WEAK SOLUTIONS TO THE GENERALIZED NAVIER-STOKES EQUATIONS WITH MIXED BOUNDARY CONDITIONS AND IMPLICIT OBSTACLE CONSTRAINTS

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Abstract. This paper is concerned with the investigation of a generalized Navier-Stokes equation for non-Newtonian fluids (GNSE, for short) involving a multivalued and nonmonotone slip boundary condition formulated by the generalized Clarke subdifferential to a locally Lipschitz superpotential, a unilateral contact condition of Signorini's type, and an implicit obstacle inequality. We obtain the weak formulation of (GNSE) which is a generalized quasi-hemivariational inequality. By introducing an Oseen model as an auxiliary (intermediated) problem and employing Kakutani-Ky Fan theorem for multivalued operators as well as the theory of nonsmooth analysis, an existence theorem to (GNSE) is established.

Dedicated to Professor Weimin Han on the occasion of his 60th birthday.

1. Introduction

The notion of hemivariational inequalities was initially introduced by the famous mathematician and physicist, Panagiotopoulos [34, 35], in order to study the nonsmooth mechanical problems with nonconvex and nonmonotone constitutive laws which intrinsically involve multivalued maps. Compared with variational inequalities, the essential difference between variational inequalities and hemivariational inequalities is that variational inequalities refer to the inequality problems with convex structure (see [3, 5, 10, 16, 20]), but hemivariational inequalities are devoted to utilize the properties of the generalized Clarke subdifferential for locally Lipschitz functions which are nonconvex in general. After that more and more scholars are attracted to promote the development of theoretical research, numerical analysis and applications to hemivariational inequalities, because a large of complex physical processes, natural phenomena, and engineering problems can be eventually modeled to various kinds of hemivariational inequalities (see e.g., [2, 6, 17, 19, 22, 23, 24, 26, 27, 29, 30, 31, 39]). For example, an obstacle plate bending problem is considered by Han [17], a sensitivity analysis of optimal control problems is studied in Li-Liu [22], and an approximation of an elastic contact problem is analysed by Zeng-Migórski-Khan [39]. On the other hand, slip and leak boundary conditions of frictional type for fluid flow problems have a long history starting with the pioneering works of Fujita [13, 14, 15]. In the meanwhile, it is fully possible that slip boundary conditions (i.e., friction constitutive laws) and dynamic pressure can be modelled by subdifferential in the sense of Clarke of nonconvex and nonsmooth potential functions. Recently, some researchers started to apply the theoretical and numerical methods of hemivariational inequalities for the study of fluid problems with nonsmooth and multivalued constitutive laws. We mention the work of Migórski-Dudek [33] where the authors apply a surjectivity theorem for multivalued pseudomontone operators to explore the existence of solutions for a class of steady state flow of incompressible fluids of Bingham type in a bounded domain in which the boundary conditions contain a generalization of the no leak condition, and a multivalued and nonmonotone version of a nonlinear Navier-Fujita frictional slip condition. Furthermore, a time-dependent Stokes fluid flow problem with nonlinear boundary conditions described by the Clarke subdifferential is considered by Fang-Han [11], and the authors in [11] invoked a limiting procedure

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based on temporally semi-discrete approximation method to establish an unique solvability result for the time-dependent Stokes fluid flow. Han-Czuprynski-Jing [18] developed and studied the mixed finite element method for a hemivariational inequality of the stationary Navier-Stokes equations modelled by a viscous incompressible fluid in a bounded domain which is subjected to a nonsmooth and nonconvex slip boundary condition. By using an equilibrium problem approach, Al-Homidan-Ansari-Chadli [1] delivered an existence result for noncoercive stationary Navier-Stokes equations of heat-conducting fluids with nonmonotone boundary conditions modeled by hemivariational inequalities. For more details with respect to the direction of fluid flow problems modeled by hemivariational inequalities, we also refer to Zhao-He-Migórski-Dudek [41], Zhao-Migórski-Dudek [42], Migórski-Paczka [32], Fang-Han [12], and Dudek-Kalita-Migórski [9].

Given a bounded and connected domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or $d = 3$) with Lipschitz boundary $\Gamma := \partial\Omega$, assume that the incompressible non-Newtonian fluid under consideration is occupied in Ω. Also, we suppose that the boundary Γ is separated into three parts Γ_a, Γ_b and Γ_c, which are all mutually disjoint such that Γ_a has positive measure. In what follows, we denote by $\boldsymbol{x} := (x_i) \in \overline{\Omega} = \Omega \cup \Gamma$, $\boldsymbol{u}\colon \Omega \to \mathbb{R}^d$ and $\boldsymbol{\nu} = (\nu_i)$ the position vector, velocity field of the incompressible non-Newtonian fluid, and the unit outward normal vector on Γ, respectively. Let $\mathbb{S}^d = \{\boldsymbol{\sigma} \in \mathbb{R}^{d\times d} \mid \boldsymbol{\sigma} \text{ is symmetric}\}$ and the inner product in $\mathbb{S}^d$ (resp. in $\mathbb{R}^d$) be defined by $\boldsymbol{\tau} : \boldsymbol{\sigma} = \tau_{ij}\sigma_{ij}$ (resp. $\boldsymbol{\xi} \cdot \boldsymbol{\eta} = \xi_i\eta_i$). The tangential and normal components of velocity field $\boldsymbol{u}$ and traction vector $\boldsymbol{\sigma}$ on the boundary Γ are defined by $u_\nu = \boldsymbol{u} \cdot \boldsymbol{\nu}$, $\boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu\boldsymbol{\nu}$, $\sigma_\nu = (\boldsymbol{\sigma\nu}) \cdot \boldsymbol{\nu}$, and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma\nu} - \sigma_\nu\boldsymbol{\nu}$, respectively. Let $\boldsymbol{S}(\boldsymbol{x}, \boldsymbol{D}(\boldsymbol{u}))$ be an extra stress tensor field of incompressible non-Newtonian fluid. In the following, by $\mathcal{T}(\boldsymbol{u}, \pi)$ we denote the total stress tensor of the fluid given by

$$\mathcal{T}(\boldsymbol{u}, \pi) := \boldsymbol{S}(\boldsymbol{x}, \boldsymbol{D}(\boldsymbol{u})) - \pi I \text{ in } \Omega,$$

