

# ESTIMATES FOR SUMS OF EIGENVALUES OF THE FREE PLATE VIA THE FOURIER TRANSFORM

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ABSTRACT. Using the Fourier transform, we obtain upper bounds for sums of eigenvalues of the free plate.

## 1. INTRODUCTION AND MAIN RESULTS

For a bounded domain  $\Omega \subset \mathbb{R}^n$  with smooth boundary, the frequencies and modes of vibration for a free membrane of shape  $\Omega$  satisfy the Neumann eigenvalue problem

$$(1) \quad \begin{cases} -\Delta u = \mu u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $\Delta$  is the Laplace operator and  $\frac{\partial u}{\partial \nu}$  is the outer normal derivative. It is well known that the free membrane problem admits a spectrum of eigenvalues

$$0 = \mu_1(\Omega) < \mu_2(\Omega) \leq \mu_3(\Omega) \leq \dots \rightarrow +\infty.$$

Estimates for the eigenvalues  $\{\mu_j(\Omega)\}$  and for their sums in terms of the geometry of  $\Omega$  have been obtained by many authors (see [3, 4, 5, 10, 11, 12, 14, 15, 16, 17, 18], for instance; see also [13, 16, 19, 20] and the references therein for analogous estimates for the fixed membrane and [1, 2, 8, 9, 21, 23, 24] for analogous estimates for the clamped plate). For the purposes of this note, we simply recall the following estimate of Kröger [14] for sums of eigenvalues:

$$(2) \quad \sum_{j=1}^m \mu_j(\Omega) \leq (2\pi)^2 \left( \frac{n}{n+2} \right) \left( \frac{1}{\omega_n |\Omega|} \right)^{\frac{2}{n}} m^{\frac{n+2}{n}}, \quad m \geq 1,$$

and also the consequential estimate for eigenvalues:

$$(3) \quad \mu_{m+1}(\Omega) \leq (2\pi)^2 \left( \frac{n+2}{2\omega_n |\Omega|} \right)^{\frac{2}{n}} m^{\frac{2}{n}}, \quad m \geq 0.$$

Here  $|\Omega|$  denotes the volume of  $\Omega$  and  $\omega_n$  denotes the volume of the unit ball in  $\mathbb{R}^n$ .

The goal of the present paper is to establish analogous estimates to (2) and (3) for the free plate problem. With  $\Omega$  as above, the frequencies and modes of vibration for a free plate of shape  $\Omega$  are governed by the eigenvalue problem

$$(4) \quad \begin{cases} \Delta^2 u - \tau \Delta u = \Lambda u & \text{in } \Omega, \\ \frac{\partial^2 u}{\partial \nu^2} = 0 & \text{on } \partial\Omega, \\ \tau \frac{\partial u}{\partial \nu} - \operatorname{div}_{\partial\Omega} \left( \operatorname{Proj}_{T_x(\partial\Omega)} [(D^2 u) \nu] \right) - \frac{\partial \Delta u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

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where  $\Delta^2 u = \Delta(\Delta u)$  is the bilaplace operator,  $\tau \in \mathbb{R}$ ,  $\text{div}_{\partial\Omega}$  denotes the divergence operator for the surface  $\partial\Omega$ ,  $D^2 u$  denotes the Hessian matrix, and  $\text{Proj}_{T_x(\partial\Omega)}$  denotes the orthogonal projection of a vector from  $T_x \mathbb{R}^n$  onto the tangent space  $T_x(\partial\Omega)$ . In this paper, we study problem (4) when the parameter  $\tau \geq 0$ ; in this case, the eigenvalue problem for the free plate exhibits a nonnegative spectrum (see [6, 7])

$$0 = \Lambda_1(\Omega) \leq \Lambda_2(\Omega) \leq \Lambda_3(\Omega) \leq \dots \rightarrow +\infty.$$

We observe that constants are solutions to problem (4) with eigenvalue zero for any parameter  $\tau$ . If  $\tau = 0$ , the coordinate functions  $x_1, \dots, x_n$  are additional solutions with eigenvalue zero, and so the lowest eigenvalue is at least  $(n+1)$ -fold degenerate. When  $\tau > 0$ , we have a *free plate under tension* and here  $\Lambda_2(\Omega) > 0$  (see [6]).

