$(p, 2)$-EQUATIONS RESONANT AT ANY VARIATIONAL EIGENVALUE

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Abstract. We consider nonlinear elliptic Dirichlet problems driven by the sum of a p-Laplacian and a Laplacian (a $(p, 2)$-equation). The reaction term at $\pm\infty$ is resonant with respect to any variational eigenvalue of the p-Laplacian. We prove two multiplicity theorems for such equations.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following nonlinear, nonhomogeneous elliptic equation

$$-\Delta_p u(z) - \Delta u(z) = f(z, u(z)) \text{ in } \Omega, \quad u\Big|_{\partial\Omega} = 0, \quad 2 < p < +\infty. \tag{1}$$

In this problem Δ_p denotes the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}\,(|\nabla u|^{p-2}\nabla u) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

When $p = 2$, then $\Delta_2 = \Delta$ is the usual Laplace differential operator. Our aim is to prove multiplicity theorems for problems which are resonant with respect to any variational eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$.

Elliptic equations driven by the sum of a p-Laplacian and a Laplacian, known as $(p, 2)$-equations, arise in problems of mathematical physics. We refer to the works of Benci-D'Avenia-Fortunato-Pisani [2] (quantum physics) and Cherfils-Il'yasov [5] (plasma physics). Recently there have been some existence and multiplicity results for such equations. We refer to the works of Cingolani-Degiovanni [6], Gasiński-Papageorgiou [10], Papageorgiou-Rădulescu [17], Papageorgiou-Rădulescu-Repovš [18, 19], Papageorgiou-Vetro [20], Papageorgiou-Vetro-Vetro [21], Papageorgiou-Winkert [22], Sun [26], Sun-Zhang-Su [27]. The distinguishing feature of our work here, is that we deal with equations which can be resonant with respect to any variational eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$. Our analysis of problem (1) combines variational tools based on the critical point theory, together with truncation and comparison techniques and the theory of critical groups.

First we deal with problems which are resonant with respect to the principal eigenvalue $\widehat{\lambda}_1(p) > 0$. The starting point for this investigation, is the work of Papageorgiou-Rădulescu [17]. In that paper, the authors deal with $(p, 2)$-equations which are resonant with respect

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to the principal eigenvalue $\widehat{\lambda}_1(p) > 0$. In [17] the resonance occurs from the left of $\widehat{\lambda}_1(p)$ in the sense that

$$\widehat{\lambda}_1(p)|x|^p - pF(z,x) \to +\infty \text{ uniformly for a.a. } z \in \Omega, \text{ as } x \to \pm\infty,$$

(here $F(z,x) = \int_0^x f(z,s)ds$). This makes the energy (Euler) functional of the problem coercive and so the direct method of the calculus of variations can be used. In contrast here we assume that the resonance occurs from the right at $\widehat{\lambda}_1(p)$ in the sense that

$$\widehat{\lambda}_1(p)|x|^p - pF(z,x) \to -\infty \text{ uniformly for a.a. } z \in \Omega, \text{ as } x \to \pm\infty.$$

As a result of this, the energy functional is indefinite and so different methods are used. Then we deal with problems which are resonant with respect to higher variational eigenvalues, extending the work of Papageorgiou-Vetro-Vetro [21].

2. Mathematical Background

Suppose X is a Banach space. By X^* we denote the topological dual of X and by $\langle \cdot, \cdot \rangle$ the duality brackets for the pair (X^*, X). Given $\varphi \in C^1(X,\mathbb{R})$, we say that φ satisfies the "Cerami condition" (the "C-condition" for short), if the following property holds:

"Every sequence $\{u_n\}_{n\geq 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and $(1 + \|u_n\|_X)\varphi'(u_n) \to 0$ in X^* as $n \to +\infty$, admits a strongly convergent subsequence".

This compactness-type condition on φ, leads to a deformation theorem, from which one can derive the minimax theory of the critical values of φ. A major result in that theory, is the so-called "mountain pass theorem", which we recall here.

Theorem 2.1. *If X is a Banach space, $\varphi \in C^1(X,\mathbb{R})$ satisfies the C-condition, $u_0, u_1 \in X$, $\|u_1 - u_0\|_X > \rho > 0$, $\max\{\varphi(u_0), \varphi(u_1)\} < \inf\{\varphi(u) : \|u - u_0\|_X = \rho\} = \eta_\rho$ and $c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} \varphi(\gamma(t))$ where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_0, \gamma(1) = u_1\}$, then $c \geq \eta_\rho$ and c is a critical value of φ.*

The following spaces will play a central role in our analysis of problem (1):

$$W_0^{1,p}(\Omega), \quad H_0^1(\Omega), \quad C_0^1(\overline{\Omega}) = \left\{ u \in C^1(\overline{\Omega}) \,:\, u\Big|_{\partial\Omega} = 0 \right\}.$$

By $\|\cdot\|$ we denote the norm of the Sobolev space $W_0^{1,p}(\Omega)$. On account of the Poincaré inequality, we have

$$\|u\| = \|\nabla u\|_p \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

The Banach space $C_0^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone given by

$$C_+ = \{u \in C_0^1(\overline{\Omega}) \,:\, u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\text{int } C_+ = \left\{ u \in C_+ \,:\, u(z) > 0 \text{ for all } z \in \Omega, \quad \frac{\partial u}{\partial n}\Big|_{\partial\Omega} < 0 \right\}.$$

Here $\dfrac{\partial u}{\partial n} = (\nabla u, n)_{\mathbb{R}^N}$ with $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

Let $A_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ $\left(\dfrac{1}{p} + \dfrac{1}{p'} = 1\right)$ be the nonlinear map defined by

$$\langle A_p(u), y\rangle = \int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla y)_{\mathbb{R}^N} dz \quad \text{for all } u, y \in W_0^{1,p}(\Omega).$$

This map has the following properties (see Gasiński-Papageorgiou [9], Problem 2.192, p. 279).

Proposition 2.2. *The map $A_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (hence maximal monotone too) and of type $(S)_+$, that is, if $u_n \xrightarrow{w} u$ in $W_0^{1,p}(\Omega)$ and $\limsup_{n\to+\infty}\langle A(u_n), u_n - u\rangle \le 0$, then $u_n \to u$ in $W_0^{1,p}(\Omega)$.*

When $p = 2$, we write $A_2 = A$ and we have $A \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$.

Suppose that $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$ $z \to f(z,x)$ is measurable and for a.a. $z \in \Omega$ $x \to f(z,x)$ is continuous) and

$$|f_0(z,x)| \le a_0(z)[1 + |x|^{r-1}] \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R},$$

with $a_0 \in L^\infty(\Omega)$ and $1 < r \le p^*$, where $p^* = \begin{cases} \dfrac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } N \le p, \end{cases}$ (the critical Sobolev exponent). We set $F_0(z,x) = \int_0^x f_0(z,s)ds$ and consider the C^1-functional $\varphi_0 : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega F_0(z,u)dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

The next result is an outgrowth of the nonlinear regularity theory of Lieberman [12] (Theorem 1) and can be found in Gasiński-Papageorgiou [8].

Proposition 2.3. *If $u_0 \in W_0^{1,p}(\Omega)$ is a local $C_0^1(\overline{\Omega})$-minimizer of φ_0, that is, there exists $\rho_0 > 0$ such that*

$$\varphi_0(u_0) \le \varphi_0(u_0 + h) \textit{ for all } h \in C_0^1(\overline{\Omega}),\ \|h\|_{C_0^1(\overline{\Omega})} \le \rho_0,$$

then $u_0 \in C_0^{1,\alpha}(\overline{\Omega})$ with $\alpha \in (0,1)$ and u_0 is also a local $W_0^{1,p}(\Omega)$-minimizer of φ_0, that is, there exists $\rho_1 > 0$ such that

$$\varphi_0(u_0) \le \varphi_0(u_0 + h) \textit{ for all } h \in W_0^{1,p}(\Omega),\ \|h\| \le \rho_1.$$

Next let us describe the spectrum of $(-\Delta_p, W_0^{1,p}(\Omega))$. So, we consider the following nonlinear eigenvalue problem

$$-\Delta_p u(z) = \widehat{\lambda}|u(z)|^{p-2}u(z) \text{ in } \Omega, \quad u\Big|_{\partial\Omega} = 0. \tag{2}$$

We say that $\widehat{\lambda} \in \mathbb{R}$ is an "eigenvalue" of $(-\Delta_p, W_0^{1,p}(\Omega))$, if problem (2) admits a nontrivial solution $\widehat{u} \in W_0^{1,p}(\Omega)$, which is known as an "eigenfunction" corresponding to $\widehat{\lambda}$. There exists a smallest eigenvalue $\widehat{\lambda}_1(p) > 0$ which has the following properties:

(a) $\widehat{\lambda}_1(p)$ is isolated (that is, we can find $\varepsilon > 0$ such that $(\widehat{\lambda}_1(p), \widehat{\lambda}_1(p) + \varepsilon)$ contains no eigenvalues of $(-\Delta_p, W_0^{1,p}(\Omega))$).

(b) $\widehat{\lambda}_1(p)$ is simple (that is, if $\widehat{u}, \widehat{v} \in W_0^{1,p}(\Omega)$ are eigenfunctions corresponding to $\widehat{\lambda}_1(p)$, then $\widehat{u} = \xi\widehat{v}$ for some $\xi \in \mathbb{R} \setminus \{0\}$).

(c)

$$\widehat{\lambda}_1(p) = \inf\left[\frac{\|\nabla u\|_p^p}{\|u\|_p^p} \,:\, u \in W_0^{1,p}(\Omega),\, u \neq 0\right]. \tag{3}$$

The infimum in (3) is realized on the corresponding one dimensional eigenspace. It is clear from the above properties, that the elements of this eigenspace do not change sign. Moreover, by the nonlinear regularity theory we have that the elements of this eigenspace belong in $C_0^1(\overline{\Omega})$. By $\widehat{u}_1(p)$ we denote the L^p-normalized (that is, $\|\widehat{u}_1(p)\|_p = 1$) positive eigenfunction corresponding to $\widehat{\lambda}_1(p) > 0$. The nonlinear maximum principle (see, for example, Gasiński-Papageorgiou [7], p. 738) implies that $\widehat{u}_1(p) \in \operatorname{int} C_+$.

