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Unified a-priori estimates for minimizers under $p, q-$growth and exponential growth

Paolo Marcellini[1], Antonella Nastasi[2], Cintia Pacchiano Camacho[3]

[1]Dipartimento di Matematica e Informatica "U. Dini", Università di Firenze
Viale Morgagni 67/A, 50134 - Firenze, Italy
paolo.marcellini@unifi.it
[2]Department of Engineering, University of Palermo
Viale delle Scienze, 90128 - Palermo, Italy
antonella.nastasi@unipa.it
[3] Theoretical Sciences Visiting Program,
Okinawa Institute of Science and Technology Graduate University
Onna, 904-0495, Japan
cintia.pacchiano@oist.jp

Abstract

We propose some *general growth conditions* on the function $f = f(x, \xi)$, including the so-called *natural growth*, or *polynomial*, or $p, q-$*growth conditions*, or even *exponential growth*, in order to obtain that any local minimizer of the energy integral $\int_\Omega f(x, Du)\,dx$ is *locally Lipschitz continuous* in Ω. In fact this is the fundamental step for further regularity: the *local boundedness of the gradient* of any Lipschitz continuous local minimizer *a-posteriori* makes irrelevant the behavior of the integrand $f(x, \xi)$ as $|\xi| \to +\infty$; i.e., the *general growth conditions* a posteriori are reduced to a standard growth, with the possibility to apply the classical regularity theory. In other words, we reduce some classes of *non-uniform* elliptic variational problems to a context of *uniform* ellipticity.

Contents

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1 Introduction

Let Ω be an open set in $\mathbb{R}^n$, $n \geq 2$, and $x \in \Omega \subset \mathbb{R}^n, \xi \in \mathbb{R}^n$ be generic vectors. Let $f = f(x, \xi)$, $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function in $\Omega \times \mathbb{R}^n$. The related *energy integral* where to look for local minimizers is

$$v \ \to \ \int_\Omega f(x, Dv)\, dx\,. \tag{1.1}$$

As well known a *local minimizer* of the energy integral (1.1) is a Sobolev function u such that $f(x,Du) \in L^1_{\mathrm{loc}}(\Omega)$ and $\int_\Omega f(x,Du)\,dx \le \int_\Omega f(x,Du+D\varphi)\,dx$ for every Sobolev function φ whose support is contained in Ω.

In this manuscript we allow examples of *anisotropic polynomial growth*, a class of variational energy integrals which may have singular (even not bounded) local minimizers. Such as for instance

$$f(x,Dv) = \sum_{i,j=1}^{n} a_{ij}(x)\, v_{x_i} u_{x_j} + |v_{x_n}|^q\,,$$

with $q \ge 2$ and (a_{ij}) being an $n \times n$ a positive definite matrix of locally Lipschitz continuous coefficients in Ω. We also allow *exponential growth* such as, for example,

$$f(x,Dv) = \exp(a(x)\,|Dv|^2).$$

For these examples *ellipticity* holds, but *not uniform ellipticity* as in the statement (1.2) below.

Our aim is to propose some *general growth conditions* on the function $f = f(x,\xi)$ (including the previous polynomial and exponential examples) in order to obtain that any *local minimizer* is *locally Lipschitz continuous* in Ω. In fact this one is the fundamental property for a local minimizer for further regularity: the *local boundedness of the gradient* of any Lipschitz continuous local minimizer *a-posteriori* makes irrelevant the behavior of the integrand $f(x,\xi)$ as $|\xi| \to +\infty$; i.e., the *general growth conditions* a posteriori are reduced to a standard growth, with the possibility to apply the classical regularity theory, valid in the uniform elliptic contexts and obtain, when possible, $C^{1,\alpha}-$regularity as a consequence. In fact, having in force the local Lipschitz continuity considered in this manuscript, *also the $C^{1,\alpha}$ regularity can be deduced* under the same assumptions made in the context of natural growth, for instance as in Ladyženskaja-Ural'ceva [42, Chapter 4, Section 61] or in Giusti [38, Sections 8.6 and 8.8]; see the $p,q-$growth cases in [44, Section 7, Theorem D] and in [45, Corollary 2.2]. See also the recent non-uniformly elliptic approach and gradient Hölder continuity by DeFilippis-Mingione [29, Section 6.4].

The *convexity* with respect to the gradient variable $\xi \in \mathbb{R}^n$ is the classical assumption on the function $f = f(x,\xi)$ when we consider existence of minimizers, once the class of competing v in (1.1) is fixed at the boundary $\partial\Omega$. The corresponding convexity condition for a C^2-function $\xi \to f(x,\xi)$, as well known, is related to the $n \times n$ matrix of second derivatives $(f_{\xi_i\xi_j})_{n\times n}$ and it is the *positivity of the quadratic form* $\lambda \in \mathbb{R}^n \to \sum_{i,j} f_{\xi_i\xi_j}(x,\xi)\,\lambda_i\lambda_j$. When a *strict qualified positivity* is required, we have the *ellipticity condition* $\sum_{i,j} f_{\xi_i\xi_j}(x,\xi)\,\lambda_i\lambda_j \ge m\,|\lambda|^2$, valid for all $\lambda,\xi \in \mathbb{R}^n$, $x \in \Omega$ and for some positive constant m. More precisely, often it is useful to emphasize the dependence of m on the gradient variable ξ; in fact more properly in general the constant m is replaced by a function of ξ. In this manuscript we assume the following *ellipticity condition*

$$\sum_{i,j} f_{\xi_i\xi_j}(x,\xi)\,\lambda_i\lambda_j \ge g_1(|\xi|)\,|\lambda|^2\,, \qquad \forall\, \lambda,\xi \in \mathbb{R}^n, x \in \Omega,$$

where $g_1 : [0, +\infty) \to [0, +\infty)$ is a nonnegative increasing function. Less standard is the bound from above for the $n \times n$ matrix of second derivatives $(f_{\xi_i \xi_j})_{n \times n}$. The classical case is the *uniformly elliptic* one. This happens when a similar bound exists from above too:

$$\textit{uniformly ellipticity:} \quad g_1(|\xi|)|\lambda|^2 \le \sum_{i,j=1}^{n} f_{\xi_i \xi_j}(x, \xi) \lambda_i \lambda_j \le M g_1(|\xi|)|\lambda|^2, \tag{1.2}$$

for all $\lambda, \xi \in \mathbb{R}^n$, $x \in \Omega$, and for a constant $M \ge 1$. The main example is the $p-$Laplacian; i.e., when the integral in (1.1) is the *$p-$Dirichlet integral*, with $f(\xi) = |\xi|^p$. In this case a computation shows

$$\sum_{i,j=1}^{n} f_{\xi_i \xi_j}(\xi) \lambda_i \lambda_j = p\,[|\xi|^2 |\lambda|^2 + (p-2)(\sum_{i=1}^{n} \xi_i \lambda_i)^2)]\; |\xi|^{p-4}$$

and thus the validity of the uniformly elliptic estimates

$$p - \textit{Laplacian:} \quad p\,|\xi|^{p-2}|\lambda|^2 \le \sum_{i,j} f_{\xi_i \xi_j}(\xi) \lambda_i \lambda_j \le p(p-1)|\xi|^{p-2}|\lambda|^2,$$

for all $\lambda, \xi \in \mathbb{R}^n$, $x \in \Omega$, when $p \ge 2$; otherwise, if $1 < p < 2$, it is necessary to interchange the first and the last sides.

Let us go back to what we said at the very beginning of this introduction. It is known that perturbations of the *$p-$Dirichlet integral* $f(\xi) = |\xi|^p$, $\xi = (\xi_1, \xi_2, \dots, \xi_n)$, for instance with either $f(\xi) = \sum_{i=1}^{n} |\xi_i|^{p_i}$, or $f(\xi) = |\xi|^p + |\xi_n|^q$ with $p < q$, give rise to energy integrals as in (1.1) which may admit not smooth local minimizers, even unbounded local minimizers; see for instance [45, Theorem 6.1]. In these cases the uniform ellipticity condition (1.2) does not hold. In the non-uniformly elliptic case regularity of local minima may not be true.

Another non-uniformly elliptic case which we like to emphasize is the exponential one, for instance with $f(\xi) = e^{|\xi|^2}$. In this example a local minimizer of the corresponding energy integral in (1.1) a-priori not necessarily satisfies the Euler's differential equation. The reason is due to the fact that if u is a local minimizer, with $\int_\Omega e^{|Du|^2}\,dx < +\infty$, then to obtain the first variation and the Euler's equation, for any fixed test function φ we need to compute the limit, as $h \to 0$ ($h \in \mathbb{R}$, $h \ne 0$), of the difference quotient

$$\tfrac{1}{h}\Big(\int_\Omega e^{|D(u+h\varphi)|^2}\,dx - \int_\Omega e^{|Du|^2}\,dx \Big);$$

if $\varphi = \eta u$ and $\eta = 1$ on a subset of Ω (as usually it happens in the regularity approach), i.e. if $\varphi = u$ and $h > 0$ on a subset Ω' of Ω, then $\int_{\Omega'} e^{|D(u+h\varphi)|^2}\,dx = \int_{\Omega'} e^{(1+h)^2|Du|^2}\,dx$; thus the summability of $e^{|Du|^2}$ in general does not implies the summability of $e^{(1+h)^2|Du|^2}$ because of the factor $(1+h)^2 > 1$ and is not possible to go directly to the Euler's equation. An energy integral of exponential type in general is more difficult to be studied from the regularity point of view. The exponential energy integrals are not uniformly elliptic and sometime they need to be treated with appropriate specific techniques (see Cellina-Staicu [9]).

In this manuscript we propose a unified approach to regularity for minimizers of nonuniformly elliptic energy integral, which include *exponential growth* and $p,q-$*growth*. This last case is related to the functions g_1,g_2 below in (1.3), when they are powers with exponents p,q; more precisely, g_1,g_2 being compared with the second derivatives of f with respect to the gradient variable ξ, the powers to compare g_1,g_2 respectively are $|\xi|^{p-2}$ and $1+|\xi|^{q-2}$, or $(1+|\xi|^2)^{(q-2)/2}$. More in general we consider the

$$\textit{non-uniformly elliptic case:}\;\; g_1\left(|\xi|\right)|\lambda|^2 \leq \sum_{i,j=1}^{n} f_{\xi_i\xi_j}\left(x,\xi\right)\lambda_i\lambda_j \leq g_2\left(|\xi|\right)|\lambda|^2, \tag{1.3}$$

for all $\lambda,\xi \in \mathbb{R}^n$, $x\in\Omega$, with $g_1,g_2:[0,+\infty)\to[0,+\infty)$ nonnegative increasing functions. Of course it is necessary to impose conditions on these functions g_1,g_2 in order to obtain a-priori estimates for regularity. Full details are stated in Section 2, together with the assumption of the local Lipschitz continuity in $\Omega\times\mathbb{R}^n$ of the gradient with respect to $\xi\in\mathbb{R}^n$ of $f=f\left(x,\xi\right)$; i.e. the local Lipschitz continuity of $D_\xi f\left(x,\xi\right)=\left(f_{\xi_i}\left(x,\xi\right)\right)_{i=1,2,\dots,n}$.

Nowadays many papers deal with regularity for non-uniformly elliptic problems. Limited to the more recent literature, for interior regularity we refer to Colombo-Mingione [15]-[17], Baroni-Colombo-Mingione [1], Eleuteri-Marcellini-Mascolo [36], Bousquet-Brasco [6], DeFilippis-Mingione [27]-[29], Mingione-Rădulescu [51], Bögelein-Duzaar-Giova-Passarelli-Scheven [4]; see also [19]-[23] and [34]-[37]. About recent boundary regularity under general growth conditions we mention Cianchi-Maz'ya [12],[13], Bögelein-Duzaar-Marcellini-Scheven [5], DeFilippis-Piccinini [30]. For *Orlicz-Sobolev spaces, variable exponents* and *double phase* see Diening-Harjulehto-Hasto-Ruzicka [32], Chlebicka [10], Chlebicka-DeFilippis [11], Byun-Oh [8], Ragusa-Tachikawa [55], Hästö-Ok [40], Crespo-Blanco-Gasiński-Winkert [18]. Higher integrability and stability of $p,q-$quasi-minimizers in double phase problems by Kinnunen-Nastasi-Pacchiano Camacho [41] and Nastasi-Pacchiano Camacho [52]-[54]. *Quasiconvex integrals* of the calculus of variations in [3],[24],[43], and about *partial regularity* by Schmidt [57], DeFilippis [25], DeFilippis-Stroffolini [31], Gmeineder-Kristensen [39].

Particularly related to this manuscript, the literature on regularity for *non-uniformly elliptic problems*, which consider at the same time *exponential growth*, is less wide. It starts from [46, Section 6] for a class of non-uniformly elliptic equations including *"slow"* exponential growth. The first regularity result specific for local minimizers in $u\in W^{1,2}_{\mathrm{loc}}\left(\Omega\right)$ such that $f\left(Du\right)\in L^1_{loc}\left(\Omega\right)$, possibly with exponential growth, is in [47], which deals with energy integrals as in (1.1) with integrand $f=f\left(\xi\right)$ independent of x, who partially inspired our research here. In the same year an approach to *the vector-valued case* was introduced in [48], later generalized by Marcellini-Papi [49], both again related to $f=f\left(\xi\right)$ independent of x. The first extensions to $f=f\left(x,\xi\right)$ are due to Mascolo-Migliorini [50] and DiMarco-Marcellini [33], who treated *the vector-valued case*, with $f := g(x, |\xi|)$ of *Uhlenbeck-type* [58], i.e. depending on the modulus $|\xi|$. Finally we mention Beck-Mingione [2], who studied energy integrals of the form $\int_\Omega \{g(|Du|) + h(x) \cdot u\}\, dx$ and they considered some sharp assumptions on the function $h(x)$, of the type $h \in L(n,1)(\Omega; \mathbb{R}^m)$ in dimension $n > 2$ (i.e., $\int_0^{+\infty} \mathrm{meas}\{\mathrm{x} \in \Omega : |\mathrm{h}(\mathrm{x})| > \lambda\}^{1/\mathrm{n}}\, \mathrm{d}\lambda < +\infty$; note that $L^{n+\varepsilon} \subset L(n,1) \subset L^n$), or $h \in L^2 (\log L)^\alpha (\Omega; \mathbb{R}^m)$ for some $\alpha > 2$ when $n = 2$. Beck-Mingione obtained the local boundedness of the gradient Du allowing exponential growth too; however the function $g(|\xi|)$ is assumed to be depending on the modulus $|\xi|$ of ξ and independent of x. Therefore the a-priori estimates of Theorem 2.2 below is new with respect to the known literature, and in particular with respect to the quoted references.

