Mediterr. J. Math. (2024) 21:138
https://doi.org/10.1007/s00009-024-02679-0

Mediterranean Journal
of Mathematics

Check for updates

Non-Nilpotent Leibniz Algebras with One-Dimensional Derived Subalgebra

Alfonso Di Bartolo, Gianmarco La Rosa and Manuel Mancini

Abstract. In this paper we study non-nilpotent non-Lie Leibniz $\mathbb{F}$-algebras with one-dimensional derived subalgebra, where $\mathbb{F}$ is a field with $\operatorname{char}(\mathbb{F}) \neq 2$. We prove that such an algebra is isomorphic to the direct sum of the two-dimensional non-nilpotent non-Lie Leibniz algebra and an abelian algebra. We denote it by L_n, where $n = \dim_{\mathbb{F}} L_n$. This generalizes the result found in Demir et al. (Algebras and Representation Theory 19:405-417, 2016), which is only valid when $\mathbb{F} = \mathbb{C}$. Moreover, we find the Lie algebra of derivations, its Lie group of automorphisms and the Leibniz algebra of biderivations of L_n. Eventually, we solve the *coquecigrue problem* for L_n by integrating it into a Lie rack.

Mathematics Subject Classification. 16W25, 17A32, 17B30, 17B40, 20M99, 22A30.

Keywords. Leibniz algebra, Lie algebra, derivation, biderivation, coquecigrue problem.

Introduction

Leibniz algebras were introduced by Loday in [19] as a non-skew symmetric version of Lie algebras. Earlier such algebraic structures were also considered by A. Blokh, who called them D-algebras [5] for their strict connection

The authors are supported by University of Palermo and by the "National Group for Algebraic and Geometric Structures, and their Applications" (GNSAGA - INdAM). The first author is also supported by UNIPA FFR 2024 VQR 2024. The second and third authors are also supported by the National Recovery and Resilience Plan (NRRP), Mission 4, Component 2, Investment 1.1, Call for tender No. 1409 published on 14/09/2022 by the Italian Ministry of University and Research (MUR), funded by the European Union—NextGenerationEU—Project Title Quantum Models for Logic, Computation and Natural Processes (QM4NP)—CUP B53D23030160001 —Grant Assignment Decree No. 1371 adopted on 01/09/2023 by the Italian Ministry of Ministry of University and Research (MUR), and by the Sustainability Decision Framework (SDF) Research Project—CUP B79J23000540005—Grant Assignment Decree No. 5486 adopted on 04/08/2023.

Birkhäuser

Published online: 07 June 2024

with derivations. Leibniz algebras play a significant role in different areas of mathematics and physics.

Many results of Lie algebras are still valid for Leibniz algebras. One of them is the *Levi decomposition*, which states that any Leibniz algebra over a field $\mathbb{F}$ of characteristic zero is the semidirect sum of its radical and a semisimple Lie algebra. This makes clear the importance of the problem of classification of solvable and nilpotent Lie/Leibniz algebras, which has been dealt with since the early 20th century (see [2–4,9–11,13] and [14], just for giving a few examples).

In [16] and [17] nilpotent Leibniz algebras L with one-dimensional derived subalgebra $[L, L]$ were studied and classified. It was proved that, up to isomorphism, there are three classes of *indecomposable* Leibniz algebras with these properties, namely the *Heisenberg* algebras $\mathfrak{l}^A_{2n+1}$, which are parameterized by their dimension $2n + 1$ and by a matrix A in canonical form, the *Kronecker* algebra $\mathfrak{k}_n$ and the *Dieudonné* algebra $\mathfrak{d}_n$, both parameterized by their dimension only. We want to complete this classification by studying non-nilpotent Leibniz $\mathbb{F}$-algebras with one-dimensional derived subalgebra, where $\mathbb{F}$ is a field with $\text{char}(\mathbb{F}) \neq 2$. Using the theory of non-abelian extensions of Leibniz algebras introduced in [18], we prove that a non-nilpotent non-Lie Leibniz algebra L with $\dim_{\mathbb{F}} L = n$ and $\dim_{\mathbb{F}}[L, L] = 1$ is isomorphic to the direct sum of the two-dimensional non-nilpotent non-Lie Leibniz algebra S_2, i.e. the algebra with basis $\{e_1, e_2\}$ and multiplication table given by $[e_2, e_1] = e_1$, and an abelian algebra of dimension $n - 2$. We denote it by L_n. This generalizes the result found in Theorem 2.6 of [11], where the authors proved that a *complex* non-split non-nilpotent non-Lie Leibniz algebra with one-dimensional derived subalgebra is isomorphic to S_2.

We study in detail the properties of the algebra L_n and we compute the Lie algebra of derivations $\text{Der}(L_n)$, its Lie group of automorphism $\text{Aut}(L_n)$ and the Leibniz algebra of biderivations $\text{Bider}(L_n)$.

Finally, we solve the *coquecigrue problem* for the Leibniz algebra L_n. We mean the problem, formulated by J.-L. Loday in [19], of finding a generalization of Lie third theorem to Leibniz algebras. Using M. K. Kinyon's results for the class of real *split Leibniz algebras* (see [15]), we show how to explicitly integrate L_n into a Lie rack defined over the vector space $\mathbb{R}^n$.

