 is still valid if we replace condition 3.4.i) with

3.4.i') f_n converges in m-measure to f.

In fact, by 3.4.i'), there exists a subsequence of $(f_{n_k})_k$ which converges m-a.e. to f. Then Theorem 3.4 is true for such subsequence. So this implies that the result of this theorem, equality (16), is still valid for the initial sequence (with convergence in m-measure) because if, absurdly, a subsequence existed in which it is not valid, there would be a contradiction.

A simple consequence of the Theorem 3.4 is the following

Theorem 3.7 *Let m and m_n, $n \in \mathbb{N}$, be measures in $\mathcal{M}(\Omega)$, with m Radon. Let $f, f_n \in \mathcal{F}(\Omega)$. Suppose that*

(3.7.i) $f_n(t) \to f(t)$, *m-a.e.;*

(3.7.ii) $f \in C_b(\Omega)$;

(3.7.iii) $(f_n)_n$ *has uniformly absolutely continuous (m_n)-integrals on Ω;*

(3.7.iv) $(m_n)_n$ *is vaguely convergent to m and uniformly absolutely continuous with respect to m.*

Then

$$\lim_n \int_A f_n dm_n = \int_A f dm, \quad \text{for every } A \in \mathcal{B}.$$

Proof The assertion follows from Theorem 3.4 since f has uniformly absolutely continuous (m_n)-integrals, in fact it is enough to take the pair $(\varepsilon, \delta(\varepsilon/M))$ where $M > \sup_{t\in\Omega} |f(t)|$. □

4 The multivalued and the vector cases

4.1 The multivalued case

Let X be a Banach space with dual X^* and let B_{X^*} be the unit ball of X^*. The symbol $cwk(X)$ denotes the family of all weakly compact and convex subsets of X. For every $C \in cwk(X)$ the *support function of C* is denoted by $s(\cdot, C)$ and defined on X^* by $s(x^*, C) = \sup\{\langle x^*, x\rangle : \ x \in C\}$. Recall that X is said to be *weakly compact generated* (briefly WCG) if it possesses a weakly compact subset K whose linear span is dense in X.
A map $\Gamma : \Omega \to cwk(X)$ is called a *multifunction.* A space $Y \subset X$ *m-determines* a multifunction Γ if $s(x^*, \Gamma) = 0$ m a.e. for every $x^* \in Y^\perp$, where the exceptional sets depend on x^*.
A multifunction Γ is said to be

- *scalarly measurable* if $t \to s(x^*, \Gamma(t))$ is measurable, for every $x^* \in X^*$;
- *scalarly m-integrable* if $t \to s(x^*, \Gamma(t))$ is m-integrable, for every $x^* \in X^*$, where $m \in \mathcal{M}(\Omega)$;
- *scalarly continuous* if for every $x^* \in X^*$, $t \to s(x^*, \Gamma(t))$ is continuous.

A multifunction $\Gamma : \Omega \to cwk(X)$ is said to be *Pettis integrable* in $cwk(X)$ with respect to a measure m (or shortly Pettis m-integrable) if Γ is scalarly m-integrable and for every measurable set A, there exists $M_\Gamma(A) \in cwk(X)$ such that

$$s(x^*, M_\Gamma(A)) = \int_A s\big(x^*, \Gamma\big)\, dm \qquad \text{for all } \; x^* \in X^*.$$

We set $\int_A \Gamma dm := M_\Gamma(A)$.
For the properties of Pettis m-integrability in the multivalued case we refer to [5–8, 26, 27], while for the vector case we refer to [25]. If Γ is single-valued we obtain the classical definition of Pettis integral for vector function.

Given a sequence of multifunctions we introduce now some definitions of uniformly absolutely continuous scalar integrability using Definition 2.5.