where I stands for the identity matrix in $\mathbb{R}^{d\times d}$ and π is the pressure field of incompressible non-Newtonian fluid. We denote by $\boldsymbol{\sigma}(\boldsymbol{u}, \pi)$, $\boldsymbol{\sigma}_\tau(\boldsymbol{u})$ and $\sigma_\nu(\boldsymbol{u}, \pi)$ the traction vector, and the normal and tangential components of the traction vector, respectively, which are defined by

$$\boldsymbol{\sigma}(\boldsymbol{u}, \pi) = \mathcal{T}(\boldsymbol{u}, \pi)\boldsymbol{\nu}, \quad \boldsymbol{\sigma}_\tau(\boldsymbol{u}) = \boldsymbol{\sigma}(\boldsymbol{u}, \pi) - \sigma_\nu(\boldsymbol{u}, \pi)\boldsymbol{\nu} \quad \text{and} \quad \sigma_\nu(\boldsymbol{u}, \pi) = \boldsymbol{\sigma}(\boldsymbol{u}, \pi) \cdot \boldsymbol{\nu} \text{ on } \Gamma.$$

In the present paper, we are interested in the investigation of the following generalized stationary Navier-Stokes problem for non-Newtonian fluid with a nonmonotone and multivalued slip boundary condition (i.e., a generalized friction constitutive law), a unilateral contact condition of Signorini's type, and an implicit obstacle effect.

Problem 1.1. *Find a velocity field $\boldsymbol{u}\colon \Omega \to \mathbb{R}^d$, a pressure field $\pi\colon \Omega \to \mathbb{R}$ and an extra stress tensor field $\boldsymbol{S}\colon \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ such that*

$$-\operatorname{Div}\boldsymbol{S} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla\pi = \boldsymbol{f} \quad \text{in } \Omega, \tag{1.1}$$

$$\boldsymbol{S} = \boldsymbol{S}(\boldsymbol{x}, \boldsymbol{D}(\boldsymbol{u})) \quad \text{in } \Omega, \tag{1.2}$$

$$\nabla \cdot \boldsymbol{u} := \operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{1.3}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_a, \tag{1.4}$$

$$\begin{cases} u_\nu = 0 \\ -\boldsymbol{\sigma}_\tau(\boldsymbol{u}) \in k(\boldsymbol{u}_\tau)\partial j(\boldsymbol{x}, \boldsymbol{u}_\tau) \end{cases} \quad \text{on } \Gamma_b, \tag{1.5}$$

$$\begin{cases} u_\nu \geq 0 \\ \sigma_\nu(\boldsymbol{u}, \pi) + h(u_\nu) \geq 0 \\ u_\nu(\sigma_\nu(\boldsymbol{u}, \pi) + h(u_\nu)) = 0 \\ \boldsymbol{\sigma}_\tau(\boldsymbol{u}) = \boldsymbol{0} \end{cases} \quad \text{on } \Gamma_c, \tag{1.6}$$

$$r(\boldsymbol{u}) \leq m(\boldsymbol{u}), \tag{1.7}$$

where $\partial j(\boldsymbol{x}, \cdot)$ stands for the generalized Clarke subgradient of locally Lipschitz function $\mathbb{R}^d \ni \boldsymbol{\xi} \mapsto j(\boldsymbol{x}, \boldsymbol{\xi}) \in$. Here, $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x})$, Div, div, $\boldsymbol{D}(\boldsymbol{u})$, $\nabla\pi$, and $(\boldsymbol{v} \cdot \nabla)\boldsymbol{u}$ are defined by

- *$\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x})$: the external force field;*
- *Div: the divergence operator for tensor, namely, Div$\boldsymbol{S} = (S_{ij,j})$;*

- *div: the divergence operator for vector, i.e.,* $\mathrm{div}\boldsymbol{u} = \nabla \cdot \boldsymbol{u} = \sum_{i=1}^{d} \frac{\partial u_i}{\partial x_i}$*;*
- $\boldsymbol{D}(\boldsymbol{u})$*: the symmetric part of the velocity gradient of incompressible non-Newtonian fluid, thus,* $\boldsymbol{D}(\boldsymbol{u}) = \dfrac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^\top)$*;*
- $\nabla\pi$*: the gradient operator of pressure field with* $\nabla\pi = \left(\frac{\partial \pi}{\partial x_i}\right)$*;*
- $(\boldsymbol{v}\cdot\nabla)\boldsymbol{u} = \left(\displaystyle\sum_{j=1}^{d} v_j \frac{\partial u_i}{\partial x_j}\right)$.

It should be mentioned that

- if the extra stress tensor $\boldsymbol{S}$ is specialized by the following constitutive function
$$\boldsymbol{S}(\boldsymbol{x},\boldsymbol{\xi}) = k(\|\boldsymbol{\xi}\|_{\mathbb{S}^d})\boldsymbol{\xi} \text{ for all } \boldsymbol{\xi}\in\mathbb{S}^d \text{ and a.e. } \boldsymbol{x}\in\Omega, \tag{1.8}$$
where $k\colon \Omega\times\mathbb{R}_+\to\mathbb{R}$ is a given positive viscosity function, then the models that involve (1.8) are called generalized Newtonian fluids;
- when $k(s) = \alpha + \beta s^{p-2}$ for all $s \geq 0$, where $p > 2$ and $\alpha, \beta > 0$ are two given constants, then the models that involve $\boldsymbol{\xi} \mapsto (\alpha + \beta\|\boldsymbol{\xi}\|_{\mathbb{S}^d}^{p-2})\,\boldsymbol{\xi}$ are called the Ladyzhenskaya fluids;
- if $k(s) = \beta s^{p-2}$ for all $s \geq 0$ with $\beta > 0$ and $p \in (1, +\infty)$, then the conservation relation $\boldsymbol{\xi} \mapsto \beta\|\boldsymbol{\xi}\|_{\mathbb{S}^d}^{p-2}\boldsymbol{\xi}$ is called the power-law;
- when $k(s) = \alpha$ for all $s \geq 0$ with $\alpha > 0$, then $\boldsymbol{\xi} \mapsto \alpha\boldsymbol{\xi}$ is called the linear Stokes law for the usual Newtonian fluids.