Using the Fadell-Rabinowitz cohomological index $\operatorname{ind}(\cdot)$ and the Ljusternik-Schnirelmann minimax scheme, we can produce a whole sequence $\{\widehat{\lambda}_k(p)\}_{k\geq 1}$ of distinct eigenvalues such that $\widehat{\lambda}_k(p) \to +\infty$ as $k \to +\infty$. Let $M = \{u \in W_0^{1,p}(\Omega) : \|u\|_p = 1\}$. We have

$$\widehat{\lambda}_k(p) = \inf\left[\max_{u\in K} \|\nabla u\|_p^p : K \subseteq M,\ K \text{ is compact, symmetric, } \operatorname{ind(K)} \geq k\right].$$

These eigenvalues are known as "variational eigenvalues" of $(-\Delta_p, W_0^{1,p}(\Omega))$. We do not know if these exhaust the spectrum of $(-\Delta_p, W_0^{1,p}(\Omega))$. This is true if $p = 2$ (linear eigenvalue problem) or if $N = 1$ (scalar eigenvalue problem). We mention that every eigenvalue $\widehat{\lambda} \neq \widehat{\lambda}_1(p)$ has nodal (that is, sign changing) eigenfunctions.

We will encounter a weighted version of the eigenvalue problem (2). So, let $m \in L^\infty(\Omega)$, $m \not\equiv 0$ and $m(z) \geq 0$ for a.a. $z \in \Omega$. We consider the following nonlinear eigenvalue problem

$$-\Delta_p u(z) = \widetilde{\lambda} m(z)|u(z)|^{p-2}u(z) \text{ in } \Omega, \quad u\Big|_{\partial\Omega} = 0.$$

This problem has a spectrum analogous to that of (2). So, we generate a whole sequence of distinct eigenvalues $\{\widetilde{\lambda}_k(p,m)\}_{k\geq 1}$ such that $\widetilde{\lambda}_k(p,m) \to +\infty$ as $k \to +\infty$. In this case the variational characterization of $\widetilde{\lambda}_1(p,m)$ has the following form:

$$\widetilde{\lambda}_1(p,m) = \inf\left[\frac{\|\nabla u\|_p^p}{\int_\Omega m(z)|u|^p dz} \,:\, u \in W_0^{1,p}(\Omega),\, u \neq 0\right]. \tag{4}$$

Using (4) we can easily prove the following strict monotonicity property for the map $m \to \widetilde{\lambda}_1(p,m)$ (see Motreanu-Motreanu-Papageorgiou [15], Proposition 9.47 (d), p. 250 and Proposition 9.51, p. 251).

PROPOSITION 2.4. *We have:*

(a) *If* $m_1, m_2 \in L^\infty(\Omega)$, $0 \leq m_1(z) \leq m_2(z)$ *for a.a.* $z \in \Omega$, $m_1 \not\equiv 0$, $m_2 \not\equiv m_1$, *then* $\widetilde{\lambda}_1(p, m_2) < \widetilde{\lambda}_1(p, m_1)$.

(b) *If* $m_1, m_2 \in L^\infty(\Omega)$, $0 \leq m_1(z) < m_2(z)$ *for a.a.* $z \in \Omega$, $m_1 \not\equiv 0$, *then* $\widetilde{\lambda}_2(p, m_2) < \widetilde{\lambda}_2(p, m_1)$.

For the linear eigenvalue problem (that is, $p = 2$), every eigenvalue $\widehat{\lambda}_k(2)$, $k \in \mathbb{N}$, has an eigenspace, denoted by $E(\widehat{\lambda}_k(2))$, which is finite dimensional. We have

$$H_0^1(\Omega) = \overline{\oplus_{k\geq 1} E(\widehat{\lambda}_k(2))}.$$

For every $k \in \mathbb{N}$, we set

$$\overline{H}_k = \oplus_{m=1}^k E(\widehat{\lambda}_m(2)) \quad \text{and} \quad \widehat{H}_k = \overline{H}_k^\perp = \overline{\oplus_{m\geq k+1} E(\widehat{\lambda}_m(2))}.$$

Of course we have $H_0^1(\Omega) = \overline{H}_k \oplus \widehat{H}_k$.

In this case all eigenvalues $\widehat{\lambda}_k(2)$, $k \in \mathbb{N}$, admit variational characterizations:

$$\widehat{\lambda}_1(2) = \inf\left[\frac{\|\nabla u\|_2^2}{\|u\|_2^2} : u \in H_0^1(\Omega),\ u \neq 0\right], \tag{5}$$

$$\begin{aligned} \widehat{\lambda}_k(2) &= \inf\left[\frac{\|\nabla u\|_2^2}{\|u\|_2^2} : u \in \widehat{H}_{k-1},\ u \neq 0\right] \\ &= \sup\left[\frac{\|\nabla u\|_2^2}{\|u\|_2^2} : u \in \overline{H}_k,\ u \neq 0\right] \quad \text{for all } k \geq 2. \end{aligned} \tag{6}$$

In (6) both the infimum and the supremum are realized on the corresponding eigenspace $E(\widehat{\lambda}_k(2))$. Each eigenspace exhibits the "unique continuation property", which says that if $u \in E(\widehat{\lambda}_k(2))$ vanishes on a set of positive measure, then $u \equiv 0$. Standard regularity theory implies that $E(\widehat{\lambda}_k(2)) \subseteq C_0^1(\overline{\Omega})$ for all $k \in \mathbb{N}$. Using the unique continuation property, we can easily have the following useful inequalities (see Marano-Papageorgiou [14])

PROPOSITION 2.5. *We have:*

(*a*) *If* $n \in \mathbb{N}$ *and* $\eta \in L^\infty(\Omega)$, $\eta(z) \geq \widehat{\lambda}_n(2)$ *for a.a.* $z \in \Omega$, $\eta \not\equiv \widehat{\lambda}_n(2)$, *then we can find* $c_0 > 0$ *such that*

$$\|\nabla u\|_2^2 - \int_\Omega \eta(z) u^2 dz \leq -c_0 \|u\|^2 \quad \textit{for all } u \in \overline{H}_n.$$

(*b*) *If* $n \in \mathbb{N}$ *and* $\eta \in L^\infty(\Omega)$, $\eta(z) \leq \widehat{\lambda}_n(2)$ *for a.a.* $z \in \Omega$, $\eta \not\equiv \widehat{\lambda}_n(2)$, *then we can find* $c_1 > 0$ *such that*

$$\|\nabla u\|_2^2 - \int_\Omega \eta(z) u^2 dz \geq c_1 \|u\|^2 \quad \textit{for all } u \in \widehat{H}_{n-1}.$$

Next let us recall some basic definitions and facts from the theory of critical groups. So, let X be a Banach space, $\varphi \in C^1(X, \mathbb{R})$ and $c \in \mathbb{R}$. We introduce the following sets

$$\varphi^c = \{u \in X : \varphi(u) \leq c\},$$
$$K_\varphi = \{u \in X : \varphi'(u) = 0\},$$
$$K_\varphi^c = \{u \in K_\varphi : \varphi(u) = c\}.$$

Let (Y_1, Y_2) be a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$ and $k \in \mathbb{N}_0$. By $H_k(Y_1, Y_2)$ we denote the kth-relative singular homology group for the pair (Y_1, Y_2) with integer coefficients. Given an isolated $u \in K_\varphi^c$, the critical groups of φ at u are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}) \quad \text{for all } k \in \mathbb{N}_0,$$

where U is a neighborhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of singular homology implies that the above definition of critical groups is independent of the particular choice of the isolating neighborhood U.

Suppose that φ satisfies the C-condition and $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. Then the critical groups of φ at infinity, are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all } k \in \mathbb{N}_0.$$

This definition is independent of the choice of the level $c < \inf \varphi(K_\varphi)$. Indeed, if $c' < c < \inf \varphi(K_\varphi)$, then $\varphi^{c'}$ is a strong deformation retract of φ^c and so

$$H_k(X, \varphi^c) = H_k(X, \varphi^{c'}) \quad \text{for all } k \in \mathbb{N}_0$$

(see Motreanu-Motreanu-Papageorgiou [15], Corollary 6.15(a), p. 145). Suppose thst $\varphi \in C^1(X, \mathbb{R})$ and assume that φ satisfies the C-condition and K_φ is finite. We set

$$M(t, u) = \Sigma_{k \geq 0} \text{rank}\, C_k(\varphi, u) t^k \quad \text{for all } t \in \mathbb{R}, \text{ all } u \in K_\varphi,$$
$$P(t, \infty) = \Sigma_{k \geq 0} \text{rank}\, C_k(\varphi, \infty) t^k \quad \text{for all } t \in \mathbb{R}.$$

The "Morse relation" says that

$$\Sigma_{u \in K_\varphi} M(t, u) = P(t, \infty) + (1 + t) Q(t) \quad \text{for all } t \in \mathbb{R}, \tag{7}$$

with $Q(t) = \Sigma_{k \geq 0} \beta_k t^k$ is a formal series in $t \in \mathbb{R}$ with nonnegative integer coefficients.

Finally we fix some basic notation. So, for $x \in \mathbb{R}$, let $x^\pm = \max\{\pm x, 0\}$. Then for $u \in W_0^{1,p}(\Omega)$, we define $u^\pm(\cdot) = u(\cdot)^\pm$. We have

$$u^\pm \in W_0^{1,p}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.$$

Also given a measurable function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ (for example, a Carathéodory function), we set

$$N_g(u)(\cdot) = g(\cdot, u(\cdot)) \quad \text{for all } u \in W_0^{1,p}(\Omega),$$

the Nemytskii (superposition) map corresponding to g. Finally, by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$.

3. Resonance with respect to $\widehat{\lambda}_1(p) > 0$

In this section we study $(p,2)$-equations which at $\pm\infty$ are resonant with respect to $\widehat{\lambda}_1(p) > 0$. The resonance occurs from the right of the principal eigenvalue $\widehat{\lambda}_1(p)$.