2 Statement of the main results

We consider a function $f = f(x, \xi)$, $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$, whose gradient with respect to the $\xi-$variable $D_\xi f(x, \xi) = (f_{\xi_i}(x, \xi))_{i=1,2,\dots,n}$ is locally Lipschitz continuous in $\Omega \times \mathbb{R}^n$. We assume that the following second partial derivatives of f (which exist in the $W^{1,\infty}_{\mathrm{loc}}-$sense) satisfy the following *ellipticity* and *growth conditions:* for every open set Ω' compactly contained in Ω there exist nonnegative increasing functions $g_1, g_2, g_3 : [0, +\infty) \to [0, +\infty)$, not identically equal to zero, such that

$$g_1(|\xi|)\, |\lambda|^2 \leq \sum_{i,j} f_{\xi_i \xi_j}(x, \xi)\, \lambda_i \lambda_j \leq g_2(|\xi|)\, |\lambda|^2 , \tag{2.1}$$

$$\sum_i |f_{\xi_i x_k}(x, \xi)| \leq g_3(|\xi|) , \tag{2.2}$$

for all $\xi, \lambda \in \mathbb{R}^n$, $k = 1, 2, \dots, n$, a.e. $x \in \Omega'$. These functions g_1, g_2, g_3 are related to each other by

$$(g_2(t))^{2\gamma-1}\, t^2 \leq M \left\{1 + \int_0^t \sqrt{g_1(s)}\, ds\right\}^\alpha , \tag{2.3}$$

$$(g_2(|\xi|))^{2\gamma-1}\, |\xi|^{2\gamma} \leq M \{1 + f(x, \xi)\}^\beta , \tag{2.4}$$

$$g_3(t) \leq M(1 + t^\gamma)(g_1(t))^{\frac{1}{2}}(g_2(t))^{\gamma - \frac{1}{2}} , \tag{2.5}$$

for a positive constant $M = M(\Omega')$ and for all $\xi \in \mathbb{R}^n$ and $t \geq 0$. Finally, the exponents α, β, γ satisfy the bounds

$$2 \leq \alpha < 2^* - 2(\gamma - 1) , \tag{2.6}$$

$$1 \leq \beta < \frac{2(\alpha + 2\gamma - 2)}{n(\alpha + 2\gamma - 4)} , \tag{2.7}$$

for some $\gamma \geq 1$. Here, as usual, we denote by 2^* the Sobolev exponent defined by $2^* = \frac{2n}{n-2}$ if $n > 2$, while 2^* is a fixed real number large enough when $n = 2$; precisely such that $\alpha < 2^*$ and $\gamma < \frac{2^* - \alpha + 2}{2}$.

Remark 2.1 *The conditions (2.6),(2.7) on the exponents* α,β *and* $\gamma \geq 1$ *are satisfied if we choose, for instance,*

$$\beta = \frac{\alpha}{2} + \delta\,, \quad \textit{and} \quad \gamma = 1 + \delta\,, \tag{2.8}$$

with $2 \leq \alpha < 2^{*} - 2(\gamma - 1) = 2^{*} - 2\delta$, $1 \leq \beta < 1 + \frac{2}{n}$ *and* $0 \leq \delta < \frac{4}{n(n-2)}$. *Indeed, with the choice* $\gamma = 1 + \delta$, *(2.7) assumes the form*

$$1 \leq \tfrac{\alpha}{2} + \delta = \tfrac{\alpha+2\delta}{2} < \tfrac{2(\alpha+2\delta)}{n(\alpha-2+2\delta)},$$

which gives $\alpha - 2 + 2\delta < \frac{4}{n}$, *that is* $\beta < 1 + \frac{2}{n}$. *Finally, we must satisfy the* $\alpha-$*bound* $\alpha < 2^{*} - 2\delta$, *i.e.* $0 \leq \delta < \frac{2^{*}-\alpha}{2}$. *Since* $\alpha = 2(\beta - \delta) \leq 2\beta < 2 + \frac{4}{n}$, *we obtain the sufficient bound for* δ

$$0 \leq \delta < \tfrac{2^{*}-2-\frac{4}{n}}{2} \underset{\textit{if } n>2}{=} \tfrac{4}{n(n-2)}\,.$$

In the particular case $\gamma = 1$ *conditions (2.6),(2.7) simplify into* $2 \leq \alpha < 2^{*}$ *and* $1 \leq \beta < \frac{2\alpha}{n(\alpha-2)}$. *For instance, for the so called natural growth conditions with, up to multiplicative constants,* $g_1(t) = t^{p-2}$, $g_2(t) = (1+t)^{p-2}$ *for some* $p \geq 2$, *(2.3) becomes*

$$(1+t)^{p-2}\, t^{2} \leq M \left\{1 + \int_0^t s^{\frac{p}{2}-1} ds\right\}^{\alpha} = M_1 \left(1 + t^{\frac{\alpha}{2}p}\right) \tag{2.9}$$

and it is satisfied for a constant M_1 *and for all* $t \geq 0$ *by choosing* $\alpha = 2$. *Similarly (2.4) holds for* $g_2(t) = (1+t)^{p-2}$ *and* $f(x,\xi) \geq \mathrm{const}\, |\xi|^{\mathrm{p}}$ *if* $\beta = 1$. *This in the natural growth conditions the simplest choice is* $\alpha = 2$, $\beta = 1$ *and* $\gamma = 1$. *Similar computations when* $\gamma = 1$ *can be done for the* $p,q-$*growth case, with* $g_1(t) = t^{p-2}$, $g_2(t) = (1+t)^{q-2}$ *and* $f(\xi) \geq \mathrm{const}\, |\xi|^{\mathrm{p}}$ *for some exponents* $q \geq p \geq 2$. *To test (2.3) we change* p *with* q *in the left hand side of (2.9) and we obtain* $q \leq \frac{\alpha}{2}p$. *To test (2.4), we consider the sufficient condition* $(1+|\xi|)^{q-2}\, |\xi|^{2} \leq M \left\{1 + |\xi|^{p}\right\}^{\beta}$, *which holds with* $q \leq \beta p$. *In this case the natural choice, as in (2.8), is to fix* $\beta = \frac{\alpha}{2}$ *when* $\gamma = 1$ *(i.e.* $\delta = 0$*), under the constraint* $\alpha - 2 < \frac{4}{n}$ *or equivalently* $\beta < 1 + \frac{2}{n}$, *which - written in the explicit form* $\frac{q}{p} < 1 + \frac{2}{n}$ *- is a natural bound for* $p,q-$*growth variational problems without* $x-$*dependence (see for instance [47, Remark 2.1] or [15],[20],[36]). Note however that in general the* $x-$*dependence and specifically the exponential growth need to choose the parameter* γ *strictly grater than* 1, *for instance as in (2.8). See details in Section 3.*

We are ready to state in Theorem 2.2 below our regularity result, whose proof, divided into several steps, is given in Section 4. With *regularity* of u we mean *interior regularity*, without fixing Dirichlet boundary conditions, or without other types of boundary conditions. Precisely, we obtain local Lipschitz continuity estimates for u and also local estimates of the L^2-norm of the $n \times n$

matrix D^2u of the second derivatives of a local minimizer u of the energy integral (1.1). A local minimizer is a Sobolev function u such that

$$\int_{\Omega'} f(x, Du)\,dx < +\infty \quad \text{and} \quad \int_{\Omega'} f(x, Du)\,dx \leq \int_{\Omega'} f(x, D(u+\psi))\,dx\,,$$

for every $\psi \in W^{1,2}(\Omega)$ with support in the open set Ω' whose closure is contained in Ω. Theorem 2.2 is a regularity result for *smooth local minimizers* u of the energy integral (1.1). These estimates are in fact *a priori estimates*: they hold for a priori *smooth* local minimizers; precisely, for local minimizers u in the class

$$\left\{u \in W^{2,2}(\Omega') : \int_{\Omega'} |Du|^{2\gamma} (g_2(|Du|))^{2\gamma-1}\,dx + \int_{\Omega'} g_2(|Du|)\left|D^2u\right|^2 dx < +\infty\right\}, \tag{2.10}$$

where these summability conditions are valid for all sets Ω' compactly contained in Ω. As well known, the natural condition is to impose *finite energy* for the local minimizer u; i.e. to belong to the class

$$\left\{u \in W^{1,1}(\Omega') : \int_{\Omega'} f(x, Du)\,dx < +\infty\right\}, \tag{2.11}$$

while (2.10) is an a priori proper condition useful to prove the estimates (2.12)-(2.13) of Theorem 2.2.

The auxiliary assumption (2.10) should be removed later. Usually, the main steps for regularity are the *a-priori estimates*; the following steps are obtained through an *approximation procedure.* A reference example of this method can be found in the first Lipschitz continuity result under *non-uniformly elliptic* and *general growth condition*, obtained in [47, Step 5] for local minimizers, however for the simpler case $f(x,\xi) = f(\xi)$, i.e. with f independent of x. A further approximation procedure to go from the a priori estimates to full regularity was done in [36]: also this one is related to a special case; precisely with *modulus-dependence* on the $\xi-$variable $f(x,\xi) = g(x, |\xi|)$, named *Uhlenbeck structure*, recalling the celebrated paper [58] published in 1977 by Karen Uhlenbeck.

Finally we observe that we do not assume *structure conditions* on the function $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$. In particular we do not assume the mentioned *Uhlenbeck structure* with the modulus dependence $f(x,\xi) = g(x, |\xi|)$ on the gradient variable, not even the so-called *double phase* structure, with $f = f(x,\xi) := |\xi|^p + a(x)|\xi|^q$, where $a(x)$ is a nonnegative coefficient which is equal to zero somewhere in Ω, neither the more general $p, q-$*growth conditions*, which are more general than the double phase case, but which of course do not include *exponential growth*, a case on the contrary that enters in our Lipschitz continuous a-priori estimates of Theorem 2.2.

Theorem 2.2 *Under the ellipticity and growth conditions (2.1)-(2.7), let u be a local minimizer of the energy integral (1.1) in the Sobolev class (2.10). Then some uniform a-priori estimates hold for the $L^\infty_{\mathrm{loc}}-$norm of the gradient Du and for the $n \times n$ matrix D^2u of the second derivatives of u. Precisely, for every* *open set Ω' compactly contained in Ω there exist exponents $\theta_1, \theta_3 > 1$, $\theta_2, \theta_4 > 0$ and a radius $R_0 > 0$ such that*

$$\|Du\|^2_{L^\infty(B_\rho;\mathbb{R}^n)} \leq \tfrac{c}{(R-\rho)^{\theta_2}} \left(\int_{B_R} (1 + f(Du))\, dx \right)^{\theta_1}, \tag{2.12}$$

$$\int_{B_\rho} g_1(|Du|) \left|D^2 u\right|^2 dx \leq \tfrac{c}{(R-\rho)^{\theta_4}} \left(\int_{B_R} (1 + f(Du))\, dx \right)^{\theta_3}, \tag{2.13}$$

for every ρ, R with $0 < \rho < R \leq R_0$ and for a positive constant c, where B_ρ, B_R are concentric balls contained in Ω' with respective radii ρ, R.

The constant c in (2.12),(2.13) depends on the dimension n and on the data, but it is independent of u itself. An explicit analytic representation of the exponent θ_1 is given below in (4.50), as well as the analytic expression of θ_2, θ_3, θ_4 in the proof of Theorem 2.2. The a priori estimate (2.13) is a $W^{2,2}_{\mathrm{loc}}(\Omega)-$bound of the local minimizer u of the energy integral (1.1) only on subsets $\Omega' \subset\subset \Omega$ of the type $\{x \in \Omega' : \; g_1(|Du(x)|) \geq m\}$ for fixed constants $m > 0$. For instance, when $g_1 : [0, +\infty) \to (0, +\infty)$ is a positive function also at $t = 0$, i.e. when $m = g_1(0) > 0$ as it happens in the nondegenerate $p-$Laplacian and $p(x)-$Laplacian cases. In fact in these cases by $(2.13)_2$ we get

$$\text{the nondegenerate case:} \quad \int_{B_\rho} \left|D^2 u\right|^2 dx \leq \tfrac{c}{m(R-\rho)^{\theta_4}} \left(\int_{B_R} (1 + f(Du))\, dx \right)^{\theta_3}. \tag{2.14}$$

As an example related to the estimate (2.13), we mention here the degenerate case recently studied by Brasco-Carlier-Santambrogio [7], Santambrogio-Vespri [56], Colombo-Figalli [14], Bögelein-Duzaar-Giova-Passarelli-Scheven [4]. It is a *very degenerate case*, with $f(x,t) = \frac{1}{p}(t-1)^p_+ - h(x)\,u$ (as before $t = |\xi|$) and thus $g_1(t) = g_2(t) = 0$ for all $t \in [0,1]$ and $g_1(t) > 0$ when $t > 1$. In this case, when $h = 0$, by Theorem 2.2 we deduce the L^∞-local gradient bound (2.12), while the $W^{2,2}_{\mathrm{loc}}(\Omega)-$bound (2.13) gives contribution only at the subset of Ω where $|Du(x)| > 1$, being indefinite the gradient $Du(x)$ and the matrix of its second derivatives $D^2u(x)$ when $|Du(x)| \leq 1$, since there any Sobolev function u, with $|Du(x)| \leq 1$, a-priori could be a minimizer.

3 Consequences and examples

We collect some examples which satisfy the hypotheses (2.1)-(2.7) and we state a specific version of Theorem 2.2 for each energy integral discussed below. For the readers' convenience we start by reporting two well-known properties which will be used to show that the models proposed below satisfy the assumptions stated in the previous section. In Proposition 3.1 we represent the quadratic form, associated to the $n \times n$ matrix $(f_{\xi_i \xi_j})$ of the second derivatives of $f(x,\xi)$ with respect to ξ, in the particular case when f depends on ξ only thought its modulus $|\xi|$. We recall that $g_1, g_2, g_3 : [0, +\infty) \to [0, +\infty)$ are nonnegative increasing functions. They are not identically equal to zero; the reason is to avoid the trivial case when the integrand $f = f(x,\xi)$ satisfies all conditions (2.1)-(2.7) with g_1, g_2, g_3 identically equal to zero; in this trivial case any $u \in W^{1,p}_{\text{loc}}(\Omega)$ comes out to be a local minimizer of the "zero integral" and of course regularity results do not hold. Thus there exists $t_0 > 0$ such that $g_2(t_0) \geq g_1(t_0) > 0$ and, up to a rescaling, we can consider $t_0 \leq 1$; therefore $g_2(1) \geq g_1(1) \geq c > 0$. If necessary we multiply f, and thus g_1, g_2, g_3 too, by the positive constant $1/c$ to have

$$g_2(1) \geq g_1(1) \geq 1\,. \tag{3.1}$$

We introduce a function $g : \Omega \times [0,+\infty) \to \mathbb{R}$, $g = g(x,t)$, $t \in [0,+\infty)$, $t = |\xi|$, g twice differentiable with respect to $t > 0$, such that

$$f(x,\xi) = g(x,|\xi|)\,, \qquad \forall\ (x,\xi) \in \Omega \times \mathbb{R}^n\,. \tag{3.2}$$

Although we explicitly remember that our Theorem 2.2 deals with a general energy integrand $f(x,\xi)$ not necessarily of the form (3.2).

Proposition 3.1 *Let $f(x,\xi)$ be represented in the form $g(x,|\xi|)$, as in (3.2), with partial derivative $g_t(x,t)$ locally Lipschitz continuous in $t \in [0,+\infty)$ for fixed $x \in \Omega$, with $g_t(x,0) = 0$. Then the quadratic form associated to the $n \times n$ matrix $(f_{\xi_i \xi_j})$ satisfies the bounds*

$$g_1(x,|\xi|)\,|\lambda|^2 \leq \sum_{i,j} f_{\xi_i\xi_j}(x,\xi)\,\lambda_i\lambda_j \leq g_2(x,|\xi|)\,|\lambda|^2\,, \tag{3.3}$$

for all $(x,\xi) \in \Omega \times \mathbb{R}^n$, with g_1 and g_2 given by

(i) in general $g_1 = \min\left\{\frac{g_t}{t}, g_{tt}\right\}$ and $g_2 = \max\left\{\frac{g_t}{t}, g_{tt}\right\}$;

(ii) $g_1 = \frac{g_t}{t}$ and $g_2 = g_{tt}$, if $\frac{g_t(x,t)}{t}$ is increasing with respect to t at $x \in \Omega$;

(iii) $g_1 = g_{tt}$ and $g_2 = \frac{g_t}{t}$, if $\frac{g_t(x,t)}{t}$ is decreasing with respect to t at $x \in \Omega$.