1. Preliminaries

We assume that $\mathbb{F}$ is a field with $\text{char}(\mathbb{F}) \neq 2$. For the general theory we refer to [1].

Definition 1.1. A *left Leibniz algebra* over $\mathbb{F}$ is a vector space L over $\mathbb{F}$ endowed with a bilinear map (called *commutator* or *bracket*) $[-, -] : L \times L \to L$ which satisfies the *left Leibniz identity*

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]], \quad \forall x, y, z \in L.$$

In the same way we can define a *right Leibniz algebra*, using the right Leibniz identity

$$[[x, y], z] = [[x, z], y] + [x, [y, z]], \quad \forall x, y, z \in L.$$

Given a left Leibniz algebra L, the multiplication $[x, y]^{\mathrm{op}} = [y, x]$ defines a right Leibniz algebra structure on L.

A Leibniz algebra that is both left and right is called *symmetric Leibniz algebra*. From now on we assume that $\dim_{\mathbb{F}} L < \infty$.

We have a full inclusion functor $i\colon \mathbf{Lie} \to \mathbf{Leib}$ that embeds Lie algebras over $\mathbb{F}$ into Leibniz algebras over $\mathbb{F}$. Its left adjoint is the functor $\pi\colon \mathbf{Leib} \to \mathbf{Lie}$, which associates to each Leibniz algebra L the quotient $L/\operatorname{Leib}(L)$, where $\operatorname{Leib}(L)$ is the smallest bilateral ideal of L such that the quotient $L/\operatorname{Leib}(L)$ becomes a Lie algebra. $\operatorname{Leib}(L)$ is defined as the subalgebra generated by all elements of the form $[x, x]$, for any $x \in L$, and it is called the *Leibniz kernel* of L.

We define the *left* and the *right center* of a Leibniz algebra

$$\mathrm{Z}_l(L) = \{x \in L \mid [x, L] = 0\}, \ \mathrm{Z}_r(L) = \{x \in L \mid [L, x] = 0\}.$$

The intersection of the left and right center is called the *center* of L and it is denoted by $\mathrm{Z}(L)$. In general for a left Leibniz algebra L, the left center $\mathrm{Z}_l(L)$ is a bilateral ideal, meanwhile the right center is not even a subalgebra. Furthermore, one can check that $\operatorname{Leib}(L) \subseteq \mathrm{Z}_l(L)$.

The definition of derivation for a Leibniz algebra is the same as in the case of Lie algebras.

Definition 1.2. A linear map $d\colon L \to L$ is a *derivation* of L if

$$d([x, y]) = [d(x), y] + [x, d(y)], \quad \forall x, y \in L.$$

An equivalent way to define a left Leibniz algebra L is to saying that the left adjoint maps $\mathrm{ad}_x = [x, -]$ are derivations. Meanwhile the right adjoint maps $\mathrm{Ad}_x = [-, x]$ are not derivations in general. The set $\operatorname{Der}(L)$ of all derivations of L is a Lie algebra with the usual bracket $[d, d'] = d \circ d' - d' \circ d$ and the set $\operatorname{Inn}(L)$ spanned by the left adjoint maps, which are called *inner derivations*, is an ideal of $\operatorname{Der}(L)$. Moreover $\operatorname{Aut}(L)$ is a Lie group and its Lie algebra is precisely $\operatorname{Der}(L)$.

In [19] Loday introduced the notion of anti-derivation and biderivation for a Leibniz algebra.

Definition 1.3. A linear map $D\colon L \to L$ is an *anti-derivation* of L if

$$D([x, y]) = [x, D(y)] - [y, D(x)], \quad \forall x, y \in L.$$

The space $\operatorname{ADer}(L)$ of anti-derivations of L has a $\operatorname{Der}(L)$-module structure with the extra multiplication $d \cdot D = d \circ D - D \circ d$, for any derivation d and for any anti-derivation D, and one can check that the right adjoint maps Ad_x are anti-derivations.

Definition 1.4. A *biderivation* of L is a pair $(d, D) \in \operatorname{Der}(L) \times \operatorname{ADer}(L)$ such that

$$[d(x) + D(x), y] = 0, \quad \forall x, y \in L.$$

The set $\mathrm{Bider}(L)$ of all biderivations of L has a Leibniz algebra structure with the bracket

$$[(d, D), (d', D')] = ([d, d'], d \cdot D')$$

and it is defined a Leibniz algebra homomorphism

$$L \to \mathrm{Bider}(L),\ x \mapsto (\mathrm{ad}_x, \mathrm{Ad}_x).$$

The pair $(\mathrm{ad}_x, \mathrm{Ad}_x)$ is called the *inner biderivation* associated with $x \in L$ and the set of all inner biderivations of L forms a Leibniz subalgebra of $\mathrm{Bider}(L)$.

We recall the definitions of solvable and nilpotent Leibniz algebras.

Definition 1.5. Let L be a left Leibniz algebra over $\mathbb{F}$ and let

$$L^0 = L,\ L^{k+1} = [L^k, L^k], \quad \forall k \geq 0$$

be the *derived series of* L. L is $n-$ *step solvable* if $L^{n-1} \neq 0$ and $L^n = 0$.