Definition 4.1 For every $n \in \mathbb{N}$, let m_n be a measure in $\mathcal{M}(\Omega)$ and let $\Gamma_n : \Omega \to cwk(X)$ be a multifunction which is scalarly m_n-integrable. We say that the sequence $(\Gamma_n)_n$ has *uniformly absolutely continuous scalar (m_n)-integrals on Ω* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for every $n \in \mathbb{N}$ and $A \in \mathcal{B}$, it is

$$m_n(A) < \delta \;\Rightarrow\; \sup\left\{\int_A |s(x^*, \Gamma_n)| dm_n \colon \; \| x^* \| \leq 1\right\} < \varepsilon. \tag{26}$$

Analogously a multifunction Γ has *uniformly absolutely continuous scalar (m_n)-integrals on Ω* if previous condition (26) holds for $\Gamma_n := \Gamma$ for every $n \in \mathbb{N}$. Moreover we say that Γ has *uniformly absolutely continuous scalar m-integrals on Ω* if, in formula (26), it is $\Gamma_n := \Gamma$ and $m_n = m$ for every $n \in \mathbb{N}$. In this case we have, for every $A \in \mathcal{B}$,

$$m(A) < \delta \;\Rightarrow\; \sup\left\{\int_A |s(x^*, \Gamma)| dm \colon \; \| x^* \| \leq 1\right\} < \varepsilon. \tag{27}$$

Theorem 4.2 *Let Γ, Γ_n, $n \in \mathbb{N}$, be scalarly measurable multifunctions. Moreover let m, m_n, $n \in \mathbb{N}$, be measures in $\mathcal{M}(\Omega)$ and let m be Radon. Suppose that*

(4.2.j) $(\Gamma_n)_n$ and Γ have uniformly absolutely continuous scalar (m_n)-integrals on Ω;

(4.2.jj)) $s(x^*, \Gamma_n) \to s(x^*, \Gamma)$ m-a.e. for each $x^* \in X^*$;

(4.2.jjj) Γ is scalar continuous;

(4.2.jv) $(m_n)_n$ is vaguely convergent to m and uniformly absolutely continuous with respect to m;

(4.2.v) each multifunction Γ_n is Pettis m_n-integrable.

Then the multifunction Γ is Pettis m-integrable in $cwk(X)$ and

$$\lim_n s\Big(x^*, \int_A \Gamma_n\, dm_n\Big) = s\Big(x^*, \int_A \Gamma\, dm\Big), \tag{28}$$

for every $x^* \in X^*$ and for every $A \in \mathcal{B}$.

Proof Let $x^* \in X^*$ be fixed. Then the sequence of functions $\big(s(x^*, \Gamma_n)\big)_n$ and the function $s(x^*, \Gamma)$ defined on Ω satisfy the assumptions of Theorem 3.4. So, for each $A \in \mathcal{B}$

$$\lim_n \int_A s(x^*, \Gamma_n)\, dm_n = \int_A s(x^*, \Gamma)\, dm. \tag{29}$$

In order to prove that Γ is Pettis m-integrable, following [26, Theorem 2.5], it is enough to show that the sublinear operator $T_\Gamma : X^* \to L^1(m)$, defined as $T_\Gamma(x^*) = s(x^*, \Gamma)$ is weakly compact (step C_w) and that Γ is determined by a WCG space $Y \subset X$ (step D).

C_w) First of all we prove that the operator T_Γ is bounded. By (4.2.jjj) Γ is scalar m-integrable. Therefore Γ is Dunford-integrable in $cw^*k(X^{**})$, where X^{**} is endowed with the w^*-topology, and for every $A \in \mathcal{B}$ let $M^D_\Gamma(A) \in cw^*k(X^{**})$ be such that

$$s(x^*, M^D_\Gamma(A)) = \int_A s(x^*, \Gamma) dm < +\infty, \tag{30}$$

for every $x^* \in X^*$. So $s(x^*, M^D_\Gamma(\cdot))$ is a scalar measure and

$$\int_\Omega |s(x^*, \Gamma)| dm \leq 2 \sup_{A \in \mathcal{B}} \left| \int_A s(x^*, \Gamma) dm \right| < +\infty.$$