Usually, it is not difficult to compute explicitly $g_1(|\xi|)$ and $g_2(|\xi|)$ and to check the monotonicity of $g_t(x,t)/t$. For instance for an integrand related to the $p-$Laplacian, of the type (3.2), with $f(x,\xi) = g(x,|\xi|) = a(x)\,|\xi|^p$ and $g(x,t) = a(x)\,t^p$, we have $g_t(x,t)/t = pa(x)\,t^{p-2}$; therefore we are in the case *(ii)* if $p \geq 2$, while *(iii)* holds when $p \in (1,2]$. In this case, if $c_1 \leq a(x) \leq c_2$ for all $x \in \Omega$ (by c_i we denote positive constants), then $c_3 g_t/t \leq g_{tt} \leq c_4 g_t/t$ for all $(x,t) \in \Omega \times (0,+\infty)$; or equivalently $c_3 \leq t g_{tt}/g_t \leq c_4$ for all $(x,t) \in \Omega \times (0,+\infty)$. Please, note the "abuse of notation", due to the fact that in Proposition 3.1 g_1 and g_2 are functions depending on $(x,t) = (x,|\xi|)$, while in the ellipticity condition (2.1) they are functions $g_1 = g_1(|\xi|)$, $g_2 = g_2(|\xi|)$ independent of x. Of course to go from g_1 in Proposition 3.1 to the g_1 in the left hand side of (2.1) we will consider the infimum of $g_1(x,|\xi|)$ with respect to x; similarly, the supremum with respect to x for g_2. In general, if $g_1 \leq g_2 \leq c_5 g_1$ for all $(x,\xi) \in \Omega \times \mathbb{R}^n$, we say that (3.3) are *uniformly elliptic conditions*. Therefore the $p-$Laplacian is an uniformly elliptic operator. However in our Theorem 2.2 we do not assume uniform elliptic conditions.

Proof of Proposition 3.1. For $f(x,\xi) = g(x,|\xi|)$, as in (3.2), a computation shows

$$f_{\xi_i} = g_t \tfrac{\xi_i}{|\xi|}\,, \qquad f_{\xi_i\xi_j} = g_{tt} \tfrac{\xi_i\xi_j}{|\xi|^2} + g_t \tfrac{\delta_{ij}|\xi| - \xi_i \frac{\xi_j}{|\xi|}}{|\xi|^2} = \xi_i\xi_j \left(\tfrac{g_{tt}}{|\xi|^2} - \tfrac{g_t}{|\xi|^3} \right) + \delta_{ij} \tfrac{g_t}{|\xi|},$$

for $i,j = 1,2,\dots,n$, where, as well known, δ_{ij} is the *Kronecker delta*: $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. Therefore

$$\sum_{i,j=1}^{n} f_{\xi_i\xi_j}(x,\xi)\,\lambda_i\lambda_j = \tfrac{1}{|\xi|^2} \left(g_{tt} - \tfrac{g_t}{|\xi|} \right) \sum_{i,j=1}^{n} \xi_i\xi_j\lambda_i\lambda_j + \tfrac{g_t}{|\xi|} \left(\lambda_1^2 + \lambda_2^2 + \dots \lambda_n^2 \right).$$

For the first sum we have $\sum_{i,j=1}^{n} \xi_i\xi_j\lambda_i\lambda_j = \left(\sum_{i=1}^{n} \xi_i\lambda_i \right)^2 = (\xi,\lambda)^2$, where (ξ,λ) is the scalar product of $\xi,\lambda \in \mathbb{R}^n$. Then we get $\sum_{i,j=1}^{n} f_{\xi_i\xi_j}(x,\xi)\,\lambda_i\lambda_j = \frac{1}{|\xi|^2} \left(g_{tt} - \frac{g_t}{|\xi|} \right) (\xi,\lambda)^2 + \frac{g_t}{|\xi|} |\lambda|^2$. By the Cauchy-Schwarz inequality $|(\xi,\lambda)| \leq |\xi|\,|\lambda|$, for all $(x,t) \in \Omega \times [0,+\infty)$ such that $g_{tt} - \frac{g_t}{|\xi|} \geq 0$ we finally deduce

$$\tfrac{g_t}{|\xi|} |\lambda|^2 \leq \sum_{i,j=1}^{n} f_{\xi_i\xi_j}(x,\xi)\,\lambda_i\lambda_j \leq \tfrac{1}{|\xi|^2} \left(g_{tt} - \tfrac{g_t}{|\xi|} \right) |\xi|^2\, |\lambda|^2 + \tfrac{g_t}{|\xi|} |\lambda|^2 = g_{tt} |\lambda|^2\,,$$

for all $\lambda \in \mathbb{R}^n$; while, if at $(x,t) \in \Omega \times [0,+\infty)$ we have $g_{tt} - \frac{g_t}{|\xi|} \leq 0$ then

$$g_{tt} |\lambda|^2 = \tfrac{1}{|\xi|^2} \left(g_{tt} - \tfrac{g_t}{|\xi|} \right) |\xi|^2\, |\lambda|^2 + \tfrac{g_t}{|\xi|} |\lambda|^2 \leq \sum_{i,j=1}^{n} f_{\xi_i\xi_j}(x,\xi)\,\lambda_i\lambda_j \leq \tfrac{g_t}{|\xi|} |\lambda|^2\,,$$

for all $\lambda \in \mathbb{R}^n$, which is equivalent to the conclusion in *(i)*.

By computing the partial derivative $\frac{\partial}{\partial t} \frac{g_t(x,t)}{t} = \frac{g_{tt}(x,t)t - g_t(x,t)}{t^2}$ we see that at every $x \in \Omega$ such that $g_{tt}(x,t)\,t - g_t(x,t) \geq 0$ for all $t \in (0,+\infty)$, then $t \to \frac{g_t(x,t)}{t}$ is increasing in $(0,+\infty)$ and vice versa. This proves *(ii)*. Similarly for *(iii)* when $g_{tt}(x,t)\,t - g_t(x,t) \leq 0$ at $x \in \Omega$, for all $t \in (0,+\infty)$. ■

The following property is elementary and does not need a proof.

Proposition 3.2 *Let $t_0 \in \mathbb{R}$. If $h(t)$ is a continuous function in $[t_0,+\infty)$ such that $\lim_{t\to+\infty} h(t) \in \mathbb{R}$, then $h(t)$ is bounded in $[t_0,+\infty)$.*

3.1 Anisotropic energy integrals

We emphasize that in this example the condition in (3.2), i.e. $f(x,\xi) = g(x,|\xi|)$, is not assumed. Not even $f(x,\xi)$ should depend on $\xi = (\xi_1,\xi_2,\dots,\xi_n) \in \mathbb{R}^n$ only through powers of its components $|\xi_1|^{p_1}$, $|\xi_2|^{p_2}$, ..., $|\xi_n|^{p_n}$, for instance as frequently represented in the recent literature on this subject, in the form $f(x,\xi) = \sum_{i=1}^{n} a_i(x)\,|u_{x_i}|^{p_i}$ for some exponents p_i, $i = 1,2,\dots,n$. .

We fix some indices in the set $\{1, 2, \ldots, n\}$ in order to form a not empty proper subset; for instance we fix the index $\{n\}$ and we consider the energy integral

$$\int_{\Omega} \left(\sum_{i,j=1}^{n} a_{ij}(x)\, u_{x_i} u_{x_j} + |u_{x_n}|^q \right) dx\,, \tag{3.4}$$

with $q \geq 2$, where $(a_{ij}(x))$ is an $n \times n$ positive definite matrix of locally Lipschitz continuous functions on Ω. Precisely, for every set Ω' compactly contained in Ω there exist positive constants c_1, c_2 such that $c_1 |\lambda|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\, \lambda_i \lambda_j \leq c_2 |\lambda|^2$ for all $\lambda \in \mathbb{R}^n$ and $x \in \Omega'$. Or more generally the anisotropic energy integrals has the form

$$\int_{\Omega} \{h(x, Du) + |u_{x_n}|^q\}\, dx\,, \tag{3.5}$$

where h is a real function defined in $\Omega \times \mathbb{R}^n$ whose gradient $D_\xi h(x, \xi)$ is locally Lipschitz continuous in $\Omega \times \mathbb{R}^n$ with respect to $\xi \in \mathbb{R}^n$ and $q \geq p \geq 2$. Moreover $h(x, Du)$ is locally uniformly elliptic and satisfies the standard (classical) $p-$ellipticity and growth conditions

$$c_1 |\xi|^{p-2} |\lambda|^2 \leq \sum_{i,j} h_{\xi_i \xi_j}(x, \xi)\, \lambda_i \lambda_j \leq c_2 \left(1 + |\xi|^{p-2}\right) |\lambda|^2\,, \tag{3.6}$$

$$\sum_{i} |h_{\xi_i x_k}(x, \xi)| \leq c_3 |\xi|^{p-1}\,, \tag{3.7}$$

for all $\xi, \lambda \in \mathbb{R}^n$ and $k = 1, 2, \ldots, n$, a.e. $x \in \Omega'$. In the particular case (3.4), with $h(x, \xi) = \sum_{i,j} a_{ij}(x)\, \xi_i \xi_j$ and $p = 2$, then $\sum_{i,j} h_{\xi_i \xi_j}(x, \xi)\, \lambda_i \lambda_j = 2 \sum_{i,j} a_{ij}(x)\, \lambda_i \lambda_j$ and (3.6) is satisfied with $p = 2$ (and different constants). It is not difficult to test the ellipticity and growth conditions of the integrand $f(x, \xi) = h(x, \xi) + |\xi_n|^q$; since $f_{\xi_i} = h_{\xi_i}$ for all $i \in \{1, 2, \ldots, n-1\}$, $f_{\xi_n} = h_{\xi_n} + q\, |\xi_n|^{q-2} \xi_n$, and $f_{\xi_i x_k} = h_{\xi_i x_k}$ for all $i \in \{1, 2, \ldots, n-1, n\}$, then

$$c_1 |\xi|^{p-2} |\lambda|^2 \leq \sum_{i,j} f_{\xi_i \xi_j}(x, \xi)\, \lambda_i \lambda_j \leq c_4 \left(1 + |\xi|^{q-2}\right) |\lambda|^2\,, \tag{3.8}$$

$$\sum_{i} |f_{\xi_i x_k}(x, \xi)| \leq c_3 |\xi|^{p-1}\,. \tag{3.9}$$

In order to test conditions (2.1)-(2.7), in this case we fix (up to the multiplicative constants) $g_1(t) = t^{p-2}$, $g_2(t) = 1 + t^{q-2}$, $g_3(t) = t^{p-1}$, under the usual notation $|\xi| = t$. We start from (2.5), which takes the form

$$h(t) = \frac{t^{p-1}}{(1+t^\gamma)(t^{p-2})^{\frac{1}{2}}(1+t^{q-2})^{\gamma-\frac{1}{2}}} = \frac{t^{p/2}}{(1+t^\gamma)(1+t^{q-2})^{\gamma-\frac{1}{2}}} \leq M\,,$$

for all $t \in [0, +\infty)$ and for a constant $M > 0$. As in Proposition 3.2, $h(t)$ is a bounded function in $[0, +\infty)$ if it has a finite limit as $t \to +\infty$. This finite limit condition $\lim_{t\to+\infty} h(t) \in \mathbb{R}$ is reduced to the exponents' inequality $\frac{p}{2} \le \gamma + (q-2)\left(\gamma - \frac{1}{2}\right)$, which, for $\gamma = 1$, simply means $p \le q$. Therefore assumption (2.5) in this case holds with $\gamma = 1$. Then (2.3) corresponds to

$$(1+t^{q-2})^{2\gamma-1}\, t^2 \le M\,(1+\int_0^t s^{\frac{p}{2}-1} ds)^\alpha = M\,(1+\tfrac{2}{p} t^{\frac{p}{2}})^\alpha, \tag{3.10}$$

which is satisfied by a positive constant M and for all $t \ge 0$, if we choose $\gamma = 1$ and α in such a way that $[(q-2)(2\gamma-1)]_{\gamma=1} + 2 = q \le \alpha\,\frac{p}{2}$; i.e., if we choose $\alpha = 2\frac{q}{p}$. To respect the conditions $2 \le \alpha < 2^* =: \frac{2n}{n-2}$ (there is not an upper bound for α when $n = 2$) we need

$$2 \le 2\tfrac{q}{p} < \tfrac{2n}{n-2} \quad \Leftrightarrow \quad 1 \le \tfrac{q}{p} < \tfrac{n}{n-2}\,.$$

Let us now discuss assumption (2.4). For $f(x,\xi)$ as in (3.5), being $|f(x,\xi)| \le c_5 (1+|\xi|^q)$, it corresponds to $\left(1+|\xi|^{q-2}\right)^{2\gamma-1} |\xi|^{2\gamma} \le M\,\{1+|\xi|^q\}^\beta$, which, in terms of exponents, gives $[(q-2)(2\gamma-1)+2]_{\gamma=1} = q \le \beta q$. Thus $\frac{q}{p} \le \beta$. To respect the conditions for β in (2.7), i.e. $1 \le \beta < \left[\frac{2(\alpha-2+2\gamma)}{n(\alpha-4+2\gamma)}\right]_{\gamma=1} = \frac{2\alpha}{n(\alpha-2)}$, we choose $\beta = \frac{\alpha}{2}$ and we assume the condition $1 \le \frac{\alpha}{2} < \frac{2\alpha}{n(\alpha-2)}$, which is equivalent to $2 \le \alpha < 2 + \frac{4}{n}$. Recalling that we already choose $\alpha = 2\frac{q}{p}$, we finally impose the bound on $\frac{q}{p}$

$$1 \le \tfrac{q}{p} < 1 + \tfrac{2}{n}\,. \tag{3.11}$$

As a consequence of Theorem 2.2, with parameters $\alpha = 2\frac{q}{p}$, $\beta = \frac{\alpha}{2} = \frac{q}{p}$, $\gamma = 1$, we have proved the following result.

Example 3.3 (Lipschitz continuity result for anisotropic energy integrals) *Let $2 \le p \le q$ satisfy (3.11). Every local minimizer in the Sobolev class (2.10) of the energy integral (3.5), or of the integral in (3.4) when $p = 2$, is locally Lipschitz continuous in Ω and satisfies the uniform gradient estimate and the estimates for the $n \times n$ matrix D^2u of the second derivatives of u stated in Theorem 2.2.*

Remark 3.4 *Similarly to the energy integrals (3.4),(3.5) considered in this Section 3.1, it is possible to deal with*

$$\int_\Omega (\sum_{i,j=1}^n a_{ij}(x)\, u_{x_i} u_{x_j} + a(x)\,|u_{x_n}|^q\,)\,dx \quad \text{or} \quad \int_\Omega \{h(x,Du) + a(x)\,|u_{x_n}|^q\}\,dx\,,$$

with a locally Lipschitz continuous coefficient $a(x) \ge 0$ in Ω. Then the proof should proceed similarly to the proof given in this section, by changing the choice of $\gamma = 1$ with $\gamma = 1+\delta$, with $\delta > 0$ as explicitly described in Remark 2.1. If $\delta > 0$ (and not $\delta = 0$ as in the above proof), then the bound on the ratio $\frac{q}{p}$ in (3.11) becomes more strict.