Since problem (4) is the ‘‘plate analogue’’ of problem (1) (see Section 2 for further discussion), it is not surprising that the spectra of the two problems share similar properties. For instance, a classical result of Szegő [22] and Weinberger [25] states that among all domains with fixed volume, the lowest nonzero Neumann eigenvalue  $\mu_2(\Omega)$  is maximized by a ball. In a relatively recent and analogous result, Chasman has shown in [7] that among all domains with prescribed volume, the first nonzero eigenvalue  $\Lambda_2(\Omega)$  for a free plate under tension is maximized by a ball. The results of our paper shed additional light on the connection between problems (1) and (4). More precisely, we prove:

**Theorem 1.** *Let  $\Omega \subset \mathbb{R}^n$  be a smooth bounded domain with  $\{\Lambda_j(\Omega)\}$  the eigenvalues of the free plate problem (4). If  $\tau \geq 0$ , then*

$$\sum_{j=1}^m \Lambda_j(\Omega) \leq (2\pi)^4 \left(\frac{n}{n+4}\right) \left(\frac{1}{\omega_n |\Omega|}\right)^{\frac{4}{n}} m^{\frac{n+4}{4}} + \tau(2\pi)^2 \left(\frac{n}{n+2}\right) \left(\frac{1}{\omega_n |\Omega|}\right)^{\frac{2}{n}} m^{\frac{n+2}{n}}, \quad m \geq 1.$$

As a consequence of Theorem 1, we obtain the following eigenvalue estimates:

**Corollary 2.** *Let  $\Omega$  and  $\{\Lambda_j(\Omega)\}$  be as in Theorem 1. If  $\tau = 0$ , then*

$$\Lambda_{m+1}(\Omega) \leq (2\pi)^4 \left(\frac{n+4}{4\omega_n |\Omega|}\right)^{\frac{4}{n}} m^{\frac{4}{n}}, \quad m \geq 0,$$

while when  $\tau > 0$ , we have

$$\Lambda_{m+1}(\Omega) \leq \min_{r > 2\pi \left(\frac{m}{\omega_n |\Omega|}\right)^{\frac{1}{n}}} \frac{n\omega_n |\Omega| \left(\frac{r^{n+4}}{n+4} + \tau \frac{r^{n+2}}{n+2}\right)}{\omega_n |\Omega| r^n - m(2\pi)^n}, \quad m \geq 0.$$

The remainder of this note is divided into two sections. Section 2 presents a discussion of the boundary conditions of the free plate problem (4) while Section 3 presents proofs of the main results and further consequences.

## 2. FREE BOUNDARY CONDITIONS

To better understand the boundary conditions appearing in the plate problem (4), we return our attention to the membrane problem (1). The bilinear form for the membrane problem is given by

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx, \quad u, v \in H^1(\Omega).$$

To say that  $u \in H^1(\Omega)$  is a weak solution to problem (1) means that

$$(5) \quad a(u, v) = \mu \int_{\Omega} uv \, dx$$

for all functions  $v \in H^1(\Omega)$ . In particular, if  $u$  is a weak solution and  $u, v \in C^\infty(\overline{\Omega})$ , integrating by parts transforms equation (5) into

$$(6) \quad - \int_{\Omega} (\Delta u)v \, dx + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v \, dS = \mu \int_{\Omega} uv \, dx.$$

Since (6) holds for functions  $v = 0$  along the boundary  $\partial\Omega$ , we see that  $-\Delta u = \mu u$  in  $\Omega$  in the classical sense. Hence (6) becomes

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu} v \, dS = 0$$

for all functions  $v \in C^\infty(\overline{\Omega})$ , and we likewise deduce that  $\frac{\partial u}{\partial \nu} = 0$  in the classical sense along the boundary  $\partial\Omega$ . The term “free” in problem (1) comes the weak formulation (5), where functions in the space  $H^1(\Omega)$  have no prescribed behavior on the boundary. The boundary condition  $\frac{\partial u}{\partial \nu} = 0$  arises naturally from the weak formulation of our eigenvalue problem.

The bilinear form associated to the free plate problem (4) is given analogously by

$$A(u, v) = \sum_{j,k=1}^n \int_{\Omega} u_{x_j x_k} v_{x_j x_k} \, dx + \tau \int_{\Omega} \nabla u \cdot \nabla v \, dx, \quad u, v \in H^2(\Omega).$$