The hypotheses on the reaction term $f(z,x)$, are the following:

H(f)$_1$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that for a.a. $z \in \Omega$ $f(z,0) = 0$, $f(z,\cdot) \in C^1(\mathbb{R})$ and

(i) $|f'_x(z,x)| \leq a(z)[1 + |x|^{r-2}]$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)$, $p \leq r < p^*$;

(ii) there exists $\eta \in L^\infty(\Omega)$ such that $\eta(z) < \widehat{\lambda}_2(p)$ for a.a. $z \in \Omega$ and

$$\widehat{\lambda}_1(p) \leq \liminf_{x \to \pm\infty} \frac{f(z,x)}{|x|^{p-2}x} \leq \limsup_{x \to \pm\infty} \frac{f(z,x)}{|x|^{p-2}x} \leq \eta(z) \text{ uniformly for a.a. } z \in \Omega;$$

(iii) there exist $\tau \in (2, p^*]$ and $\beta_0 > 0$ such that

$$\beta_0 \leq \liminf_{x \to \pm\infty} \frac{pF(z,x) - f(z,x)x}{|x|^\tau} \quad \text{uniformly for a.a. } z \in \Omega,$$

with $F(z,x) = \int_0^x f(z,s)ds$;

(iv) there exist $m \in \mathbb{N}$ and $\delta_0 > 0$ such that

$$f'_x(z,0) \leq \widehat{\lambda}_{m+1}(2) \quad \text{for a.a. } z \in \Omega, \; f'_x(\cdot,0) \not\equiv \widehat{\lambda}_{m+1}(2),$$
$$\widehat{\lambda}_m(2)x^2 \leq f(z,x)x \quad \text{for a.a. } z \in \Omega, \; \text{all } |x| \leq \delta_0.$$

Remark 3.1. Hypothesis $H(f)_1(ii)$ permits asymptotically at $\pm\infty$ resonance with respect to the principal eigenvalue $\widehat{\lambda}_1(p) > 0$. The resonance occurs from the right of $\widehat{\lambda}_1(p)$ in the sense that

$$pF(z,x) - \widehat{\lambda}_1(p)|x|^p \to +\infty \text{ uniformly for a.a. } z \in \Omega, \text{ as } x \to \pm\infty. \tag{8}$$

To see this, note that hypothesis $H(f)_1(iii)$ implies that given any $\xi > 0$ we can find $M_1 = M_1(\xi) > 0$ such that

$$pF(z,x) - f(z,x)x \geq \xi \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \geq M_1. \tag{9}$$

We have

$$\begin{aligned} \frac{d}{dx}\left[\frac{F(z,x)}{|x|^p}\right] &= \frac{f(z,x)|x|^p - p|x|^{p-2}xF(z,x)}{|x|^{2p}} \\ &= \frac{f(z,x)x - pF(z,x)}{|x|^p x} \begin{cases} \leq \frac{-\xi}{x^{p+1}} & \text{if } x \geq M_1, \\ \geq \frac{-\xi}{|x|^p x} & \text{if } x \leq -M_1 \end{cases} \text{(see (9))}, \end{aligned}$$

$$\Rightarrow \quad \frac{F(z,y)}{|y|^p} - \frac{F(z,x)}{|x|^p} \leq -\frac{\xi}{p}\left[\frac{1}{|x|^p} - \frac{1}{|y|^p}\right] \tag{10}$$

$$\text{for a.a. } z \in \Omega, \text{ all } |y|, |x| \geq M_1.$$

Hypothesis $H(f)_1(ii)$ implies that

$$\widehat{\lambda}_1(p) \le \liminf_{x\to\pm\infty} \frac{pF(z,x)}{|x|^p} \le \limsup_{x\to\pm\infty} \frac{pF(z,x)}{|x|^p} \le \eta(z) \quad \text{uniformly for a.a. } z \in \Omega. \tag{11}$$

So, if in (10) we let $y \to \pm\infty$ and we use (11), then

$$pF(z,x) - \widehat{\lambda}_1(p)|x|^p \ge \xi \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \ge M_1. \tag{12}$$

Since $\xi > 0$ is arbitrary, from (12) we infer that (8) holds.

Finally hypothesis $H(f)_1(iv)$ permits also resonance at zero with respect to any eigenvalue of $(-\Delta, H_0^1(\Omega))$.

Let $\varphi : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the energy functional for problem (1) defined by

$$\varphi(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega F(z,u)dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Evidently $\varphi \in C^2(W_0^{1,p}(\Omega))$.

PROPOSITION 3.2. *If hypotheses $H(f)_1$ hold, then the energy functional φ satisfies the C-condition.*

PROOF. Let $\{u_n\}_{n\ge1} \subseteq W_0^{1,p}(\Omega)$ be a sequence such that

$$|\varphi(u_n)| \le M_2 \quad \text{for some } M_2 > 0, \text{ all } n \in \mathbb{N}, \tag{13}$$

$$(1 + \|u_n\|)\varphi'(u_n) \to 0 \quad \text{in } W^{-1,p'}(\Omega) \text{ as } n \to +\infty. \tag{14}$$

From (14) we have

$$\left|\langle A_p(u_n), h\rangle + \langle A(u_n), h\rangle - \int_\Omega f(z,u_n)h dz\right| \le \frac{\varepsilon_n\|h\|}{1 + \|u_n\|} \tag{15}$$

for all $h \in W_0^{1,p}(\Omega)$, with $\varepsilon_n \to 0^+$. In (15) we choose $h = u_n \in W_0^{1,p}(\Omega)$. Then

$$\|\nabla u_n\|_p^p + \|\nabla u_n\|_2^2 - \int_\Omega f(z,u_n)u_n dz \le \varepsilon_n \quad \text{for all } n \in \mathbb{N}. \tag{16}$$

On the other hand, from (13) we have

$$-\|\nabla u_n\|_p^p - \frac{p}{2}\|\nabla u_n\|_2^2 + \int_\Omega pF(z,u_n)dz \le pM_2 \quad \text{for all } n \in \mathbb{N}. \tag{17}$$

Adding (16) and (17), we obtain

$$\begin{aligned}
&\int_\Omega [pF(z,u_n) - f(z,u_n)u_n]dz \\
\le{}& c_2 + \left(\frac{p}{2} - 1\right)\|\nabla u_n\|_2^2 \\
\le{}& c_3[1 + \|u_n\|^2] \text{ for some } c_2, c_3 > 0, \text{ all } n \in \mathbb{N} \text{ (recall that } p > 2).
\end{aligned} \tag{18}$$

Suppose that $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ is unbounded. By passing to a suitable subsequence if necessary, we may assume that

$$\|u_n\| \to +\infty. \tag{19}$$

Set $y_n = \dfrac{u_n}{\|u_n\|}$, $n \in \mathbb{N}$. We have $\|y_n\| = 1$ for all $n \in \mathbb{N}$ and so, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega) \text{ and } y_n \to y \text{ in } L^p(\Omega) \text{ as } n \to +\infty. \tag{20}$$

From (15) we have

$$\langle A_p(y_n), h\rangle + \frac{1}{\|u_n\|^{p-2}}\langle A(y_n), h\rangle - \int_\Omega \frac{N_f(u_n)}{\|u_n\|^{p-1}}\, h\, dz \leq \frac{\varepsilon_n \|h\|}{(1+\|u_n\|)\|u_n\|^{p-1}} \tag{21}$$

for all $n \in \mathbb{N}$. Hypotheses $H(f)_1(i), (ii)$ imply that

$$\left\{\frac{N_f(u_n)}{\|u_n\|^{p-1}}\right\}_{n\geq 1} \subseteq L^{p'}(\Omega) \text{ is bounded.}$$

This fact and hypothesis $H(f)_1(ii)$ imply that, at least for a subsequence, we have

$$\frac{N_f(u_n)}{\|u_n\|^{p-1}} \xrightarrow{w} \widehat{\eta}(z)|y|^{p-2}y \text{ in } L^{p'}(\Omega) \text{ as } n \to +\infty, \tag{22}$$

$\widehat{\lambda}_1(p) \leq \widehat{\eta}(z) \leq \eta(z)$ for a.a. $z \in \Omega$, (see Aizicovici-Papageorgiou-Staicu [1], proof of Proposition 30).

In (21) we choose $h = y_n - y \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (19), (20), (22) and the fact that $p > 2$. We obtain

$$\lim_{n\to+\infty} \langle A_p(y_n), y_n - y\rangle = 0,$$

$$\Rightarrow \quad y_n \to y \text{ in } W_0^{1,p}(\Omega) \text{ and so } \|y\| = 1, \text{ (see Proposition 2.2)} \tag{23}$$

So, if in (21) we pass to limit as $n \to +\infty$ and use (23) and (22), then

$$\langle A_p(y), h\rangle = \int_\Omega \widehat{\eta}(z)|y|^{p-2}yh\, dz \quad \text{for all } h \in W_0^{1,p}(\Omega),$$

$$\Rightarrow \quad -\Delta_p y(z) = \widehat{\eta}(z)|y(z)|^{p-2}y(z) \quad \text{for a.a. } z \in \Omega, \quad y\Big|_{\partial\Omega} = 0. \tag{24}$$

Recall that

$$\widehat{\lambda}_1(p) \leq \widehat{\eta}(z) \leq \eta(z) \quad \text{for a.a. } z \in \Omega \text{ (see (22))}.$$

First assume that $\widehat{\eta} \not\equiv \widehat{\lambda}_1(p)$. We have

$$\widetilde{\lambda}_1(p, \widehat{\eta}) < \widetilde{\lambda}_1(p, \widehat{\lambda}_1(p)) = 1 \text{ and } 1 = \widetilde{\lambda}_2(p, \widehat{\lambda}_2(p)) < \widetilde{\lambda}_2(p, \widehat{\eta})$$

(see Proposition 2.4). So, from (24), we infer that $y = 0$, which contradicts (23).