Definition 1.6. Let L be a left Leibniz algebra over $\mathbb{F}$ and let

$$L^{(0)} = L,\ L^{(k+1)} = [L, L^{(k)}], \quad \forall k \geq 0$$

be the *lower central series of* L. L is $n-$ *step nilpotent* if $L^{(n-1)} \neq 0$ and $L^{(n)} = 0$.

When L is two-step nilpotent, it lies in different varieties of non-associative algebras, such as associative, alternative and Zinbiel algebras. In this case we refer at L as a *two-step nilpotent algebra* and we have the following.

Proposition 1.7. (i) *If L is a two-step nilpotent algebra, then $L^{(1)} = [L, L] \subseteq \mathrm{Z}(L)$ and L is a symmetric Leibniz algebra.*

(ii) *If L is a left nilpotent Leibniz algebra with* $\dim_{\mathbb{F}}[L, L] = 1$, *then L is two-step nilpotent.*

In [16] the classification of nilpotent Leibniz algebras with one-dimensional derived subalgebra was established. The classification revealed that, up to isomorphism, there exist only three classes of indecomposable nilpotent Leibniz algebras of this type.

Definition 1.8 [16]. Let $f(x) \in \mathbb{F}[x]$ be a monic irreducible polynomial. Let $k \in \mathbb{N}$ and let $A = (a_{ij})_{i,j}$ be the companion matrix of $f(x)^k$. The *Heisenberg* algebra $\mathfrak{l}_{2n+1}^A$ is the $(2n+1)$-dimensional Leibniz algebra with basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n, z\}$ and the brackets are given by

$$[e_i, f_j] = (\delta_{ij} + a_{ij})z, \quad [f_j, e_i] = (-\delta_{ij} + a_{ij})z, \quad \forall i, j = 1, \ldots, n.$$

When A is the zero matrix, then we obtain the $(2n+1)-$dimensional Heisenberg Lie algebra $\mathfrak{h}_{2n+1}$.

Definition 1.9 [16]. Let $n \in \mathbb{N}$. The *Kronecker* algebra $\mathfrak{k}_n$ is the $(2n+1)$-dimensional Leibniz algebra with basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n, z\}$ and the brackets are given by

$$[e_i, f_i] = [f_i, e_i] = z, \quad \forall i = 1, \ldots, n$$
$$[e_i, f_{i-1}] = z, [f_{i-1}, e_i] = -z, \quad \forall i = 2, \ldots, n.$$

Definition 1.10 [16] Let $n \in \mathbb{N}$. The *Dieudonné* algebra $\mathfrak{d}_n$ is the $(2n+2)$-dimensional Leibniz algebra with basis $\{e_1, \ldots, e_{2n+1}, z\}$ and the brackets are given by

$$\begin{aligned}
&[e_1, e_{n+2}] = z, \\
&[e_i, e_{n+i}] = [e_i, e_{n+i+1}] = z, \quad \forall i = 2, \ldots, n, \\
&[e_{n+1}, e_{2n+1}] = z, \\
&[e_i, e_{i-n}] = z, \quad [e_i, e_{i-n-1}] = -z, \quad \forall i = n+2, \ldots, 2n+1.
\end{aligned}$$

We want to extend this classification by studying non-nilpotent Leibniz algebras with one-dimensional derived subalgebra.

2. Non-nilpotent Leibniz algebras with one-dimensional derived subalgebra

Let L be a non-nilpotent left Leibniz algebra over $\mathbb{F}$ with $\dim_{\mathbb{F}} L = n$ and $\dim_{\mathbb{F}} [L, L] = 1$. We observe that such an algebra is two-step solvable since the derived subalgebra $[L, L]$ is abelian.

It is well known that a non-nilpotent Lie algebra with one-dimensional derived subalgebra is isomorphic to the direct sum of the two-dimensional non-abelian Lie algebra and an abelian algebra (see [12, Sect. 3]). Thus we are interested in the classification of non-Lie Leibniz algebras with these properties.

In [11, Theorem 2.6] the authors prove that a *complex* non-split non-nilpotent non-Lie Leibniz algebra with one-dimensional derived subalgebra is isomorphic to the two-dimensional algebra with basis $\{e_1, e_2\}$ and multiplication table $[e_2, e_1] = [e_2, e_2] = e_1$. Here we generalize this result when $\mathbb{F}$ is a general field with $\mathrm{char}(\mathbb{F}) \neq 2$.