We say that  $u$  is a weak solution to problem (4) if

$$A(u, v) = \Lambda \int_{\Omega} uv \, dx$$

for each  $v \in H^2(\Omega)$ . Thus, if  $u$  is a weak solution and  $u, v \in C^\infty(\overline{\Omega})$ , integration by parts transforms the above equation into

$$(7) \quad \begin{aligned} A(u, v) &= \int_{\Omega} (\Delta^2 u - \tau \Delta u)v \, dx + \int_{\partial\Omega} \frac{\partial^2 u}{\partial \nu^2} \frac{\partial v}{\partial \nu} \, dS \\ &\quad + \int_{\partial\Omega} \left( \tau \frac{\partial u}{\partial \nu} - \operatorname{div}_{\partial\Omega} \left( \operatorname{Proj}_{T_x(\partial\Omega)}[(D^2 u)\nu] \right) - \frac{\partial \Delta u}{\partial \nu} \right) v \, dS \\ &= \Lambda \int_{\Omega} uv \, dx. \end{aligned}$$

For the details of this calculation, see [6]. Taking  $v \in C_c^\infty(\Omega)$  to be a test function, we see that  $\Delta^2 u - \tau \Delta u = \Lambda u$  in  $\Omega$  in the classical sense. Thus, equation (7) becomes

$$(8) \quad 0 = \int_{\partial\Omega} \frac{\partial^2 u}{\partial \nu^2} \frac{\partial v}{\partial \nu} \, dS + \int_{\partial\Omega} \left( \tau \frac{\partial u}{\partial \nu} - \operatorname{div}_{\partial\Omega} \left( \operatorname{Proj}_{T_x(\partial\Omega)}[(D^2 u)\nu] \right) - \frac{\partial \Delta u}{\partial \nu} \right) v \, dS.$$

Observe that any smooth function  $v \in C^\infty(\partial\Omega)$  can be extended to  $C^\infty(\overline{\Omega})$  with  $\frac{\partial v}{\partial \nu} = 0$  along the boundary  $\partial\Omega$ . Such an extension can be constructed, for example, by first extending  $v$  to be constant along the inner normal direction (for a small fixed distance) and then using a  $C^\infty$  bump function to extend  $v$  to the rest of  $\overline{\Omega}$ . This observation implies that each boundary integral in (8) vanishes separately, and arguing as before, we have that

$$\frac{\partial^2 u}{\partial \nu^2} = 0 \quad \text{and} \quad \tau \frac{\partial u}{\partial \nu} - \operatorname{div}_{\partial\Omega} \left( \operatorname{Proj}_{T_x(\partial\Omega)}[(D^2 u)\nu] \right) - \frac{\partial \Delta u}{\partial \nu} = 0$$

in the classical sense along  $\partial\Omega$ .

## 3. MAIN RESULTS

We begin this section with a proof of Theorem 1.

*Proof of Theorem 1.* We use some of the ideas contained in [14]. Let  $\phi_1, \dots, \phi_m$  denote an orthonormal set of eigenfunctions for  $\Lambda_1, \dots, \Lambda_m$  and define

$$\Phi(x, y) = \sum_{j=1}^m \phi_j(x) \phi_j(y), \quad x, y \in \Omega.$$

Let  $\widehat{\Phi}(z, y)$  denote the Fourier transform of  $\Phi$  in the variable  $x$ , so that

$$\widehat{\Phi}(z, y) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\Omega} \Phi(x, y) e^{ix \cdot z} dx.$$

Observe that

$$(2\pi)^{\frac{n}{2}} \widehat{\Phi}(z, y) = \sum_{j=1}^m \phi_j(y) \int_{\Omega} \phi_j(x) e^{ix \cdot z} dx$$

is the orthogonal projection of the function

$$h_z(x) = e^{ix \cdot z}$$

onto the subspace of  $L^2(\Omega)$  spanned by  $\phi_1, \dots, \phi_m$ . Using  $\rho(z, y) = h_z(y) - (2\pi)^{\frac{n}{2}} \widehat{\Phi}(z, y)$  as a trial function in the Rayleigh quotient for  $\Lambda_{m+1}$ , we have that

$$\Lambda_{m+1}(\Omega) \leq \frac{\sum_{j,k=1}^n \int_{\Omega} |\rho(z, y)_{y_j y_k}|^2 dy dz + \tau \sum_{j=1}^n \int_{\Omega} |\rho(z, y)_{y_j}|^2 dy dz}{\int_{\Omega} |\rho(z, y)|^2 dy}.$$

Multiplying both sides of the above inequality by the denominator and integrating over  $B_r = \{z \in \mathbb{R}^n : |z| < r\}$ , we see that

$$(9) \quad \Lambda_{m+1}(\Omega) \leq \inf_r \left\{ \frac{\sum_{j,k=1}^n \int_{B_r} \int_{\Omega} |\rho(z, y)_{y_j y_k}|^2 dy dz + \tau \sum_{j=1}^n \int_{B_r} \int_{\Omega} |\rho(z, y)_{y_j}|^2 dy dz}{\int_{B_r} \int_{\Omega} |\rho(z, y)|^2 dy dz} \right\} \\ = \inf_r \left\{ \frac{N}{D} \right\},$$

where the inf is taken over  $r > 2\pi \left( \frac{m}{\omega_n |\Omega|} \right)^{\frac{1}{n}}$ .