It should be pointed out that Problem 1.1 can be reduced to various existing problems, for example, when the convection $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$ is replaced by the one $(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}$ and nonlocal functions k and h are specialized by $k(\boldsymbol{u}_\tau)(\boldsymbol{x}) = \widetilde{k}(\boldsymbol{x}, \boldsymbol{u}_\tau(\boldsymbol{x}))$ for a.e. $\boldsymbol{x}\in\Gamma_b$ and $h(u_\nu)(\boldsymbol{x}) = \widetilde{h}(\boldsymbol{x}, u_\nu(\boldsymbol{x}))$ for a.e. $\boldsymbol{x}\in\Gamma_c$, then Problem 1.1 becomes the one, which has been studied by Migórski-Dudek [28], where $\boldsymbol{b}$ is a given divergence free convection field with $b_\nu = 0$ on Γ_c and $\widetilde{k}\colon \Gamma_b\times\mathbb{R}^d\to\mathbb{R}$ and $\widetilde{h}\colon \Gamma_c\mathbb{R}\to\mathbb{R}$ are two Carathéodory functions. Because of the presence of convection $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$, we observe that the term $\displaystyle\int_\Omega (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{u}\,d\boldsymbol{x}$ does not vanish. On the other side, the nonlinear and nonlocal functions h and k lead to the invalidity of classical methods and analysis techniques for hemivariatinal inequalities. In order to overcome the difficulties mentioned above, the main goal of this paper is to develop a new framework for establishing the existence of solutions to the generalized stationary Navier-Stokes problem, namely Problem 1.1.

The outline of this paper is as follows. In Section 2, some necessary notation and preliminary results will be recalled. Also, we impose the mild hypotheses on the data of Problem 1.1 and establish the weak formulation of Problem 1.1 which is, exactly, a quasi-hemivariational inequality. In Section 3, we introduce an Oseen problem for non-Newtonian fluid, namely Problem 3.2, and prove that the solution set of Oseen problem is nonempty, closed and convex by using an existence theorem for variational inequalities. Moreover, we consider two multivalued variational selections and apply truncation approach as well as the Kakutani-Ky Fan theorem for multivalued maps to examine the solvability of Problem 1.1.

2. Preliminaries, hypotheses and variational formula

In this section we shall pay our attention to review some necessary basic notation and preliminary results which can be found in [6, 7, 8, 38], to impose assumptions on the data of Problem 1.1, and to deliver the weak variational formula of Problem 1.1.

Given a normed space X, in what follows, we denote by X^*, $\|\cdot\|_X$, and $\langle\cdot,\cdot\rangle_{X^*\times X}$ the dual space, the norm and the duality pairing between X^* and X, respectively. Concerning the duality pairing, if no confusion arises, we always drop the subscript. Also, we will apply the symbols " $\xrightarrow{w}$ " and "$\to$" to stand for the weak convergence and the strong convergence in various spaces. For any subset D of X, we write $\|D\|_X := \sup\{\|v\|_X \mid v\in X\}$. Let $p\in(1,+\infty)$ and $d\in(1,+\infty)$. In the sequel, we adopt the symbols p', p^* and p_* for representing the conjugate exponent, Sobolev critical exponent in the domain and Sobolev critical exponent on the boundary of p, namely we have:

$$\frac{1}{p}+\frac{1}{p'}=1,$$

$$p^*:=\begin{cases}\frac{dp}{d-p} & \text{if } p<d,\\ +\infty & \text{if } p\geq d,\end{cases} \tag{2.1}$$

$$p_*:=\begin{cases}\frac{(d-1)p}{d-p} & \text{if } p<d,\\ +\infty & \text{if } p\geq d.\end{cases} \tag{2.2}$$

Assume that Y, Z are two topological spaces and D is a nonempty subset of Y. We say that a multi-valued mapping $F\colon D\to 2^Z$ is

(i): upper semicontinuous (u.s.c., for short) in D, if for every $y\in D$ and any open set $A\subset Z$ with $F(y)\subset A$, we can find a neighborhood $N(y)$ of y such that $F(N(y)):=\cup_{z\in N(y)}F(z)\subset O$.

(ii): closed in D, if for each $y\in D$ and sequences $\{y_n\}\subset D$ and $\{z_n\}\subset Z$ with $z_n\in F(y_n)$ such that $(y_n,z_n)\to(y,z)$ in $Y\times Z$, then we have $z\in F(y)$.

More particularly, we say that $F\colon D\to 2^Z$ is sequentially strongly-weakly closed, if F is sequentially closed from Y endowed with the strong topology into the subsets of Z with the weak topology.

The following proposition provides an equivalent condition to determinate the upper semicontinuity to multi-valued operators.

Proposition 2.1. *Let $F\colon X\to 2^Y$ with X and Y topological spaces. The following are equivalent:*

(i): *F is upper semicontinuous.*

(ii): *For each closed set $C\subset Y$, $F^-(C):=\{x\in X\mid F(x)\cap C\neq\emptyset\}$ is closed in X.*

Next, let us review some basic concepts concerning nonsmooth analysis. Given a Banach space X with dual X^* and norm $\|\cdot\|_X$, we say that $J\colon X\to\mathbb{R}$ is locally Lipschitz continuous at $u\in X$, if there are a neighborhood $N(u)$ of u and a constant $L_u>0$ satisfying

$$|J(w)-J(v)|\leq L_u\|w-v\|_X \quad\text{for all}\quad w,v\in N(u).$$

For a given locally Lipschitz function $J\colon X\to\mathbb{R}$, we denote by $J^0(u;v)$ the generalized (Clarke) directional derivative of J at the point $u\in X$ in the direction $v\in X$ defined by

$$J^0(u;v)=\limsup_{\lambda\to 0^+,\, w\to u}\frac{J(w+\lambda v)-J(w)}{\lambda}.$$

In addition, the generalized subgradient of $J\colon X\to\mathbb{R}$ at $u\in X$ is defined by

$$\partial J(u)=\{\,\xi\in X^*\mid J^0(u;v)\geq\langle\xi,v\rangle_{X^*\times X}\quad\text{for all}\quad v\in X\,\}.$$

The generalized subgradient and generalized directional derivative of a locally Lipschitz function enjoy nice properties and rich calculus. Here we summarize some basic results (see for example [30, Proposition 3.23]).