Next assume that $\widehat{\eta}(z) = \widehat{\lambda}_1(p)$ for a.a. $z \in \Omega$. Then from (24) and (23) it follows that

$$y = \beta\widehat{u}_1(p) \text{ with } \beta \in \mathbb{R} \setminus \{0\}.$$

Without any loss of generality we may assume that $\beta > 0$ (the reasoning is similar if $\beta < 0$). We have $y(z) > 0$ for all $z \in \Omega$ and so

$$u_n(z) \to +\infty \quad \text{for a.a. } z \in \Omega \text{ (see (19))}.$$

On account of hypothesis $H(f)_1(iii)$ we have that

$$\begin{aligned}
&\beta_0 \le \liminf_{n\to+\infty} \frac{pF(z,u_n(z)) - f(z,u_n(z))u_n(z)}{|u_n(z)|^\tau} \quad \text{for a.a. } z \in \Omega,\\
\Rightarrow\ & 0 < c_4 \le \liminf_{n\to+\infty} \int_\Omega \frac{pF(z,u_n) - f(z,u_n)u_n}{|u_n|^\tau} y_n^\tau dz\\
&\qquad \text{(by Fatou's lemma, see hypothesis } H(f)_1(iii)),\\
\Rightarrow\ & 0 < \liminf_{n\to+\infty} \frac{1}{\|u_n\|^\tau} \int_\Omega [pF(z,u_n) - f(z,u_n)u_n]dz. \qquad (25)
\end{aligned}$$

On the other hand from (18) we have

$$\frac{1}{\|u_n\|^\tau} \int_\Omega [pF(z,u_n) - f(z,u_n)u_n]dz \le c_3 \left[\frac{1}{\|u_n\|^\tau} + \frac{1}{\|u_n\|^{\tau-2}}\right] \text{ for all } n \in \mathbb{N},$$

Since $\tau > 2$, using (19), we conclude that

$$\limsup_{n\to+\infty} \frac{1}{\|u_n\|^\tau} \int_\Omega [pF(z,u_n) - f(z,u_n)u_n]dz \le 0. \qquad (26)$$

Comparing (25) and (26), we have a contradiction. This proves that $\{u_n\}_{n\ge1} \subseteq W_0^{1,p}(\Omega)$ is bounded. So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega) \text{ and } u_n \to u \text{ in } L^p(\Omega). \qquad (27)$$

In (15) we choose $h = u_n - u \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (27). Then

$$\begin{aligned}
&\lim_{n\to+\infty} [\langle A_p(u_n), u_n - u\rangle + \langle A(u_n), u_n - u\rangle] = 0,\\
\Rightarrow\ & \limsup_{n\to+\infty} [\langle A_p(u_n), u_n - u\rangle + \langle A(u), u_n - u\rangle] \le 0\\
&\qquad \text{(exploiting the monotonicity of } A(\cdot)),\\
\Rightarrow\ & \limsup_{n\to+\infty} \langle A_p(u_n), u_n - u\rangle \le 0 \text{ (see (27))},\\
\Rightarrow\ & u_n \to u \text{ in } W_0^{1,p}(\Omega) \text{ (see Proposition 2.2)}.
\end{aligned}$$

This proves that φ satisfies the C-condition. □

PROPOSITION 3.3. *If hypotheses $H(f)_1$ hold, then $C_k(\varphi, 0) = \delta_{k,d_m}\mathbb{Z}$ for all $k \in \mathbb{N}_0$ with $d_m = \dim \oplus_{i=1}^m E(\widehat{\lambda}_i(2))$.*

PROOF. We consider the C^2-functional $\widehat{\psi}_0 : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\psi}_0(u) = \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega F(z,u)dz \quad \text{for all } u \in H_0^1(\Omega).$$

Hypothesis $H(f)_1(iv)$ implies that given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that

$$F(z,x) \leq \frac{1}{2}[f_x'(z,0)+\varepsilon]x^2 \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \leq \delta. \tag{28}$$

We consider the following orthogonal direct sum decomposition

$$H_0^1(\Omega) = \overline{H}_m \oplus \widehat{H}_m,$$

where $\overline{H}_m = \oplus_{i=1}^m E(\widehat{\lambda}_i(2), \ \widehat{H}_m = \overline{H}_m^{\perp} = \overline{\oplus_{i\geq m+1} E(\widehat{\lambda}_i(2))}$. The space $\overline{H}_m$ is finite dimensional. So, all norms are equivalent. Hence we can find $\rho_0 > 0$ such that

$$u \in \overline{H}_m, \ \|u\| \leq \rho_0 \ \Rightarrow |u(z)| \leq \delta_0 \quad \text{for all } z \in \overline{\Omega}.$$

Here $\delta_0 > 0$ is as postulated by hypothesis $H(f)_1(iv)$. Then for $u \in \overline{H}_m$ with $\|u\| \leq \rho_0$, on account of hypothesis $H(f)_1(iv)$ we have

$$\widehat{\psi}_0(u) \leq \frac{1}{2}\|\nabla u\|_2^2 - \frac{\widehat{\lambda}_m(2)}{2}\|u\|_2^2 \leq 0 \quad (\text{see } (6)). \tag{29}$$

On the other hand, from (28) and hypothesis $H(f)_1(i)$, we have

$$F(z,x) \leq \frac{1}{2}[f_x'(z,0)+\varepsilon]x^2 + c_5|x|^q \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}$$

with $c_5 > 0$, $2 < q \leq 2^*$. Then for $u \in \widehat{H}_m$, we have

$$\begin{aligned} \widehat{\psi}_0(u) &\geq \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2}\int_\Omega f_x'(z,0)u^2dz - \frac{\varepsilon}{2\widehat{\lambda}_1(2)}\|u\|^2 - c_6\|u\|^q \\ &\qquad\qquad \text{for some } c_6 > 0 \text{ (see (3) and (5))} \\ &\geq \frac{1}{2}\left[c_7 - \frac{\varepsilon}{\widehat{\lambda}_1(2)}\right]\|u\|^2 - c_6\|u\|^q \text{ for some } c_7 > 0 \\ &\qquad\qquad \text{(see Proposition 2.5(b)).} \end{aligned}$$

Choosing $\varepsilon \in (0, \widehat{\lambda}_1(2)c_7)$, we obtain

$$\widehat{\psi}_0(u) \geq c_8\|u\|^2 - c_6\|u\|^q \quad \text{for some } c_8 > 0, \text{ all } u \in W_0^{1,p}(\Omega). \tag{30}$$

Since $q > 2$ from (30) we see that we can find $\rho_1 \in (0,1)$ small such that

$$\widehat{\psi}_0(u) > 0 \quad \text{for all } u \in \widehat{H}_m, \text{ with } 0 < \|u\| \leq \rho_1. \tag{31}$$

From (29) and (31), we infer that $\widehat{\psi}_0$ has a local linking at $u = 0$ with respect to decomposition $H_0^1(\Omega) = \overline{H}_m \oplus \widehat{H}_m$. Invoking Proposition 2.3 of Su [25], we have

$$C_k(\widehat{\psi}_0, 0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{32}$$

Let $\psi_0 = \widehat{\psi}_0\big|_{W_0^{1,p}(\Omega)}$. The space $W_0^{1,p}(\Omega)$ is dense in $H_0^1(\Omega)$. So, from Palais [16] (see also Chang [3], p. 14) we have

$$C_k(\psi_0, 0) = C_k(\widehat{\psi}_0, 0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0 \quad (\text{see } (32)). \tag{33}$$

We have

$$
\begin{aligned}
& |\varphi(u)-\psi_0(u)| = \frac{1}{p}\|\nabla u\|_p^p, \\
& |\langle \varphi'(u)-\psi_0'(u), h\rangle| = \int_\Omega |\nabla u|^{p-1}(\nabla u, \nabla h)_{\mathbb{R}^N} dz, \\
\Rightarrow\quad & |\langle \varphi'(u)-\psi_0'(u), h\rangle| \le \|\nabla u\|_p^{p-1}\|\nabla h\|_p, \\
\Rightarrow\quad & \|\varphi'(u)-\psi_0'(u)\|_* \le \|\nabla u\|_p^{p-1}.
\end{aligned}
$$

Then from the C^1-continuity of critical groups (see Gasiński-Papageorgiou [9], Theorem 5.126, p. 836), we have

$$
\begin{aligned}
& C_k(\varphi,0) = C_k(\psi_0,0) \text{ for all } k\in\mathbb{N}_0, \\
\Rightarrow\quad & C_k(\varphi,0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k\in\mathbb{N}_0 \quad \text{(see (33))}.
\end{aligned}
$$

□

Proposition 3.4. *If hypotheses $H(f)_1$ hold, then $C_k(-\varphi,\infty)=\delta_{k,1}\mathbb{Z}$ for all $k\in\mathbb{N}_0$ and $C_k(\varphi,u)=C_k(-\varphi,u)$ for all $u\in K_\varphi=K_{-\varphi}$, all $k\in\mathbb{N}_0$.*

Proof. Let $\lambda\in(\widehat{\lambda}_1(p),\widehat{\lambda}_2(p))$ and consider the C^1-functional $\gamma: W_0^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\gamma(u) = \frac{1}{p}\|\nabla u\|_p^p - \frac{\lambda}{p}\|u\|_p^p \quad \text{for all } u\in W_0^{1,p}(\Omega).$$

We consider the homotopy

$$h(t,u) = (1-t)(-\varphi)(u) + t\gamma(u) \quad \text{for all } t\in[0,1], \text{ all } u\in W_0^{1,p}(\Omega).$$

Claim: We can find $\eta\in\mathbb{R}$ and $\delta>0$ such that

$$h(t,u)\le\eta \Rightarrow (1+\|u\|)\|h_u'(t,u)\|_*\ge\delta \text{ for all } t\in[0,1]. \tag{34}$$

Arguing by contradiction, suppose that the Claim is not true. Since $h(\cdot,\cdot)$ maps bounded sets to bounded sets, we can find $\{t_n\}_{n\ge1}\subseteq[0,1]$ and $\{u_n\}_{n\ge1}\subseteq W_0^{1,p}(\Omega)$ such that

$$t_n\to t,\ \|u_n\|\to+\infty,\ h(t_n,u_n)\to-\infty \text{ and } (1+\|u_n\|)h_u'(t_n,u_n)\to 0 \text{ in } W^{-1,p'}(\Omega) \text{ as } n\to+\infty. \tag{35}$$

From the last convergence in (35), we have

$$
\begin{aligned}
& \Big|\langle A_p(u_n),h\rangle + (1-t_n)\langle A(u_n),h\rangle - (1-t_n)\int_\Omega f(z,u_n)h dz \\
& - t_n\lambda\int_\Omega |u_n|^{p-2}u_n h dz\Big| \le \frac{\varepsilon_n\|h\|}{1+\|u_n\|} \quad \text{for all } h\in W_0^{1,p}(\Omega) \text{ with } \varepsilon_n\to 0^+.
\end{aligned} \tag{36}
$$

In (36) we choose $h=u_n\in W_0^{1,p}(\Omega)$. Then

$$\|\nabla u_n\|_p^p + (1-t_n)\|\nabla u_n\|_2^2 - (1-t_n)\int_\Omega f(z,u_n)u_n dz - t_n\lambda\|u_n\|_p^p \le \varepsilon_n \tag{37}$$

for all $n \in \mathbb{N}$. Also, from the third convergence in (35), we have

$$-\|\nabla u_n\|_p^p - \frac{p}{2}(1-t_n)\|\nabla u_n\|_2^2 + (1-t_n)\int_\Omega pF(z,u_n)dz + t_n\lambda\|u_n\|_p^p \leq -1 \tag{38}$$

for all $n \geq n_0$. We add (37) and (38). Since $\varepsilon_n \to 0^+$, we obtain

$$(1-t_n)\int_\Omega [pF(z,u_n) - f(z,u_n)u_n]dz \leq (1-t_n)\left[\frac{p}{2}-1\right]\|\nabla u_n\|_2^2 \tag{39}$$

for all $n \geq n_1 \geq n_0$. Note that $h(1,u) = \gamma(u)$ and since $\lambda \in (\widehat{\lambda}_1(p), \widehat{\lambda}_2(p))$, we have

$$K_\gamma = \{0\}.$$

It follows that we may assume that $t_n \neq 1$ for all $n \in \mathbb{N}$. So, from (39) we have

$$\int_\Omega [pF(z,u_n) - f(z,u_n)u_n]dz \leq \left[\frac{p}{2}-1\right]\|\nabla u_n\|_2^2 \text{ for all } n \geq n_1. \tag{40}$$

