Some recent results on singular p-Laplacian systems

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Abstract

Some recent existence, multiplicity, and uniqueness results for singular p-Laplacian systems either in bounded domains or in the whole space are presented, with a special attention to the case of convective reactions. A extensive bibliography is also provided.

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1 Introduction

This survey paper can be divided into two parts. The first one (cf. Section 3) treats *singular quasi-linear Dirichlet systems in bounded domains.* So, we

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study problems of the type

$$\begin{cases} -\Delta_p u = f(x,u,v,\nabla u,\nabla v) \ \text{ in } \Omega, \ u>0 \ \text{ in } \Omega, \ u\lfloor_{\partial\Omega}= 0, \\ -\Delta_q v = g(x,u,v,\nabla u,\nabla v) \ \text{ in } \Omega, \ v>0 \ \text{ in } \Omega, \ v\lfloor_{\partial\Omega}= 0, \end{cases} \tag{1.1}$$

where $1 < p, q < \infty$, the symbol Δ_r denotes the r-Laplace operator, namely

$$\Delta_r u := \operatorname{div}(|\nabla u|^{r-2}\nabla u),$$

Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 3$, having a smooth boundary $\partial\Omega$, while $f, g : \Omega \times (\mathbb{R}^+)^2 \times (\mathbb{R}^N)^2 \to \mathbb{R}$ fulfill Carathéodory's conditions and are singular at zero with respect the solution (u, v) or even its gradient $(\nabla u, \nabla v)$.

If $p = q = 2$ then various special, often non-convective, cases of (1.1) have been thoroughly investigated; see Section 3.1 below. In particular, the monograph [33] gives a nice introduction to so-called singular Gierer-Meinhardt systems. Here, we simply make a short account on some recent existence, multiplicity, or uniqueness results when $(p, q) \neq (2, 2)$, as well as the relevant technical approaches. Regarding this, let us also point out Chapter 7 of [57].

The second part (cf. Section 4) carries out a similar analysis for *singular quasi-linear systems in the whole space*. Hence, it deals with situations like

$$\begin{cases} -\Delta_p u = f(x,u,v,\nabla u,\nabla v) \ \text{ in } \mathbb{R}^N, \ u>0 \ \text{ in } \mathbb{R}^N, \\ -\Delta_q v = g(x,u,v,\nabla u,\nabla v) \ \text{ in } \mathbb{R}^N, \ v>0 \ \text{ in } \mathbb{R}^N. \end{cases} \tag{1.2}$$

The literature on problem (1.2) looks not fully exhaustive. For instance, two basic questions seem to be still open. Precisely,

- the existence of multiple solutions, even for semi-linear non-convective systems, and
- the uniqueness of solutions.

Moreover, as far as we know, no previous book or survey is already available.

Both parts contain four sub-sections. The first represents a historical sketch of the semi-linear setting. The next two address existence, multiplicity, and uniqueness in the non-convective case. The fourth is devoted to singular systems with convection.

It may be useful to emphasize that [38] treats non-local singular elliptic problems while [66, 71] pertain singular parabolic systems. For the sake of brevity, they are not examined here.

We apologize in advance for possibly forgetting significant works, but the production on (1.1)–(1.2) is by now extensive and our knowledge is somewhat limited.

2 Basic notation

Let $X(\Omega)$ be a real-valued function space on a nonempty measurable set $\Omega \subseteq \mathbb{R}^N$. If $u_1, u_2 \in X(\Omega)$ and $u_1(x) < u_2(x)$ a.e. in Ω then we simply write $u_1 < u_2$. The meaning of $u_1 \leq u_2$, etc. is analogous. Put

$$X(\Omega)_+ := \{u \in X(\Omega) : u \geq 0\}.$$

The symbol $u \in X_{\text{loc}}(\Omega)$ means that $u : \Omega \to \mathbb{R}$ and $u\lfloor_K \in X(K)$ for all nonempty compact subset K of Ω. Given $1 < r < \infty$, define

$$r' := \frac{r}{r-1}.$$

We denote by $\lambda_{1,r}$ the first eigenvalue of the operator $-\Delta_r$ in $W_0^{1,r}(\Omega)$. If $r < N$ then

$$r^* := \frac{Nr}{N-r}.$$

Let us next recall the notion and some relevant properties of the so-called Beppo Levi spaces $\mathcal{D}_0^{1,r}(\mathbb{R}^N)$, systematically studied for the first time by Deny and Lions [25]. Set

$$\mathcal{D}^{1,r} := \left\{z \in L^1_{\text{loc}}(\mathbb{R}^N) : |\nabla z| \in L^r(\mathbb{R}^N)\right\}$$

and write $\mathcal{R}$ for the equivalence relation that identifies two elements in $\mathcal{D}^{1,r}$ whose difference is a constant. The quotient set $\dot{\mathcal{D}}^{1,r}$, endowed with the norm

$$\|u\|_{1,r} := \left(\int_{\mathbb{R}^N} |\nabla u(x)|^r \mathrm{d}x\right)^{1/r},$$

turns out complete. Indicate with $\mathcal{D}_0^{1,r}(\mathbb{R}^N)$ the subspace of $\dot{\mathcal{D}}^{1,r}$ defined as the closure of $C_0^\infty(\mathbb{R}^N)$ under $\|\cdot\|_{1,r}$, namely

$$\mathcal{D}_0^{1,r}(\mathbb{R}^N) := \overline{C_0^\infty(\mathbb{R}^N)}^{\|\cdot\|_{1,r}}.$$

$\mathcal{D}_0^{1,r}(\mathbb{R}^N)$, usually called Beppo Levi space, is reflexive and continuously embeds in $L^{r^*}(\mathbb{R}^N)$, i.e.,

$$\mathcal{D}_0^{1,r}(\mathbb{R}^N) \hookrightarrow L^{r^*}(\mathbb{R}^N). \tag{2.1}$$

Consequently, if $u \in \mathcal{D}_0^{1,r}(\mathbb{R}^N)$ then u vanishes at infinity, meaning that the set $\{x \in \mathbb{R}^N : |u(x)| \geq \varepsilon\}$ has finite measure for any $\varepsilon > 0$. In fact, by Chebichev's inequality and (2.1), one has

$$|\{x \in \mathbb{R}^N : |u(x)| \geq \varepsilon\}| \leq \varepsilon^{-r^*} \|u\|_{r^*}^{r^*} \leq (c\varepsilon^{-1}\|u\|_{1,r})^{r^*} < +\infty,$$

where $c > 0$ is the best constant related to (2.1). The monographs [28, 52, 69] provide an exhaustive introduction on this topic.

Finally, $\mathbb{R}^+ :=]0, \infty[$ and

$$a \vee b := \max\{a, b\}, \quad a^+ := a \vee 0, \quad a \wedge b := \min\{a, b\} \qquad \forall\, a, b \in \mathbb{R}.$$

If Ω is a bounded domain in $\mathbb{R}^N$ then

$$d(x) := \operatorname{dist}(x, \partial\Omega), \quad x \in \overline{\Omega}, \tag{2.2}$$

while $\Omega' \subset\subset \Omega$ means $\overline{\Omega'} \subseteq \Omega$.