Hence, the set $\bigcup_{A \in \mathcal{B}} M^D_\Gamma(A) \subset X^{**}$ is bounded, by the Banach– Steinhaus Theorem, and

$$\sup_{\|x^*\| \leq 1} \int_\Omega |s(x^*, \Gamma)| dm \leq 2 \sup \left\{ \| x \| : x \in \bigcup_{A \in \mathcal{B}} M^D_\Gamma(A) \right\} < +\infty.$$

Since the set $\{s(x^*, \Gamma) : \| x^* \| \leq 1\}$ is bounded in $L^1(m)$, the operator T_Γ is bounded. In order to obtain the weak compactness of the operator T_Γ it is enough to prove that Γ has absolutely continuous scalar m-integrals on Ω. Let $x^* \in B_{X^*}$ be fixed. Now fix $\varepsilon > 0$ and let $\sigma(\varepsilon) > 0$ satisfy (4.2.j). Moreover let $\delta(\sigma) > 0$ verify (4.2.jv). Let $E \in \mathcal{B}$ be such that $m(E) < \delta$ and set

$$E^+ = \{t \in E : s(x^*, \Gamma(t)) \geq 0\} \quad E^- = \{t \in E : s(x^*, \Gamma(t)) < 0\}.$$

By (29) let now $N_{x^*} \in \mathbb{N}$ be an integer such that for every $n \geq N_{x^*}$

$$\left| \int_{E^\pm} s(x^*, \Gamma) dm \right| < \left| \int_{E^\pm} s(x^*, \Gamma_n) dm_n \right| + \frac{\varepsilon}{2}.$$

So, for every $n \geq N_{x^*}$,

$$\begin{aligned} \int_E |s(x^*, \Gamma)| dm &= \int_{E^+} s(x^*, \Gamma) dm + \left| \int_{E^-} s(x^*, \Gamma) dm \right| \\ &< \left| \int_{E^+} s(x^*, \Gamma_n) dm_n \right| + \left| \int_{E^-} s(x^*, \Gamma_n) dm_n \right| + \varepsilon. \end{aligned}$$

Since, by (4.2.jv), it is in particular $m_n(E) < \sigma$ for every $n \geq N_{x^*}$, we get

$$\int_E |s(x^*, \Gamma)| dm \leq \int_E |s(x^*, \Gamma_n)| dm_n + \varepsilon < 2\varepsilon$$

so Γ has uniformly absolutely continuous scalar m-integral on Ω.

D) We have to show the existence of a WCG subspace of X which determines Γ. Since Γ_n is Pettis m_n-integrable, for every $n \in \mathbb{N}$, let $Y_n \subseteq X$ be a WCG space generated by a set $W_n \in cwk(B_{X^*})$ which m_n-determines Γ_n, by [26, Theorem 2.5]. We may suppose, withous loss of generality that each W_n is absolutely convex, by Krein-Smulian's Theorem. Let Y be the WCG space generated by $W := \sum 2^{-n} W_n$. We want to prove that Γ is m-determined by Y.
If $y^* \in Y^\perp$, then $y^* \in Y_n^\perp$ for each n, hence $s(y^*, \Gamma_n) = 0$ m_n-a.e. Applying (29) with $A = \Omega^+ := \{t : s(y^*, \Gamma(t)) \geq 0\}$ ($A = \Omega^- := \{t : s(y^*, \Gamma(t)) < 0\}$) we get

$$\int_{\Omega^\pm} s(y^*, \Gamma)\, dm = \lim_n \int_{\Omega^\pm} s(y^*, \Gamma_n)\, dm_n = 0.$$

Therefore $s(y^*, \Gamma(t)) = 0$ m-a.e. on the set Ω. Thus, Y m-determines the multifunction Γ and the Pettis m-integrability of Γ follows. Moreover (28) follows from (29).