We argue as in the proof of Proposisition 3.2. So, suppose that $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ is unbounded. We may assume, at least for a subsequence, that

$$\|u_n\| \to +\infty. \tag{41}$$

Let $y_n = \dfrac{u_n}{\|u_n\|}$, $n \in \mathbb{N}$. Then $\|y_n\| = 1$ for all $n \in \mathbb{N}$ and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega) \text{ and } y_n \to y \text{ in } L^p(\Omega) \text{ as } n \to +\infty. \tag{42}$$

From (36) we have

$$\begin{aligned}&\Big|\langle A_p(y_n), h\rangle + \frac{1-t_n}{\|u_n\|^{p-2}}\langle A(y_n), h\rangle - (1-t_n)\int_\Omega \frac{N_f(u_n)}{\|u_n\|^{p-1}}hdz \\ &- t_n\lambda\int_\Omega |y_n|^{p-2}y_nhdz\Big| \leq \frac{\varepsilon_n\|h\|}{(1+\|u_n\|)\|u_n\|^{p-1}} \quad \text{for all } n \in \mathbb{N}.\end{aligned} \tag{43}$$

Choosing $h = y_n - y \in W_0^{1,p}(\Omega)$, passing to the limit as $n \to +\infty$ and using (22), (41), (42) and the fact that $2 < p$, from (43) we obtain

$$\begin{aligned}&\lim_{n\to+\infty}\langle A_p(y_n), y_n - y\rangle = 0 \\ \Rightarrow\quad & y_n \to y \text{ in } W_0^{1,p}(\Omega) \text{ and so } \|y\| = 1.\end{aligned} \tag{44}$$

Passing to the limit as $n \to +\infty$ in (43) and using (22), (41), (44) and the fact that $2 < p$, we obtain

$$\begin{aligned}&\langle A_p(u), h\rangle = \int_\Omega [(1-t)\widehat{\eta}(z) + t\lambda]|y|^{p-2}yhdz \text{ for all } h \in W_0^{1,p}(\Omega), \\ \Rightarrow\quad & -\Delta_p y(z) = [(1-t)\widehat{\eta}(z) + t\lambda]|y(z)|^{p-2}y(z) \text{ for a.a. } z \in \Omega,\ y\Big|_{\partial\Omega} = 0.\end{aligned} \tag{45}$$

If $t \neq 0$, then $(1-t)\widehat{\eta}(z) + t\lambda \geq \widehat{\lambda}_1(p)$ for a.a. $z \in \Omega$, $(1-t)\widehat{\eta} + t\lambda \not\equiv \widehat{\lambda}_1(p)$. Then from (45) and Proposition 2.4, we infer that $y \equiv 0$, a contradiction (see (44)).

If $t = 0$ and $\widehat{\eta} \not\equiv \widehat{\lambda}_1(p)$, then the above argument remains valid. Finally, if $t = 0$ and $\widehat{\eta}(z) = \widehat{\lambda}_1(p)$, for a.a. $z \in \Omega$, then we argue as in the proof of Proposition 3.2 and using hypothesis $H(f)_1(iii)$, we contradict (40).

Therefore the Claim is true and (34) holds. Invoking Theorem 5.1.21, p. 334 of Chang [4] (see also Theorem 3.2 of Liang-Su [13]), we have

$$C_k(-\varphi,\infty) = C_k(\gamma,\infty) \quad \text{for all } k \in \mathbb{N}_0. \tag{46}$$

Since $\lambda \in (\widehat{\lambda}_1(p), \widehat{\lambda}_2(p))$, we have

$$C_k(\gamma,\infty) = C_k(\gamma,0) \quad \text{for all } k \in \mathbb{N}_0. \tag{47}$$

But from Perera [23], we have

$$\begin{aligned} & C_k(\gamma,0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0, \\ \Rightarrow\ & C_k(-\varphi,\infty) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0 \text{ (see (46), (47))}. \end{aligned} \tag{48}$$

Clearly, $K_\varphi = K_{-\varphi}$. We assume that this set is finite or otherwise we already have an infinity of solutions and so we are done.

Consider the homotopy

$$\widehat{h}(t,u) = (1-t)\varphi(u) + t(-\varphi)(u) = (1-2t)\varphi(u).$$

Since K_φ is finite, the homotopy invariance property of critical groups (see Gasiński-Papageorgiou [9], Theorem 5.125, p. 836), implies that

$$C_k(\varphi,u) = C_k(-\varphi,u) \quad \text{for all } k \in \mathbb{N}_0, \text{ all } u \in K_\varphi = K_{-\varphi}.$$

□

Now we are ready for the first multiplicity theorem for problem (1). It covers the case of problems resonant with respect to the principal eigenvalue $\widehat{\lambda}_1(p) > 0$ (resonance from the right).

THEOREM 3.5. *If hypotheses $H(f)_1$ hold, then problem* (1) *admits at least two nontrivial solutions $u_0, \widehat{u} \in C_0^1(\overline{\Omega})$.*

PROOF. From Proposition 3.4, we known that

$$C_k(-\varphi,\infty) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0.$$

So, we can find $u_0 \in K_{-\varphi} = K_\varphi$ such that

$$0 \neq C_1(-\varphi,u_0) = C_1(\varphi,u_0) \quad \text{(see Proposition 3.4)}.$$

Since $\varphi \in C^2(W_0^{1,p}(\Omega))$, from Papageorgiou-Rădulescu [17], we have

$$C_k(\varphi,u_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{49}$$

From Proposition 3.3 we have

$$C_k(\varphi,0)=\delta_{k,d_m}\mathbb{Z} \quad \text{for all } k\in\mathbb{N}_0. \tag{50}$$

From the Morse relation (see (7)), with $t=-1$, we have

$$\begin{aligned}\Sigma_{k\in\mathbb{N}_0}\operatorname{rank} C_k(\varphi,\infty) &= \Sigma_{u\in K_\varphi}\Sigma_{k\in\mathbb{N}_0}\operatorname{rank} C_k(\varphi,u)\\ &= \Sigma_{u\in K_{-\varphi}}\Sigma_{k\in\mathbb{N}_0}\operatorname{rank} C_k(-\varphi,u) \text{ (see Proposition 3.4)}\\ &= \Sigma_{k\in\mathbb{N}_0}\operatorname{rank} C_k(-\varphi,\infty)=1 \text{ (see (48)).} \qquad (51)\end{aligned}$$

Suppose that $K_\varphi=\{0,u_0\}$. Then from (49), (50), (51), we have

$$(-1)^1+(-1)^{d_m}=1,$$

a contradiction. Therefore there exists $\widehat{u}\in K_\varphi$, $\widehat{u}\notin\{0,u_0\}$. Evidently $\widehat{u}$ is the second nontrivial solution of (1). From Ladyzhenskaya-Ural′tseva [11] (Theorem 7.1, p. 286), we have $u_0,\widehat{u}\in L^\infty(\Omega)$.

Invoking Theorem 1 of Lieberman [12], we conclude that $u_0,\widehat{u}\in C_0^1(\overline{\Omega})$. □

4. Resonance with respect to higher eigenvalues

In this section we investigate what happens when we have resonance with respect to a nonprincipal variational eigenvalue $\widehat{\lambda}_m(p)$, $m\geq 2$.

The hypotheses on the reaction term $f(z,x)$, are the following:

H(f)$_2$: $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a measurable function such that for a.a. $z\in\Omega$ $f(z,0)=0$, $f(z,\cdot)\in C^1(\mathbb{R})$ and

(i) $|f'_x(z,x)|\leq a(z)[1+|x|^{r-2}]$ for a.a. $z\in\Omega$, all $x\in\mathbb{R}$, with $a\in L^\infty(\Omega)$, $p\leq r<p^*$;

(ii) there exists integer $m\geq 2$ such that

$$\lim_{x\to\pm\infty}\frac{f(z,x)}{|x|^{p-2}x}=\widehat{\lambda}_m(p) \text{ uniformly for a.a. } z\in\Omega;$$

(iii) $f(z,x)x-pF(z,x)\to+\infty$ uniformly for a.a. $z\in\Omega$ as $x\to\pm\infty$;

(iv) $f'_x(z,0)=\lim_{x\to 0}\dfrac{f(z,x)}{x}$ uniformly for a.a. $z\in\Omega$, $f'_x(z,0)\geq\widehat{c}>0$ for a.a. $z\in\Omega$ and $f'_x(z,0)\leq\widehat{\lambda}_1(2)$ for a.a. $z\in\Omega$, $f'_x(\cdot,0)\not\equiv\widehat{\lambda}_1(2)$.

We introduce the positive and negative truncations of $f(z,\cdot)$, namely the Carathéodory functions

$$f_\pm(z,x)=f(z,\pm x^\pm).$$

We set $F_\pm(z,x)=\int_0^x f_\pm(z,s)ds$ and consider the C^1-functionals $\varphi_\pm: W_0^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\varphi_\pm(u)=\frac{1}{p}\|\nabla u\|_p^p+\frac{1}{2}\|\nabla u\|_2^2-\int_\Omega F_\pm(z,u)dz \quad \text{for all } u\in W_0^{1,p}(\Omega).$$

As before $\varphi : W_0^{1,p}(\Omega) \to \mathbb{R}$ is the energy functional for problem (1) defined by

$$\varphi(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega F(z,u)dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

We have $\varphi \in C^2(W_0^{1,p}(\Omega))$.

First we show that these functionals satisfy the compactness condition.