Proposition 2.1. *Let L be a non-nilpotent left Leibniz algebra over $\mathbb{F}$ with $\dim_{\mathbb{F}} [L, L] = 1$. Then L has a two-dimensional bilateral ideal S which is isomorphic to one of the following Leibniz algebras:*

(i) $S_1 = \langle e_1, e_2 \rangle$ *with* $[e_2, e_1] = -[e_1, e_2] = e_1$*;*
(ii) $S_2 = \langle e_1, e_2 \rangle$ *with* $[e_2, e_1] = [e_2, e_2] = e_1$*.*

Proof. Let $[L, L] = \mathbb{F}z$. Since L is not nilpotent, then

$$[L, [L, L]] \neq 0,$$

i.e. $z \notin \mathrm{Z}_r(L)$. Since $[L, L]$ is an abelian algebra, there exists a vector $x \in L$, which is linearly independent than z, such that $[x, z] \neq 0$. Thus

$$[x, z] = \gamma z,$$

for some $\gamma \in \mathbb{F}^*$. The subspace $S = \langle x, z \rangle$ is an ideal of L and it is not nilpotent: in fact

$$0 \neq \gamma z = [x, z] \in [S, [S, S]].$$

Thus S is a non-nilpotent Leibniz algebra. Using the classification of two-dimensional Leibniz algebras given by C. Cuvier in [8], S is isomorphic either to S_1 or to S_2. □

Remark 2.1. The algebras S_1 and S_2 are respectively the Leibniz algebras L_2 and L_4 of Sect. 3.1 in [1]. We observe that S_1 is a Lie algebra, meanwhile S_2 is a non-right left Leibniz algebra.

One can see L as an extension of the abelian algebra $L_0 = L/S \cong \mathbb{F}^{n-2}$ by S [18]

$$0 \longrightarrow S \xrightarrow{\ i\ } L \underset{s}{\overset{\pi}{\rightleftarrows}} L_0 \longrightarrow 0\ . \tag{1}$$

It turns out that there exists an equivalence of Leibniz algebra extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S & \xrightarrow{i_2} & L_0 \ltimes_\omega S & \underset{i_1}{\overset{\pi_1}{\rightleftarrows}} & L_0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}_S} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \mathrm{id}_B} & & \\ 0 & \longrightarrow & S & \xrightarrow{i} & L & \underset{\sigma}{\overset{\pi}{\rightleftarrows}} & L_0 & \longrightarrow & 0 \end{array}$$

where $L_0 \ltimes_\omega S$ is the Leibniz algebra defined on the direct sum of vector spaces $L_0 \oplus S$ with the bilinear operation given by

$$[(x,a),(y,b)]_{(l,r,\omega)} = (0, [a,b] + l_x(b) + r_y(a) + \omega(x,y)),$$

where

$$\omega(x,y) = [\sigma(x),\sigma(y)]_L - \sigma([x,y]_{L_0}) = [\sigma(x),\sigma(y)]_L$$

is the Leibniz algebra 2-cocycle associated with (1) and

$$l_x(b) = [\sigma(x), i(b)]_L, \ \ r_y(a) = [i(a), \sigma(y)]_L$$

define the action of L_0 on S; i_1, i_2, π_1 are the canonical injections and projection. The Leibniz algebra isomorphism θ is defined by $\theta(x,a) = \sigma(x) + i(a)$, for every $(x,a) \in L_0 \oplus S$.

By [18, Proposition 4.2], the 2-cocycle $\omega\colon L_0 \times L_0 \to S$ and the linear maps $l, r\colon L_0 \to \mathrm{gl}(S)$ must satisfy the following set of equations

(L1) $l_x([a,b]) = [l_x(a), b] + [x, l_x(b)]$;
(L2) $r_x([a,b]) = [a, r_x(b)] - [b, r_x(a)]$;
(L3) $[l_x(a) + r_x(a), b] = 0$;
(L4) $[l_x, l_y]_{\mathrm{gl}(S)} - l_{[x,y]_{L_0}} = \mathrm{ad}_{\omega(x,y)}$;
(L5) $[l_x, r_y]_{\mathrm{gl}(S)} - r_{[x,y]_{L_0}} = \mathrm{Ad}_{\omega(x,y)}$;
(L6) $r_y(r_x(a) + l_x(a)) = 0$;
(L7) $l_x(\omega(y,z)) - l_y(\omega(x,z)) - r_z(\omega(x,y))$
$= \omega([x,y]_{L_0}, z) - \omega(x, [y,z]_{L_0}) + \omega(y, [x,z]_{L_0})$

for any $x, y \in L_0$ and for any $a, b \in S$. Notice that these equations where also studied in [6] in the case of Leibniz algebra *split extensions*.

Remark 2.2. The first three equations state that the pair (l_x, r_x) is a biderivation of the Leibniz algebra S, for any $x \in L_0$. Biderivations of low-dimensional Leibniz algebras were classified in [20] and it turns out that

- $\mathrm{Bider}(S_1) = \{(d, -d) \mid d \in \mathrm{Der}(S_1)\}$ and
  $\mathrm{Der}(S_1) = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} \middle| \ \alpha, \beta \in \mathbb{F} \right\}$;

- $\mathrm{Bider}(S_2) = \left\{ \left(\begin{pmatrix} \alpha & \alpha \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \right) \middle| \alpha, \beta \in \mathbb{F} \right\}$.

We study now in detail the non-abelian extension (1) in both cases that S is isomorphic either to S_1 or to S_2.