3.2 Exponential growth

We consider the energy integral

$$\int_\Omega f(x, Du)\, dx = \int_\Omega e^{a(x)|Du|^\tau}\, dx \tag{3.12}$$

related to the *exponential growth case* $f(x,\xi) = e^{a(x)|\xi|^\tau}$, with $a(x)$ positive locally Lipschitz function and $\tau \geq 2$. For simplicity, here we consider $\tau = 2$. Under the notations of Proposition 3.1 we have $f(x,\xi) = g(x, |\xi|)$ and

$$g(x,t) = e^{a(x)t^2}, \;\; g_t(x,t) = 2tae^{at^2}, \;\; g_{tt}(x,t) = 4t^2a^2e^{at^2} + 2ae^{at^2}.$$

Then $g_t(x,t)/t = 2a(x)\, e^{a(x)t^2}$, being product of positive increasing functions, is increasing with respect to $t \in [0, +\infty)$ for all $x \in \Omega$. By *(ii)* of Proposition 3.1, (3.3) holds with $g_1(x,t) = g_t(x,t)/t$ and $g_2(x,t) = g_{tt}(x,t)$.

Let Ω' be an open set compactly contained in Ω and B_R a generic ball in Ω'. The coefficient belongs to $W^{1,\infty}_{\mathrm{loc}}(\Omega)$ and thus is Lipschitz continuous in Ω' with a Lipschitz constant L uniform for all $B_R \subset \Omega'$. We define

$$p = \min\{a(x) : x \in \overline{B_R}\} \quad \text{and} \quad q = \max\{a(x) : x \in \overline{B_R}\}. \tag{3.13}$$

For every $\theta > 1$ we can choose the radius $R = R(\theta)$ of the ball B_R small enough such that $q \leq \theta p$; in fact $q - p = a(x_2) - a(x_1) \leq L|x_2 - x_1| \leq 2R$ and thus $q \leq p + 2R < \theta p$ if we choose, for instance, $\theta = 1 + 3R/p$. Therefore

$$g_t(x,t)/t = 2a(x)\, e^{a(x)t^2} \geq 2pe^{pt^2}, \quad \forall\ (x,t) \in \overline{B_R} \times [0, +\infty)$$

and similarly, being $q \leq \theta p$, $g_{tt}(x,t) \leq 4t^2[\theta p]^2 e^{\theta pt^2} + 2\theta pe^{\theta pt^2}$. Then, by considering the infimum with respect to $x \in B_R$ of $g_t(x,t)/t$ and the supremum with respect to $x \in B_R$ of $g_{tt}(x,t)$, the ellipticity condition (2.1) is satisfied in B_R with

$$g_1(t) = c_1 e^{pt^2} \quad \text{and} \quad g_2(t) = c_2\left(1 + t^2\right) e^{qt^2}, \tag{3.14}$$

and $c_1 = 2p$, $c_2 = (4p)^2$ (with $\theta > 1$ also bounded above by 2). Firstly, we verify (2.3). Then for $t \geq t_0 \geq 1$,

$$\int_0^t \sqrt{g_1(s)}\, ds \geq \int_0^{t_0} \sqrt{g_1(s)}\, ds + p\int_{t_0}^t e^{\frac{ps^2}{2}}\, s\, ds = c + p\, e^{\frac{p}{2}t^2}.$$

We apply Proposition 3.2 with $h(t) := ((e^{qt^2}\left(1+t^2\right))^{2\gamma-1}\, t^2) \,/\, (1 + e^{\frac{p}{2}t^2})^\alpha$ and we require

$$q(2\gamma - 1) \leq \alpha\tfrac{p}{2} \tag{3.15}$$

in order to get a finite limit of $h(t)$ as $t \to +\infty$. We use here the possibility stated after the definition (3.13) of p, q; precisely, $\frac{q}{p} = \theta$, with θ arbitrarily close to 1. Then (3.15) becomes $\theta p(2\gamma - 1) \leq \alpha p/2$. As in Remark 2.1 we pose $\gamma = 1 + \delta$, therefore we get $\theta p\, 2\delta \leq p\left(\frac{\alpha}{2} - \theta\right)$. Note that θp is a bounded quantity, since $1 \leq \theta p \leq 2p$. We can choose $\alpha > 2$ and θ sufficiently close to 1, so that the right hand side is positive. Therefore we can also fix δ sufficiently close to 0 in order to obtain the validity of the inequality in (3.15). For these values of α and γ assumption (2.3) holds. To verify (2.4), it is enough to prove that

$$(e^{qt^2}\left(1+t^2\right))^{2\gamma-1}\, t^{2\gamma} \leq M\,(1+e^{pt^2})^{\beta}. \tag{3.16}$$

Proceeding as in the previous lines, (3.16) holds if $\theta(2\delta+1) \leq \beta$. The previous inequality is trivially true for $\beta > 1$, δ sufficiently close to 0 and θ sufficiently close to 1. Now, we calculate $g_{tx_k}(x,\xi) = 2a_{x_k}(x)te^{a(x)t^2}\left(1+a(x)t^2\right)$. Since $a(x)$ is a locally Lipschitz function, then there exists a constant $L > 0$ such that $|a_{x_k}(x)| \leq L$, so hypothesis (2.2) is satisfied with

$$g_3(t) = c_3 e^{\theta p t^2} t\left(1+t^2\right), \tag{3.17}$$

where $c_3 = (1+2p)2L$ (here we have used that $q \leq \theta p$ and $1 < \theta < 2$). To verify (2.5), we should prove that

$$e^{\theta p t^2} t\left(1+t^2\right) \leq M(1+t^{\gamma})\,(e^{pt^2})^{\frac{1}{2}}\,(e^{\theta p t^2}\left(1+t^2\right))^{\gamma-\frac{1}{2}}.$$

If $t \to 0^+$, the left hand side of the inequality goes to 0, while the right hand side converges to a positive number, so (2.5) holds true. When $t \to +\infty$, (2.5) is reduced to the exponents' inequality $\theta\left(\frac{1}{2}-\delta\right) \leq \frac{1}{2}$, which is true by choosing δ sufficiently close to 0 and θ sufficiently close to 1.

As a consequence, we get the following consequence of Theorem 2.2, with parameters $\beta = \frac{\alpha}{2}+\delta$ and $\gamma = 1+\delta$, where $2 < \alpha < 2^*$, $1 < \beta < 1+\frac{2}{n}$ and $0 < \delta < \frac{4}{n(n-2)}$ (see Remark 2.1).

Example 3.5 (Lipschitz continuity result under exponential growth) *Every smooth local minimizer of the energy integral (3.12) is locally Lipschitz continuous in Ω and satisfies the uniform gradient estimate (2.12) and the estimates (2.13) for the $n \times n$ matrix D^2u of the second derivatives of u.*

3.3 $p(x)-$Laplacian and logarithm $p(x)-$Laplacian

We consider the energy integral

$$\int_{\Omega} f\,(x, Du)\;dx = \int_{\Omega} |Du|^{p(x)}\;dx \tag{3.18}$$

related to the $p(x)-$*Laplacian degenerate case* $f(x,\xi) = |\xi|^{p(x)}$, with variable exponent $p(x) \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$, $p(x) \geq 2$. Under the notations of Proposition 3.1 we have $f(x,\xi) = g\,(x, |\xi|)$ and

$$g\,(x,t) = t^{p(x)}, \quad g_t\,(x,t) = p(x)t^{p(x)-1}, \quad g_{tt}\,(x,t) = p(x)\,(p(x)-1)\,t^{p(x)-2}.$$

Then $g_t\,(x,t)\,/t = p(x)t^{p(x)-2}$ is locally Lipschitz continuous with respect to $t \in [0,+\infty)$ for all $x \in \Omega$. By *(i)* of Proposition 3.1, (3.3) holds with

$$g_1\,(x,t) = p(x)t^{p(x)-2} \quad \text{and} \quad g_2\,(x,t) = p(x)\,(p(x)-1)\,t^{p(x)-2}.$$

Let Ω' be an open set compactly contained in Ω and B_R a generic ball in Ω'. We define $p = \min\{p(x) : x \in \overline{B_R}\}$, $q = \max\{p(x) : x \in \overline{B_R}\}$. Given $\theta > 1$, we choose the radius R of the ball B_R small enough such that $q \leq \theta p$. Therefore, for every $x \in \overline{B_R}$ and $t > 1$, $g_t(x,t)/t \geq pt^{p-2}$, and analogously, $g_{tt}(x,t) \leq \theta p(\theta p - 1)t^{\theta p-2}$. Then, the ellipticity condition (2.1) is satisfied in B_R with

$$g_1(t) = c_1 t^{p-2} \quad \text{and} \quad g_2(t) = c_2 t^{\theta p-2}, \tag{3.19}$$

and $c_1 = p$, $c_2 = 2p(2p-1)$ (where we used that $\theta > 1$ is also bounded above by 2). Similarly for $t \in [0,1]$, indeed in this case it is sufficient to interchange the supremum with the infimum and viceversa. So that we obtain $g_1(t) = c_1 t^{\theta p-2}$ and $g_2(t) = c_2 t^{p-2}$. We observe that hypothesis (2.3) is automatically satisfied when $0 \leq t \leq 1$, since the left hand side is bounded from above and the right hand side is bounded by a positive constant from below. In order to verify (2.3) for $t > 1$, we need to prove that $\left(c_2 t^{\theta p-2}\right)^{2\gamma-1} t^2 \leq M\left(1 + \frac{2\sqrt{c_1}}{p} t^{\frac{p}{2}\alpha}\right)$ which, by Proposition 3.2, follows from the condition $(\theta p - 2)(2\gamma - 1) + 2 \leq p\frac{\alpha}{2}$. As in Remark 2.1 we pose $\gamma = 1 + \delta$, which gives

$$(\theta p - 2)\, 2\delta \leq p\left(\tfrac{\alpha}{2} - \theta\right). \tag{3.20}$$

Note that $(\theta p - 2)$ is a bounded quantity, since $0 \leq \theta p - 2 \leq 2(p-1)$. The right hand side is positive if we fix $\alpha > 2$ and θ sufficiently close to 1. Therefore we can also fix δ sufficiently close to 0 in order to obtain the validity of the inequality in (3.20). For these values of α and γ assumption (2.3) holds.
In order to verify (2.4), it is enough to show that

$$\left(c_2 t^{\theta p-2}\right)^{2\gamma-1} t^{2\gamma} \leq M\{1 + t^p\}^{\beta}. \tag{3.21}$$

Using Proposition 3.2, inequality (3.21) is true by imposing the following condition $(\theta p - 2)(2\gamma - 1) + 2\gamma \leq p\beta$. As previously done, by Remark 2.1 we pose $\gamma = 1 + \delta$, obtaining

$$(\theta p - 1)\, 2\delta \leq p(\beta - \theta). \tag{3.22}$$

Note that $(\theta p - 1)$ is a bounded quantity, since $0 \leq \theta p - 1 \leq 2p - 1$. The right hand side is positive if we fix $\beta > 1$ and θ sufficiently close to 1. Therefore we can also fix δ sufficiently close to 0 in order to obtain the validity of the inequality in (3.22). For these values of β and γ assumption (2.4) holds. Now we compute $g_{tx_k}(x,t) = p_{x_k}(x)t^{p(x)-1}(1 + p(x)\log t)$. We observe that, even if $|p_{x_k}(x)|$ is locally bounded, because of the logarithmic factor, we cannot bound $|g_{tx_k}(x,t)|$ only with the exponential term $t^{p(x)-1}$. However, since $\log t < t^{\omega}/\omega$ for every $\omega > 0$, we can find a constant c_3, which depends on ω and on the $L^{\infty}(B_R)$-bounds of $p(x)$ and $p_{x_k}(x)$, such that $|g_{tx_k}(x,t)| \leq c_3\left(1 + t^{\theta p-1+\omega}\right)$. So we fix

$$g_3(t) = c_3\left(1 + t^{\theta p-1+\omega}\right). \tag{3.23}$$

In order to verify (2.5), we need to find a constant M such that

$$c_3\left(1 + t^{\theta p-1+\omega}\right) \leq M(1 + t^{\gamma})\left(c_1 t^{p-2}\right)^{\frac{1}{2}}\left(c_2 t^{\theta p-2}\right)^{\gamma-\frac{1}{2}}. \tag{3.24}$$

We apply Proposition 3.2, imposing the following inequality on the exponents

$$\theta p \left(\tfrac{3}{2} - \gamma\right) + \omega \leq \tfrac{p}{2} + 1 - \gamma \,. \tag{3.25}$$

We can verify that $\gamma = 1$ is not a possible choice, since $\frac{1}{2}q + \omega \leq \frac{1}{2}\theta p + \omega \leq \frac{1}{2}p$ is a false inequality even if $q = p$. Then we follow Remark 2.1 by posing $\gamma = 1 + \delta$ (and $\beta = \frac{\alpha}{2} + \delta$); from (3.25) we get $\theta p \left(\frac{1}{2} - \delta\right) + \omega \leq \frac{p}{2} - \delta$, or equivalently $\frac{1}{2}\theta p + \omega \leq \frac{p}{2} + \delta\,(q-1)$, which in terms of the parameter δ means (we choose for δ the minimum possible value; i.e., when the equality sign holds)

$$\delta = \tfrac{(\theta-1)p+2\omega}{2(\theta p-1)} \,. \tag{3.26}$$

Finally we observe that the value of $\delta > 0$ in (3.26) can be fixed as small as we like, by choosing θ close to 1 and ω close to 0. Therefore we can fix $\theta > 1$ and $\omega > 0$ such that the condition $0 \leq \delta < \frac{4}{n(n-2)}$ in Remark 2.1 is satisfied.

At this point, we can state the following consequence of Theorem 2.2.

Example 3.6 (Lipschitz continuity for $p(x)$ Laplacian integral) *There exist $\alpha > 2$, $\beta > 1$, $\gamma > 1$ and functions g_1, g_2, g_3 defined as (3.19) and (3.23) satisfying the ellipticity and growth conditions (2.1)-(2.7), such that every smooth local minimizer of the energy integral (3.18) is locally Lipschitz continuous in Ω and satisfies the uniform gradient estimate (2.12) and the estimates (2.13) for the $n \times n$ matrix D^2u of the second derivatives of u.*

Similar computations can be carried out for the *Orlicz type energy functionals* (*logarithm $p(x)-$Laplacian*)

$$f\,(x,\xi) = |\xi|^{p(x)} \log(1 + |\xi|^2),$$

where $p(x) \geq 2$ is a local Lipschitz continuous exponent. In this case we have to test the hypotheses set under the choice

$$g_1\,(t) = t^{p-2} \log(1+t^2), \quad g_2\,(t) = t^{\theta p-2}\left(\log(1+t^2)+1\right) \quad g_3(t) = \left(1 + t^{\theta p-1+\omega}\right),$$

where we have omitted multiplicative constants for simplicity.

3.4 Double phase case

We consider the energy integral

$$\int_\Omega f\,(x, Du)\; dx = \int_\Omega \{|Du|^p + a(x)\,|Du|^q\}\; dx \tag{3.27}$$

related to the *double phase degenerate case* $f\,(x,\xi) = |\xi|^p + a(x)\,|\xi|^q$, with $2 \leq p \leq q$ and a nonnegative locally Lipschitz continuous function in Ω. We fix

$$g_1\,(|\xi|) = |\xi|^{p-2,} \quad g_2\,(|\xi|) = |\xi|^{q-2} \quad \text{and} \quad g_3\,(|\xi|) = |\xi|^{q-1},$$

where we have omitted multiplicative constants for simplicity.