PROPOSITION 4.1. *If hypotheses $H(f)_2$ hold, then the functionals φ and $\varphi_\pm$ satisfy the C-condition.*

PROOF. Let $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ be a sequence such that

$$|\varphi(u_n)| \leq M_3 \quad \text{for some } M_3 > 0, \text{ all } n \in \mathbb{N}, \tag{52}$$

$$(1 + \|u_n\|)\varphi'(u_n) \to 0 \quad \text{in } W^{-1,p'}(\Omega) \text{ as } n \to +\infty. \tag{53}$$

From (53) we have

$$\left|\langle A_p(u_n), h\rangle + \langle A(u_n), h\rangle - \int_\Omega f(z,u_n)h dz\right| \leq \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \tag{54}$$

for all $h \in W_0^{1,p}(\Omega)$, with $\varepsilon_n \to 0^+$. In (54) we choose $h = u_n \in W_0^{1,p}(\Omega)$. Then

$$-\|\nabla u_n\|_p^p - \|\nabla u_n\|_2^2 + \int_\Omega f(z,u_n)u_n dz \leq \varepsilon_n \quad \text{for all } n \in \mathbb{N}. \tag{55}$$

On the other hand, from (52) we have

$$\|\nabla u_n\|_p^p + \frac{p}{2}\|\nabla u_n\|_2^2 - \int_\Omega pF(z,u_n)dz \leq pM_3 \quad \text{for all } n \in \mathbb{N}. \tag{56}$$

We add (55) and (56). Since $2 < p$, we obtain

$$\int_\Omega [f(z,u_n)u_n - pF(z,u_n)]dz \leq M_4 \text{ for some } M_4 > 0, \text{ all } n \in \mathbb{N}. \tag{57}$$

Suppose that

$$\|u_n\| \to +\infty \text{ as } n \to +\infty. \tag{58}$$

Let $y_n = \dfrac{u_n}{\|u_n\|}$, $n \in \mathbb{N}$. Then $\|y_n\| = 1$ for all $n \in \mathbb{N}$ and so, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega) \text{ and } y_n \to y \text{ in } L^p(\Omega). \tag{59}$$

From (54) we have

$$\left|\langle A_p(y_n), h\rangle + \frac{1}{\|u_n\|^{p-2}}\langle A(y_n), h\rangle - \int_\Omega \frac{N_f(u_n)}{\|u_n\|^{p-1}}\, h\, dz\right| \leq \frac{\varepsilon_n\|h\|}{(1 + \|u_n\|)\|u_n\|^{p-1}} \tag{60}$$

for all $n \in \mathbb{N}$. Hypotheses $H(f)_2(i), (ii)$ imply that

$$|f(z,x)| \leq c_9[1 + |x|^{p-1}] \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ some } c_9 > 0. \tag{61}$$

From (61) it follows that

$$\left\{\frac{N_f(u_n)}{\|u_n\|^{p-1}}\right\}_{n\geq 1} \subseteq L^{p'}(\Omega) \text{ is bounded.} \tag{62}$$

So, by passing to a subsequence if necessary and using hypothesis $H(f)_2(ii)$, we have

$$\frac{N_f(u_n)}{\|u_n\|^{p-1}} \xrightarrow{w} \widehat{\lambda}_m(p)|y|^{p-2}y \text{ in } L^{p'}(\Omega).$$

In (60) we choose $h = y_n - y \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (58), (59), (62) and the fact that $p > 2$. Then

$$\lim_{n\to+\infty} \langle A_p(y_n), y_n - y\rangle = 0,$$

$$\Rightarrow \quad y_n \to y \text{ in } W_0^{1,p}(\Omega) \text{ and so } \|y\| = 1, \text{ (see Proposition 2.2).} \tag{63}$$

Let $\Omega_0 = \{z \in \Omega : y(z) \neq 0\}$. Evidently $|\Omega_0|_N > 0$ (see (63)) and

$$|u_n(z)| \to +\infty \quad \text{ for a.a. } z \in \Omega_0.$$

Then hypothesis $H(f)_2(iii)$ implies that

$$f(z, u_n(z))u_n(z) - pF(z, u_n(z)) \to +\infty \text{ for a.a. } z \in \Omega_0,$$

$$\Rightarrow \quad \int_{\Omega_0} [f(z, u_n)u_n - pF(z, u_n)]dz \to +\infty \text{ (by Fatou's lemma).} \tag{64}$$

Hypotheses $H(f)_2(i), (iii)$ imply that

$$f(z, x)x - pF(z, x) \geq -c_{10} \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ some } c_{10} > 0. \tag{65}$$

Then we have

$$\begin{aligned}
&\int_\Omega [f(z, u_n)u_n - pF(z, u_n)]dz \\
&= \int_{\Omega_0} [f(z, u_n)u_n - pF(z, u_n)]dz + \int_{\Omega\setminus\Omega_0} [f(z, u_n)u_n - pF(z, u_n)]dz \\
&\geq \int_{\Omega_0} [f(z, u_n)u_n - pF(z, u_n)]dz - c_{10}|\Omega|_N \text{ (see (65)),}
\end{aligned}$$

$$\Rightarrow \quad \int_\Omega [f(z, u_n)u_n - pF(z, u_n)]dz \to +\infty \text{ as } n \to +\infty \text{ (see (64)).} \tag{66}$$

Comparing (57) and (66), we have a contradiction.

Therefore we can say that

$$\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega) \quad \text{ is bounded.}$$

So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega) \text{ and } u_n \to u \text{ in } L^p(\Omega). \tag{67}$$

In (54) we choose $h = u_n - u \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (67). Then

$$\begin{aligned}
&\lim_{n\to+\infty} [\langle A_p(u_n), u_n - u\rangle + \langle A(u_n), u_n - u\rangle] = 0,\\
\Rightarrow\ &\limsup_{n\to+\infty} [\langle A_p(u_n), u_n - u\rangle + \langle A(u), u_n - u\rangle] \leq 0 \text{ (since } A(\cdot) \text{ is monotone)},\\
\Rightarrow\ &\limsup_{n\to+\infty} \langle A_p(u_n), u_n - u\rangle \leq 0 \text{ (see (67))},\\
\Rightarrow\ &u_n \to u \text{ in } W_0^{1,p}(\Omega) \text{ (see Proposition 2.2)}.
\end{aligned}$$

Therefore the functional φ satisfies the C-condition.

Next we consider the functional φ_+.

We consider a sequence $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ such that

$$|\varphi_+(u_n)| \leq M_5 \quad \text{for some } M_5 > 0, \text{ all } n \in \mathbb{N}, \tag{68}$$

$$(1 + \|u_n\|)\varphi_+'(u_n) \to 0 \quad \text{in } W^{-1,p'}(\Omega) \text{ as } n \to +\infty. \tag{69}$$

From (69) we have

$$\left|\langle A_p(u_n), h\rangle + \langle A(u_n), h\rangle - \int_\Omega f_+(z, u_n) h dz\right| \leq \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \tag{70}$$

for all $h \in W_0^{1,p}(\Omega)$, with $\varepsilon_n \to 0^+$. In (70) we choose $h = -u_n^- \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned}
&\|\nabla u_n^-\|_p^p + \|\nabla u_n^-\|_2^2 \leq \varepsilon_n \quad \text{for all } n \in \mathbb{N},\\
\Rightarrow\ &u_n^- \to 0 \quad \text{in } W_0^{1,p}(\Omega). \qquad (71)
\end{aligned}$$

From (68) and (71), we have

$$\|\nabla u_n^+\|_p^p + \frac{p}{2}\|\nabla u_n^+\|_2^2 - \int_\Omega pF(z, u_n^+) dz \leq M_6 \quad \text{for some } M_6 > 0, \text{ all } n \in \mathbb{N}. \tag{72}$$

Also, from (70) with $h = u_n^+ \in W_0^{1,p}(\Omega)$, we have

$$-\|\nabla u_n^+\|_p^p - \|\nabla u_n^+\|_2^2 + \int_\Omega f(z, u_n^+) u_n^+ dz \leq \varepsilon_n \quad \text{for all } n \in \mathbb{N}. \tag{73}$$

Adding (72) and (73) and recalling that $2 < p$, we obtain

$$\int_\Omega [f(z, u_n^+) u_n^+ - pF(z, u_n^+)] dz \leq M_7 \text{ for some } M_7 > 0, \text{ all } n \in \mathbb{N}. \tag{74}$$

We use (74) and a contradiction argument and as before (see the part of the proof after (57) until (66)), we show that

$$\begin{aligned}
&\{u_n^+\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega) \quad \text{is bounded},\\
\Rightarrow\ &\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega) \quad \text{is bounded (see (71))}.
\end{aligned}$$

Using this fact and Proposition 2.2, as before we conclude that φ_+ satisfies the C-condition.

Similarly we show that φ_- satisfies the C-condition. $\square$

PROPOSITION 4.2. *If hypotheses $H(f)_2$ hold, then $u = 0$ is a local minimizer for the functionals $\varphi_\pm$ and φ.*

PROOF. We do the proof for the functional φ_+, the proofs for the functionals φ_- and φ being similar.

On account of hypothesis $H(f)_2(iv)$, given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that

$$F(z,x) \leq \frac{1}{2}[f'_x(z,0)+\varepsilon]x^2 \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \leq \delta. \tag{75}$$

Let $u \in C^1_0(\overline{\Omega})$ with $\|u\|_{C^1_0(\overline{\Omega})} \leq \delta$. We have

$$\begin{aligned}
\varphi_+(u) &\geq \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2}\int_\Omega f'_x(z,0)(u^+)^2 dz - \frac{\varepsilon}{2}\|u^+\|_2^2 \text{ (see (75))} \\
&\geq \frac{1}{2}\left[\|\nabla u^+\|_2^2 - \int_\Omega f'_x(z,0)(u^+)^2 dz - \frac{\varepsilon}{\widehat{\lambda}_1(2)}\|\nabla u^+\|_2^2\right] + \frac{1}{2}\|\nabla u^-\|_2^2 \\
&\qquad + \frac{1}{p}\|\nabla u\|_p^p \text{ (see (5))} \\
&\geq \frac{1}{p}\|\nabla u\|_p^p \text{ (using Proposition 2.5(b) and choosing } \varepsilon > 0 \text{ small)},
\end{aligned}$$

$\Rightarrow u = 0$ is a local $C^1_0(\overline{\Omega})$-minimizer of φ_+,

$\Rightarrow u = 0$ is a local $W^{1,p}_0(\Omega)$-minimizer of φ_+ (see Proposition 2.3).