Proposition 2.2. *Let $J\colon X\to\mathbb{R}$ be a locally Lipschitz function. Then, the following features hold:*

(i): *for every $x\in X$, the function $X\ni v\mapsto J^0(x;v)\in\mathbb{R}$ is positively homogeneous and subadditive, that is, $J^0(x;\lambda v)=\lambda J^0(x;v)$ for all $\lambda\geq 0$, $v\in X$, and $J^0(x;v_1+v_2)\leq J^0(x;v_1)+J^0(x;v_2)$ for all v_1, $v_2\in X$, respectively;*

(ii): *for every $x\in X$, the set $\partial J(x)$ is a nonempty, convex, and weakly* compact subset of X^*, which is bounded by the Lipschitz constant $K_x>0$ of J near x;*

(iii): *the graph of the generalized subdifferential operator ∂J of J is closed in $X\times(w^*\text{–}X^*)$ topology, i.e., if $\{x_n\}\subset X$ and $\{\xi_n\}\subset X^*$ are sequences such that $\xi_n\in\partial J(x_n)$ and $x_n\to x$ in X, $\xi_n\xrightarrow{w}\xi$ in X^*, then $\xi\in\partial J(x)$, where $(w^*\text{–}X^*)$ denotes the spaces X^* equipped with weak* topology.*

Moreover, we recall the celebrated Kakutani-Ky Fan theorem for multivalued operators, see for example [36, Theorem 2.6.7].

Theorem 2.3. *Let Y be a reflexive Banach space and $D \subseteq Y$ be a nonempty, bounded, closed and convex set. Let $\Lambda\colon D \to 2^D$ be a multi-valued map with nonempty, closed and convex values such that its graph is sequentially closed in $Y_w \times Y_w$ topology. Then Λ has a fixed point.*

The basic function space to Problem 1.1 is the one, E, which is the closure of V in $W^{1,p}(\Omega;\mathbb{R}^d)$ topology, i.e., $E = \overline{V}^{W^{1,p}(\Omega;\mathbb{R}^d)}$, where V is defined by

$$V = \left\{ \boldsymbol{u} \in C^\infty(\overline{\Omega};\mathbb{R}^d) \mid \nabla\cdot\boldsymbol{u} = 0 \text{ in } \Omega,\, \boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma_a \text{ and } u_\nu = 0 \text{ on } \Gamma_b \right\}.$$

Obviously, we can see that E, equipped with the usual norm $\|\boldsymbol{v}\| = \|\boldsymbol{v}\|_{W^{1,p}(\Omega;\mathbb{R}^d)}$ for $\boldsymbol{v} \in E$, becomes a separable and reflexive Banach space. However, due to the positive measure of Γ_a, we are able to invoke Korn inequality for finding a constant $m_K > 0$ such that

$$\|\boldsymbol{v}\|_{W^{1,p}(\Omega;\mathbb{R}^d)} \le m_K \|\boldsymbol{D}(\boldsymbol{u})\|_{L^p(\Omega;\mathbb{S}^d)} \text{ for all } \boldsymbol{u} \in E.$$

This means that $\|\cdot\|_E$ with $\|\boldsymbol{v}\|_E = \|\boldsymbol{D}(\boldsymbol{u})\|_{L^p(\Omega;\mathbb{R}^d)}$ for all $\boldsymbol{v} \in E$ is an equivalent norm to $\|\cdot\|_{W^{1,p}(\Omega;\mathbb{R}^d)}$ in E. Throughout this paper, we use $\|\cdot\|_E$ as the norm of E and denote by $\langle\cdot,\cdot\rangle$ the inner product in E with

$$\langle \boldsymbol{v}, \boldsymbol{w}\rangle = \int_\Omega \boldsymbol{D}(\boldsymbol{v}) : \boldsymbol{D}(\boldsymbol{w})\, d\boldsymbol{x} \text{ for all } \boldsymbol{v}, \boldsymbol{w} \in E.$$

Because of the unilateral contact condition of Signorini's type (see (1.6)), we need the following admissible set of velocity fields

$$K := \{\boldsymbol{v} \in E \mid v_\nu \ge 0 \text{ on } \Gamma_c\}.$$

Whereas, the incompressible non-Newtonian fluid is considered to satisfy an implicit obstacle inequality, so it requires us to introduce the following multivalued mapping $C\colon K \to 2^K$ for the velocity fields defined by

$$C(\boldsymbol{v}) := \{\boldsymbol{w} \in K \mid r(\boldsymbol{w}) \le m(\boldsymbol{v})\} \text{ for all } \boldsymbol{v} \in K.$$

Moreover, we make the following hypotheses on the data of Problem 1.1.

<u>$H(\boldsymbol{S})$</u>**:** $\boldsymbol{S}\colon \Omega\times\mathbb{S}^d \to \mathbb{S}^d$ is such that

- **(i):** $\boldsymbol{x} \mapsto \boldsymbol{S}(\boldsymbol{x},\boldsymbol{\xi})$ is measurable in Ω for all $\boldsymbol{\xi} \in \mathbb{S}^d$;
- **(ii):** for a.e. $\boldsymbol{x} \in \Omega$, the function $\boldsymbol{\xi} \mapsto \boldsymbol{S}(\boldsymbol{x},\boldsymbol{\xi})$ is continuous and monotone in $\mathbb{S}^d$;
- **(iii):** there exist $\alpha_S \in L^{p'}(\Omega)$, $\beta_S > 0$ and $r \in (1, p^*)$ satisfying

$$\|\boldsymbol{S}(\boldsymbol{x},\boldsymbol{\xi})\|_{\mathbb{S}^d} \le \alpha_S(\boldsymbol{x}) + \beta_S \|\boldsymbol{\xi}\|_{\mathbb{S}^d}^{r-1}$$

for all $\boldsymbol{\xi} \in \mathbb{S}^d$ and a.e. $\boldsymbol{x} \in \Omega$, where p^* is the critical exponent of p and p is given in $H(\boldsymbol{S})$(iv) below;
- **(iv):** there exist $a_S > 0$, $b_S \ge 0$, $c_S \in L^1(\Omega)$ and $1 \le \tau < p$ with $p \ge 2$ such that

$$\boldsymbol{S}(\boldsymbol{x},\boldsymbol{\xi}) : \boldsymbol{\xi} \ge a_S \|\boldsymbol{\xi}\|_{\mathbb{S}^d}^p - b_S \|\boldsymbol{\xi}\|_{\mathbb{S}^d}^\tau + c_S(\boldsymbol{x})$$

for all $\boldsymbol{\xi} \in \mathbb{S}^d$ and a.e. $\boldsymbol{x} \in \Omega$.