We introduce a further exponent, greater than q, at the same distance $q-p$ from q; precisely $q+(q-p)=2q-p$. Then for any fixed $b \geq 0$, we consider the auxiliary variational problem, related to a *multiple phase*, of the form

$$f\left(x,\xi\right)=\left|\xi\right|^{p}+a\left(x\right)\left|\xi\right|^{q}+b\left|\xi\right|^{2q-p} \tag{3.28}$$

and $g_1\left(\left|\xi\right|\right) = \left|\xi\right|^{p-2}$, $g_2\left(\left|\xi\right|\right) = \left|\xi\right|^{2q-p-2}$, $g_3\left(\left|\xi\right|\right) = \left|\xi\right|^{q-1}$. We apply the regularity Theorem 2.2. In particular assumption (2.5) trivially holds with $\gamma = 1$. With the aim to test assumption (2.3), we note that

$$\int_0^t \sqrt{g_1(s)}ds = \int_0^t s^{\frac{p-2}{2}}ds = \tfrac{2\,t^{\frac{p}{2}}}{p};$$

therefore in order to obtain the validity of (2.3) we must impose $2q-p-2+2 = 2q-p \leq \frac{p}{2}\alpha$, that is $\frac{q}{p} \leq 1+\frac{\alpha}{2}$. Taking into account the bound $2 \leq \alpha < 2^* \underset{\text{if } n>2}{=} \frac{2n}{n-2}$ for α in (2.6) and letting α converge to the limit value 2^* we finally obtain

$$\tfrac{q}{p} < 1+\tfrac{2^*}{2} \underset{\text{if } n>2}{=} \tfrac{n}{n-2} = 1+\tfrac{2}{n-2}\,. \tag{3.29}$$

Assumption (2.4) requires that $t^{2q-p} \leq M\left\{1+t^p+a(x)t^q+bt^{2q-p}\right\}^{\beta}$, that is $\beta \geq 1$, which is true by assumption (2.7).

We remark that in the above considerations we used a nonnegative constant coefficient b which can be chosen equal to zero too, the case which corresponds to the original energy integral (3.27). Thus we have proved the following result, as a consequence of the general Theorem 2.2.

Example 3.7 *Under the bound (3.29) on the ratio q/p every local minimizer $u \in W^{1,2q-p}_{\mathrm{loc}}(\Omega)$ to the double phase energy integral in (3.27) (or, when $b \neq 0$, a minimizer of the integral related to the auxiliary integrand (3.28) as well) is locally Lipschitz continuous in Ω and satisfies the uniform gradient estimate (2.12) and the estimates (2.13) for the $n\times n$ matrix D^2u of the second derivatives of u.*

4 Proof of Theorem 2.2

4.1 Step 1. Second variation

We consider a local minimizer u of the energy integral (1.1) under the supplementary assumptions (2.10). The local minimizer u satisfies the Euler's first variation in the weak form

$$\int_\Omega \sum_{i=1}^n f_{\xi_i}\left(x,Du\right)\psi_{x_i}\,dx = 0\,,$$

for every $\psi \in W_0^{1,2}(\Omega)$. Since $u \in W_{\mathrm{loc}}^{2,2}(\Omega)$, we consider a generic integer $k \in \{1,2,\dots,n\}$ and $\psi = -\frac{\partial \varphi}{\partial x_k} = -\varphi_{x_k}$, where $\varphi \in W_0^{1,2}(\Omega)$ is a generic test function. By integrating by parts we get

$$\int_\Omega \left(\sum_{i,j=1}^n f_{\xi_i \xi_j}(x, Du)\, u_{x_j x_k} + \sum_{i=1}^n f_{\xi_i x_k}(x, Du) \right) \varphi_{x_i}\, dx = 0\,. \tag{4.1}$$

We make a further choice of the test function φ, by posing $\varphi = \eta^2 u_{x_k} \Phi(|Du|)$, where $\eta \in W_0^{1,2}(\Omega)$ and $\Phi : (0,+\infty) \to (0,+\infty)$ is a nonnegative, increasing, locally Lipschitz continuous function in $(0,+\infty)$, to be chosen later. Then

$$\varphi_{x_i} = 2\eta\eta_{x_i} u_{x_k} \Phi(|Du|) + \eta^2 u_{x_k x_i} \Phi(|Du|) + \eta^2 u_{x_k} \Phi'(|Du|)\, |Du|_{x_i}\,.$$

4.1.1 Terms to estimate from $\boxed{\mathbf{1}}$ to $\boxed{\mathbf{6}}$

The left hand side in (4.1) now splits into six addenda

$$\begin{aligned}
&\boxed{\mathbf{1}} \quad \int_\Omega 2\eta\Phi(|Du|) \sum_{i,j=1}^n f_{\xi_i \xi_j}(x, Du)\, u_{x_j x_k} \eta_{x_i} u_{x_k}\, dx \\
&\boxed{\mathbf{2}} \quad + \int_\Omega \eta^2 \Phi(|Du|) \sum_{i,j=1}^n f_{\xi_i \xi_j}(x, Du)\, u_{x_j x_k} u_{x_k x_i}\, dx \\
&\boxed{\mathbf{3}} \quad + \int_\Omega \eta^2 \Phi'(|Du|) \sum_{i,j=1}^n f_{\xi_i \xi_j}(x, Du)\, u_{x_j x_k} u_{x_k}\, |Du|_{x_i}\, dx \\
&\boxed{\mathbf{4}} \quad + \int_\Omega 2\eta\Phi(|Du|) \sum_{i=1}^n f_{\xi_i x_k}(x, Du)\, \eta_{x_i} u_{x_k}\, dx \\
&\boxed{\mathbf{5}} \quad + \int_\Omega \eta^2 \Phi(|Du|) \sum_{i=1}^n f_{\xi_i x_k}(x, Du)\, u_{x_k x_i}\, dx \\
&\boxed{\mathbf{6}} \quad + \int_\Omega \eta^2 \Phi'(|Du|) \sum_{i=1}^n f_{\xi_i x_k}(x, Du)\, u_{x_k}\, |Du|_{x_i}\, dx = 0\,.
\end{aligned} \tag{4.2}$$

In the following sections we estimate (4.2) term by term.

4.2 Step 2. Main estimates

In the proof of Theorem 2.2 we use several times the elementary inequality $\left(\sqrt{\varepsilon} a - \frac{1}{2\sqrt{\varepsilon}} b\right)^2 \geq 0$; i.e., $ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2$, valid for all $a, b \in \mathbb{R}$ and $\varepsilon > 0$.

4.2.1 Estimate of the term in $\boxed{\mathbf{1}}$

In order to estimate of the term in $\boxed{\mathbf{1}}$ we use here the *Cauchy-Schwarz inequality* for symmetric quadratic forms

$$
\begin{aligned}
&\left|\int_\Omega 2\eta\Phi\left(|Du|\right)\sum_{i,j=1}^{n} f_{\xi_i\xi_j}\left(x,Du\right)u_{x_jx_k}\eta_{x_i}u_{x_k}\,dx\right| \\
&\leq \int_\Omega 2\Phi\left(|Du|\right)\left(\eta^2\sum_{i,j=1}^{n} f_{\xi_i\xi_j}\left(x,Du\right)u_{x_ix_k}u_{x_jx_k}\right)^{1/2} \\
&\quad\cdot\left(\left(u_{x_k}\right)^2\sum_{i,j=1}^{n} f_{\xi_i\xi_j}\left(x,Du\right)\eta_{x_i}\eta_{x_j}\right)^{1/2}\,dx \\
&\leq 2\varepsilon\int_\Omega \eta^2\Phi\left(|Du|\right)\sum_{i,j=1}^{n} f_{\xi_i\xi_j}\left(x,Du\right)u_{x_ix_k}u_{x_jx_k}\,dx \\
&\quad+\frac{1}{2\varepsilon}\int_\Omega \Phi\left(|Du|\right)\left(u_{x_k}\right)^2\sum_{i,j=1}^{n} f_{\xi_i\xi_j}\left(x,Du\right)\eta_{x_i}\eta_{x_j}\,dx.
\end{aligned}
$$

By the *growth condition* in the right hand side of the (2.1), for $\varepsilon=\frac{1}{4}$ we obtain

$$
\begin{aligned}
&\left|\int_\Omega 2\eta\Phi\left(|Du|\right)\sum_{i,j=1}^{n} f_{\xi_i\xi_j}\left(x,Du\right)u_{x_jx_k}\eta_{x_i}u_{x_k}\,dx\right| \\
&\leq \tfrac{1}{2}\int_\Omega \eta^2\Phi\left(|Du|\right)\sum_{i,j=1}^{n} f_{\xi_i\xi_j}\left(x,Du\right)u_{x_ix_k}u_{x_jx_k}\,dx \\
&\quad+2\int_\Omega \Phi\left(|Du|\right)\left(u_{x_k}\right)^2 g_2\left(|Du|\right)\,|D\eta|^2\,dx\,.
\end{aligned}
\tag{4.3}
$$

We insert this inequality in the equation (4.2) obtained in the previous Section 4.1.1. We chose to put some addenda in the left hand side and some others in the opposite side; the reason is clear by taking into account the sign of all addenda and/or their absolute values. In particular $\boxed{\mathbf{2}}$ and $\boxed{\mathbf{3}}$ are nonnegative as consequence of the ellipticity condition in the left hand side of (2.1). Precisely $\boxed{\mathbf{3}}$ is nonnegative after a sum with respect to $k\in\{1,2,\dots,n\}$; see Section 4.2.3. Since $\left|\boxed{\mathbf{1}}\right|\leq\frac{1}{2}\boxed{\mathbf{2}}$ plus the last term in (4.3), we get

$$
\frac{1}{2}\boxed{\mathbf{2}}+\boxed{\mathbf{3}}\leq\left|\boxed{\mathbf{4}}\right|+\left|\boxed{\mathbf{5}}\right|+\left|\boxed{\mathbf{6}}\right|+2\int_\Omega |D\eta|^2\,\Phi\left(|Du|\right)\left(u_{x_k}\right)^2 g_2\left(|Du|\right)\,dx\,.
\tag{4.4}
$$

We observe that, when we sum both sides over $k\in\{1,2,\dots,n\}$, in the right hand side of (4.4) we have $\sum_{k=1}^{n}\left(u_{x_k}\right)^2=|Du|^2$.

4.2.2 Use of ellipticity in the term in $\boxed{2}$

This is the simplest step among these estimates. We bound from below the addendum in $\boxed{2}$ by mean of the *ellipticity condition* in the left hand side of (2.1)

$$\int_\Omega \eta^2 \Phi\left(|Du|\right) \sum_{i,j=1}^{n} f_{\xi_i \xi_j}\left(x, Du\right) u_{x_j x_k} u_{x_k x_i}\, dx$$

$$\geq \int_\Omega \eta^2 \Phi\left(|Du|\right) g_1\left(|Du|\right) \left|Du_{x_k}\right|^2\, dx\,,$$

by noting that for Sobolev functions $u_{x_j x_k} = u_{x_k x_j}$ a.e. in Ω. When we sum both sides over $k = 1, 2, \ldots, n$ we observe that $\sum_{k=1}^{n} \left|Du_{x_k}\right|^2 = \left|D^2 u\right|^2$.

4.2.3 Use of ellipticity in the term in $\boxed{3}$

In order to estimate the term in $\boxed{3}$, we explicitly represent $|Du|_{x_i}$, recalling that for Sobolev functions the second order mixed derivatives coincide

$$|Du|_{x_i} = \frac{\partial}{\partial x_i}\left(\sum_{k=1}^{n} \left(u_{x_k}\right)^2\right)^{1/2} = \frac{\sum_{k=1}^{n} u_{x_k} u_{x_k x_i}}{|Du|}\,. \tag{4.5}$$

Thus,

$$\frac{1}{|Du|} \sum_{i,j,k=1}^{n} f_{\xi_i \xi_j}\left(x, Du\right) u_{x_j x_k} u_{x_k} (|Du|)_{x_i} = \sum_{i,j=1}^{n} f_{\xi_i \xi_j}\left(x, Du\right) (|Du|)_{x_j} (|Du|)_{x_i}. \tag{4.6}$$

Under the notation $D\left(|Du|\right) = \left(|Du|_{x_1}, |Du|_{x_2}, \ldots, |Du|_{x_n}\right)$ and $\sum_{i=1}^{n} \left(|Du|_{x_i}\right)^2 = \left|D\left(|Du|\right)\right|^2$, we sum over k in $\boxed{3}$ and we get

$$\begin{aligned} &\int_\Omega \eta^2 \Phi'\left(|Du|\right) \sum_{i,j,k=1}^{n} f_{\xi_i \xi_j}\left(x, Du\right) u_{x_j x_k} u_{x_k} (|Du|)_{x_i}\, dx \\ &= \int_\Omega \eta^2 \Phi'\left(|Du|\right) |Du| \sum_{i,j=1}^{n} f_{\xi_i \xi_j}\left(x, Du\right) (|Du|)_{x_j} (|Du|)_{x_i}\, dx \\ &\geq \int_\Omega \eta^2 \Phi'\left(|Du|\right) |Du|\, g_1\left(|Du|\right) \left|D\left(|Du|\right)\right|^2\, dx, \end{aligned} \tag{4.7}$$

where in the last inequality we have used again the *ellipticity condition* in the left hand side of (2.1).