3 Problems in bounded domains

3.1 The case $p = 2$

As far as we know, the study of singular semi-linear elliptic systems in bounded domains started with the paper [14], devoted to Gierer-Meinhardt's type problem (see [37])

$$\begin{cases} -\Delta u = -u + \dfrac{u}{v} \ \text{ in } \Omega, \ u > 0 \ \text{ in } \Omega, \ u\lfloor_{\partial\Omega} = 0, \\ -\Delta v = -\alpha v + \dfrac{u}{v} \ \text{ in } \Omega, \ v > 0 \ \text{ in } \Omega, \ v\lfloor_{\partial\Omega} = 0, \end{cases} \tag{3.1}$$

where Ω denotes a smooth bounded domain in $\mathbb{R}^N$, $N \geq 1$, while $\alpha \in \mathbb{R}^+$. More general right-hand sides were then considered in [15, 48, 32]. The work [56] investigates the system

$$\begin{cases} -\Delta u = \lambda u^{q_1} - \dfrac{u^{p_1}}{v^{\beta_1}} \ \text{ in } \Omega, \ u > 0 \ \text{ in } \Omega, \ u\lfloor_{\partial\Omega} = 0, \\ -\Delta v = \mu v^{q_2} - \dfrac{u^{p_2}}{v^{\beta_2}} \ \text{ in } \Omega, \ v > 0 \ \text{ in } \Omega, \ v\lfloor_{\partial\Omega} = 0. \end{cases} \tag{3.2}$$

Here, $q_1, q_2, \beta_1, \beta_2 \in]0, 1[$, $p_1, p_2 \in \mathbb{R}^+$, and λ, μ are two positive parameters. Define

$$\sigma := \frac{p_2(1 - q_2)}{(1 + \beta_2)(1 - q_1)}, \quad \eta := \frac{\beta_1(1 - q_1)}{(1 - p_1)(1 - q_2)}.$$

An adequate sub-super-solution method yields the following

Theorem 3.1 ([56], Theorem 1.1)**.** *If $q_1 < p_1$ then there exists $c_1 > 0$ such that* (3.2) *admits a solution $(u, v) \in C^{1,\alpha}(\overline{\Omega})^2$ for all $\lambda > 0$, $\mu \geq c_1 \lambda^\sigma$.*
If $q_1 \geq p_1$ then there exists $c_2 > 0$ such that (3.2) *has no solution once $\mu > 0$ and $\lambda < c_2 \mu^{-\eta}$.*

Two years before, in 2008, Hernandez, Mancebo, and Vega [45] established the existence of classical solutions to the problem

$$\begin{cases} \mathcal{L}_1 u = f(x,u,v) \text{ in } \Omega, \ u > 0 \text{ in } \Omega, \ u\lfloor_{\partial\Omega} = 0, \\ \mathcal{L}_2 v = g(x,u,v) \text{ in } \Omega, \ v > 0 \text{ in } \Omega, \ v\lfloor_{\partial\Omega} = 0, \end{cases} \tag{3.3}$$

where $\mathcal{L}_i$ denotes a linear, second-order, uniformly elliptic operator in non-divergence form while $f, g : \Omega \times (\mathbb{R}^+)^2 \to \mathbb{R}$ are smooth enough. They examined both the cooperative and the non-cooperative situations; see [45, Theorems 2.1–2.2]. Here, cooperative means that the reaction $f(x,\cdot,\cdot)$ and $g(x,\cdot,\cdot)$ turn out increasing on $\mathbb{R}^+$ with respect to each variable separately. A *uniqueness result* is also obtained provided (3.3) turns out cooperative and concave, namely

$$f(x,\tau s,\tau t) > \tau f(x,s,t), \quad g(x,\tau s,\tau t) > \tau g(x,s,t)$$

for all $\tau \in]0,1[$ and $(x,s,t) \in \Omega \times (\mathbb{R}^+)^2$; cf. [45, Theorem 2.3]. Many special cases are finally discussed. Singular systems driven by other linear, second-order, elliptic operators in divergence form were treated in [8].

Let us next point out that the paper [44] contains a uniqueness result (cf. [44, Theorem 2.2]) where no cooperative structure is supposed.

In 2015, Ghergu [31] thoroughly investigated both existence and uniqueness of classical solutions to the model system

$$\begin{cases} -\Delta u = u^{\alpha_1} + v^{\beta_1} \text{ in } \Omega, \ u > 0 \text{ in } \Omega, \ u\lfloor_{\partial\Omega} = 0, \\ -\Delta v = v^{\alpha_2} + u^{\beta_2} \text{ in } \Omega, \ v > 0 \text{ in } \Omega, \ \ v\lfloor_{\partial\Omega} = 0 \end{cases} \tag{3.4}$$

for $\alpha_i \vee \beta_i < 0$, $i = 1,2$. The work [55] (see also [30]) contains a similar analysis on the problem

$$\begin{cases} -\Delta u = K_1(x) u^{\alpha_1} v^{\beta_1} \text{ in } \Omega, \ u > 0 \text{ in } \Omega, \ u\lfloor_{\partial\Omega} = 0, \\ -\Delta v = K_2(x) v^{\alpha_2} u^{\beta_2} \text{ in } \Omega, \ v > 0 \text{ in } \Omega, \ v\lfloor_{\partial\Omega} = 0, \end{cases} \tag{3.5}$$

where, as before, $\alpha_i \vee \beta_i < 0$ and $K_i \in C^{0,\alpha}(\Omega)$, $i = 1,2$, with $\alpha \in (0,1)$.

The existence of solutions to singular convective elliptic systems was firstly studied by Alves and Moussaoui [2] in 2014.

Theorem 3.2 ([2], Sections 3–4)**.** *Let* $-\alpha_i, \beta_i \in [0,1[$ *and* $g_i \in C^0(\mathbb{R}^{2N}, \mathbb{R}^+)$ *be bounded,* $i = 1,2$*. Then the problem*

$$\begin{cases} -\Delta u = v^{\alpha_1} \pm v^{\beta_1} + g_1(\nabla u, \nabla v) \ \textit{in } \Omega, \ u > 0 \ \textit{in } \Omega, \ u\lfloor_{\partial\Omega} = 0, \\ -\Delta v = u^{\alpha_2} \pm u^{\beta_2} + g_2(\nabla u, \nabla v) \ \textit{in } \Omega, \ v > 0 \ \textit{in } \Omega, \ v\lfloor_{\partial\Omega} = 0 \end{cases}$$

admits a solution $(u,v) \in H_0^1(\Omega)^2 \cap C^2(\Omega)^2$.

Finally, the very recent work [18] treats quasi-linear Schrödinger elliptic problems with both singular and convective reactions, while [11] concerns singular systems having quadratic gradient terms.

3.2 Existence and multiplicity

To shorten notation, given $p, q \in]1, \infty[$, write

$$\begin{aligned} X^{p,q}(\Omega) &:= W^{1,p}(\Omega) \times W^{1,q}(\Omega), \\ X^{p,q}_{\mathrm{loc}}(\Omega) &:= W^{1,p}_{\mathrm{loc}}(\Omega) \times W^{1,q}_{\mathrm{loc}}(\Omega), \\ X^{p,q}_0(\Omega) &:= W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega). \end{aligned}$$

Moreover, if $(u_1, v_1), (u_2, v_2) \in X^{p,q}_{\mathrm{loc}}(\Omega)$ then $(u_1, v_1) \leq (u_2, v_2)$ means both $u_1 \leq u_2$ and $v_1 \leq v_2$.

To the best of our knowledge, the first existence result for singular quasi-linear problems dates back to 2007 and examines the case

$$\begin{cases} -\Delta_p u = v^{\alpha_1} + v^{\beta_1} \text{ in } \Omega, \ u > 0 \text{ in } \Omega, \ u\lfloor_{\partial\Omega} = 0, \\ -\Delta_q v = u^{\alpha_2} + u^{\beta_2} \text{ in } \Omega, \ v > 0 \text{ in } \Omega, \ v\lfloor_{\partial\Omega} = 0. \end{cases} \tag{3.6}$$

Here, $\alpha_i < 0 < \beta_i$, while $N \geq 2$.

Theorem 3.3 ([1], Theorem 1.1)**.** *If* $\frac{2N}{N+1} \leq p, q < N$ *and* $0 < -\alpha_i, \beta_i < \theta_i$, $i = 1, 2$, *with*

$$\theta_1 := \min \left\{ 1, p-1, \frac{q'}{p'} \right\}, \quad \theta_2 := \min \left\{ 1, q-1, \frac{p'}{q'} \right\},$$

then (3.6) *has a weak solution* $(u, v) \in X^{p,q}_0(\Omega)$.