<u>$H(\boldsymbol{f})$</u>**:** $\boldsymbol{f} \in L^{p'}(\Omega;\mathbb{R}^d)$.

<u>$H(k)$</u>**:** $k\colon L^{s_1}(\Gamma_b;\mathbb{R}^d) \to L^\infty(\Gamma_b;(0,+\infty))$ with $s_1 \in [1, p_*)$ satisfies the following conditions, where p_* is the critical exponent of p on the boundary (see (2.2)):

- **(i):** k is continuous in $L^{s_1}(\Gamma_b;\mathbb{R}^d)$;
- **(ii):** there exists $k_0 > 0$ such that $k(\boldsymbol{u})(\boldsymbol{x}) \le k_0$ for all $\boldsymbol{u} \in L^{s_1}(\Gamma_b;\mathbb{R}^d)$ and a.e. $\boldsymbol{x} \in \Gamma_b$.

<u>$H(j)$</u>**:** $j\colon \Gamma_b\times\mathbb{R}^d \to \mathbb{R}^d$ is such that

- **(i):** $\boldsymbol{x} \mapsto j(\boldsymbol{x},\boldsymbol{u})$ is measurable on Γ_b for all $\boldsymbol{u} \in \mathbb{R}^d$;
- **(ii):** for a.e. $\boldsymbol{x} \in \Gamma_b$, the function $\boldsymbol{u} \mapsto j(\boldsymbol{x},\boldsymbol{u})$ is locally Lipschitz continuous;

(iii): there exists $\theta \in [1,p]$ such that for all $\boldsymbol{u} \in \mathbb{R}^d$ and a.e. $\boldsymbol{x} \in \Gamma_b$, it holds

$$\|\partial j(\boldsymbol{x},\boldsymbol{u})\|_{\mathbb{R}^d} \le \alpha_j(\boldsymbol{x}) + \beta_j \|\boldsymbol{u}\|_{\mathbb{R}^d}^{\theta-1},$$

where $\alpha_j \in L^{\theta'}(\Gamma_b)$ and $\beta_j > 0$ is such that

$$\begin{cases} \beta_j \in (0,+\infty) & \text{if } \theta < p, \\ k_0 2^{\frac{1}{\theta}} \beta_j \|\gamma\|^p < \alpha_S & \text{if } \theta = p, \end{cases} \tag{2.3}$$

where γ is the trace operator from E into $L^p(\Gamma_b;\mathbb{R}^d)$ and α_S is given in $H(\boldsymbol{S})$(iv).

$\underline{H(h)}$**:** $h\colon L^{s_2}(\Gamma_c;\mathbb{R}^d) \to L^{s_3'}(\Gamma_c;(0,+\infty))$ with $s_2, s_3 \in [1,p_*)$ is such that

(i): $h\colon L^{s_2}(\Gamma_c;\mathbb{R}^d) \to L^{s_3'}(\Gamma_c;(0,+\infty))$ is continuous;

(ii): it holds

$$\frac{\|h(l)\|_{L^{s_3'}(\Gamma_c;(0,+\infty))}}{\|l\|_{L^{s_2}(\Gamma_c;\mathbb{R}^d)}} \to 0 \text{ as } \|l\|_{L^{s_2}(\Gamma_c;\mathbb{R}^d)} \to +\infty.$$

$\underline{H(r)}$**:** $r\colon E \to \mathbb{R}$ is positively homogeneous, subadditive and lower semicontinuous.

$\underline{H(m)}$**:** $m\colon L^p(\Omega;\mathbb{R}^d) \to \mathbb{R}$ is continuous such that $\inf_{\boldsymbol{u}\in L^p(\Omega;\mathbb{R}^d)} m(\boldsymbol{u}) > 0$.

We end this section to establish the weak variational formula of Problem 1.1. Let $\boldsymbol{u}$, π and $\boldsymbol{S}$ be smooth functions which satisfy (1.1)–(1.7), and let $\boldsymbol{w} \in C(\boldsymbol{u})$ be arbitrary. We use $\boldsymbol{w}-\boldsymbol{u}$ as test function in (1.1) and then apply Green identity for tensors to get

$$\begin{aligned}\int_\Omega \boldsymbol{S}(\boldsymbol{x},\boldsymbol{D}(\boldsymbol{u})) : \boldsymbol{D}(\boldsymbol{w}-\boldsymbol{u})\,d\boldsymbol{x} - \int_{\partial\Omega} (\boldsymbol{S}(\boldsymbol{x},\boldsymbol{D}(\boldsymbol{u}))\boldsymbol{\nu})\cdot(\boldsymbol{w}-\boldsymbol{u})\,d\Gamma + \int_\Omega ((\boldsymbol{u}\cdot\nabla)\boldsymbol{u})\cdot(\boldsymbol{w}-\boldsymbol{u})\,d\boldsymbol{x} \\ + \int_\Omega \nabla\pi\cdot(\boldsymbol{w}-\boldsymbol{u})\,d\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{w}-\boldsymbol{u})\,d\boldsymbol{x}.\end{aligned}$$

The divergence theorem implies

$$\int_\Omega \nabla\pi\cdot(\boldsymbol{w}-\boldsymbol{u})\,d\boldsymbol{x} = -\int_\Omega (\operatorname{div}(\boldsymbol{w}-\boldsymbol{u}))\pi\,d\boldsymbol{x} + \int_{\partial\Omega} \pi(w_\nu - u_\nu)\,d\Gamma.$$

The latter combined with the solenoidal condition (1.3) and boundary conditions (1.4)–(1.5) derives

$$\int_\Omega \nabla\pi\cdot(\boldsymbol{w}-\boldsymbol{u})\,d\boldsymbol{x} = \int_{\Gamma_c} \pi(w_\nu - u_\nu)\,d\Gamma.$$