2.1. S is a Lie algebra

When $S \cong S_1$, we have that $r_y = -l_y$, for any $y \in L_0$ and the bilinear operation of $L_0 \ltimes_\omega S_1$ becomes

$$[(x, a), (y, b)]_{(l,\omega)} = (0, [a, b] + l_x(b) - l_y(a) + \omega(x, y)).$$

The linear map l_x is represented by a 2×2 matrix

$$\begin{pmatrix} \alpha_x & \beta_x \\ 0 & 0 \end{pmatrix}$$

with α_x,$\beta_x \in \mathbb{F}$. From equations (L4)-(L5) it turns out that

$$\omega(x, y) = (\alpha_x \beta_y - \alpha_y \beta_x)e_1, \quad \forall x, y \in L_0$$

and the 2-cocycle ω is skew-symmetric. Moreover, equations (L6)-(L7) are automatically satisfied and the resulting algebra $L_0 \ltimes_\omega S_1 \cong L$ is a Lie algebra. We conclude that L is isomorphic to the direct sum of S_1 and $L_0 \cong \mathbb{F}^{n-2}$.

2.2. S is not a Lie algebra

With the change of basis $e_2 \mapsto e_2 - e_1$, S_2 becomes the Leibniz algebra with basis $\{e_1, e_2\}$ and the only non-trivial bracket given by $[e_2, e_1] = e_1$. Now a biderivation of S_1 is represented by a pair of matrices

$$\left(\begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \right)$$

with $\alpha, \beta \in \mathbb{F}$ and the pair $(l_x, r_x) \in \mathrm{Bider}(S_2)$ is defined by $l_x(e_1) = \alpha_x e_1$ and $r_x(e_2) = \beta_x e_1$, for any $x \in L_0$.

Equation (L4) states that $[l_x, l_y]_{\mathrm{gl}(S_2)} = [\omega(x, y), -]$, with

$$\begin{aligned} [l_x, l_y]_{\mathrm{gl}(S_2)} = l_x \circ l_y - l_y \circ l_x &= \begin{pmatrix} \alpha_x & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_y & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \alpha_y & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_x & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \alpha_x \alpha_y & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \alpha_x \alpha_y & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

for any $x, y \in L_0$. Thus $\omega(x, y) \in \mathrm{Z}_l(S_2) = \mathbb{F}e_1$.

From equation (L5) we have $[l_x, r_y]_{\mathrm{gl}(S_2)} = [-, \omega(x, y)]_{S_2}$, with

$$[l_x, r_y]_{\mathrm{gl}(S_2)} = l_x \circ r_y - r_y \circ l_x = \begin{pmatrix} 0 & \alpha_x \beta_y \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha_x \beta_y \\ 0 & 0 \end{pmatrix}.$$

Thus, for every $a = a_1 e_1 + a_2 e_2 \in S_2$ and for every $x, y \in L_0$, we have

$$[a, \omega(x, y)] = [l_x, r_y](a) = \alpha_x \beta_y a_2 e_1,$$

i.e. $\omega(x, y) = \alpha_x \beta_y e_1$. Finally, equations (L6) and (L7) are identically satisfied.

Summarizing we have

$$\begin{cases} l_x \equiv \begin{pmatrix} \alpha_x & 0 \\ 0 & 0 \end{pmatrix} \\ r_y \equiv \begin{pmatrix} 0 & \beta_y \\ 0 & 0 \end{pmatrix} \\ \omega(x,y) = \alpha_x \beta_y e_1 \end{cases}$$

for every $x, y \in L_0$ and the bilinear operation $[-,-]_{(l,r,\omega)}$ becomes

$$[(x,a),(y,b)]_{(l,r,\omega)} = (0, (a_2 b_1 + \alpha_x b_1 + \beta_y a_2 + \alpha_x \beta_y) e_1),$$

for any $x, y \in L_0$ and for any $a = a_1 e_1 + a_2 e_2$, $b = b_1 e_1 + b_2 e_2 \in S_2$.

If we fix a basis $\{f_3, \ldots, f_n\}$ of L_0 and we denote by

$$\alpha_i = \alpha_{f_i}, \quad \beta_i = \beta_{f_i}, \quad \forall i = 3, \ldots, n$$

then L is isomorphic to the Leibniz algebra with basis $\{e_1, e_2, f_3, \ldots, f_n\}$ and non-zero brackets

$$\begin{aligned} &[e_2, e_1] = e_1 \\ &[e_2, f_i] = \beta_i e_1, \quad \forall i = 3, \ldots, n \\ &[f_i, e_1] = \alpha_i e_1, \quad \forall i = 3, \ldots, n \\ &[f_i, f_j] = \alpha_i \beta_j e_1, \quad \forall i, j = 3, \ldots, n. \end{aligned}$$

With the change of basis $f_i \mapsto f_i' = \dfrac{f_i}{\beta_i} - e_1$, if $\beta_i \neq 0$, we obtain that

$$\begin{aligned} &[e_2, f_i'] = e_1 - [e_2, e_1] = 0, \\ &[f_i', e_1] = \gamma_i e_1, \quad \text{where } \gamma_i = \frac{\alpha_i}{\beta_i}, \\ &[f_i, f_j'] = \alpha_i e_1 - [f_i, e_1] = 0, \\ &[f_i', f_j'] = \gamma_i e_1 - \frac{1}{\beta_i}[f_i, e_1] = 0. \end{aligned}$$