Its proof employs a result on nonlinear eigenvalue problems with lack of bifurcation due to Rabinowitz [65] and a Hardy–Sobolev type inequality [46].

Now, consider the general problem

$$\begin{cases} -\Delta_p u = f(x, u, v) \text{ in } \Omega, \ u > 0 \text{ in } \Omega, \ u\lfloor_{\partial\Omega} = 0, \\ -\Delta_q v = g(x, u, v) \text{ in } \Omega, \ v > 0 \text{ in } \Omega, \ v\lfloor_{\partial\Omega} = 0, \end{cases} \tag{3.7}$$

where $f, g : \Omega \times (\mathbb{R}^+)^2 \to \mathbb{R}$ fulfill Carathéodory's conditions. In 2012, El Manouni, Perera, and Shivaji [27] investigated (3.7) under the assumptions below.

(a_1) The functions $t \mapsto f(x, s, t)$ and $s \mapsto g(x, s, t)$ are increasing in $\mathbb{R}^+$, namely system (3.7) is *cooperative*.

(a$_2$) For every $0 < s_0 \leq s_1$ and $0 < t_0 \leq t_1$ one has

$$\sup_{\Omega\times[s_0,s_1]\times]0,t_1]} f(x,s,t) < \infty, \qquad \sup_{\Omega\times]0,s_1]\times[t_0,t_1]} g(x,s,t) < \infty,$$

as well as

$$\sup_{\Omega\times[s_0,s_1]\times[t_0,t_1]} \max\{|f(x,s,t)|, |g(x,s,t)|\} < \infty.$$

They seek solutions $(u, v) \in X^{p,q}_{\mathrm{loc}}(\Omega)$ that satisfy the differential equations in the sense of distributions, i.e.,

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \,\mathrm{d}x = \int_\Omega f(\cdot, u, v)\varphi \,\mathrm{d}x \quad \forall\, \varphi \in C_0^\infty(\Omega),$$
$$\int_\Omega |\nabla v|^{q-2}\nabla v \cdot \nabla\psi \,\mathrm{d}x = \int_\Omega g(\cdot, u, v)\psi \,\mathrm{d}x \quad \forall\, \psi \in C_0^\infty(\Omega),$$

and, moreover, $u, v \in C^0(\overline{\Omega})$.

A pair $(\underline{u}, \underline{v}) \in X^{p,q}(\Omega)$ is called a sub-solution of problem (3.7) provided $f(\cdot, \underline{u}, \underline{v}) \in L^{p'}(\Omega)$, $g(\cdot, \underline{u}, \underline{v}) \in L^{q'}(\Omega)$, and

$$-\Delta_p \underline{u} \leq f(\cdot, \underline{u}, \underline{v}) \text{ in } \Omega, \quad -\Delta_p \underline{v} \leq g(\cdot, \underline{u}, \underline{v}) \text{ in } \Omega, \ \underline{u} \vee \underline{v} \leq 0 \text{ on } \partial\Omega.$$

A super-solution $(\overline{u}, \overline{v})$ is defined similarly, by reversing all the above inequalities.

Let $\{\varepsilon_n\} \subseteq \mathbb{R}^+$ satisfy $\varepsilon_n \to 0$. Set, for every $n \in \mathbb{N}$, $(x, s, t) \in \Omega \times (\mathbb{R}^+)^2$,

$$f_n(x,s,t) := f(x, s \vee \varepsilon_n, t \vee \varepsilon_n), \quad g_j(x,s,t) := g(x, s \vee \varepsilon_n, t \vee \varepsilon_n),$$

and consider the sequence of regularized systems

$$\begin{cases} -\Delta_p u = f_n(x,u,v) \text{ in } \Omega, \ u\lfloor_\Omega = 0, \\ -\Delta_q v = g_n(x,u,v) \text{ in } \Omega, \ v\lfloor_\Omega = 0. \end{cases} \tag{3.8}$$

Theorem 3.4 ([27], Theorem 3.1)**.** *Suppose* (a$_1$)–(a$_2$) *hold. If, for each* $n \in \mathbb{N}$*, there exist a sub-solution* $(\underline{u}_n, \underline{v}_n)$ *and a super-solution* $(\overline{u}_n, \overline{v}_n)$ *to* (3.8) *such that* $(\underline{u}_n, \underline{v}_n) \leq (\overline{u}_n, \overline{v}_n)$,

$$\inf_{n\in\mathbb{N}} \operatorname*{ess\,inf}_{\Omega'} (\underline{u}_n \wedge \underline{v}_n) > 0$$

whenever $\Omega' \subset\subset \Omega$*, and*

$$\sup_{n\in\mathbb{N}} \operatorname*{ess\,sup}_{\Omega} (\overline{u}_n \vee \overline{v}_n) < \infty,$$

then (3.7) *admits a distributional solution* $(u, v) \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)^2 \cap C^0(\overline{\Omega})^2$.

Sufficient conditions for the existence of sub-super-solution pairs to (3.8) are given in [27, Sections 4–5]. As an example, via Theorem 3.4 one can show that the model problem (cf. (3.4))

$$\begin{cases} -\Delta_p u = u^{\alpha_1} + \mu v^{\beta_1} \text{ in } \Omega, \ u > 0 \text{ in } \Omega, \ u\lfloor_{\partial\Omega} = 0, \\ -\Delta_q v = v^{\alpha_2} + \mu u^{\beta_2} \text{ in } \Omega, \ v > 0 \text{ in } \Omega, \ v\lfloor_{\partial\Omega} = 0, \end{cases}$$

where $\alpha_1 \vee \alpha_2 < 0$ and $\beta_1 \wedge \beta_2 \geq 0$, possesses a solution provided $\mu \geq 0$ is small enough. Moreover, singular *semi-positone systems*, which means both $\lim_{s\to 0^+} f(x,s,t) = -\infty$ uniformly in (x,t) and $\lim_{t\to 0^+} g(x,s,t) = -\infty$ uniformly with respect to (x,s), are investigated. Let us also mention the papers [51], [47], and [16]. In particular, [16] deals with a non-cooperative system.

One year later, Giacomoni, Hernandez, and Sauvy [35] obtained the general results below, where $f, g \in C^1(\Omega \times (\mathbb{R}^+)^2)$, through a different notion of sub-super-solution.

We say that $(\underline{u}, \underline{v}), (\overline{u}, \overline{v}) \in X^{p.q}_{\text{loc}}(\Omega) \cap C^0(\overline{\Omega})^2$ are a sub-super-solution pair to (3.7) when $\underline{u}$, $\underline{v}$, $\overline{u}$, $\overline{v}$ are locally uniformly positive, $(\underline{u}, \underline{v}) \leq (\overline{u}, \overline{v})$, and for any $(u,v) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$ one has

$$-\Delta_p \underline{u} \leq f(\cdot, \underline{u}, v), \ -\Delta_p \underline{v} \leq g(\cdot, u, \underline{v}), \ -\Delta_p \overline{u} \geq f(\cdot, \overline{u}, v), \ -\Delta_p \overline{v} \geq g(\cdot, u, \overline{v})$$

in Ω. The set $\mathcal{C} := [\underline{v}, \overline{v}] \times [\underline{u}, \overline{u}]$ is usually called *trapping region*; cf. [10].