We first simplify the denominator  $D$  in the formula above. We observe that

$$D = I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= \int_{B_r} \int_{\Omega} |h_z(y)|^2 dy dz, \\ I_2 &= -2(2\pi)^{\frac{n}{2}} \operatorname{Re} \left\{ \int_{B_r} \int_{\Omega} h_z(y) \widehat{\Phi}(z, y) dy dz \right\}, \\ I_3 &= (2\pi)^n \int_{B_r} \int_{\Omega} |\widehat{\Phi}(z, y)|^2 dy dz. \end{aligned}$$

We evaluate each integral separately. Since  $|h_z(y)| = 1$ , we have

$$I_1 = \omega_n |\Omega| r^n.$$

Noting

$$(10) \quad \widehat{\Phi}(z, y) = \sum_{j=1}^m \phi_j(y) \widehat{\phi}_j(z),$$

we have

$$\begin{aligned} I_2 &= -2(2\pi)^{\frac{n}{2}} \operatorname{Re} \left\{ \sum_{j=1}^m \int_{B_r} \int_{\Omega} e^{iy \cdot z} \phi_j(y) \overline{\widehat{\phi}_j(z)} dy dz \right\} \\ &= -2(2\pi)^n \sum_{j=1}^m \int_{B_r} |\widehat{\phi}_j(z)|^2 dz. \end{aligned}$$

Invoking (10) again, the final denominator term simplifies to

$$\begin{aligned} I_3 &= (2\pi)^n \int_{B_r} \int_{\Omega} |\widehat{\Phi}(z, y)|^2 dy dz \\ &= (2\pi)^n \sum_{j,l=1}^m \int_{B_r} \int_{\Omega} \phi_j(y) \phi_l(y) \widehat{\phi}_j(z) \overline{\widehat{\phi}_l(z)} dy dz \\ &= (2\pi)^n \sum_{j=1}^m \int_{B_r} |\widehat{\phi}_j(z)|^2 dz. \end{aligned}$$

Thus,

$$(11) \quad D = \omega_n |\Omega| r^n - (2\pi)^n \sum_{j=1}^m \int_{B_r} |\widehat{\phi}_j(z)|^2 dz.$$

We next turn our attention to the numerator of (9). Observe that

$$N = J_1 + J_2 + J_3,$$

where

$$\begin{aligned}
J_1 &= \sum_{j,k=1}^n \int_{B_r} \int_{\Omega} |h_z(y)_{y_j y_k}|^2 dy dz + \tau \sum_{j=1}^n \int_{B_r} \int_{\Omega} |h_z(y)_{y_j}|^2 dy dz, \\
J_2 &= -2(2\pi)^{\frac{n}{2}} \operatorname{Re} \left\{ \sum_{j,k=1}^n \int_{B_r} \int_{\Omega} h_z(y)_{y_j y_k} \overline{\widehat{\Phi}(z, y)_{y_j y_k}} dy dz + \tau \sum_{j=1}^n \int_{B_r} \int_{\Omega} h_z(y)_{y_j} \overline{\widehat{\Phi}(z, y)_{y_j}} dy dz \right\}, \\
J_3 &= (2\pi)^n \left\{ \sum_{j,k=1}^n \int_{B_r} \int_{\Omega} |\widehat{\Phi}(z, y)_{y_j y_k}|^2 dy dz + \tau \sum_{j=1}^n \int_{B_r} \int_{\Omega} |\widehat{\Phi}(z, y)_{y_j}|^2 dy dz \right\}.
\end{aligned}$$

Since  $|h_z(y)_{y_j}| = |z_j|$  and  $|h_z(y)_{y_j y_k}| = |z_j||z_k|$ , we have

$$J_1 = \int_{B_r} \int_{\Omega} (|z|^4 + \tau|z|^2) dy dz = n\omega_n |\Omega| \left( \frac{r^{n+4}}{n+4} + \tau \frac{r^{n+2}}{n+2} \right).$$