If we denote again $f_i \equiv f_i'$ and $\alpha_i \equiv \gamma_i$ when $\beta_i \neq 0$, then L has basis $\{e_1, e_2, f_3, \ldots, f_n\}$ and non-trivial brackets

$$[e_2, e_1] = e_1, \quad [f_i, e_1] = \alpha_i e_1, \quad \forall i = 3, \ldots, n.$$

Finally, when $\alpha_i \neq 0$, we can operate the change of basis

$$f_i \mapsto \frac{f_i}{\alpha_i} - e_2.$$

One can check that the only non-trivial bracket now is $[e_2, e_1] = e_1$ and L is isomorphic to the direct sum of S_2 and the abelian algebra $L_0 \cong \mathbb{F}^{n-2}$. This allows us to conclude with the following.

Theorem 2.2. *Let* $\mathbb{F}$ *be a field with* $\mathrm{char}(\mathbb{F}) \neq 2$. *Let* L *be a non-nilpotent non-Lie left Leibniz algebra over* $\mathbb{F}$ *with* $\dim_{\mathbb{F}} L = n$ *and* $\dim_{\mathbb{F}}[L, L] = 1$. *Then* L *is isomorphic to the direct sum of the two-dimensional non-nilpotent non-Lie Leibniz algebra* S_2 *and an abelian algebra of dimension* $n - 2$. *We denote this algebra by* L_n. □

If we suppose that L is a *non-split* algebra, i.e. L cannot be written as the direct sum of two proper ideals, then we obtain the following result, that is a generalization of [11, Theorem 2.6] and which is valid over a general field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$.

Corollary 2.3. *Let L be a non-split non-nilpotent non-Lie left Leibniz algebra over $\mathbb{F}$ with $\dim_{\mathbb{F}} L = n$ and $\dim_{\mathbb{F}}[L, L] = 1$. Then $n = 2$ and $L \cong S_2$.* □

Now we study in detail the algebra $L_n = S_2 \oplus \mathbb{F}^{n-2}$ by describing the Lie algebra of derivations, its Lie group of automorphisms and the Leibniz algebra of biderivations. Moreover, when $\mathbb{F} = \mathbb{R}$, we solve the *coquegigrue problem* (see [7] and [15]) for L_n by integrating it into a Lie rack.

2.3. Derivations, Automorphisms and Biderivations of L_n

Let $n \geq 2$ and let $L_n = S_2 \oplus \mathbb{F}^{n-2}$. We fix the basis $\mathcal{B}_n = \{e_1, e_2, f_3, \ldots, f_n\}$ of L_n and we recall that the only non-trivial commutator is $[e_2, e_1] = e_1$. A straightforward application of the algorithm proposed in [20] for finding derivations and anti-derivations of a Leibniz algebra as pair of matrices with respect to a fixed basis produces the following.

Theorem 2.4. (i) *A derivation of L_n is represented, with respect to the basis $\mathcal{B}_n$, by a matrix*

$$\left(\begin{array}{cc|cccc} \alpha & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \hline 0 & a_3 & & & & \\ 0 & a_4 & & & & \\ \vdots & \vdots & & & A & \\ 0 & a_n & & & & \end{array}\right)$$

where $A \in \mathrm{M}_{n-2}(\mathbb{F})$.

(ii) *The group of automorphisms $\mathrm{Aut}(L_n)$ is the Lie subgroup of $\mathrm{GL}_n(\mathbb{F})$ of matrices of the form*

$$\left(\begin{array}{cc|cccc} \beta & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ \hline 0 & b_3 & & & & \\ 0 & b_4 & & & & \\ \vdots & \vdots & & & B & \\ 0 & b_n & & & & \end{array}\right)$$

where $\beta \neq 0$ and $B \in \mathrm{GL}_{n-2}(\mathbb{F})$.

(iii) *The Leibniz algebra of biderivations of L_n consists of the pairs (d, D) of linear endomorphisms of L_n which are represented by the pair of* *matrices*

$$\left(\left(\begin{array}{cc|cccc} \alpha & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \hline 0 & a_3 & & & & \\ 0 & a_4 & & & & \\ \vdots & \vdots & & & A & \\ 0 & a_n & & & & \end{array}\right), \left(\begin{array}{cc|cccc} 0 & \alpha' & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \hline 0 & a'_3 & & & & \\ 0 & a'_4 & & & & \\ \vdots & \vdots & & & A' & \\ 0 & a'_n & & & & \end{array}\right)\right)$$

where $A, A' \in \mathrm{M}_{n-2}(\mathbb{F})$. □

3. The Integration of the Leibniz Algebra L_n

The *coquecigrue problem* is the problem formulated by Loday in [19] of finding a generalization of Lie third theorem to Leibniz algebras. Given a real Leibniz algebra L, one wants to find a manifold endowed with a smooth map, which plays the role of the adjoint map for Lie groups, such that the tangent space at a distinguished element, endowed with the differential of this map, gives a Leibniz algebra isomorphic to L. Moreover, when L is a Lie algebra, we want to obatin the simply connected Lie group associated with L. From now on, we assume that the underlying field of any algebra is $\mathbb{F} = \mathbb{R}$.