$\square$

Let $W_+ = \{u \in W^{1,p}_0(\Omega) : u(z) \geq 0 \text{ for a.a. } z \in \Omega\}$. We can easily verify that

$$K_{\varphi_+} \subseteq W_+ \quad \text{and} \quad K_{\varphi_-} \subseteq -W_+.$$

So, we may assume that both K_{φ_+} and K_{φ_-} are finite. Otherwise we already have an infinity of nontrivial positive and negative solutions for problem (1).

PROPOSITION 4.3. *If hypotheses $H(f)_2$ hold, then problem* (1) *admits at least two nontrivial constant sign solutions $u_0 \in \operatorname{int} C_+$ and $v_0 \in -\operatorname{int} C_+$.*

PROOF. Proposition 4.2 implies that we can find $\rho \in (0,1)$ small such that

$$\varphi_+(0) = 0 < \inf\,[\varphi_+(u) : \|u\| = \rho] = m_+ \tag{76}$$

(see Aizicovici-Papageorgiou-Staicu [1], proof of Proposition 29). From Proposition 4.1 we know that

$$\varphi_+ \text{ satisfies the } C\text{-condition.} \tag{77}$$

Hypotheses $H(f)_2(i),(ii)$ imply that given $\varepsilon > 0$, we can find $c_{11} = c_{11}(\varepsilon) > 0$ such that

$$F(z,x) \geq \frac{1}{p}\left[\widehat{\lambda}_m(p) - \varepsilon\right]|x|^p - c_{11} \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{78}$$

Recall that $\widehat{u}_1(p) \in \operatorname{int} C_+$. For $t > 0$, we have

$$\varphi_+(t\widehat{u}_1(p)) \leq \frac{\widehat{\lambda}_1(p)}{p} t^p + \frac{t^2}{2} \|\nabla \widehat{u}_1(p)\|_2^2 - \frac{\widehat{\lambda}_m(p) - \varepsilon}{p} t^p + c_{11}|\Omega|_N$$

$$\text{(see (78) and recall that } \|\widehat{u}_1(p)\|_p = 1)$$

$$= \frac{t^p}{p}\left[\widehat{\lambda}_1(p) + \varepsilon - \widehat{\lambda}_m(p)\right] + \frac{t^2}{2}\|\nabla \widehat{u}_1(p)\|_2^2 + c_{11}|\Omega|_N. \tag{79}$$

We choose $0 < \varepsilon < \widehat{\lambda}_m(p) - \widehat{\lambda}_1(p)$ (recall that $m \geq 2$). Since $2 < p$, from (79) it follows that

$$\varphi_+(t\widehat{u}_1(p)) \to -\infty \quad \text{as } t \to +\infty. \tag{80}$$

Then (76), (77) and (80) permit the use of Theorem 2.1 (the mountain pass theorem). So, we can find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$u_0 \in K_{\varphi_+} \subseteq W_+ \text{ and } m_+ \leq \varphi_+(u_0) = \varphi(u_0),\ u_0 \neq 0 \text{ (see (76))}.$$

We have

$$\langle A_p(u_0), h\rangle + \langle A(u_0), h\rangle = \int_\Omega f(z, u_0) h dz \text{ for all } h \in W_0^{1,p}(\Omega),$$

$$\Rightarrow \quad -\Delta_p u_0(z) - \Delta u_0(z) = f(z, u_0(z)) \text{ for a.a. } z \in \Omega,\ u_0\Big|_{\partial\Omega} = 0. \tag{81}$$

Hypotheses $H(f)_2(i),(iv)$ imply that given $\rho > 0$, we can find $\widehat{\xi}_\rho > 0$ such that

$$f(z,x)x + \widehat{\xi}_\rho |x|^p \geq 0 \text{ for a.a. } z \in \Omega, \text{ all } |x| \leq \rho. \tag{82}$$

From (81) and the nonlinear regularity theory, we have

$$u_0 \in C_+ \setminus \{0\}.$$

Let $\rho = \|u_0\|_\infty$ and let $\widehat{\xi}_\rho > 0$ be as postulated by (82). Then from (81) and (82), we have

$$\Delta_p u_0(z) + \Delta u_0(z) \leq \widehat{\xi}_\rho u_0(z)^{p-1} \quad \text{for a.a. } z \in \Omega. \tag{83}$$

Let $a(y) = |y|^{p-2}y + y$ for all $y \in \mathbb{R}$. We have

$$\operatorname{div} a(\nabla u) = \Delta_p u + \Delta u \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Note that $a \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ (recall that $2 < p$). We have

$$\nabla a(y) = |y|^{p-2}\left[\mathrm{id} + (p-1)\frac{y \otimes y}{|y|^2}\right] + \mathrm{id},$$

$$\Rightarrow \quad (\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \geq |\xi|^2 \quad \text{for all } y, \xi \in \mathbb{R}^N.$$

So, using the tangency principle of Pucci-Serrin [24] (p. 35), we have

$$u_0(z) > 0 \quad \text{for all } z \in \Omega.$$

From (83) and the boundary point theorem of Pucci-Serrin [24] (p. 120), we have

$$u_0 \in \operatorname{int} C_+.$$

Similarly working with φ_-, we produce a negative solution

$$v_0 \in -\operatorname{int} C_+.$$

□

Next using the theory of critical groups, we will produce a third nontrivial smooth solution.

PROPOSITION 4.4. *If hypotheses $H(f)_2$ hold, then $C_m(\varphi, \infty) \neq 0$ with $m \in \mathbb{N}$ as in hypothesis $H(f)_2(ii)$.*

PROOF. Let $\widehat{\sigma}(p)$ denote the spectrum of $(-\Delta_p, W_0^{1,p}(\Omega))$. Pick $\lambda \in (\widehat{\lambda}_m(p), \widehat{\lambda}_{m+1}(p)) \setminus \widehat{\sigma}(p)$ and consider the C^1-functional $\psi : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi(u) = \frac{1}{p}\|\nabla u\|_p^p - \frac{\lambda}{p}\|u\|_p^p \text{ for all } u \in W_0^{1,p}(\Omega).$$

We introduce the homotopy $h(t, u)$ defined by

$$h(t, u) = (1 - t)\varphi(u) + t\psi(u) \quad \text{for all } t \in [0, 1], \text{ all } u \in W_0^{1,p}(\Omega).$$

Claim: There exist $\eta \in \mathbb{R}$ and $\delta > 0$ such that

$$h(t, u) \leq \eta \Rightarrow (1 + \|u\|)\|h'_u(t, u)\|_* \geq \delta \text{ for all } t \in [0, 1].$$

As in the proof of Proposition 3.4, we argue by contradiction. So, suppose that the Claim is not true. Since $h(\cdot, \cdot)$ maps bounded sets to bounded sets, we can find $\{t_n\}_{n\geq1} \subseteq [0, 1]$ and $\{u_n\}_{n\geq1} \subseteq W_0^{1,p}(\Omega)$ such that

$$t_n \to t,\ \|u_n\| \to +\infty,\ h(t_n, u_n) \to -\infty \text{ and } (1 + \|u_n\|)h'_u(t_n, u_n) \to 0 \tag{84}$$

as $n \to +\infty$. From the last convergence in (84), we have

$$\begin{aligned} &\big|\langle A_p(u_n), h\rangle + (1 - t_n)\langle A(u_n), h\rangle - (1 - t_n)\int_\Omega f(z, u_n)hdz \\ &- t_n\lambda\int_\Omega |u_n|^{p-2}u_n hdz\Big| \leq \frac{\varepsilon_n\|h\|}{1 + \|u_n\|} \quad \text{for all } h \in W_0^{1,p}(\Omega) \text{ with } \varepsilon_n \to 0^+. \end{aligned} \tag{85}$$

Choosing $h = u_n \in W_0^{1,p}(\Omega)$ in (85), we obtain

$$-\|\nabla u_n\|_p^p - (1 - t_n)\|\nabla u_n\|_2^2 + (1 - t_n)\int_\Omega f(z, u_n)u_n dz + t_n\lambda\|u_n\|_p^p \leq \varepsilon_n \tag{86}$$

for all $n \in \mathbb{N}$. From the third convergence in (84), we have

$$\|\nabla u_n\|_p^p + \frac{p}{2}(1 - t_n)\|\nabla u_n\|_2^2 - (1 - t_n)\int_\Omega pF(z, u_n)dz - t_n\lambda\|u_n\|_p^p \leq -1 \tag{87}$$

for all $n \geq n_0$. We add (86), (87). Since $2 < p$, we obtain

$$(1 - t_n)\int_\Omega [f(z, u_n)u_n - pF(z, u_n)]dz \leq 0 \text{ for all } n \geq n_1 \geq n_0. \tag{88}$$

We claim that $t_n < 1$ for all $n \geq n_2 \geq n_1$. Otherwise we can find a subsequence $\{t_{n_k}\}_{k\geq 1}$ of $\{t_n\}_{n\geq 1}$ such that $t_{n_k} = 1$ for all $k \in \mathbb{N}$. We have

$$h(t_{n_k}, u) = h(1, u) = \psi(u) \text{ for all } k \in \mathbb{N}, \text{ all } u \in W_0^{1,p}(\Omega).$$

Let $y_k = \dfrac{u_{n_k}}{\|u_{n_k}\|}$, $k \in \mathbb{N}$. We have $\|y_k\| = 1$ for all $k \in \mathbb{N}$ and so we may assume that

$$y_k \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega) \text{ and } y_k \to y \text{ in } L^p(\Omega) \text{ as } k \to +\infty. \tag{89}$$

From (86) and since $t_{n_k} = 1$ for all $k \in \mathbb{N}$, we have

$$\left| \langle A_p(y_k), h\rangle - \lambda \int_\Omega |y_k|^{p-2} y_k h dz \right| \leq \frac{\varepsilon_{n_k}\|h\|}{(1 + \|u_{n_k}\|)\|u_{n_k}\|^{p-1}} \quad \text{for all } k \in \mathbb{N}. \tag{90}$$

In (90) we choose $h = y_k - y \in W_0^{1,p}(\Omega)$, pass to the limit as $k \to +\infty$ and use (89). Then

$$\lim_{k\to+\infty} \langle A_p(y_k), y_k - y\rangle = 0,$$
$$\Rightarrow \quad y_k \to y \text{ in } W_0^{1,p}(\Omega) \text{ and so } \|y\| = 1 \text{ (see Proposition 2.2).} \tag{91}$$