Theorem 3.5 ([35], Theorem 2.1)**.** *Let* $(\underline{u}, \underline{v}), (\overline{u}, \overline{v}) \in X^{p,q}_0(\Omega)$ *be a sub-super-solution pair to* (3.7) *fulfilling:*

- (a$_3$) $\overline{u} \leq c_1 d^{\gamma_1}$ *and* $\overline{v} \leq c_2 d^{\gamma_2}$*, with appropriate* $c_1, c_2, \gamma_1, \gamma_2 \in \mathbb{R}^+$*.*
- (a$_4$) *There exist* $c_3, c_4 \in \mathbb{R}^+$*,* $\delta_1, \delta_2 \in \mathbb{R}$ *such that*

$$|f(\cdot, u, v)| \leq c_3 d^{\delta_1}, \ \ |g(\cdot, u, v)| \leq c_4 d^{\delta_2} \ \ \forall\, (u,v) \in \mathcal{C}.$$

- (a$_5$) *For suitable* $c_5, c_6, \sigma_1, \sigma_2 \in \mathbb{R}^+$ *one has*

$$\left|\frac{\partial f}{\partial s}(\cdot, u, v)\right| \leq c_5 d^{\delta_1 - \sigma_1}, \ \left|\frac{\partial g}{\partial t}(\cdot, u, v)\right| \leq c_6 d^{\delta_2 - \sigma_2}, \ (u,v) \in \mathcal{C}.$$

Here, d *is given by* (2.2)*. Assume further that*

$$\delta_1 > -2 + \frac{1}{p} + (\sigma_1 - \gamma_1)^+, \ \ \delta_2 > -2 + \frac{1}{q} + (\sigma_2 - \gamma_2)^+.$$

Then (3.7) *admits a weak solution* $(u,v) \in \mathcal{C}$*.*

Theorem 3.6 ([35], Theorem 2.3)**.** *Suppose* $(\underline{u}, \underline{v}), (\overline{u}, \overline{v})$ *is a sub-super-solution pair to* (3.7) *complying with* (a_3)*. If*

$$\frac{\partial f}{\partial t}(x, s, t) > 0, \ \ \frac{\partial g}{\partial s}(x, s, t) > 0 \quad \forall\, (x, s, t) \in \Omega \times (\mathbb{R}^+)^2 \tag{3.9}$$

and there exist $c_7, c_8 \in \mathbb{R}^+$*,* $\eta_1, \eta_2 \in \mathbb{R}$ *such that*

$$\left|\frac{\partial f}{\partial s}(\cdot, u, v)\right| \le c_7 d^{\eta_1}, \quad \left|\frac{\partial g}{\partial t}(\cdot, u, v)\right| \le c_8 d^{\eta_2} \ \ \forall\, (u, v) \in \mathcal{C},$$

for d *as in* (2.2)*, then* (3.7) *possesses a distributional solution* $(u, v) \in \mathcal{C}$*.*

Proofs are based on a very nice, non-trivial use of Schauder's fixed point theorem. Applications to the model problem

$$\begin{cases} -\Delta_p u = K_1(x) u^{\alpha_1}\, v^{\beta_1} \ \text{ in } \Omega, \ u > 0 \ \text{ in } \Omega, \ u\lfloor_{\partial\Omega} = 0, \\ -\Delta_q v = K_2(x) v^{\alpha_2}\, u^{\beta_2} \ \text{ in } \Omega, \ v > 0 \ \text{ in } \Omega, \ v\lfloor_{\partial\Omega} = 0, \end{cases} \tag{3.10}$$

where K_1, K_2 satisfy appropriate conditions while

$$\alpha_1 < p - 1, \ \ \alpha_2 < q - 1, \ \ (p - 1 - \alpha_1)(q - 1 - \alpha_2) > |\beta_1 \beta_2| > 0, \tag{3.11}$$

are furnished. Evidently, (3.10) becomes (3.5) for $p = q = 2$. See also [21], where $p = q < N$, $K_1 \in L^\infty(\Omega)_+$, $K_2 \in L^\gamma(\Omega)_+$ for some $\gamma > \frac{N}{p}$,

$$0 < \beta_1 < 1 \wedge (p - 1), \ \ -1 < \alpha_1 < p - \beta_1 - 1, \ \ \alpha_2 := \beta_1 - 1, \ \ \beta_2 := \alpha_1 + 1\,.$$

A special case of (3.10) was treated in [58] (cf. in addition [20]), namely

$$\begin{cases} -\Delta_p u = u^{\alpha_1}\, v^{\beta_1} \ \text{ in } \Omega, \ u > 0 \ \text{ in } \Omega, \ u\lfloor_{\partial\Omega} = 0, \\ -\Delta_q v = v^{\alpha_2}\, u^{\beta_2} \ \text{ in } \Omega, \ v > 0 \ \text{ in } \Omega, \ v\lfloor_{\partial\Omega} = 0. \end{cases} \tag{3.12}$$

Theorem 3.7 ([58], Theorem 1.1)**.** *Let* $1 < p, q \le N$ *and let* α_i, β_i *satisfy*

$$\begin{gathered} -2 + \frac{1}{p} \le \alpha_1 < 0, \ \ -2 + \frac{1}{q} \le \alpha_2 < 0, \\ 0 < \beta_1 < \frac{q}{p}(p - 1 - \alpha_1), \ \ 0 < \beta_2 < \frac{p}{q}(q - 1 - \alpha_2). \end{gathered} \tag{3.13}$$

Then (3.12) *has a weak solution* $(u, v) \in X_0^{p,q}(\Omega) \cap L^\infty(\Omega)^2$*.*

Remark 3.8. It should be noted that (3.13) forces (3.11). Moreover, both (3.9) and (3.13) basically entail (a_1).

System (3.12), with a *competitive* interaction between the two components u and v was thoroughly studied by Giacomoni, Schindler, and Takáč [36]. Recall that (3.7) is said to be competitive if

(a'_1) The functions $t \mapsto f(x,s,t)$ and $s \mapsto g(x,s,t)$ are decreasing on $\mathbb{R}^+$.

In the situation above this means $\beta_1 \vee \beta_2 < 0$. Under the *sub-homogeneity condition*

$$(p-1-\alpha_1)(q-1-\alpha_2) > \beta_1\beta_2 \tag{3.14}$$

and suitable upper bounds regarding α_i, β_i, they proved that (3.12) admits a solution $(u,v) \in X_0^{p,q}(\Omega) \cap C^{0,\alpha}(\overline{\Omega})^2$. When $p = q$, see also [70].

Existence results for problem (3.7) where the competitive structure (a'_1) is allowed can be found in [59, 60].

The recent work [4] examines singular $(p(x), q(x))$-Laplacian problems with singularity coming through logarithmic reactions that involve variable exponents growth conditions; cf. also [3, 63].

As far as we know, till today, much less attention has been paid to *multiplicity of solutions*. Actually, we can only mention the papers [68, 12, 26, 5, 50]. The first deals with singular $p(x)$-Laplacian systems while the second is devoted to quasi-linear problems driven by (Φ_1, Φ_2)-Laplace operators. Theorem 1 in [26] considers the case when f, g do not depend on x, are positive, entail a cooperative structure, and fulfill

$$\lim_{s\to+\infty} \frac{f(s,t)}{s^{p-1}} = J_1 > \lambda_{1,p}, \quad \lim_{t\to+\infty} \frac{g(s,t)}{t^{q-1}} = J_2 > \lambda_{1,q}.$$

Two smooth solutions are obtained combining sub-super-solution methods with the Leray-Schauder topological degree. A different approach is adopted in [5]. The differential operators, which include the r-Laplacian as a special case, are neither homogeneous nor linear, while

$$f(x,s,t) := K_1(x)s^{\alpha_1} + \frac{\partial h}{\partial s}(x,s,t), \quad g(x,s,t) := K_2(x)t^{\alpha_2} + \frac{\partial h}{\partial t}(x,s,t),$$

where $K_i \in L^\infty(\Omega)$, $\alpha_i < 0$, and $h \in C^1(\overline{\Omega} \times \mathbb{R}^2)$ satisfy appropriate conditions that permit the use of variational methods. Theorem 1.7 of [5] gives two positive solutions provided $\max_{i=1,2} \|K_i\|_\infty$ is sufficiently small. Finally, [50] addresses singular $p(x)$-Laplacian systems with nonlinear boundary conditions.