In [15] M. K. Kinyon shows that it is possible to define an algebraic structure, called *rack*, whose operation, differentiated twice, defines on its tangent space at the unit element a Leibniz algebra structure.

Definition 3.1. A *rack* is a set X with a binary operation $\rhd \colon X \times X \to X$ which is left autodistributive

$$x \rhd (y \rhd z) = (x \rhd y) \rhd (x \rhd z), \quad \forall x, y, z \in X$$

and such that the left multiplications $x \rhd -$ are bijections.

A rack is *pointed* if there exists an element $1 \in X$ such that $1 \rhd x = x$ and $x \rhd 1 = 1$, for any $x \in X$.

A rack is a *quandle* if the binary operation $\rhd$ is idempotent.

The first example of a rack is any group G endowed with its conjugation

$$x \rhd y = xyx^{-1}, \quad \forall x, y \in G.$$

We denote this rack by $\mathrm{Conj}(G)$ and we observe that it is a quandle.

Definition 3.2. A pointed rack $(X, \rhd, 1)$ is said to be a *Lie rack* if X is a smooth manifold, $\rhd$ is a smooth map and the left multiplications are diffeomorphisms.

M. K. Kinyon proved that the tangent space $\mathrm{T}_1 X$ at the unit element 1 of a Lie rack X, endowed with the bilinear operation

$$[x, y] = \left.\frac{\partial^2}{\partial s \partial t}\right|_{s,t=0} \gamma_1(s) \rhd \gamma_2(t)$$

where $\gamma_1, \gamma_2 \colon [0, 1] \to X$ are smooth paths such that $\gamma_1(0) = \gamma_2(0) = 1$, $\gamma_1'(0) = x$ and $\gamma_2'(0) = y$, is a Leibniz algebra.

He also solved the coquecigrue problem for the class of *split Leibniz algebras*. Here a Leibniz algebra is said to be *split* if there exists an ideal

$$\mathrm{Leib}(L) \subseteq I \subseteq \mathrm{Z}_l(L)$$

and a Lie subalgebra M of L such that $L \cong (M \oplus I, \{-,-\})$, where the bilinear operation $\{-,-\}$ is defined by

$$\{(x,a),(y,b)\} = ([x,y], \rho_x(b))$$

and $\rho\colon M \times I \to I$ is the action on the M-module I. L is said to be the *demisemidirect product* of M and I. More precisely, we have the following.

Theorem 3.3 [15]. *Let L be a split Leibniz algebra. Then a Lie rack integrating L is $X = (H \oplus I, \rhd)$, where H is the simply connected Lie group integrating M and the binary operation is defined by*

$$(g,a) \rhd (h,b) = (ghg^{-1}, \phi_g(b)),$$

where ϕ is the exponentiation of the Lie algebra action ρ.

Some years later S. Covez generalized M. K. Kinyon's results proving that every real Leibniz algebra admits an integration into a *Lie local rack* (see [7]). More recently it was showed in [16] that the integration proposed by S. Covez is global for any nilpotent Leibniz algebra. Moreover, when a Leibniz algebra L is integrated into a Lie quandle X, it turns out that L is a Lie algebra and $X = \mathrm{Conj}(G)$, where G is the simply connected Lie group integrating L.

Our aim here is to solve the coquecigrue problem for the non-nilpotent Leibniz algebra $L_n = S_2 \oplus \mathbb{F}^{n-2}$. One can check that S_2 is a split Leibniz algebra, in the sense of M. K. Kinyon, with $I = \mathrm{Z}_l(S_2) \cong \mathbb{R}$ and $M \cong \mathbb{R}$. Thus $L \cong (\mathbb{R}^2, \{-,-\})$ with the bilinear operation defined by

$$\{(x_1,x_2),(y_1,y_2)\} = (0, \rho_{x_1}(y_2))$$

and $\rho_{x_1}(y_2) = x_1 y_2$, for any $x_1, y_2 \in \mathbb{R}$. It turns out that a Lie rack integrating S_2 is $(\mathbb{R}^2, \rhd)$, where

$$(x_1,x_2) \rhd (y_1,y_2) = (y_1, y_2 + e^{x_1} y_2).$$

and the unit element is $(0,0)$. Finally, one can check that the binary operation

$$(x_1,x_2,x_3,\dots,x_n) \rhd (y_1,y_2,y_3,\dots,y_n) = (y_1, y_2 + e^{x_1} y_2, y_3, \dots, y_n)$$

defines on $\mathbb{R}^n$ a Lie rack structure with unit element $1 = (0,\dots,0)$, such that $(\mathrm{T}_1\,\mathbb{R}^n, \rhd)$ is a Leibniz algebra isomorphic to L_n. This result, combined with the ones of [16, Section 4], completes the classification of Lie racks whose tangent space at the unit element gives a Leibniz algebra with one-dimensional derived subalgebra.

Author contributions All the authors contributed equally to the manuscript.

Funding Open access funding provided by Università degli Studi di Palermo within the CRUI-CARE Agreement. Open access funding provided by Università degli Studi di Palermo within the CRUI-CARE Agreement.

Data Availability No datasets were generated or analysed during the current study.