This means that

$$\begin{aligned}\int_\Omega \boldsymbol{S}(\boldsymbol{x},\boldsymbol{D}(\boldsymbol{u})) : \boldsymbol{D}(\boldsymbol{w}-\boldsymbol{u})\,d\boldsymbol{x} - \int_{\partial\Omega} (\boldsymbol{S}(\boldsymbol{x},\boldsymbol{D}(\boldsymbol{u}))\boldsymbol{\nu})\cdot(\boldsymbol{w}-\boldsymbol{u})\,d\Gamma + \int_\Omega ((\boldsymbol{u}\cdot\nabla)\boldsymbol{u})\cdot(\boldsymbol{w}-\boldsymbol{u})\,d\boldsymbol{x} \\ + \int_{\Gamma_c} \pi(w_\nu - u_\nu)\,d\Gamma = \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{w}-\boldsymbol{u})\,d\boldsymbol{x}.\end{aligned}$$

Noticing that $(\boldsymbol{S}\boldsymbol{\nu})\cdot(\boldsymbol{w}-\boldsymbol{u}) = S_\nu(\boldsymbol{u})(w_\nu - u_\nu) + \boldsymbol{S}_\tau(\boldsymbol{u})\cdot(\boldsymbol{w}_\tau - \boldsymbol{u}_\tau)$ on Γ, we utilize the definitions of $\boldsymbol{\sigma}_\tau(\boldsymbol{u})$ and $\sigma_\nu(\boldsymbol{u},\pi)$ to deduce that

$$(\boldsymbol{S}\boldsymbol{\nu})\cdot(\boldsymbol{w}-\boldsymbol{u}) = \boldsymbol{\sigma}_\tau(\boldsymbol{u})\cdot(\boldsymbol{w}_\tau-\boldsymbol{u}_\tau) + (\sigma_\nu(\boldsymbol{u},\pi)+\pi)(w_\nu-u_\nu) \text{ on } \Gamma.$$

Therefore, we have

$$\begin{aligned}\int_\Omega \boldsymbol{S}(\boldsymbol{x},\boldsymbol{D}(\boldsymbol{u})) : \boldsymbol{D}(\boldsymbol{w}-\boldsymbol{u})\,d\boldsymbol{x} - \int_{\Gamma_b} \boldsymbol{\sigma}_\tau(\boldsymbol{u})\cdot(\boldsymbol{w}_\tau-\boldsymbol{u}_\tau)\,d\Gamma - \int_{\Gamma_c} \sigma_\nu(\boldsymbol{u},\pi)(w_\nu-u_\nu)\,d\Gamma \\ + \int_\Omega ((\boldsymbol{u}\cdot\nabla)\boldsymbol{u})\cdot(\boldsymbol{w}-\boldsymbol{u})\,d\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{w}-\boldsymbol{u})\,d\boldsymbol{x}.\end{aligned} \tag{2.4}$$

By (1.5), there exists an element $\boldsymbol{\xi} \in \partial j(\boldsymbol{x},\boldsymbol{u}_\tau)$ such that $\boldsymbol{\sigma}_\tau(\boldsymbol{u}) = -k(\boldsymbol{u}_\tau)\boldsymbol{\xi}$. So, it follows from the Clarke subgradient that the following is the case

$$-\int_{\Gamma_b} \boldsymbol{\sigma}_\tau(\boldsymbol{u})\cdot(\boldsymbol{w}_\tau-\boldsymbol{u}_\tau)\,d\Gamma = \int_{\Gamma_b} k(\boldsymbol{u}_\tau)\boldsymbol{\xi}\cdot(\boldsymbol{w}_\tau-\boldsymbol{u}_\tau)\,d\Gamma \le \int_{\Gamma_b} k(\boldsymbol{u}_\tau) j^0(\boldsymbol{x},\boldsymbol{u}_\tau;\boldsymbol{w}_\tau-\boldsymbol{u}_\tau)\,d\Gamma. \tag{2.5}$$

In addition, (1.6) points out that

$$\begin{aligned}
&\int_{\Gamma_c} \sigma_\nu(\boldsymbol{u},\pi)(w_\nu - u_\nu)\, d\Gamma \\
&= \int_{\Gamma_c} (\sigma_\nu(\boldsymbol{u},\pi) + h(u_\nu))w_\nu - (\sigma_\nu(\boldsymbol{u},\pi) + h(u_\nu))u_\nu - h(u_\nu)(w_\nu - u_\nu)\, d\Gamma \\
&= \int_{\Gamma_c} (\sigma_\nu(\boldsymbol{u},\pi) + h(u_\nu))w_\nu - h(u_\nu)(w_\nu - u_\nu)\, d\Gamma \\
&\geq -\int_{\Gamma_c} h(u_\nu)(w_\nu - u_\nu)\, d\Gamma. \qquad (2.6)
\end{aligned}$$

Taking into account (2.4), (2.5) and (2.6), it yields

$$\int_\Omega \boldsymbol{S}(\boldsymbol{x},\boldsymbol{D}(\boldsymbol{u})) : \boldsymbol{D}(\boldsymbol{w}-\boldsymbol{u})\, d\boldsymbol{x} + \int_\Omega ((\boldsymbol{u}\cdot\nabla)\boldsymbol{u})\cdot(\boldsymbol{w}-\boldsymbol{u})\, d\boldsymbol{x} + \int_{\Gamma_b} k(\boldsymbol{u}_\tau) j^0(\boldsymbol{x},\boldsymbol{u}_\tau;\boldsymbol{w}_\tau - \boldsymbol{u}_\tau)\, d\Gamma$$
$$+ \int_{\Gamma_c} h(u_\nu)(w_\nu - u_\nu)\, d\Gamma \geq \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{w}-\boldsymbol{u})\, d\boldsymbol{x}.$$

By the arbitrariness of $\boldsymbol{w} \in C(\boldsymbol{u})$, we have the following variational formulation to Problem 1.1.