Remark 4.1 *We use in (4.5) the Cauchy-Schwarz inequality*

$$|Du|_{x_i} = \frac{\sum_{k=1}^{n} u_{x_k} u_{x_k x_i}}{|Du|} \leq \frac{|Du| \left(\sum_{k=1}^{n} \left(u_{x_k x_i}\right)^2\right)^{\frac{1}{2}}}{|Du|} = \left(\sum_{k=1}^{n} \left(u_{x_i x_k}\right)^2\right)^{\frac{1}{2}} = \left|Du_{x_i}\right|.$$

Therefore

$$|D(|Du|)|^2 = \sum_{i=1}^{n} \left(|Du|_{x_i}\right)^2 \le \sum_{i=1}^{n} |Du_{x_i}|^2 \tag{4.8}$$

$$= \sum_{i=1}^{n} (\sum_{j=1}^{n} \left(u_{x_i x_j}\right)^2) = \sum_{i,j=1}^{n} \left(u_{x_i x_j}\right)^2 = \left|D^2 u\right|^2 .$$

For the term in $\boxed{\mathit{3}}$*, as represented in (4.7), by the growth condition in the right hand side of (2.1) we also deduce the estimate*

$$0 \le \int_\Omega \eta^2 \Phi'(|Du|)\,|Du| \sum_{i,j=1}^{n} f_{\xi_i \xi_j}(x, Du)\,(|Du|)_{x_j} (|Du|)_{x_i}\, dx$$

$$\le \int_\Omega \eta^2 \Phi'(|Du|)\,|Du|\, g_2(|Du|)\ |D(|Du|)|^2\ dx$$

$$\le \int_\Omega \eta^2 \Phi'(|Du|)\,|Du|\, g_2(|Du|)\ \left|D^2 u\right|^2\ dx\,. \tag{4.9}$$

At the beginning of Section 4.5, for the first step of the iteration procedure, we fix $\Phi(t) = t$ *for all* $t \in [0, +\infty)$*, thus* $\Phi'(t) = 1$*. Therefore, as a consequence of (2.10), (4.9) shows that the term in* $\boxed{\mathit{3}}$ *is finite, as well as all the other addenda listed in Section 4.1.1, when the iteration procedure starts with* $\Phi(t) \equiv t$*.*

4.2.4 Estimate of the term in $\boxed{4}$

For any fixed $k \in \{1, 2, \dots, n\}$, by the growth conditions (2.2) and assumption (2.5) we obtain

$$\left| \int_\Omega 2\eta \Phi(|Du|) \sum_{i=1}^{n} f_{\xi_i x_k}(x, Du)\, \eta_{x_i} u_{x_k}\, dx \right|$$

$$\le \int_\Omega 2\eta\, |D\eta|\, \Phi(|Du|)\, |Du| \sum_{i=1}^{n} |f_{\xi_i x_k}(x, Du)|\ dx$$

$$\le \int_\Omega 2\eta\, |D\eta|\, \Phi(|Du|)\, |Du|\, g_3(|Du|)\ dx$$

$$\le 2M \int_\Omega \eta\, |D\eta|\, \Phi(|Du|)\, |Du|\, (1 + |Du|^\gamma)\ (g_1(|Du|))^{\frac{1}{2}}\ (g_2(|Du|))^{\gamma - \frac{1}{2}}\ dx$$

$$\le 2M \int_\Omega \eta\, |D\eta|\, \Phi(|Du|)\, |Du|\, (1 + |Du|^\gamma)\, (g_2(|Du|))^\gamma\ dx.$$

We sum over k and, since the right hand side does not depend on k, we get

$$\sum_{k=1}^{n} \left| \int_\Omega 2\eta \Phi(|Du|) \sum_{i=1}^{n} f_{\xi_i x_k}(x, Du)\, \eta_{x_i} u_{x_k}\, dx \right|$$

$$\le 2nM \int_\Omega \eta\, |D\eta|\, \Phi(|Du|)\, |Du|\, (1 + |Du|^\gamma)\, (g_2(|Du|))^\gamma\ dx.$$

4.2.5 Estimate of the term in $\boxed{5}$

We use the growth assumptions (2.2),(2.5) to get

$$
\begin{aligned}
&\left|\int_{\Omega} \eta^2 \Phi\left(|Du|\right) \sum_{i=1}^{n} f_{\xi_i x_k}\left(x, Du\right) u_{x_k x_i}\, dx\right| \\
&\quad \leq \int_{\Omega} \eta^2 \Phi\left(|Du|\right) g_3\left(|Du|\right) \left|Du_{x_k}\right|\, dx \\
&\quad \leq M \int_{\Omega} \eta^2 \Phi\left(|Du|\right) \left(g_1\left(|Du|\right)\right)^{\frac{1}{2}} \left(1+|Du|^{\gamma}\right) \left(g_1\left(|Du|\right)\right)^{\frac{\gamma-1}{2}} \left(g_2\left(|Du|\right)\right)^{\frac{\gamma}{2}} \left|Du_{x_k}\right|\, dx \\
&\quad \leq M \int_{\Omega} \eta^2 \Phi\left(|Du|\right) \left(g_1\left(|Du|\right)\right)^{\frac{1}{2}} \left(1+|Du|^{\gamma}\right) \left(g_2\left(|Du|\right)\right)^{\gamma-\frac{1}{2}} \left|Du_{x_k}\right|\, dx \\
&\quad \leq \varepsilon M \int_{\Omega} \eta^2 \Phi\left(|Du|\right) g_1\left(|Du|\right)\, \left|Du_{x_k}\right|^2\, dx \\
&\qquad + \frac{M}{4\varepsilon} \int_{\Omega} \eta^2 \Phi\left(|Du|\right) \left(1+|Du|^{\gamma}\right)^2 \left(g_2\left(|Du|\right)\right)^{2\gamma-1}\, dx\,.
\end{aligned}
$$

We sum over k and we observe that $\sum_{k=1}^{n} \left|Du_{x_k}\right|^2 = \left|D^2 u\right|^2$

$$
\begin{aligned}
&\sum_{k=1}^{n} \left|\int_{\Omega} \eta^2 \Phi\left(|Du|\right) \sum_{i=1}^{n} f_{\xi_i x_k}\left(x, Du\right) u_{x_k x_i}\, dx\right| \\
&\quad \leq \varepsilon M \int_{\Omega} \eta^2 \Phi\left(|Du|\right) g_1\left(|Du|\right)\, \left|D^2 u\right|^2\, dx \\
&\quad + \frac{nM}{4\varepsilon} \int_{\Omega} \eta^2 \Phi\left(|Du|\right) \left(1+|Du|^{\gamma}\right)^2 \left(g_2\left(|Du|\right)\right)^{2\gamma-1}\, dx\,.
\end{aligned}
$$

4.2.6 Estimate of the term in $\boxed{6}$

Similarly to the previous Sections, under the notation $D\left(|Du|\right) = \left(|Du|_{x_i}\right)_{i=1,2,\ldots,n}$ and $\sum_{i=1}^{n} \left(|Du|_{x_i}\right)^2 = \left|D\left(|Du|\right)\right|^2$, by the Cauchy-Schwarz inequality and (2.2), (2.5), we obtain

$$
\left|\int_{\Omega} \eta^2 \Phi'\left(|Du|\right) \sum_{i=1}^{n} f_{\xi_i x_k}\left(x, Du\right) u_{x_k} \left|Du\right|_{x_i}\, dx\right| \tag{4.10}
$$

$$
\begin{aligned}
&\int_{\Omega} \eta^2 \Phi'\left(|Du|\right) \left|u_{x_k}\right| \Big(\sum_{i=1}^{n} \left(f_{\xi_i x_k}\left(x, Du\right)\right)^2\Big)^{\frac{1}{2}} \Big(\sum_{i=1}^{n} \left(|Du|_{x_i}\right)^2\Big)^{\frac{1}{2}}\, dx \\
&\quad \leq \sqrt{n} \int_{\Omega} \eta^2 \Phi'\left(|Du|\right) |Du|\, g_3\left(|Du|\right)\, \left|D\left(|Du|\right)\right|\, dx \\
&\leq \sqrt{n} M \int_{\Omega} \eta^2 \Phi'\left(|Du|\right) |Du| \left(g_1\left(|Du|\right)\right)^{\frac{1}{2}} \left(1+|Du|^{\gamma}\right) \left(g_1\left(|Du|\right)\right)^{\frac{\gamma-1}{2}} \left(g_2\left(|Du|\right)\right)^{\frac{\gamma}{2}}\, \left|D\left(|Du|\right)\right|\, dx
\end{aligned}
$$

$$
\begin{aligned}
&\le \varepsilon\sqrt{n}M\int_\Omega \eta^2\Phi'\left(|Du|\right)|Du|\,g_1\left(|Du|\right)\,\left|D\left(|Du|\right)\right|^2\,dx\\
&+\frac{\sqrt{n}M}{4\varepsilon}\int_\Omega \eta^2\Phi'\left(|Du|\right)|Du|\left(1+|Du|^\gamma\right)^2\left(g_2\left(|Du|\right)\right)^{2\gamma-1}\,dx\,.
\end{aligned}
$$

4.3 Step 3. Collecting together the previous estimates

First we make the previous estimates uniform each other with a sum over $k\in\{1,2,\ldots,n\}$ where this has been not jet done. Then we collect them, starting from the estimate of the term in $\boxed{\mathbf{1}}$ with the inequality (4.4), with the nonnegative terms $\boxed{\mathbf{2}}$ and $\boxed{\mathbf{3}}$ in the left hand side. By also taking the $\varepsilon-$addenda of $\boxed{\mathbf{5}}$ and $\boxed{\mathbf{6}}$ in the left hand side, we obtain

$$
\begin{aligned}
&\Big(\underset{\boxed{1}}{\tfrac{1}{2}}-\underset{\boxed{5}}{\varepsilon M}\Big)\int_\Omega \eta^2\Phi\left(|Du|\right)\underset{\boxed{2}}{g_1\left(|Du|\right)}\left|D^2u\right|^2\,dx\\
&+\Big(1-\underset{\boxed{6}}{\varepsilon\sqrt{n}M}\Big)\int_\Omega \eta^2\Phi'\left(|Du|\right)|Du|\,\underset{\boxed{3}}{g_1\left(|Du|\right)}\,\left|D\left(|Du|\right)\right|^2\,dx\\
&\le 2nM\int_\Omega \eta\,|D\eta|\,\Phi\left(|Du|\right)|Du|\,\underset{\boxed{4}}{\left(1+|Du|^\gamma\right)}\left(g_2\left(|Du|\right)\right)^\gamma\,dx\\
&+\frac{nM}{4\varepsilon}\int_\Omega \eta^2\Phi\left(|Du|\right)\underset{\boxed{5}}{\left(1+|Du|^\gamma\right)^2}\left(g_2\left(|Du|\right)\right)^{2\gamma-1}\,dx\\
&+\frac{\sqrt{n}M}{4\varepsilon}\int_\Omega \eta^2\Phi'\left(|Du|\right)|Du|\underset{\boxed{6}}{\left(1+|Du|^\gamma\right)^2}\left(g_2\left(|Du|\right)\right)^{2\gamma-1}\,dx\\
&+2\int_\Omega |D\eta|^2\,\Phi\left(|Du|\right)\underset{\boxed{1}}{|Du|^2}\,g_2\left(|Du|\right)\,dx\,.
\end{aligned}
$$

We chose $\varepsilon=\frac{1}{4M}\min\left\{1;\frac{3}{\sqrt{n}}\right\}$, so that $\frac{1}{2}-\varepsilon M\ge\frac{1}{4}$ and $1-\varepsilon\sqrt{n}M\ge\frac{1}{4}$ too. Form the previous estimate we obtain the existence of a positive constant $c_0=c_0\left(n,M\right)$ such that

$$
\begin{aligned}
&\tfrac{1}{c_0}\int_\Omega \eta^2 g_1\left(|Du|\right)\left(\Phi\left(|Du|\right)\left|D^2u\right|^2+\Phi'\left(|Du|\right)|Du|\,\left|D\left(|Du|\right)\right|^2\right)dx\\
&\le\int_\Omega \eta\,|D\eta|\,\Phi\left(|Du|\right)|Du|\left(1+|Du|^\gamma\right)\left(g_2\left(|Du|\right)\right)^\gamma\,dx\\
&+\int_\Omega \eta^2\left(\Phi\left(|Du|\right)+\Phi'\left(|Du|\right)|Du|\right)\left(1+|Du|^\gamma\right)^2\left(g_2\left(|Du|\right)\right)^{2\gamma-1}\,dx\\
&+\int_\Omega |D\eta|^2\,\Phi\left(|Du|\right)|Du|^2\,g_2\left(|Du|\right)\,dx\,.
\end{aligned}
$$

We make use of the inequality (4.8); i.e., $|D(|Du|)| \leq |D^2 u|$. Therefore finally we get

$$\frac{1}{c_0}\int_\Omega \eta^2 g_1(|Du|)\left(\Phi(|Du|)+\Phi'(|Du|)|Du|\right)|D(|Du|)|^2\,dx \tag{4.11}$$

$$\leq \int_\Omega \eta\,|D\eta|\,\Phi(|Du|)\,|Du|\,(1+|Du|^\gamma)\,(g_2(|Du|))^\gamma\,dx$$

$$+\int_\Omega \eta^2\left(\Phi(|Du|)+\Phi'(|Du|)|Du|\right)(1+|Du|^\gamma)^2\,(g_2(|Du|))^{2\gamma-1}\,dx$$

$$+\int_\Omega |D\eta|^2\,\Phi(|Du|)\,|Du|^2\,g_2(|Du|)\,dx\,.$$

4.4 Step 4. Use of the Sobolev inequality

We define the integral function $G:[0,+\infty)\to[1,+\infty)$ by

$$G(t)=1+\int_0^t\sqrt{\Phi(s)g_1(s)}\,ds. \tag{4.12}$$

Since g_1 and Φ are increasing, then $\int_0^t\sqrt{\Phi(s)g_1(s)} \leq \sqrt{\Phi(t)g_1(t)}\cdot t$ and

$$[G(t)]^2 \leq 2\left(1+\left(\int_0^t\sqrt{\Phi(s)g_1(s)}\right)^2\right) \leq 2\left(1+\Phi(t)g_1(t)t^2\right)\,.$$

We deduce

$$|D(\eta G(|Du|)|^2 \leq 2|D\eta|^2[G(|Du|)]^2+2\eta^2[G'(|Du|)]^2|D(|Du|)|^2$$

$$\leq 4|D\eta|^2(1+\Phi(|Du|)g_1(|Du|)|Du|^2)+2\eta^2\Phi(|Du|)g_1(|Du|)\,|D(|Du|)|^2.$$

By integrating we get

$$\int_\Omega |D(\eta G(|Du|)|^2\,dx \leq 4\int_\Omega |D\eta|^2(1+\Phi(|Du|)g_1(|Du|)|Du|^2)\,dx \tag{4.13}$$

$$+2\int_\Omega \eta^2\Phi(|Du|)g_1(|Du|)\,|D(|Du|)|^2\,dx\,.$$

When compared with the estimate (4.11), we note that the last addendum in the right hand side of (4.13), is less that or equal to the left hand side of (4.11). By combining (4.11), (4.13) we obtain

$$\int_\Omega |D(\eta G(|Du|)|^2\,dx \leq \int_\Omega |D\eta|^2(4+2c_0\Phi(|Du|)g_1(|Du|)|Du|^2)\,dx \tag{4.14}$$

$$+2c_0\int_\Omega \eta\,|D\eta|\,\Phi(|Du|)\,|Du|\,(1+|Du|^\gamma)\,(g_2(|Du|))^\gamma\,dx$$

$$+2c_0 \int_\Omega \eta^2 \left(\Phi\left(|Du|\right) + \Phi'\left(|Du|\right)|Du|\right)\left(1+|Du|^\gamma\right)^2 \left(g_2\left(|Du|\right)\right)^{2\gamma-1}\, dx\,.$$

Recall that $2^* = \frac{2n}{n-2}$ if $n \geq 3$, while 2^* is any fixed real number greater than α if $n=2$. By the Sobolev inequality, there exists a constant c_1 such that

$$\left(\int_\Omega [\eta G(|Du|)]^{2^*}\, dx\right)^{2/2^*} \leq c_1 \int_\Omega |D(\eta G(|Du|)|^2\, dx\,, \tag{4.15}$$

which naturally can be combined with (4.14), as done in the next section.