□

As an immediate consequence of the previous theorem we have a result for the vector case:

Corollary 4.3 Let $g, g_n : \Omega \to X$, $n \in \mathbb{N}$, be scalarly measurable functions. Moreover let m, m_n, $n \in \mathbb{N}$, be measures in $\mathcal{M}(\Omega)$ and let m be Radon. Suppose that

(4.3.j) $(g_n)_n$ and g have scalarly uniformly absolutely continuous (m_n)-integrals on *Ω*;

(4.3.jj)) $g_n \to g$ scalarly m-a.e. where the null set depends on $x^* \in X^*$;

(4.3.jjj) g is scalar continuous;

(4.3.jv) $(m_n)_n$ is vaguely convergent to m and uniformly absolutely continuous with respect to m;

(4.3.v) each g_n is Pettis m_n-integrable.

Then g is Pettis m-integrable in X and

$$\lim_n x^* \left(\int_A g_n\, dm_n \right) = x^* \left(\int_A g\, dm \right),$$

for every $x^* \in X^*$ and $A \in \mathcal{B}$.

We conclude with the following result that holds in a general measure space without any topology on the space Ω.

Proposition 4.4 *Let Ω be a measurable space on a σ-algebra $\mathcal{A}$ and let Γ, Γ_n, $n \in \mathbb{N}$, be scalarly measurable multifunctions. Moreover let m, m_n, $n \in \mathbb{N}$, be measures in $\mathcal{M}(\Omega)$. Suppose that*

(4.4.j) $(\Gamma_n)_n$ *have scalarly uniformly absolutely continuous* (m_n)*-integrals on Ω;*

(4.4.jj) Γ *is scalarly* m*-integrable;*

(4.4.jjj) $(m_n)_n$ *is uniformly absolutely continuous with respect to* m*;*

(4.4.jv) *each multifunction* Γ_n *is Pettis* m_n*-integrable.*

(4.4.v) for every $A \in \mathcal{A}$ and for every $x^* \in X^*$ it is

$$\lim_n \int_A s(x^*, \Gamma_n)\, dm_n = \int_A s(x^*, \Gamma)\, dm.$$

Then the multifunction Γ is Pettis m-integrable.

Proof The weak compactness of the sublinear operator $T_\Gamma : X^* \to L^1(m)$, defined as $T_\Gamma(x^*) = s(x^*, \Gamma)$ can be proved as Theorem 4.2, taking into account hypotheses (4.4.jj), (4.4.jjj) and (4.4.v).
Moreover, the proof that Γ is determined by a WCG space $Y \subset X$ follows as in Theorem 4.2, taking into account hypotheses (4.4.j), (4.4.jjj), (4.4.jv) and (4.4.v). Therefore Γ is Pettis m-integrable.

□

At this point it is worth to observe that a similar result has been proved in [13, Theorem 3.2] under the hypothesis of the setwise convergence of the measures. Here instead of the setwise convergence we assume the uniform absolute continuity of $(m_n)_n$ with respect to m.

For the vector case, as before, we have:

Corollary 4.5 Let Ω be a measurable space on a σ-algebra $\mathcal{A}$ and let $g, g_n : \Omega \to X$, $n \in \mathbb{N}$, be scalarly measurable functions. Moreover let m, m_n, $n \in \mathbb{N}$, be measures in $\mathcal{M}(\Omega)$. Suppose that

(4.5.j) $(g_n)_n$ have scalarly uniformly absolutely continuous (m_n)-integrals on Ω;

(4.5.jj) g is scalar m-integrable;

(4.5.jjj) $(m_n)_n$ is uniformly absolutely continuous with respect to m;

(4.5.jv) each function g_n is Pettis m_n-integrable;

(4.5.v) for every $A \in \mathcal{A}$ and for every $x^* \in X^*$ it is

$$\lim_n \int_A x^* g_n\, dm_n = \int_A x^* g\, dm.$$

Then the function g is Pettis m-integrable.

5 Conclusion

Some limit theorems for the sequences $\left(\int f_n\, dm_n\right)_n$ are presented for vector and multivalued Pettis integrable functions when the sequence $(m_n)_n$ vaguely converges to a measure m. The results are obtained thanks to a limit result obtained for the scalar case (Theorem 3.4).

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