Problem 2.4. *Find a velocity field $\boldsymbol{u}\colon \Omega \to \mathbb{R}^d$ such that $\boldsymbol{u} \in C(\boldsymbol{u})$ and*

$$\int_\Omega \boldsymbol{S}(\boldsymbol{x},\boldsymbol{D}(\boldsymbol{u})) : \boldsymbol{D}(\boldsymbol{w}-\boldsymbol{u})\, d\boldsymbol{x} + \int_\Omega ((\boldsymbol{u}\cdot\nabla)\boldsymbol{u})\cdot(\boldsymbol{w}-\boldsymbol{u})\, d\boldsymbol{x} + \int_{\Gamma_b} k(\boldsymbol{u}_\tau) j^0(\boldsymbol{x},\boldsymbol{u}_\tau;\boldsymbol{w}_\tau - \boldsymbol{u}_\tau)\, d\Gamma$$
$$+ \int_{\Gamma_c} h(u_\nu)(w_\nu - u_\nu)\, d\Gamma \geq \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{w}-\boldsymbol{u})\, d\boldsymbol{x} \qquad (2.7)$$

for all $\boldsymbol{w} \in C(\boldsymbol{u})$.

3. Existence result

This section is concerned with the study of existence of solutions to Problem 2.4 in which our main methods are based on the Kakutani-Ky Fan theorem for multivalued operators, truncation techniques, theory of nonsmooth analysis and an existence theorem for variational inequalities.

The existence theorem for Problem 2.4 is stated as follows.

Theorem 3.1. *Assume that $H(\boldsymbol{S})$, $H(\boldsymbol{f})$, $H(k)$, $H(j)$, $H(h)$, $H(r)$ and $H(m)$ hold. Then, the solution set of Problem 2.4 is nonempty and bounded.*

To this end, for any $\boldsymbol{v} \in K$ and $\boldsymbol{\xi} \in L^{\theta'}(\Gamma_b;\mathbb{R}^d)$ fixed, we consider the following Oseen model for non-Newtonian fluid.

Problem 3.2. *Find a velocity field $\boldsymbol{u}\colon \Omega \to \mathbb{R}^d$, a pressure field $\pi\colon \Omega \to \mathbb{R}$ and an extra stress tensor field $\boldsymbol{S}\colon \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ such that*

$$\begin{aligned}
&-\operatorname{Div}\boldsymbol{S} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{u} + \nabla\pi = \boldsymbol{f} && \text{in} \quad \Omega,\\
&\boldsymbol{S} = \boldsymbol{S}(\boldsymbol{x},\boldsymbol{D}(\boldsymbol{u})) && \text{in} \quad \Omega,\\
&\nabla\cdot\boldsymbol{u} := \operatorname{div}\boldsymbol{u} = 0 && \text{in} \quad \Omega,\\
&\boldsymbol{u} = \boldsymbol{0} && \text{on} \quad \Gamma_a,\\
&\begin{cases} u_\nu = 0 \\ -\boldsymbol{\sigma}_\tau(\boldsymbol{u}) = k(\boldsymbol{v}_\tau)\boldsymbol{\xi} \end{cases} && \text{on} \quad \Gamma_b,\\
&\begin{cases} u_\nu \geq 0 \\ \sigma_\nu(\boldsymbol{u},\pi) + h(v_\nu) \geq 0 \\ u_\nu(\sigma_\nu(\boldsymbol{u},\pi) + h(v_\nu)) = 0 \\ \boldsymbol{\sigma}_\tau(\boldsymbol{u}) = \boldsymbol{0} \end{cases} && \text{on} \quad \Gamma_c,\\
&r(\boldsymbol{u}) \leq m(\boldsymbol{v}).
\end{aligned}$$

Employing the same arguments as in Section 2, we obtain the weak formulation of Problem 3.2 as follows.

Problem 3.3. *Find a velocity field $\boldsymbol{u}\colon \Omega \to \mathbb{R}^d$ such that $\boldsymbol{u} \in C(\boldsymbol{v})$ and*

$$\int_\Omega \boldsymbol{S}(\boldsymbol{x}, \boldsymbol{D}(\boldsymbol{u})) : \boldsymbol{D}(\boldsymbol{w}-\boldsymbol{u})\, d\boldsymbol{x} + \int_\Omega ((\boldsymbol{v}\cdot\nabla)\boldsymbol{u})\cdot(\boldsymbol{w}-\boldsymbol{u})\, d\boldsymbol{x} + \int_{\Gamma_b} k(\boldsymbol{v}_\tau)\boldsymbol{\xi}\cdot(\boldsymbol{w}_\tau - \boldsymbol{u}_\tau)\, d\Gamma$$

$$+ \int_{\Gamma_c} h(v_\nu)(w_\nu - u_\nu)\, d\Gamma \geq \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{w}-\boldsymbol{u})\, d\boldsymbol{x} \tag{3.1}$$

for all $\boldsymbol{w} \in C(\boldsymbol{v})$.

The following lemma is a direct consequence of Lemma 3.3 in [40].

Lemma 3.4. *Assume that r and m satisfy hypotheses* $\mathrm{H}(r)$ *and* $\mathrm{H}(m)$*, respectively. Then, we have*

(i) *$C(\boldsymbol{u})$ is nonempty (particularly, $\boldsymbol{0} \in C(\boldsymbol{u})$), closed and convex in E for each $\boldsymbol{u} \in K$;*
(ii) *the graph of C is sequentially closed in $E_w \times E_w$;*
(iii) *suppose that sequence $\{\boldsymbol{v}_n\} \subset K$ satisfy $\boldsymbol{v}_n \xrightarrow{w} \boldsymbol{v}$ in E as $n \to \infty$, then for each $\boldsymbol{w} \in C(\boldsymbol{v})$ we are able to find a sequence $\{\boldsymbol{w}_n\} \subset E$ with $\boldsymbol{w}_n \in C(\boldsymbol{v}_n)$ for every $n \in \mathbb{N}$ such that $\boldsymbol{w}_n \to \boldsymbol{w}$ in E as $n \to \infty$.*

We next prove that the solution set of Problem 3.3 is nonempty, closed and convex.