4.5 Step 5. Choice of the test function Φ

Let us define $\Phi(t) = t^{2\lambda}$ for every $t \geq 0$, with $\lambda \geq 1$. Making use of [46, Lemma 3.4 (v)] we can estimate the integral function $G(t)$ defined in (4.12) by

$$G(t) = 1 + \int_0^t \sqrt{\Phi(s) g_1(s)}\, ds \geq 1 + \frac{t^\lambda}{\lambda+1}\int_0^t \sqrt{g_1(s)}\, ds. \tag{4.16}$$

Recall that the function $g_1 : [0, +\infty) \to [0, +\infty)$ is increasing and not identically zero. Then, there exists $t_0 > 0$ such that $g_1(t) > 0$, for every $t \geq t_0$. Up to a rescaling, we can assume that $t_0 \leq 1$, so that

$$g_1(1) = c_2 > 0, \quad \int_0^1 \sqrt{g_1(s)}\, ds = c_3 > 0. \tag{4.17}$$

Then, for every $t \geq 1$, we have

$$\begin{aligned} 2\int_0^t \sqrt{g_1(s)}\, ds &\geq c_3 + \int_0^t \sqrt{g_1(s)}\, ds \\ &\geq \min\{c_3, 1\}\left(1 + \int_0^t \sqrt{g_1(s)}\, ds\right). \end{aligned} \tag{4.18}$$

Now, we use assumptions (2.4) and (2.7). By (2.4) and by (4.18), there exists a constant c_4 such that

$$\left(\int_0^t \sqrt{g_1(s)}\, ds\right)^\alpha \geq c_4 (g_2(t))^{2\gamma-1} t^2, \quad \forall t \geq 1. \tag{4.19}$$

Recall that by (2.6), $2 \leq \alpha < 2^*$. Then, by (4.17), for every $t \geq 1$ we obtain

$$\begin{aligned} \left(\int_0^t \sqrt{g_1(s)}\, ds\right)^{2^*} &\geq \left(\int_0^t \sqrt{g_1(s)}\, ds\right)^\alpha (c_2(t-1)+c_3)^{2^*-\alpha} \\ &\geq (\min\{c_2, c_3\})^{2^*-\alpha} c_4 (g_2(t))^{2\gamma-1} t^{2^*-\alpha+2}, \quad \forall t \geq 1. \end{aligned} \tag{4.20}$$

We note that for the previous step, it's fundamental assuming α strictly less then 2^*. By (4.20) and (4.16), we deduce that for every $t \geq 1$

$$\begin{aligned} [G(t)]^{2^*} &\geq \frac{1}{2}\left(1 + \left(\frac{t^\lambda}{(\lambda+1)}\int_0^t \sqrt{g_1(s)}\, ds\right)^{2^*}\right) \\ &\geq \frac{1}{2} + \frac{(\min\{c_2, c_3\})^{2^*-\alpha} c_4}{2(\lambda+1)^2} (g_2(t))^{2\gamma-1} t^{2^*(\lambda+1)-\alpha+2}. \end{aligned}$$

That is, there exists a constant c_5 such that

$$c_5(\lambda+1)^{2^*}[G(t)]^{2^*} \geq 1 + (g_2(t))^{2\gamma-1} t^{2^*(\lambda+1)-\alpha+2},$$

for every $t \geq 1$, and also for $t \in [0,1)$, since the left-hand side is bounded for $t \in (0,1)$ by the constant (independent of γ) $1+g_2(1)$, while the right-hand side is bounded from below away from zero. So, for every $t \geq 0$,

$$c_5(\lambda+1)^{2^*}[G(t)]^{2^*} \geq 1 + (g_2(t))^{2\gamma-1} t^{2^*(\lambda+1)-\alpha+2}. \tag{4.21}$$

Since $\Phi(t) = t^{2\lambda}$ and $g_1(t) \leq g_2(t)$, from (4.14), (4.15) and (4.21), with positive constants c_6, c_7, ..., we obtain

$$\begin{aligned}
&\left(\int_\Omega \eta^{2^*}\left\{1+|Du|^{2^*(\lambda+1)-\alpha+2}(g_2(|Du|))^{2\gamma-1}\right\}dx\right)^{2/2^*} \\
&\leq c_6\,(\lambda+1)^2\left(\int_\Omega \eta^{2^*}[G(|Du|)]^{2^*}\,dx\right)^{2/2^*} \\
&\leq c_7\,(\lambda+1)^2\left\{\int_\Omega |D\eta|^2(1+|Du|^{2\lambda}\,g_2(|Du|)\;|Du|^2)\,dx\right. \\
&\quad+\int_\Omega \eta\,|D\eta|\,|Du|^{2\lambda+1}\,(1+|Du|^\gamma)\,(g_2\,(|Du|))^\gamma\;dx \\
&\quad\left.+\int_\Omega \eta^2\left(|Du|^{2\lambda}+2\lambda|Du|^{2\lambda}\right)(1+|Du|^\gamma)^2\,(g_2\,(|Du|))^{2\gamma-1}\;dx\right\}.
\end{aligned}$$

Then we uniform as much as possible the right had side

$$\begin{aligned}
&\left(\int_\Omega \eta^{2^*}\left\{1+|Du|^{2^*(\lambda+1)-\alpha+2}(g_2(|Du|))^{2\gamma-1}\right\}dx\right)^{2/2^*} \qquad (4.22)\\
&\leq c_8\,(\lambda+1)^3\left\{\int_\Omega |D\eta|^2(1+|Du|^{2\lambda}\,g_2(|Du|)\;|Du|^2)\,dx\right. \\
&\quad+\int_\Omega \eta\,|D\eta|\,|Du|^{2\lambda+1}\,(1+|Du|^\gamma)\,(g_2\,(|Du|))^\gamma\;dx \\
&\quad\left.+\int_\Omega \eta^2\left(|Du|^{2\lambda}+|Du|^{2\lambda}\right)(1+|Du|^\gamma)^2\,(g_2\,(|Du|))^{2\gamma-1}\;dx\right\}.
\end{aligned}$$

With the aim to reduce the previous estimate to a unique addendum in the right hand side we recall that $\gamma \geq 1$ and $\lambda \geq 1$. Moreover $g_2 : (0,+\infty) \to (0,+\infty)$ is an increasing function such that $g_2\,(1) \geq 1$, and thus also $g_2\,(t) \geq 1$ for all

$t \geq 1$. Thus for every $t \geq 0$ we have

$$t^{2\lambda}\left(1+t^{\gamma}\right)^{2}\left(g_{2}\left(t\right)\right)^{2\gamma-1} \leq \left\{ \begin{array}{ll} 4\,t^{2(\lambda+\gamma)}\left(g_{2}\left(t\right)\right)^{2\gamma-1} & \text{if } \ t\geq 1 \\ 4\left(g_{2}\left(1\right)\right)^{2\gamma-1} & \text{if } \ 0\leq t\leq 1 \end{array} \right\}$$
$$\leq \max\left\{4\,\left(g_{2}\left(1\right)\right)^{2\gamma-1};4\,t^{2(\lambda+\gamma)}\left(g_{2}\left(t\right)\right)^{2\gamma-1}\right\}$$
$$\leq 4\,\left(g_{2}\left(1\right)\right)^{2\gamma-1}+4\,t^{2(\lambda+\gamma)}\left(g_{2}\left(t\right)\right)^{2\gamma-1}$$
$$\leq 4\,\left(g_{2}\left(1\right)\right)^{2\gamma-1}+4\,\left(g_{2}\left(1\right)\right)^{2\gamma-1}\,t^{2(\lambda+\gamma)}\left(g_{2}\left(t\right)\right)^{2\gamma-1}$$
$$= 4\,\left(g_{2}\left(1\right)\right)^{2\gamma-1}\left(1+\,t^{2(\lambda+\gamma)}\left(g_{2}\left(t\right)\right)^{2\gamma-1}\right). \tag{4.23}$$

By proceeding in a similar way we get

$$t^{2\lambda+1}(1+t)^{\gamma}\left(g_{2}(t)\right)^{\gamma} \leq \left\{ \begin{array}{ll} 2\,t^{2\lambda+\gamma+1}\left(g_{2}(1)\right)^{\gamma} & \text{if } \ t\geq 1 \\ 2\left(g_{2}(1)\right)^{\gamma} & \text{if } \ 0\leq t\leq 1 \end{array} \right\}$$
$$\leq \left\{ \begin{array}{ll} 2\,t^{2(\lambda+\gamma)}\left(g_{2}(1)\right)^{\gamma} & \text{if } \ t\geq 1 \\ 2\left(g_{2}(1)\right)^{\gamma} & \text{if } \ 0\leq t\leq 1 \end{array} \right\}$$
$$\leq 2\left(g_{2}\left(1\right)\right)^{2\gamma-1}\left(1+t^{2(\lambda+\gamma)}\left(g_{2}(t)\right)^{2\gamma-1}\right), \quad \forall\, t\geq 0\,, \tag{4.24}$$

and

$$t^{2(\lambda+1)}g_{2}(t) \leq g_{2}(1)\left(1+t^{2(\lambda+\gamma)}\left(g_{2}(t)\right)^{2\gamma-1}\right)$$
$$\leq \left(g_{2}\left(1\right)\right)^{2\gamma-1}\left(1+t^{2(\lambda+\gamma)}\left(g_{2}(t)\right)^{2\gamma-1}\right), \quad \forall\, t\geq 0. \tag{4.25}$$

By using (4.23), (4.24), (4.25), from (4.22) we get the final estimate with only one addendum in the right hand side

$$\left(\int_{\Omega}\eta^{2^{*}}[1+|Du|^{2^{*}(\lambda+1)-\alpha+2}(g_{2}(|Du|))^{2\gamma-1}]\,dx\right)^{2/2^{*}}$$
$$\leq c(2\lambda+1)^{3}\int_{\Omega}(\eta+|D\eta|)^{2}\left(1+|Du|^{2(\lambda+\gamma)}\left(g_{2}(|Du|)\right)^{2\gamma-1}\right)\,dx\,. \tag{4.26}$$

Note that the constant c in (4.26) depends on the dimension n, on the other constants m, M, γ, on the values $g_1(1)$, $g_2(1)$, but it is independent of λ.

Let us denote by B_R and B_ρ balls compactly contained in Ω, of radii respectively R, ρ, with the same center. Let η be a test function equal to 1 in B_ρ, whose support is contained in B_R, such that $|D\eta| \le \frac{2}{R-\rho}$. Thus, we obtain

$$\left(\int_{B_\rho}[1+|Du|^{2^*(\lambda+1)-\alpha+2}(g_2(|Du|))^{2\gamma-1}]\,dx\right)^{2/2^*}$$
$$\le c(2\lambda+1)^3\int_{B_R}\left(\eta^2+|D\eta|^2\right)\left(1+|Du|^{2(\lambda+\gamma)}\,(g_2(|Du|))^{2\gamma-1}\right)\,dx$$
$$\le c(2\lambda+1)^3\left(1+\frac{4}{(R-\rho)^2}\right)\int_{B_R}\left(1+|Du|^{2(\lambda+\gamma)}\,(g_2(|Du|))^{2\gamma-1}\right)\,dx\,. \tag{4.27}$$

4.6 Step 6. Iteration

We define by induction a sequence λ_k in the following way:

$$\lambda_1=0,\quad \lambda_{k+1}=\frac{2^*}{2}\lambda_k+\frac{2^*-\alpha+2}{2}-\gamma,\quad \forall k\in\mathbb{N}; \tag{4.28}$$

we note in particular that λ_k satisfies the property

$$2^*(\lambda_k+1)-\alpha+2=2(\lambda_{k+1}+\gamma),\qquad \forall k\in\mathbb{N}. \tag{4.29}$$

It is easy to prove by induction the following representation formula for λ_k:

$$\lambda_k=\frac{2^*-\alpha-2(\gamma-1)}{2^*-2}[(\frac{2^*}{2})^{k-1}-1],\qquad \forall k\in\mathbb{N}. \tag{4.30}$$

Since $\alpha\ge 2$, from (4.29), (4.30), we deduce the inequality

$$(\lambda_k+1)2^*-\alpha+2\ge 2\,\frac{2^*-a-2(\gamma-1)}{2^*-2}\left(\frac{2^*}{2}\right)^k. \tag{4.31}$$

For fixed R_0 and ρ_0, for all $k\in\mathbb{N}$, we rewrite (4.27) with $R=\rho_{k-1}$ and $\rho=\rho_k$, where $\rho_k=\rho_0+\frac{R_0-\rho_0}{2^k}$; moreover, for $k=1,2,3,...,i$, with i fixed in $\mathbb{N}$, we put λ equal to λ_k. By iterating (4.27), by (4.29) we obtain

$$\left(\int_{B_{\rho_i}}\left(1+|Du|^{(\lambda_{i+1})2^*-a+2}\right)\,(g_2(|Du|))^{2\gamma-1}\,dx\right)^{\frac{2}{2^*}}$$
$$\le c_7\int_{B_{R_0}}\left(1+|Du|^{2\gamma}\,(g_2(|Du|))^{2\gamma-1}\right)\,dx. \tag{4.32}$$

Since $R-\rho=\rho_{k-1}-\rho_k=\frac{R_0-\rho_0}{2^k}$ for all $k\in\mathbb{N}$, if $n\ge 3$; otherwise, if $n=2$, then for every $\epsilon>0$ we can choose 2^* so that

$$c_7=\frac{c_8}{(R_0-\rho_0)^{-2-\epsilon}},\quad \text{for some constant } c_8. \tag{4.33}$$

Since g_2 is increasing, then $g_2(t) \geq g_1(t) \geq g_1(1) > 0$, for all $t \geq 1$. Therefore for $r \geq s \geq 0$, we have

$$g_2(t)t^r + 1 \geq g_1(1)t^s, \quad \text{if } t \geq 1,$$

$$g_2(t)t^r + 1 \geq 1 \geq t^s, \quad \text{if } 0 \leq t \leq 1.$$

Thus, by posing $c_9 = \min\{g_1(1), 1\}$, we obtain

$$g_2(t)t^r + 1 \geq c_9 t^s, \quad \forall t \geq 0, \forall r \geq s \geq 0. \tag{4.34}$$

Now, we go to the limit in (4.32) as $i \to +\infty$. We use the inequalities (4.31), (4.32), (4.34) and we obtain

$$\begin{aligned}
&\sup\{|Du(x)|^{2\left(\frac{2^*-a-2(\gamma-1)}{2^*-2}\right)} : x \in B_{\rho_0}\} \\
&= \lim_{i\to+\infty} \left(\int_{B_{\rho_0}} |Du|^{2\left(\frac{2^*-a-2(\gamma-1)}{2^*-2}\right)\left(\frac{2^*}{2}\right)^i} dx \right)^{\left(\frac{2}{2^*}\right)^i} \\
&\leq \limsup_{i\to+\infty} \left(\frac{1}{c_9} \int_{B_{\rho_i}} \left(1 + |Du|^{2^*(\lambda_i+1)-a+2} \left(g_2(|Du|)\right)^{2\gamma-1}\right) dx \right)^{\left(\frac{2}{2^*}\right)^i} \\
&\leq \limsup_{i\to+\infty} \frac{c_7}{c_9} \int_{B_{R_0}} \left(1 + |Du|^{2\gamma} \left(g_2(|Du|)\right)^{2\gamma-1}\right) dx
\end{aligned}$$

and by the representation of c_7 in (4.33), we finally obtain

$$\|Du\|_{L^\infty(B_{\rho_0},\mathbb{R}^n)}^{2\left(\frac{2^*-a-2(\gamma-1)}{2^*-2}\right)} \leq \frac{c_{10}}{(R_0-\rho_0)^n} \int_{B_{R_0}} \left(1 + |Du|^{2\gamma} \left(g_2(|Du|)\right)^{2\gamma-1}\right) dx. \tag{4.35}$$

4.7 Step 7. A priori gradient estimate

We start putting together the inequalities (4.26), (4.14) and (4.15), with $\lambda = 0$, that is $\Phi = 1$, we get

$$\begin{aligned}
&\left(\int_\Omega (\eta G(|Du|))^{2^*} dx \right)^{2/2^*} \\
&\quad \leq c \int_\Omega \left(\eta^2 + |D\eta|^2\right) \left(1 + |Du|^{2\gamma} \left(g_2(|Du|)\right)^{2\gamma-1}\right) dx\,.
\end{aligned} \tag{4.36}$$