To compute  $J_2$ , we combine identity (10) with the integration by parts formula in (7) to deduce

$$\begin{aligned}
J_2 &= -2(2\pi)^{\frac{n}{2}} \operatorname{Re} \left\{ \int_{B_r} \int_{\Omega} h_z(y) \overline{\Delta_y^2 \widehat{\Phi}(z, y)} dy dz - \tau \int_{B_r} \int_{\Omega} h_z(y) \overline{\Delta_y \widehat{\Phi}(y, z)} dy dz \right\} \\
&= -2(2\pi)^{\frac{n}{2}} \operatorname{Re} \left\{ \sum_{j=1}^m \Lambda_j \int_{B_r} \int_{\Omega} e^{iy \cdot z} \phi_j(y) \overline{\widehat{\phi}_j(z)} dy dz \right\} \\
&= -2(2\pi)^n \sum_{j=1}^m \Lambda_j \int_{B_r} |\widehat{\phi}_j(z)|^2 dz.
\end{aligned}$$

We finally compute  $J_3$  again using (7):

$$\begin{aligned}
J_3 &= (2\pi)^n \sum_{j,k=1}^n \int_{B_r} \int_{\Omega} \left( \sum_{l_1=1}^m \phi_{l_1}(y)_{y_j y_k} \widehat{\phi}_{l_1}(z) \right) \left( \sum_{l_2=1}^m \phi_{l_2}(y)_{y_j y_k} \overline{\widehat{\phi}_{l_2}(z)} \right) dy dz \\
&\quad + (2\pi)^n \tau \sum_{j=1}^n \int_{B_r} \int_{\Omega} \left( \sum_{l_1=1}^m \phi_{l_1}(y)_{y_j} \widehat{\phi}_{l_1}(z) \right) \left( \sum_{l_2=1}^m \phi_{l_2}(y)_{y_j} \overline{\widehat{\phi}_{l_2}(z)} \right) dy dz \\
&= (2\pi)^n \sum_{j,k=1}^n \sum_{l_1, l_2=1}^m \int_{B_r} \int_{\Omega} \phi_{l_1}(y)_{y_j y_k} \widehat{\phi}_{l_1}(z) \phi_{l_2}(y)_{y_j y_k} \overline{\widehat{\phi}_{l_2}(z)} dy dz \\
&\quad + (2\pi)^n \tau \sum_{j=1}^n \sum_{l_1, l_2=1}^m \int_{B_r} \int_{\Omega} \phi_{l_1}(y)_{y_j} \widehat{\phi}_{l_1}(z) \phi_{l_2}(y)_{y_j} \overline{\widehat{\phi}_{l_2}(z)} dy dz \\
&= (2\pi)^n \sum_{l_1, l_2=1}^m \Lambda_{l_1} \int_{B_r} \int_{\Omega} \phi_{l_1}(y) \widehat{\phi}_{l_1}(z) \phi_{l_2}(y) \overline{\widehat{\phi}_{l_2}(z)} dy dz \\
&= (2\pi)^n \sum_{l_1=1}^m \Lambda_{l_1} \int_{B_r} |\widehat{\phi}_{l_1}(z)|^2 dz.
\end{aligned}$$

We conclude that the numerator in (9) simplifies to

$$N = n\omega_n|\Omega| \left( \frac{r^{n+4}}{n+4} + \tau \frac{r^{n+2}}{n+2} \right) - (2\pi)^n \sum_{j=1}^m \Lambda_j \int_{B_r} |\widehat{\phi}_j(z)|^2 dz.$$

Combining the expression above for  $N$  with the expression for  $D$  in (11), we see that (9) becomes

$$(12) \quad \Lambda_{m+1}(\Omega) \leq \inf_r \left\{ \frac{\frac{n\omega_n|\Omega|}{(2\pi)^n} \left( \frac{r^{n+4}}{n+4} + \tau \frac{r^{n+2}}{n+2} \right) - \sum_{j=1}^m \Lambda_j \int_{B_r} |\widehat{\phi}_j(z)|^2 dz}{\frac{\omega_n|\Omega|r^n}{(2\pi)^n} - \sum_{j=1}^m \int_{B_r} |\widehat{\phi}_j(z)|^2 dz} \right\},$$

where we remind the reader that the inf is taken over  $r > 2\pi \left( \frac{m}{\omega_n|\Omega|} \right)^{\frac{1}{n}}$ . By Plancherel's Theorem,

$$(13) \quad \int_{B_r} |\widehat{\phi}_j(z)|^2 dz \leq 1$$

for each  $j$ . Moreover, since  $\tau \geq 0$ , all the eigenvalues  $\Lambda_j$  are nonnegative. Hence we may apply Lemma A1 in the Appendix to deduce

$$\sum_{j=1}^m \Lambda_j(\Omega) \leq \frac{n\omega_n|\Omega|}{(2\pi)^n} \left( \frac{r^{n+4}}{n+4} + \tau \frac{r^{n+2}}{n+2} \right), \quad r > 2\pi \left( \frac{m}{\omega_n|\Omega|} \right)^{\frac{1}{n}}.$$