3.3 Uniqueness

Except for the semi-linear case, uniqueness of solutions looks a difficult matter, even for problem (3.12). In fact, as pointed out by Giacomoni, Hernandez, and Moussaoui [34], equations with quasi-linear elliptic operators exhibit

additional troubles to obtain the validity of the strong comparison principle, which requires the C^1-regularity of solutions. If it cannot be obtained (as in the strongly singular setting $\alpha_i + \beta_i < -1$) then one can still try to use a suitable variant of the well known Krasnoleskii's argument [49].
Let us first examine the cooperative case $\beta_1 \wedge \beta_2 > 0$. Proposition 3.1 of [34] basically yields

Theorem 3.9 ([34], Theorem 3.2)**.** *If* $-1 < \alpha_i + \beta_i < 0 < \beta_i$*,* $i = 1, 2$*, then* (3.12) *admits a unique weak solution* $(u, v) \in X_0^{p,q}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})^2$*. Moreover,* $u, v \geq c_1 d$ *for some* $c_1 > 0$*.*

Suppose now

$$\alpha_1 + \beta_i < -1, \ i = 1, 2, \ \ 0 < \beta_1 < p - 1, \ \ 0 < \beta_2 < q - 1, \tag{3.15}$$

and denote by $\gamma, \theta \in]0, 1[$ the unique solution to the system

$$\begin{cases} (\gamma - 1)(p - 1) - 1 = \alpha_1 \gamma + \beta_1 \theta \\ (\theta - 1)(q - 1) - 1 = \beta_2 \gamma + \alpha_2 \theta. \end{cases} \tag{3.16}$$

Theorem 3.10 ([34], Theorem 3.3)**.** *Let* (3.15)–(3.16) *be satisfied. Assume also that*

$$\alpha_1 \gamma + \beta_1 \theta > \frac{1}{p} - 2, \ \ \beta_2 \gamma + \alpha_2 \theta > \frac{1}{q} - 2. \tag{3.17}$$

Then (3.12) *possesses a unique weak solution* $(u, v) \in X_0^{p,q}(\Omega)$ *fulfilling*

$$c_2(\varphi_{1,p}^{\gamma}, \varphi_{1,q}^{\theta}) \leq (u, v) \leq c_3(\varphi_{1,p}^{\gamma}, \varphi_{1,q}^{\theta})$$

with appropriate $c_2, c_3 \in \mathbb{R}^+$*.*

We next examine the competitive case $\beta_i < 0$. Via the sub-homogeneity condition (3.14), Theorem 2.2 of [36] considers various possible choices of exponents. Here, for the sake of brevity, we will present only one.

Theorem 3.11 ([36], Theorem 2.2)**.** *Suppose* $\alpha_i \vee \beta_i < 0$*. Under* (3.14)*,* (3.16)*, and* (3.17)*, the same conclusion of Theorem 3.10 is true.*

3.4 Systems with convection terms

In 2017, Motreanu, Moussaoui, and Zhang [61] treated the general problem

$$\begin{cases} -\Delta_p u = f(u, v, \nabla u, \nabla v) \ \text{ in } \Omega, \ u > 0 \ \text{ in } \Omega, \ u\lfloor_{\partial\Omega} = 0, \\ -\Delta_q v = g(u, v, \nabla u, \nabla v) \ \text{ in } \Omega, \ v > 0 \ \text{ in } \Omega, \ v\lfloor_{\partial\Omega} = 0, \end{cases} \tag{3.18}$$

where $f, g : (\mathbb{R}^+)^2 \times (\mathbb{R}^N \setminus \{0\})^2 \to \mathbb{R}^+$ are continuous functions satisfying the growth conditions below.

(a_6) There exist $\mu_i, \hat{\mu}_i \in \mathbb{R}^+$ and $\alpha_i, \beta_i \in \mathbb{R}$, such that

$$\mu_1(s+|\xi_1|)^{\alpha_1}(t+|\xi_2|)^{\beta_1} \leq f(s,t,\xi_1,\xi_2) \leq \hat{\mu}_1(s+|\xi_1|)^{\alpha_1}(t+|\xi_2|)^{\beta_1},$$
$$\mu_2(s+|\xi_1|)^{\beta_2}(t+|\xi_2|)^{\alpha_2} \leq g(s,t,\xi_1,\xi_2) \leq \hat{\mu}_2(s+|\xi_1|)^{\beta_2}(t+|\xi_2|)^{\alpha_2}$$

for all $(s,t,\xi_1,\xi_2) \in (\mathbb{R}^+)^2 \times (\mathbb{R}^N \setminus \{0\})^2$.

(a_7) One has

$$\beta_1 \vee \beta_2 < 0, \quad \alpha_1\alpha_2 > 0, \quad |\alpha_1| < p-1+\beta_1, \quad |\alpha_2| < q-1+\beta_2.$$

Since $\beta_1 \vee \beta_2 < 0$, system (3.18) has a competitive structure. Moreover, *the right-hand sides may exhibit singularities in both the solution and its gradient.*

Combining comparison arguments with a priori estimates yields the next

Theorem 3.12 ([61], Theorem 1.1)**.** *If* (a_6)–(a_7) *hold then* (3.18) *admits a solution* $(u,v) \in C^{1,\alpha}(\overline{\Omega})^2$.

Three years later, also the cooperative case was examined; see [9]. Now, $1 < p, q < N$ while $f, g : (\mathbb{R}^+)^2 \times \mathbb{R}^{2N} \to \mathbb{R}^+$ are continuous and fulfill the following conditions.

(a_8) There exist $\mu_i, \hat{\mu}_i, \beta_i, \gamma_i, \theta_i \in \mathbb{R}^+$, $\alpha_i \in]-1, 0[$, $i = 1, 2$, such that

$$\mu_1 s^{\alpha_1} t^{\beta_1} \leq f(s,t,\xi_1,\xi_2) \leq \hat{\mu}_1 s^{\alpha_1} t^{\beta_1} + |\xi_1|^{\gamma_1} + |\xi_2|^{\theta_1},$$
$$\mu_2 s^{\beta_2} t^{\alpha_2} \leq g(s,t,\xi_1,\xi_2) \leq \hat{\mu}_2 s^{\beta_2} t^{\alpha_2} + |\xi_1|^{\gamma_2} + |\xi_2|^{\theta_2}$$

for all $(s,t,\xi_1,\xi_2) \in (\mathbb{R}^+)^2 \times \mathbb{R}^{2N}$.

(a_9) One has $\alpha_i + \beta_i \geq 0$, as well as

$$\max\{-\alpha_1+\beta_1, \gamma_1, \theta_1\} < p-1, \;\; \max\{-\alpha_2+\beta_2, \gamma_2, \theta_2\} < q-1.$$

Theorem 3.13 ([9], Theorem 1)**.** *Under* (a_8)–(a_9)*, problem* (3.18) *possesses a solution* $(u,v) \in C_0^1(\overline{\Omega})^2$*. Moreover,* $c_1 d \leq u, v \leq c_2 d$ *for suitable* $c_1, c_2 > 0$*, with* d *given by* (2.2).

A further interesting contribution in contained in [22].

The very recent paper [40] (see also [41]) establishes the existence of infinitely many solutions to the Neumann problem

$$\begin{cases} -\Delta_p u = f(x,u,v,\nabla u,\nabla v) \;\text{ in } \Omega, \;\; u > 0 \;\text{ in } \Omega, \;\; \dfrac{\partial u}{\partial \nu} = 0 \;\text{ on } \partial\Omega, \\ -\Delta_q v = g(x,u,v,\nabla u,\nabla v) \;\text{ in } \Omega, \;\; v > 0 \;\text{ in } \Omega, \;\; \dfrac{\partial v}{\partial \nu} = 0 \;\text{ on } \partial\Omega, \end{cases} \tag{3.19}$$

where $1 < p, q < \infty$, $f, g : \Omega \times (\mathbb{R}^+)^2 \times \mathbb{R}^{2N} \to \mathbb{R}$ satisfy Caratheodory's conditions while ν denotes the outer unit normal to $\partial\Omega$.
The *sub-linear case* is first investigated using the next assumptions.