Declarations

Conflict of interest Not applicable. There is no Conflict of interest.

References

[1] Ayupov, S., Omirov, B., Rakhimov, I.: Leibniz Algebras: Structure and Classification. CRC Press (2019). (**ISBN: 9781000740004**)

[2] Bartolone, C., Di Bartolo, A., Falcone, G.: Nilpotent Lie algebras with 2-dimensional commutator ideals. Linear Algebra and its Applications **434**(3), 650–656 (2011). https://doi.org/10.1016/j.laa.2010.09.036

[3] Bartolone, C., Di Bartolo, A., Falcone, G.: Solvable extensions of nilpotent complex Lie algebras of type n,1,1}. Moscow Math. J. **18**(4), 607–616 (2018). https://doi.org/10.17323/1609-4514-2018-18-4-607-616

[4] Biggs, R., Falcone, G.: A class of nilpotent Lie algebras admitting a compact subgroup of automorphisms. Differ. Geometry Appl. **54**, 251–263 (2017). https://doi.org/10.1016/j.difgeo.2017.04.009.

[5] Blokh, A.: A generalization of the concept of a Lie algebra. Dokl. Akad. Nauk SSSR **165**(3), 471–473 (1965)

[6] Cigoli, A.S., Mancini, M., Metere, G.: On the representability of actions of Leibniz algebras and Poisson algebras. Proc Edinburgh Math. Soc **66**(4), 998–1021 (2023). https://doi.org/10.1017/S0013091523000548

[7] Covez, S.: The local integration of Leibniz algebras. Annales de l'Institut Fourier **63**(1), 1–35 (2013). https://doi.org/10.5802/aif.2754

[8] Cuvier, C.: Algèbres de Leibnitz: définitions, propriétés. Annales scientifiques de l' É cole Normale Sup é rieure **27**(1), 1–45 (1994)

[9] Demir, I.: Classification of some subclasses of 6-dimensional nilpotent Leibniz algebras. Turkish J. Math. **44**(5), 1012–1018 (2020). https://doi.org/10.3906/mat-2002-69

[10] Demir, I., Kailash, K.C., Misra, C., Stitzinger, E.: On classification of four-dimensional nilpotent Leibniz algebras. Commun. Algebra **45**(3), 1012–1018 (2017). https://doi.org/10.1080/00927872.2016.1172626

[11] Demir, I., Misra, K.C., Stitzinger, E.: Classification of some solvable Leibniz algebras. Algebras Representation Theory **19**, 405–417 (2016). https://doi.org/10.1007/s10468-015-9580-5

[12] Erdmann, K., Wildon, M.J.: Introduction to Lie Algebras. Springer, London (2006). (**ISBN: 9781846284908**)

[13] Ignatyev, M.V., Kaygorodov, I., Popov, Y.: The geometric classification of 2-step nilpotent algebras and applications. Revista Matem á tica Complutense **35**(3), 907–922 (2022). https://doi.org/10.1007/s13163-021-00411-0

[14] Khudoyberdiyev, A. Kh., Rakhimov, I. S., Said Husain, Sh. K.: On classification of 5-dimensional solvable Leibniz algebras. Linear Algebra and its Applications **457**(27), 428–454 (2014). https://doi.org/10.1016/j.laa.2014.05.034

[15] Kinyon, M.K.: Leibniz algebras, Lie racks, and digroups. J. Lie Theory **17**(1), 99–114 (2007)

[16] La Rosa, G., Mancini, M.: Two-step nilpotent Leibniz algebras. Linear Algebra Appl. **637**(7), 119–137 (2022). https://doi.org/10.1016/j.laa.2021.12.013

[17] La Rosa, G., Mancini, M.: Derivations of two-step nilpotent algebras. Commun. Algebra **51**(12), 4928–4948 (2023). https://doi.org/10.1080/00927872.2023.2222415

[18] Liu, J., Sheng, Y., Wang, Q.: On non-abelian extensions of Leibniz algebras. Communications in Algebra **46**(2), 574–587 (2018). https://doi.org/10.1080/00927872.2017.1324870

[19] Loday, J.-L.: Une version non commutative des algebres de Lie: les algebres de Leibniz. L'Enseignement Math é matique **39**(3–4), 269–293 (1993)

[20] Mancini, M.: Biderivations of low-dimensional Leibniz algebras. In: Albuquerque, H., Brox, J., Mart í nez, C., Saraiva (eds.), P.: Non-Associative Algebras and Related Topics. NAART 2020. Springer Proceedings in Mathematics & Statistics 427.8 , pp. 127–136. (2023). https://doi.org/10.1007/978-3-031-32707-0_8

Alfonso Di Bartolo, Gianmarco La Rosa and Manuel Mancini
Dipartimento di Matematica e Informatica
Università degli Studi di Palermo
Via Archirafi 34
90123 Palermo
Italy
e-mail: alfonso.dibartolo@unipa.it

Gianmarco La Rosa
e-mail: gianmarco.larosa@unipa.it

Manuel Mancini
e-mail: manuel.mancini@unipa.it

Received: April 17, 2024.
Revised: April 17, 2024.
Accepted: May 19, 2024.