We note that, by assumption (2.6), $\frac{2^*}{\alpha-2+2\gamma} > 1$. Then we can define ν such that

$$1 \leq \nu < \frac{2^*}{\alpha-2+2\gamma}. \tag{4.37}$$

Following the lines of Step 5, we have, for every $t \geq 1$

$$\left(\int_0^t \sqrt{g_1(s)}\,ds\right)^{\frac{2^*}{\nu}} \geq \left(\int_0^t \sqrt{g_1(s)}\,ds\right)^{\alpha} (c_2(t-1)+c_3)^{\frac{2^*}{\nu}-\alpha}$$
$$\geq (\min\{c_2,c_3\})^{\frac{2^*}{\nu}-\alpha} c_4 (g_2(t))^{2\gamma-1} t^{\frac{2^*}{\nu}-\alpha+2}. \tag{4.38}$$

By (4.38) and (4.16), we deduce that for every $t \geq 1$

$$[G(t)]^{\frac{2^*}{\nu}} \geq \frac{1}{2}\left(1+\left(\int_0^t \sqrt{g_{1(s)}}\,ds\right)^{\frac{2^*}{\nu}}\right)$$
$$\geq \frac{1}{2} + \frac{(\min\{c_2,c_3\})^{\frac{2^*}{\nu}-\alpha} c_4}{2} (g_2(t))^{2\gamma-1} t^{\frac{2^*}{\nu}-\alpha+2}.$$

That is, there exists a constant c such that

$$[G(t)]^{\frac{2^*}{\nu}} \geq c\left(1+(g_2(t))^{2\gamma-1} t^{\frac{2^*}{\nu}-\alpha+2}\right),$$

for every $t \geq 1$, and also for $t \in [0,1)$, since the left-hand side is bounded for $t \in (0,1)$ by the constant (independent of γ) $1+g_2(1)$, while the right-hand side is bounded from below away from zero. So, for every $t \geq 0$,

$$[G(t)]^{\frac{2^*}{\nu}} \geq c\left(1+(g_2(t))^{2\gamma-1} t^{\frac{2^*}{\nu}-\alpha+2}\right). \tag{4.39}$$

Since $\nu < \frac{2^*}{\alpha-2+2\gamma}$, we have $\frac{2^*}{\nu}-\alpha+2 > 2\gamma$; by (4.36), (4.39), we get

$$\left(\int_\Omega \eta^{2^*}[1+|Du|^{2\gamma}(g_2(|Du|))^{2\gamma-1}]^{\nu}\,dx\right)^{2/2^*}$$
$$\leq \left(\int_\Omega \eta^{2^*}[1+|Du|^{\frac{2^*}{\nu}-\alpha+2}(g_2(|Du|))^{2\gamma-1}]^{\nu}\,dx\right)^{2/2^*}$$
$$\leq c\int_\Omega \left(\eta^2+|D\eta|^2\right)\left(1+|Du|^{2\gamma}\,(g_2(|Du|))^{2\gamma-1}\right)dx\,. \tag{4.40}$$

Then, under the notation

$$V = V(x) = 1+|Du|^{2\gamma}\,(g_2(|Du|))^{2\gamma-1}\,,$$

inequality (4.40) becomes

$$\left(\int_\Omega \eta^{2^*} V^{\nu}\,dx\right)^{2/2^*} \leq c\int_\Omega \left(\eta^2+|D\eta|^2\right) V\,dx\,. \tag{4.41}$$

We consider a test function $\eta = 1$ in B_ρ with $|D\eta| \leq \frac{2}{R-\rho}$ as in the previous step, thus we get

$$\left(\int_{B_\rho} V^{\nu}\,dx\right)^{2/2^*} \leq \frac{c}{(R-\rho)^2}\int_{B_R} V\,dx\,. \tag{4.42}$$

Let $\mu > \frac{2^*}{2}$ to be chosen later. By Hölder inequality we have

$$\left(\int_{B_\rho} V^\nu \, dx\right)^{2/2^*} \leq \frac{c}{(R-\rho)^2} \int_{B_R} V^{\frac{\nu}{\mu}} V^{1-\frac{\nu}{\mu}} \, dx$$
$$\leq \left(\int_{B_R} V^\nu \, dx\right)^{\frac{1}{\mu}} \left(\int_{B_R} V^{\frac{\mu-\nu}{\mu-1}} \, dx\right)^{\frac{\mu-1}{\mu}}. \tag{4.43}$$

Let R_0 and ρ_0 be fixed. For any $i \in \mathbb{N}$ we consider (4.43) with $R = \rho_i$ and $\rho = \rho_{i-1}$, where $\rho_i = R_0 - \frac{R_0-\rho_0}{2^i}$. By iterating (4.43) since $R - \rho = \frac{R_0-\rho_0}{2^i}$, we obtain

$$\int_{B_{\rho_0}} V^\nu \, dx \leq \left(\int_{B_{\rho_i}} V^\nu \, dx\right)^{\left(\frac{2^*}{2\mu}\right)^i} \prod_{i=1}^{\infty} \left(\frac{c4^{i+1}}{(R_0-\rho_0)^2}\right)^{\mu\left(\frac{2^*}{2\mu}\right)^i}$$
$$\cdot \left(\int_{B_{\rho_0}} V^{\frac{\mu-\nu}{\mu-1}} \, dx\right)^{(\mu-1)\left(\frac{2^*}{2\mu}\right)^i}$$
$$\leq \left(\int_{B_{\rho_i}} V^\nu \, dx\right)^{\left(\frac{2^*}{2\mu}\right)^i} c \left(\frac{1}{(R_0-\rho_0)^2}\right)^{\frac{2^*\mu}{2\mu-2^*}}$$
$$\cdot \left(\int_{B_{\rho_0}} V^{\frac{\mu-\nu}{\mu-1}} \, dx\right)^{2^* \frac{\mu-1}{2\mu-2^*}}. \tag{4.44}$$

Now we use assumption (2.4):

$$V = 1 + |Du|^{2\gamma} \left(g_2(|Du|)\right)^{2\gamma-1} \leq 2\max\{1, c\}(1 + f(Du))^\beta, \tag{4.45}$$

where, since $\nu \geq 1$,

$$\beta = \frac{\mu-1}{\mu-\nu} \geq 1. \tag{4.46}$$

We recall that (4.37) holds, so we have

$$1 \leq \beta < \frac{\mu-1}{\mu - \frac{2^*}{\alpha-2+2\gamma}}.$$

We compute the limit as $\mu \to \frac{2^*}{2}$ of the right hand side, which, by computations, is equal to $\frac{2(\alpha-2+2\gamma)}{n(\alpha-4+2\gamma)}$. While the limit as $\mu \to +\infty$ is equal to 1. Then it is possible to choose $\mu \in \left(\frac{2^*}{2}, +\infty\right)$ so that the definition of β in (4.46) is compatible with assumption (2.7).
We go to the limit in (4.44) as $i \to +\infty$ and we use (4.45) and (4.46) to obtain

$$\int_{B_{\rho_0}} V^\nu \, dx \leq c \left(\frac{1}{(R_0-\rho_0)^2}\right)^{\frac{2^*\mu}{2\mu-2^*}} \left(\int_{B_{R_0}} (1 + f(Du)) \, dx\right)^{\frac{2^*(\mu-1)}{2\mu-2^*}}.$$

Then we get

$$\int_{B_{\rho_0}} V\,dx \le |B_{\rho_0}|^{1-\frac{1}{\nu}} \left(\int_{B_{\rho_0}} V^{\nu}\,dx\right)^{\frac{1}{\nu}}$$

$$\le c\left(\frac{1}{(R_0-\rho_0)^2}\right)^{\frac{2^*\mu}{(2\mu-2^*)\nu}} \left(\int_{B_{R_0}} (1+f(Du))\,dx\right)^{\frac{2^*(\mu-1)}{(2\mu-2^*)\nu}}. \quad (4.47)$$

Therefore, with the notation $\theta_0 = 2\frac{2^*\mu}{(2\mu-2^*)\nu}$, there exists a constant $c = c(n,\alpha,\beta,\gamma)$ such that

$$\int_{B_{\rho_0}} \left\{1+|Du|^{2\gamma}\left(g_2(|Du|)\right)^{2\gamma-1}\right\} dx \quad (4.48)$$

$$\le \frac{c}{(R_0-\rho_0)^{\theta_0}} \left(\int_{B_{R_0}} \{1+f(Du)\}\,dx\right)^{\frac{2^*(\mu-1)}{(2\mu-2^*)\nu}}.$$

By (4.35) finally we deduce that for all ρ_0, R_0 with $0<\rho_0<R_0$

$$\|Du\|^2_{L^\infty\left(B_{\rho_0};\mathbb{R}^n\right)} \le \frac{c}{\left(R_0-\rho_0\right)^{\theta_2}} \left(\int_{B_{R_0}} \{1+f\left(Du\right)\}\,dx\right)^{\theta_1}, \quad (4.49)$$

where we have used the notation

$$\theta_1 = \tfrac{2^*(\mu-1)}{(2\mu-2^*)\nu}\cdot\tfrac{2^*-2}{2^*-\alpha-2(\gamma-1)} \quad \text{and} \quad \theta_2 = \theta_0\tfrac{2^*-2}{2^*-\alpha-2(\gamma-1)}. \quad (4.50)$$

We observe that $\theta_1>1$; indeed $\frac{2^*-2}{2^*-\alpha-2(\gamma-1)} \ge 1$ since $\alpha\ge 2$ and $\gamma\ge 1$. The other factor of θ_1 is strictly greater than 1. In fact, being $\alpha\ge 2$ and $\gamma\ge 1$, we have that $\nu < \frac{2^*}{\alpha-2+2\gamma} \le \frac{2^*}{2}$. Then we deduce

$$\tfrac{2^*(\mu-1)}{(2\mu-2^*)\nu} > \tfrac{2(\mu-1)}{2\mu-2^*} > 1\,. \quad (4.51)$$

4.8 Step 8. $W^{2,2}$−estimate

We are ready to prove the $W^{2,2}$−estimate (2.13). We start from (4.11). At the beginning of our proof we choose the test function $\varphi = \eta^2 u_{x_k}\Phi\left(|Du|\right)$, with $\eta\in C^1\left(\Omega\right)$ with compact support in Ω, $\Phi:[0,+\infty)\to[0,+\infty)$ is a generic nonnegative, increasing, locally Lipschitz continuous function in $[0,+\infty)$. Here, in the estimate in (4.11), we simply consider the case when Φ is the constant identically equal to 1. We obtain

$$\begin{aligned}\int_\Omega \eta^2 g_1\left(|Du|\right)\left|D^2u\right|^2\,dx \le & 8Mn\int_\Omega \eta\,|D\eta|\,|Du|^{1+\gamma}\left(g_2\left(|Du|\right)\right)^{\gamma}\,dx \\ & + 8(Mn)^2\int_\Omega \eta^2\left(1+|Du|^{\gamma}\right)^2\left(g_2\left(|Du|\right)\right)^{2\gamma-1}\,dx \\ & + 8\int_\Omega |D\eta|^2\,g_2\left(|Du|\right)|Du|^2\,dx.\end{aligned}$$

Since $\gamma \geq 1$, in the integrands in the right hand side we can use the inequalities for a constant $c \geq 1$

$$\begin{cases} |Du|^{1+\gamma}, \ \ (1+|Du|^{\gamma})^2 \ \text{ and } \ |Du|^2 \leq c\left(1+|Du|^{2\gamma}\right) \\ (g_2(|Du|))^{\gamma} \ \text{ and } \ g_2(|Du|) \leq 1+(g_2(|Du|))^{2\gamma-1} \\ \eta\,|D\eta|\,, \ \eta^2 \ \text{ and } \ |D\eta|^2 \leq (\eta+|D\eta|)^2 \end{cases}$$

and, for a new positive constant c, we get

$$\int_{\Omega} \eta^2 g_1(|Du|)\left|D^2u\right|^2 dx \leq c\int_{\Omega}(\eta+|D\eta|)^2\left(1+|Du|^{2\gamma}\right)\left(1+(g_2(|Du|))^{2\gamma-1}\right) dx\,. \tag{4.52}$$

The function $g_2 : [0,+\infty) \to [0,+\infty)$ is nonnegative, increasing and, as in (3.1), $g_2(t) \geq g_2(1) \geq 1$ for all $t \geq 1$; then

$$|Du|^{2\gamma} \ \text{ and } \ (g_2(|Du|))^{2\gamma-1} \leq 1+|Du|^{2\gamma}(g_2(|Du|))^{2\gamma-1}\,.$$

Thus from (4.52), for a further constants c' we deduce

$$\int_{\Omega} \eta^2 g_1(|Du|)\left|D^2u\right|^2 dx \leq c'\int_{\Omega}(\eta+|D\eta|)^2\left(1+|Du|^{2\gamma}(g_2(|Du|))^{2\gamma-1}\right) dx\,. \tag{4.53}$$

We consider concentric balls B_R, $B_{(R+\rho)/2}$ and B_ρ compactly contained in Ω, with $0 < \rho < \frac{R+\rho}{2} < R < R_1$. As usual we also consider a test function $\eta \in C_0^1(B_R)$, $0 \leq \eta \leq 1$ in B_R, $\eta = 1$ in B_ρ and $\eta = 0$ out of $B_{(R+\rho)/2}$, with pointwise gradient bound $|D\eta| \leq 4/(R-\rho)$ in $B_{(R+\rho)/2} \backslash B_\rho$ and of course in all B_R too. By (4.53) we have

$$\int_{B_\rho} g_1(|Du|)\left|D^2u\right|^2 dx \leq c'\left(1+\tfrac{4}{R-\rho}\right)^2\int_{B_{\frac{R+\rho}{2}}}\left(1+|Du|^{2\gamma}(g_2(|Du|))^{2\gamma-1}\right) dx\,.$$

Since $R \leq R_1$, then $\frac{4}{R-\rho} \geq \frac{4}{R_1}$ and thus $1 \leq \frac{R_1}{R-\rho}$. We obtain

$$\int_{B_\rho} g_1(|Du|)\left|D^2u\right|^2 dx \leq \frac{c''\left(R_1^2+4\right)^2}{(R-\rho)^2}\int_{B_{\frac{R+\rho}{2}}}\left(1+|Du|^{2\gamma}(g_2(|Du|))^{2\gamma-1}\right) dx. \tag{4.54}$$

We rewrite (4.48) in the balls $B_{(R+\rho)/2}$ and B_R, being (with the notation there) $R_0-\rho_0 = R-\frac{R+\rho}{2} = \frac{R-\rho}{2}$,

$$\int_{B_{\frac{R+\rho}{2}}}\left[1+|Du|^{2\gamma}(g_2(|Du|))^{2\gamma-1}\right] dx \leq \frac{2^{\theta_0}c}{(R-\rho)^{\theta_0}}\left(\int_{B_R}(1+f(Du))\,dx\right)^{\frac{2^*(\mu-1)}{(2\mu-2^*)\nu}}. \tag{4.55}$$

By combining (4.54),(4.55) we finally get

$$\int_{B_\rho} g_1(|Du|)\,\left|D^2u\right|^2\,dx \le \frac{c'''}{(R-\rho)^{2+\theta_0}}\left(\int_{B_R}(1+f(Du))\,dx\right)^{\frac{2^*(\mu-1)}{(2\mu-2^*)\nu}}.$$

for a constant c'''. Therefore the $W^{2,2}_{\mathrm{loc}}(\Omega)-$bound stated in (2.13) is obtained, with exponent $\theta_3=\frac{2^*(\mu-1)}{(2\mu-2^*)\nu}$, which is greater than 1 (see (4.51)).

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