(a_{10}) There exist $\alpha_i < 0 < \beta_i$, $\gamma_1, \delta_1 \in [0, p-1[$, $\gamma_2, \delta_2 \in [0, q-1[$, and $a_i, b_i, c_i \in L^\infty(\Omega)$ such that

$$|f(x,s,t,\xi_1,\xi_2)| \le a_1(x) s^{\alpha_1} t^{\beta_1} + b_1(x)(|\xi_1|^{\gamma_1} + |\xi_2|^{\delta_1}) + c_1(x),$$
$$|g(x,s,t,\xi_1,\xi_2)| \le a_2(x) s^{\beta_2} t^{\alpha_2} + b_2(x)(|\xi_1|^{\gamma_2} + |\xi_2|^{\delta_2}) + c_2(x)$$

for all $(x,s,t,\xi_1,\xi_2) \in \Omega \times (\mathbb{R}^+)^2 \times \mathbb{R}^{2N}$.

(a_{11}) There are $\{h_n\}, \{\hat{h}_n\}, \{k_n\}, \{\hat{k}_n\}, \{C_n\} \subseteq \mathbb{R}^+$, with $C_n \to +\infty$, satisfying $h_n < k_n < h_{n+1}$, $\hat{h}_n < \hat{k}_n < \hat{h}_{n+1}$,

$$\begin{aligned} &f(x,k_n,t,\xi_1,\xi_2) \le 0 \le f(x,h_n,t,\xi_1,\xi_2), \\ &g(x,s,\hat{k}_n,\xi_1,\xi_2) \le 0 \le g(x,s,\hat{h}_n,\xi_1,\xi_2) \end{aligned} \tag{3.20}$$

for all $(x,s,t,\xi_1,\xi_2) \in \Omega \times [h_n,k_n] \times [\hat{h}_n,\hat{k}_n] \times B_{\mathbb{R}^N}(C_n)^2$, $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \frac{h_n^{\alpha_1} \hat{k}_n^{\beta_1}}{C_n^{p-1}} = 0, \quad \lim_{n\to\infty} \frac{\hat{h}_n^{\alpha_2} k_n^{\beta_2}}{C_n^{q-1}} = 0. \tag{3.21}$$

Theorem 3.14 ([40], Theorem 4.2)**.** *If* (a_{10})–(a_{11}) *hold then* (3.19) *has a sequence of solutions* $\{(u_n,v_n)\} \subseteq C^1(\overline{\Omega})^2$ *such that* $(u_n,v_n) < (u_{n+1},v_{n+1})$ *for every* $n \in \mathbb{N}$. *Moreover,* $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = +\infty$ *uniformly in* $\overline{\Omega}$ *once* $h_n, \hat{h}_n \to +\infty$.

As regards the *super-linear case*, denote by (a'_{10}) condition (a_{10}) written for $\gamma_1, \delta_1 > p-1$ and $\gamma_2, \delta_2 > q-1$. Similarly,

(a'_{11}) There exist $\{h_n\}, \{\hat{h}_n\}, \{k_n\}, \{\hat{k}_n\}, \{C_n\} \subseteq \mathbb{R}^+$, with $C_n \to 0$, satisfying $k_{n+1} < h_n < k_n$, $\hat{k}_{n+1} < \hat{h}_n < \hat{k}_n$ for all $n \in \mathbb{N}$ and (3.20)–(3.21).

Theorem 3.15 ([40], Theorem 4.3)**.** *Under* (a'_{10})–(a'_{11}), *problem* (3.19) *possesses a sequence of solutions* $\{(u_n,v_n)\} \subseteq C^1(\overline{\Omega})^2$ *such that* $(u_{n+1},v_{n+1}) < (u_n,v_n)$ *for every* $n \in \mathbb{N}$. *Moreover,* $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = 0$ *uniformly in* $\overline{\Omega}$ *once* $k_n, \hat{k}_n \to 0$.

An easy example of nonlinearities, with both singular and convective terms, that fulfill (3.20)–(3.21) is the following.

Example 3.16. Set, for every $(x,s,t,\xi_1,\xi_2) \in \Omega \times (\mathbb{R}^+)^2 \times \mathbb{R}^{2N}$,

$$f(x,s,t,\xi_1,\xi_2) = \sin\frac{1}{s}\left(s^{\alpha_1} t^{\beta_1} - |\xi_1|^{\gamma_1} - |\xi_2|^{\delta_1}\right),$$
$$g(x,s,t,\xi_1,\xi_2) = \cos\frac{1}{t}\left(s^{\beta_2} t^{\alpha_2} - |\xi_1|^{\gamma_2} - |\xi_2|^{\delta_2}\right),$$

where

$$\gamma_1 \wedge \delta_1 > \alpha_1 + \beta_1 > p - 1, \quad \gamma_2 \wedge \delta_2 > \alpha_2 + \beta_2 > q - 1.$$

To check (3.20)–(3.21) one can pick $C_n = \frac{1}{n}$,

$$h_n = \frac{1}{\pi/2 + 2\pi n}, \; k_n = \frac{1}{-\pi/2 + 2\pi n}, \; \hat{h}_n = \frac{1}{2\pi + 2\pi n}, \; \hat{k}_n = \frac{1}{\pi + 2\pi n}.$$

4 Problems on the whole space

4.1 The case $p = 2$

In 2009, Moussaoui, Khodja, and Tas [62] studied the following singular, semi-linear elliptic, Gierer-Meinhardt's type system (see [37]):

$$\begin{cases} -\Delta u + \alpha_1(x) u = a_1(x) \dfrac{1}{v^q} \;\text{ in } \mathbb{R}^N, \; u > 0 \;\text{ in } \mathbb{R}^N, \\ -\Delta v + \alpha_2(x) v = a_2(x) \dfrac{u^r}{v^s} \;\text{ in } \mathbb{R}^N, \; v > 0 \;\text{ in } \mathbb{R}^N, \\ \qquad u(x) \to 0, \; v(x) \to 0 \;\text{ as } |x| \to \infty, \end{cases} \tag{4.1}$$

where, roughly speaking, $\alpha_i, a_i \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)_+$, a_i satisfies suitable integrability conditions, $q, r, s > 0$, and $r \leq s + 1 < 2$. A solution $(u, v) \in \mathcal{D}^{1,2}_0(\mathbb{R}^N)^2$ is obtained via Schauder's fixed point theorem. Previous papers on the same subject are [23, 24], whilst, excepting [6, 7], we were not able to find more recent contributions.

Even for the semi-linear case, *the question whether a singular elliptic problem on the whole space admits multiple positive solutions is an open question.*

Finally, as regards uniqueness, we mention [6, Theorem 1.1], which deals with the system

$$\begin{cases} -\Delta u + \alpha(x) u^2 = a(x) \dfrac{v^{1-p}}{u^p} \;\text{ in } \mathbb{R}^N, \; u > 0 \;\text{ in } \mathbb{R}^N, \\ -\Delta v + \alpha(x) v^2 = a(x) \dfrac{u^{1-p}}{v^p} \;\text{ in } \mathbb{R}^N, \; v > 0 \;\text{ in } \mathbb{R}^N, \end{cases} \tag{4.2}$$

where $N \geq 3$, $p \in]\frac{1}{2}, 1[$, $\alpha \in L^1(\mathbb{R}^N) \cap L^{\frac{q}{q-2}}(\mathbb{R}^N)$ for some $q \in]2, 2^*[$, $\alpha > 0$, and $0 \leq a \leq \alpha$.