Letting  $r \rightarrow 2\pi \left( \frac{m}{\omega_n|\Omega|} \right)^{\frac{1}{n}}$  gives the result.  $\square$

We next establish the estimate of Corollary 2.

*Proof of Corollary 2.* We return our attention to the estimate of (12). Combining with (13) we deduce

$$(14) \quad \Lambda_{m+1}(\Omega) \leq \frac{n\omega_n|\Omega| \left( \frac{r^{n+4}}{n+4} + \tau \frac{r^{n+2}}{n+2} \right)}{\omega_n|\Omega|r^n - m(2\pi)^n} = F(r), \quad r > 2\pi \left( \frac{m}{\omega_n|\Omega|} \right)^{\frac{1}{n}}.$$

Since

$$\lim_{r \rightarrow 2\pi \left( \frac{m}{\omega_n|\Omega|} \right)^{\frac{1}{n}}} F(r) = \lim_{r \rightarrow +\infty} F(r) = +\infty,$$

our first claim immediately follows.

In the case  $\tau = 0$ , it is easy to check that the derivative  $F'(r)$  vanishes precisely when

$$(\omega_n|\Omega|r^n - m(2\pi)^n) (n\omega_n|\Omega|r^{n+3}) - \left( n\omega_n|\Omega| \frac{r^{n+4}}{n+4} \right) (n\omega_n|\Omega|r^{n-1}) = 0$$

and solving this equation for  $r$  gives

$$r = 2\pi \left( \frac{m(n+4)}{4\omega_n|\Omega|} \right)^{\frac{1}{n}}.$$

Substituting this value of  $r$  into (14) gives the result.  $\square$

**Remark 3.** We make two observations when the parameter  $\tau = 0$ . First, our proof of Corollary 2 gives an alternative and elementary proof of Corollary 3.3 from [16] for the case  $l = 2$  without appealing to trace inequalities for convex functions of operators. Second, if  $\Lambda_{M+1}(\Omega)$  denotes the lowest nonzero free plate eigenvalue, then the estimate of Corollary 2 shows

$$C(n, |\Omega|) \sum_{m=M}^{\infty} \frac{1}{m^{\frac{4}{n}}} \leq \sum_{m=M}^{\infty} \frac{1}{\Lambda_{m+1}(\Omega)},$$

where  $C(n, |\Omega|)$  is a positive constant that depends on the dimension and volume of  $\Omega$ . Thus, the sum of the reciprocals of the nonzero eigenvalues for the free plate problem diverges when the dimension  $n$  is at least 4.

#### APPENDIX

In this section we establish a lemma used in the proof of Theorem 1. This lemma appears in [18]; we provide a proof so that our paper is self contained.

**Lemma A1.** Say  $0 \leq \Lambda_1 \leq \Lambda_2 \leq \dots \leq \Lambda_{m+1}$  are such that

$$(15) \quad \Lambda_{m+1} \leq \frac{a - \sum_{j=1}^m \Lambda_j c_j}{b - \sum_{j=1}^m c_j},$$

where  $a, b, c, c_j$  are positive numbers with  $c_j \leq c$ . If  $b > mc$ , then

$$c \sum_{j=1}^m \Lambda_j \leq a.$$

*Proof.* Inequality (15) becomes

$$\Lambda_{m+1} \left( b - \sum_{j=1}^m c_j \right) = a - \sum_{j=1}^m \Lambda_j c_j$$

and rearranging terms we have

$$\Lambda_{m+1} b - a = \sum_{j=1}^m (\Lambda_{m+1} - \Lambda_j) c_j \leq c \sum_{j=1}^m (\Lambda_{m+1} - \Lambda_j).$$

Solving the above inequality for  $c \sum_{j=1}^m \Lambda_j$  we see

$$c \sum_{j=1}^m \Lambda_j \leq a + (mc - b) \Lambda_{m+1}.$$

The result now follows from the assumption  $b > mc$ . □

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