Passing to the limit as $k \to +\infty$ in (90) and using (91), we obtain

$$\langle A_p(y), h\rangle = \lambda \int_\Omega |y|^{p-2} y h dz \text{ for all } h \in W_0^{1,p}(\Omega),$$
$$\Rightarrow \quad -\Delta_p y(z) = \lambda |y(z)|^{p-2} y(z) \text{ for a.a. } z \in \Omega, \ y\Big|_{\partial\Omega} = 0,$$
$$\Rightarrow \quad y = 0 \text{ (since } \lambda \notin \widehat{\sigma}(p)\text{), a contradiction to (91).}$$

Therefore $t_n < 1$ for all $n \geq n_2 \geq n_1$. Then from (88) we have

$$\int_\Omega [f(z, u_n)u_n - pF(z, u_n)]dz \leq 0 \text{ for all } n \geq n_2 \geq n_1. \tag{92}$$

We let $y_n = \dfrac{u_n}{\|u_n\|}$, $n \in \mathbb{N}$. Then $\|y_n\| = 1$ for all $n \in \mathbb{N}$ and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega) \text{ and } y_n \to y \text{ in } L^p(\Omega). \tag{93}$$

From (85) we have

$$\begin{aligned} &\Big| \langle A_p(y_n), h\rangle + \frac{1 - t_n}{\|u_n\|^{p-2}} \langle A(y_n), h\rangle - (1 - t_n) \int_\Omega \frac{N_f(u_n)}{\|u_n\|^{p-1}} h dz \\ &- t_n \lambda \int_\Omega |y_n|^{p-2} y_n h dz \Big| \leq \frac{\varepsilon_n \|h\|}{(1 + \|u_n\|)\|u_n\|^{p-1}} \quad \text{for all } n \in \mathbb{N}. \end{aligned} \tag{94}$$

Hypotheses $H(f)_2(i), (ii)$ imply that

$$|f(z, x)| \leq c_{12}[1 + |x|^{p-1}] \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ some } c_{12} > 0,$$
$$\Rightarrow \quad \left\{ \frac{N_f(u_n)}{\|u_n\|^{p-1}} \right\}_{n\geq 1} \subseteq L^{p'}(\Omega) \text{ is bounded.} \tag{95}$$

So, by passing to a subsequence if necessary and using hypothesis $H(f)_2(ii)$, we have

$$\frac{N_f(u_n)}{\|u_n\|^{p-1}} \xrightarrow{w} \widehat{\lambda}_m(p)|y|^{p-2}y \text{ in } L^{p'}(\Omega) \text{ as } n \to +\infty. \tag{96}$$

In (94) we choose $h = y_n - y \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (84), (93), (95) and the fact that $2 < p$. Then

$$\begin{aligned} &\lim_{n\to+\infty} \langle A_p(y_n), y_n - y\rangle = 0, \\ \Rightarrow \quad & y_n \to y \text{ in } W_0^{1,p}(\Omega) \text{ and } \|y\| = 1 \text{ (see Proposition 2.2).} \end{aligned} \tag{97}$$

So, if in (94) we pass to the limit as $n \to +\infty$ and use (97) and (96), then

$$\begin{aligned} &\langle A_p(y), h\rangle = \int_\Omega [(1-t)\widehat{\lambda}_m(p) + t\lambda]|y|^{p-2}yhdz \text{ for all } h \in W_0^{1,p}(\Omega), \\ \Rightarrow & -\Delta_p y(z) = [(1-t)\widehat{\lambda}_m(p) + t\lambda]|y(z)|^{p-2}y(z) \text{ for a.a. } z \in \Omega,\ y\Big|_{\partial\Omega} = 0. \end{aligned} \tag{98}$$

We set $\lambda_t = (1-t)\widehat{\lambda}_m(p) + t\lambda$. If $\lambda_t \notin \widehat{\sigma}(p)$, then from (98) il follows that

$$y = 0, \text{ a contradiction to (97).}$$

Otherwise from (97) and if $\Omega_0 = \{z \in \Omega : y(z) \neq 0\}$, we have $|\Omega_0|_N > 0$. Then

$$\begin{aligned} & |u_n(z)| \to +\infty \text{ for a.a. } z \in \Omega_0, \\ \Rightarrow \quad & f(z, u_n(z))u_n(z) - pF(z, u_n(z)) \to +\infty \text{ for a.a. } z \in \Omega_0 \\ & \qquad\qquad \text{(see hypothesis } H(f)_2(iii)), \\ \Rightarrow \quad & \int_{\Omega_0} [f(z,u_n)u_n - pF(z,u_n)]dz \to +\infty \text{ by Fatou's lemma),} \\ \Rightarrow \quad & \int_\Omega [f(z,u_n)u_n - pF(z,u_n)]dz \to +\infty \text{ (as before using } H(f)_2(iii)). \end{aligned} \tag{99}$$

Comparing (92) and (99), we have a contradiction. So the Claim is true. Using the Claim and Theorem 5.1.21, p. 334, of Chang [3] (see also Proposition 3.2 of Liang-Su [13]), we have

$$C_k(\varphi, \infty) = C_k(\psi, \infty) \text{ for all } k \in \mathbb{N}_0. \tag{100}$$

Note that $K_\psi = \{0\}$. So, using Proposition 1.1 of Perera [23], we have

$$\begin{aligned} & C_m(\psi, \infty) = C_m(\psi, 0) \neq 0, \\ \Rightarrow \quad & C_m(\varphi, \infty) \neq 0 \text{ (see (100)).} \end{aligned}$$

□

Now we are ready for our second multiplicity theorem. As usual we assume that K_φ is finite (otherwise, we already have an infinity of nontrivial solutions).

THEOREM 4.5. *If hypothesis $H(f)_2$ hold, then problem* (1) *admits at least three nontrivial solutions* $u_0 \in \operatorname{int} C_+$, $v_0 \in -\operatorname{int} C_+$, $y_0 \in C_0^1(\overline{\Omega})$.

PROOF. From Proposition 4.3, we already have two nontrivial constant sign solutions

$$u_0 \in \operatorname{int} C_+ \text{ and } v_0 \in -\operatorname{int} C_+.$$

From the proof of Proposition 4.3, we know that

$$\begin{aligned}&u_0 \in \operatorname{int} C_+ \text{ is a critical point of } \varphi_+ \text{ of mountain pass type,}\\ &v_0 \in -\operatorname{int} C_+ \text{ is a critical point of } \varphi_- \text{ of mountain pass type.}\end{aligned}$$

It follows that

$$C_1(\varphi_+, u_0) \neq 0 \text{ and } C_1(\varphi_-, v_0) \neq 0 \tag{101}$$

(see Motreanu-Motreanu-Papageorgiou [15], Corollary 6.81, p. 168). We introduce the homotopy $h_+(t,u)$ defined by

$$h_+(t,u) = (1-t)\varphi(u) + t\varphi_+(u) \quad \text{for all } t \in [0,1], \text{ all } u \in W_0^{1,p}(\Omega).$$

Suppose we could find $\{t_n\}_{n\geq 1} \subseteq [0,1]$ and $\{y_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ such that

$$t_n \to t \in [0,1],\ y_n \to u_0 \in W_0^{1,p}(\Omega),\ (h_+)'_u(t_n, y_n) = 0 \text{ for all } n \in \mathbb{N}. \tag{102}$$

From the equality in (102), we have

$$\begin{aligned}&\langle A_p(y_n), h\rangle + \langle A(y_n), h\rangle = (1-t_n)\int_\Omega f(z,y_n)h dz + t_n \int_\Omega f(z, y_n^+) h dz\\ &\qquad\qquad\qquad\qquad\qquad\qquad\text{for all } h \in W_0^{1,p}(\Omega), \text{ all } n \in \mathbb{N},\\ \Rightarrow\quad & -\Delta_p y_n(z) - \Delta y_n(z) = (1-t_n) f(z, y_n(z)) + t_n f(z, y_n^+(z))\\ &\qquad\qquad\qquad\qquad\text{for a.a. } z \in \Omega,\ y_n\Big|_{\partial\Omega} = 0 \text{ for all } n \in \mathbb{N}.\end{aligned}$$

Invoking Theorem 7.1, p. 286, of Ladyzhenskaya-Ural′tseva [11], we can find $c_{13} > 0$ such that

$$\|y_n\|_\infty \leq c_{13} \text{ for all } n \in \mathbb{N}.$$

Then on account of Theorem 1 of Lieberman [12], we can find $\alpha \in (0,1)$ and $c_{14} > 0$ such that

$$y_n \in C_0^{1,\alpha}(\overline{\Omega}) \text{ and } \|y_n\|_{C_0^{1,\alpha}(\overline{\Omega})} \leq c_{14} \text{ for all } n \in \mathbb{N}.$$

Exploiting the compact embedding of $C_0^{1,\alpha}(\overline{\Omega})$ into $C_0^1(\overline{\Omega})$ and using (102), we have

$$y_n \to u_0 \text{ in } C_0^1(\overline{\Omega}).$$

Recall that $u_0 \in \operatorname{int} C_+$. So, we can find $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}&y_n \in C_+ \setminus \{0\} \text{ for all } n \geq n_0,\\ \Rightarrow\quad &\{y_n\}_{n\geq n_0} \subseteq K_\varphi,\end{aligned}$$

which contradicts our hypothesis that K_φ is finite. Hence (102) can not occur and so we can use the homotopy invariance property of critical groups (see Gasiński-Papageorgiou [9], Theorem 5.125, p. 836) and have

$$C_k(\varphi, u_0) = C_k(\varphi_+, u_0) \quad \text{for all } k \in \mathbb{N}_0,$$

$$\Rightarrow \quad C_1(\varphi, u_0) \neq 0 \quad \text{(see (101))}. \tag{103}$$

Similarly we show that

$$C_1(\varphi, v_0) \neq 0. \tag{104}$$

We know that $\varphi \in C^2(W_0^{1,p}(\Omega))$. Therefore from (103), (104) and Papageorgiou-Rădulescu [17], we have

$$C_k(\varphi, u_0) = C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0. \tag{105}$$

From Proposition 4.2, we have

$$C_k(\varphi, 0) = \delta_{k,0}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0. \tag{106}$$

Also, from Proposition 4.4 we know that $C_m(\varphi, \infty) \neq 0$. Hence we can find $y_0 \in K_\varphi$ such that

$$C_m(\varphi, y_0) \neq 0. \tag{107}$$

Since $m \geq 2$, from (105), (106), (107) it follows that

$$y_0 \notin \{u_0, v_0, 0\}.$$

Therefore y_0 is the third nontrivial solution of (1). The nonlinear regularity theory implies that $y_0 \in C_0^1(\overline{\Omega})$. □

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