Existence and uniqueness of solutions to singular convective elliptic problems was firstly studied by Benrhouma [7] in 2017, who considered the system

$$\begin{cases} -\Delta u + \alpha_1 \dfrac{|\nabla u|^2}{u} = \dfrac{p}{p+q} a(x) u^{p-1} v^q + b_1(x) \ \text{ in } \mathbb{R}^N, \ u > 0 \ \text{ in } \mathbb{R}^N, \\ -\Delta v + \alpha_2 \dfrac{|\nabla v|^2}{v} = \dfrac{q}{p+q} a(x) u^p v^{q-1} + b_2(x) \ \text{ in } \mathbb{R}^N, \ v > 0 \ \text{ in } \mathbb{R}^N, \end{cases}$$

with $p \wedge q > 1$, $p+q < 2^* - 1$, a fulfilling appropriate integrability conditions, $\alpha_i > \frac{N+2}{4}$, and $b_i \in L^{2^*}(\mathbb{R}^N)_+ \cap L^\infty(\mathbb{R}^N)$, $i = 1, 2$.

4.2 Existence and multiplicity

Henceforth, given $p, q \in]1, \infty[$, we will write

$$X^{p,q}(\mathbb{R}^N) := \mathcal{D}_0^{1,p}(\mathbb{R}^N) \times \mathcal{D}_0^{1,q}(\mathbb{R}^N).$$

To the best of our knowledge, until 2019, singular elliptic systems in the whole space were investigated only for $p = q = 2$, essentially exploiting the linearity of involved differential operators. The paper [54] considers the problem

$$\begin{cases} -\Delta_p u = a_1(x) f(u,v) \ \text{ in } \mathbb{R}^N, \ u > 0 \ \text{ in } \mathbb{R}^N, \\ -\Delta_q v = a_2(x) g(u,v) \ \text{ in } \mathbb{R}^N, \ v > 0 \ \text{ in } \mathbb{R}^N, \end{cases} \tag{4.3}$$

where $N \geq 3$ while $p, q \in]1, N[$. The nonlinearities $f, g : (\mathbb{R}^+)^2 \to \mathbb{R}^+$ are continuous and fulfill the condition

(a_{12}) There exist $\mu_i, \hat{\mu}_i \in \mathbb{R}^+$, $i = 1, 2$, such that

$$\mu_1 s^{\alpha_1} \leq f(s,t) \leq \hat{\mu}_1 s^{\alpha_1}(1 + t^{\beta_1}), \ \ \mu_2 t^{\alpha_2} \leq g(s,t) \leq \hat{\mu}_2 (1 + s^{\beta_2}) t^{\alpha_2}$$

for all $s, t \in \mathbb{R}^+$, with $-1 < \alpha_i < 0 < \beta_i$,

$$\alpha_1 + \beta_2 < p - 1, \ \ \alpha_2 + \beta_1 < q - 1, \tag{4.4}$$

as well as

$$\beta_1 < \frac{q^*}{p^*} \min\{p-1, p^* - p\}, \ \ \beta_2 < \frac{p^*}{q^*} \min\{q-1, q^* - q\}.$$

The coefficients $a_i : \mathbb{R}^N \to \mathbb{R}^+$ satisfy the assumption

(a_{13}) $a_i \in L^1(\mathbb{R}^N) \cap L^{\zeta_i}(\mathbb{R}^N)$, where

$$\frac{1}{\zeta_1} \leq 1 - \frac{p}{p^*} - \frac{\beta_1}{q^*}, \ \ \frac{1}{\zeta_2} \leq 1 - \frac{q}{q^*} - \frac{\beta_2}{p^*}.$$

A pair $(u,v) \in X^{p,q}(\mathbb{R}^N)$ is called a (weak) solution to (4.3) provided $u, v > 0$ and

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla\varphi \, \mathrm{d}x = \int_{\mathbb{R}^N} a_1 f(u,v)\varphi \, \mathrm{d}x \quad \forall\, \varphi \in \mathcal{D}_0^{1,p}(\mathbb{R}^N),$$
$$\int_{\mathbb{R}^N} |\nabla v|^{q-2}\nabla v \nabla\psi \, dx = \int_{\mathbb{R}^N} a_2 g(u,v)\psi \, dx \quad \forall\, \psi \in \mathcal{D}_0^{1,q}(\mathbb{R}^N).$$

Variational methods do not work, at least in a direct way, because (4.3) has, in general, no variational structure. A similar comment holds for sub-super-solution techniques, that are usually employed in the case of bounded domains. So, one is naturally led to apply fixed point results. An a priori estimate in $L^\infty(\mathbb{R}^N) \times L^\infty(\mathbb{R}^N)$ for solutions of (4.3) is first established by a Moser's type iteration procedure and an adequate truncation, which, due to singular terms, require a specific treatment. Problem (4.3) is next perturbed by introducing a parameter $\varepsilon > 0$. This produces the family of regularized systems

$$\begin{cases} -\Delta_p u = a_1(x) f(u+\varepsilon, v) \ \text{ in } \mathbb{R}^N, \ u > 0 \ \text{ in } \mathbb{R}^N, \\ -\Delta_q v = a_2(x) g(u, v+\varepsilon) \ \text{ in } \mathbb{R}^N, \ v > 0 \ \text{ in } \mathbb{R}^N, \end{cases} \tag{4.5}$$

whose study yields useful information on the original problem. In fact, the previous L^∞-boundedness still holds for solutions to (4.5), regardless of ε. Thus, via Schauder's fixed point theorem, one gets a solution $(u_\varepsilon, v_\varepsilon)$ lying inside a rectangle given by positive lower bounds, where ε does not appear, and positive upper bounds, that may instead depend on ε. Finally, letting $\varepsilon \to 0^+$ and using the $(\mathrm{S})_+$-property of the negative r-Laplacian in $\mathcal{D}_0^{1,r}(\mathbb{R}^N)$ (see [54, Proposition 2.2]) yields a weak solution to (4.3).

Theorem 4.1 ([54], Theorem 5.1)**.** *Let* (a_{12}) *and* (a_{13}) *be satisfied. Then* (4.3) *has a weak solution* $(u,v) \in X^{p,q}(\mathbb{R}^N)$*, which is essentially bounded.*

Very recently, the parametric system

$$\begin{cases} -\Delta_p u = a_1(x) f_1(u) + \lambda b_1(x) g_1(u) h_1(v) \ \text{ in } \mathbb{R}^N, \ u > 0 \ \text{ in } \mathbb{R}^N, \\ -\Delta_p v = a_2(x) f_2(v) + \mu b_2(x) g_2(v) h_2(u) \ \text{ in } \mathbb{R}^N, \ v > 0 \ \text{ in } \mathbb{R}^N, \\ \qquad u(x) \to 0, \ v(x) \to 0 \ \text{ as } |x| \to \infty \end{cases} \tag{4.6}$$

where $N \geq 3$, $1 < p < N$, $a_i, b_i \in C^0(\mathbb{R}^N)$, $f_i, g_i, h_i \in C^0(\mathbb{R}^+, \mathbb{R}^+)$, f_i is singular at zero, and $\lambda, \mu \in \mathbb{R}^+$, was thoroughly investigated in [67]. Under suitable hypotheses, it is shown that there exists an open set $\Theta \subseteq (\mathbb{R}^+)^2$, whose part of its boundary contained in $(\mathbb{R}^+)^2$, say Γ, turns out to be a continuous monotone curve, such that (4.6) admits a C^1-solution if $(\lambda, \mu) \in \Theta$ and has no solution when $(\lambda, \mu) \in (\mathbb{R}^+)^2 \setminus (\Theta \cup \Gamma)$.

4.3 Uniqueness

As far as we know, *uniqueness of solutions to singular quasi-linear elliptic systems in the whole space is still an open problem.* Taking inspiration from [13], a first result has been obtained by Gambera and Guarnotta [29].

4.4 Systems with convection terms

The very recent paper [43] treats the problem

$$\begin{cases} -\Delta_p u = f(x,u,v,\nabla u,\nabla v) \ \text{ in } \mathbb{R}^N, \ u > 0 \ \text{ in } \mathbb{R}^N, \\ -\Delta_q v = g(x,u,v,\nabla u,\nabla v) \ \text{ in } \mathbb{R}^N, \ v > 0 \ \text{ in } \mathbb{R}^N, \end{cases} \tag{4.7}$$

where $N \geq 3$, $p,q \in]2-\frac{1}{N}, N[$, while $f,g : \mathbb{R}^N \times (\mathbb{R}^+)^2 \times \mathbb{R}^{2N} \to \mathbb{R}^+$ are Carathéodory's functions satisfying assumptions (a_{14})–(a_{15}) below. Since f,g depend on the gradient of solutions and equations are set in the whole space, neither variational methods can be exploited nor compactness for Sobolev embedding holds. The research started in [54], where convective terms did not appear, is continued here, along the works [9, 40, 42], which address analogous questions, but concerning a bounded domain.

A pair $(u,v) \in X^{p,q}(\mathbb{R}^N)$ such that $u,v > 0$ is called:

1) *distributional solution* to (4.7) if for every $(\varphi,\psi) \in C_0^\infty(\mathbb{R}^N)^2$ one has

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla\varphi \,\mathrm{d}x &= \int_{\mathbb{R}^N} f(\cdot,u,v,\nabla u,\nabla v)\varphi \,\mathrm{d}x, \\ \int_{\mathbb{R}^N} |\nabla v|^{q-2}\nabla v \nabla\psi \,\mathrm{d}x &= \int_{\mathbb{R}^N} g(\cdot,u,v,\nabla u,\nabla v)\psi \,\mathrm{d}x; \end{aligned} \tag{4.8}$$

2) *(weak) solution* of (4.7) when (4.8) holds for all $(\varphi,\psi) \in X^{p,q}(\mathbb{R}^N)$;

3) *'strong' solution* to (4.7) if $|\nabla u|^{p-2}\nabla u, |\nabla v|^{q-2}\nabla v \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^N)$ and the differential equations are satisfied a.e. in $\mathbb{R}^N$.

Obviously, both 2) and 3) force 1), whilst reverse implications turn out generally false; see also [43, Remark 4.5]. Moreover, as observed at p. 48 of [69], problems in unbounded domains may admit strong solutions that are not weak or vice-versa.

Roughly speaking, the technical approach proceeds as follows. The auxiliary problem

$$\begin{cases} -\Delta_p u = f(x,u+\varepsilon,v,\nabla u,\nabla v) \ \text{ in } \mathbb{R}^N, \ u > 0 \ \text{ in } \mathbb{R}^N, \\ -\Delta_q v = g(x,u,v+\varepsilon,\nabla u,\nabla v) \ \text{ in } \mathbb{R}^N, \ v > 0 \ \text{ in } \mathbb{R}^N, \end{cases} \tag{4.9}$$

$\varepsilon > 0$, obtained by shifting appropriate variables of reactions, which avoids singularities, is first solved. To do this, nonlinear regularity theory, a priori estimates, Moser's iteration, trapping region, and fixed point arguments are employed. Unfortunately, bounds from above alone do not allow to get a solution of (4.7): treating singular terms additionally requires some estimates from below. Theorem 3.1 in [19] ensures that solutions to (4.9) turn out locally greater than a positive constant regardless of ε. Thus, under the hypotheses below, one can construct a sequence $\{(u_\varepsilon, v_\varepsilon)\} \subseteq X^{p,q}(\mathbb{R}^N)$ such that $(u_\varepsilon, v_\varepsilon)$ solves (4.9) for all $\varepsilon > 0$ and whose weak limit as $\varepsilon \to 0^+$ is a distributional solution to (4.7). Next, a localization-regularization reasoning shows that

$$(u, v) \text{ distributional solution} \implies (u, v) \text{ weak solution.}$$

Through the recent differentiability result [17, Theorem 2.1], one then has

$$(u, v) \text{ distributional solution} \implies (u, v) \text{ strong solution.}$$

The assumptions below will be posited.

(a_{14}) There exist $\alpha_i \in]-1, 0]$, $\beta_i, \gamma_i, \delta_i \in \mathbb{R}^+_0$, as well as $\mu_i, \hat{\mu}_i \in \mathbb{R}^+$ such that

$$\mu_1 a_1(x) s^{\alpha_1} t^{\beta_1} \le f(x, s, t, \xi_1, \xi_2) \le \hat{\mu}_1 a_1(x) \left(s^{\alpha_1} t^{\beta_1} + |\xi_1|^{\gamma_1} + |\xi_2|^{\delta_1} \right),$$
$$\mu_2 a_2(x) s^{\beta_2} t^{\alpha_2} \le g(x, s, t, \xi_1, \xi_2) \le \hat{\mu}_2 a_2(x) \left(s^{\beta_2} t^{\alpha_2} + |\xi_1|^{\gamma_2} + |\xi_2|^{\delta_2} \right)$$

in $\mathbb{R}^N \times (\mathbb{R}^+)^2 \times \mathbb{R}^{2N}$. Moreover,

$$\beta_1 \vee \delta_i < q - 1, \quad \beta_2 \vee \gamma_i < p - 1, \quad i = 1, 2,$$

$a_1 \in L^{s_p}_{\mathrm{loc}}(\mathbb{R}^N)$, with $s_p > p'N$, $a_2 \in L^{s_q}_{\mathrm{loc}}(\mathbb{R}^N)$, with $s_q > q'N$, and $\operatorname*{ess\,inf}_{B_\rho} a_i > 0$ for all $\rho > 0$.

(a_{15}) There exist $\zeta_1, \zeta_2 \in]N, \infty]$ such that $a_i \in L^1(\mathbb{R}^N) \cap L^{\zeta_i}(\mathbb{R}^N)$, where

$$\frac{1}{\zeta_1} < 1 - \frac{p}{p^*} - \theta_1, \quad \frac{1}{\zeta_2} < 1 - \frac{q}{q^*} - \theta_2,$$

with

$$\theta_1 := \max \left\{ \frac{\beta_1}{q^*}, \frac{\gamma_1}{p}, \frac{\delta_1}{q} \right\} < 1 - \frac{p}{p^*}, \quad \theta_2 := \max \left\{ \frac{\beta_2}{p^*}, \frac{\gamma_2}{p}, \frac{\delta_2}{q} \right\} < 1 - \frac{q}{q^*}.$$

Further,

$$(\beta_1 \vee \delta_1)(\beta_2 \vee \gamma_2) < (p - 1 - \gamma_1)(q - 1 - \delta_2),$$

$$\frac{1}{s_p} + \left(\frac{\gamma_1}{p} \vee \frac{\delta_1}{q} \right) \le \frac{1}{2}, \qquad \frac{1}{s_q} + \left(\frac{\gamma_2}{p} \vee \frac{\delta_2}{q} \right) \le \frac{1}{2}.$$

Example 4.2. Condition (a_{15}) is fulfilled once $a_1, a_2 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and

$$\max\left\{\frac{\beta_1}{q^*}, \frac{\gamma_1}{p}, \frac{\delta_1}{q}\right\} < 1 - \frac{p}{p^*}, \quad \max\left\{\frac{\beta_2}{p^*}, \frac{\gamma_2}{p}, \frac{\delta_2}{q}\right\} < 1 - \frac{q}{q^*}.$$

In fact, it suffices to choose $\zeta_1 := \zeta_2 := \infty$.

Theorem 4.3 ([43], Theorem 1.3)**.** *Under hypotheses* (a_{14})–(a_{15})*, problem* (4.7) *admits a weak and strong solution* $(u, v) \in X^{p,q}(\mathbb{R}^N)$.

Remark 4.4. If we merely seek weak solutions to (4.7) then the request $p, q \in]1, N[$ and a weaker integrability property of a_i suffice; cf. [39, Section 4.2.2].

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