Lemma 3.5. *Suppose that $H(\boldsymbol{S})$, $H(\boldsymbol{f})$, $H(k)$, $H(h)$, $H(r)$ and $H(m)$ are fulfilled. Then, for every pair $(\boldsymbol{v}, \boldsymbol{\xi}) \in K \times L^{\theta'}(\Gamma_b; \mathbb{R}^d)$, the solution set of Problem 3.3 is nonempty, closed and convex.*

Proof. Let $(\boldsymbol{v}, \boldsymbol{\xi}) \in K \times L^{\theta'}(\Gamma_b; \mathbb{R}^d)$ be any fixed. We introduce the following operators $A\colon E \to E^*$ and $\Psi\colon E \to \overline{\mathbb{R}}$ defined by

$$\langle A\boldsymbol{u}, \boldsymbol{w}\rangle := \int_\Omega \boldsymbol{S}(\boldsymbol{x}, \boldsymbol{D}(\boldsymbol{u})) : \boldsymbol{D}(\boldsymbol{w})\, d\boldsymbol{x} + \int_\Omega ((\boldsymbol{v}\cdot\nabla)\boldsymbol{u})\cdot\boldsymbol{w}\, d\boldsymbol{x},$$

$$\Psi(\boldsymbol{w}) := \int_{\Gamma_b} k(\boldsymbol{v}_\tau)\boldsymbol{\xi}\cdot\boldsymbol{w}_\tau\, d\Gamma + \int_{\Gamma_c} h(v_\nu) w_\nu\, d\Gamma - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{w}\, d\boldsymbol{x}.$$

So, it is obvious that $\boldsymbol{u}$ is a solution of Problem 3.3, if and only if, it solves the following variational inequality: find $\boldsymbol{u} \in C(\boldsymbol{v})$ such that

$$\langle A\boldsymbol{u}, \boldsymbol{w}-\boldsymbol{u}\rangle + \Psi(\boldsymbol{w}) - \Psi(\boldsymbol{u}) \geq 0 \tag{3.2}$$

for all $\boldsymbol{w} \in C(\boldsymbol{v})$.

We are going to apply Theorem 3.2 of [25] for obtaining the desired conclusion. It is obvious from the definition of Ψ that Ψ is linear and continuous in E. We assert that $A\colon E \to E^*$ is monotone and continuous. For any $\boldsymbol{w} \in E$, by divergence free condition, we have

$$\int_\Omega ((\boldsymbol{v}\cdot\nabla)\boldsymbol{w})\cdot\boldsymbol{w}\, d\boldsymbol{x} = \int_\Omega \sum_{i,j=1}^d v_i w_{j,i} w_j\, d\boldsymbol{x} = \int_\Omega \sum_{i,j=1}^d v_i \Big(\frac{w_j^2}{2}\Big)_{,i}\, d\boldsymbol{x}$$

$$= -\frac{1}{2}\int_\Omega (\nabla\cdot\boldsymbol{v})\sum_{i=1}^d [w_i]^2\, d\boldsymbol{x} + \frac{1}{2}\int_\Gamma v_\nu \sum_{i=1}^d [w_i]^2\, d\boldsymbol{x} = \frac{1}{2}\int_{\Gamma_c} v_\nu \sum_{i=1}^d [w_i]^2\, d\boldsymbol{x} \geq 0, \tag{3.3}$$

where the last inequality is obtained via using the inequality $v_\nu \geq 0$ on Γ_c. Hence,

$$\int_\Omega ((\boldsymbol{v}\cdot\nabla)\boldsymbol{u})\cdot(\boldsymbol{u}-\boldsymbol{w})\, d\boldsymbol{x} - \int_\Omega ((\boldsymbol{v}\cdot\nabla)\boldsymbol{w})\cdot(\boldsymbol{u}-\boldsymbol{w})\, d\boldsymbol{x} = \int_\Omega ((\boldsymbol{v}\cdot\nabla)(\boldsymbol{u}-\boldsymbol{w}))\cdot(\boldsymbol{u}-\boldsymbol{w})\, d\boldsymbol{x} \geq 0$$

for any $\boldsymbol{u}, \boldsymbol{w} \in E$. The latter together with the monotonicity of $\boldsymbol{\xi} \mapsto \boldsymbol{S}(\boldsymbol{x}, \boldsymbol{\xi})$ (see hypothesis $H(\boldsymbol{S})$(ii)) implies that A is monotone as well. Whereas, the continuity of $\boldsymbol{\xi} \mapsto \boldsymbol{S}(\boldsymbol{x}, \boldsymbol{\xi})$, hypothesis $H(\boldsymbol{S})$(iii) and Lebesgue dominated convergence theorem reveal that A is continuous.

Finally, we show that A is coercive. For any $\boldsymbol{u} \in E$, it follows from hypothesis $H(\boldsymbol{S})$(iv) and estimate (3.3) that

$$\langle A\boldsymbol{u}, \boldsymbol{u}\rangle \geq \int_\Omega a_S \|\boldsymbol{D}(\boldsymbol{u})\|_{\mathbb{S}^d}^p - b_S \|\boldsymbol{D}(\boldsymbol{u})\|_{\mathbb{S}^d}^\tau + c_S(\boldsymbol{x})\, d\boldsymbol{x} \geq a_S \|\boldsymbol{u}\|_E^p - b_S \|\boldsymbol{u}\|_E^\tau |\Omega|^{\frac{p-\tau}{p}} - \|c_S\|_{L^1(\Omega)},$$

namely, A is coercive. So, we obtain

$$\frac{\langle A\boldsymbol{u}, \boldsymbol{u}\rangle + \Psi(\boldsymbol{u})}{\|\boldsymbol{u}\|_V} \to +\infty \text{ as } \|\boldsymbol{u}\|_V \to +\infty,$$

where we have used [4, Proposition 1.10]) and the fact that $\Psi(\mathbf{0}) = 0$.

We are now in a position to invoke Theorem 3.2 of [25] to conclude that the solution set of Problem 3.3 is nonempty, closed and convex. □

This allows us to introduce the variational map $M\colon K \times L^{\theta'}(\Gamma_b; \mathbb{R}^d) \to 2^K$ defined by

$$M(\boldsymbol{v}, \boldsymbol{\xi}) := \{\boldsymbol{u} \in K \mid \boldsymbol{u} \text{ is a solution to Problem 3.3 corresponding to } (\boldsymbol{v}, \boldsymbol{\xi})\} \tag{3.4}$$

for all $(\boldsymbol{v}, \boldsymbol{\xi}) \in K \times L^{\theta'}(\Gamma_b; \mathbb{R}^d)$. Also, we consider the multivalued mapping $J\colon L^\theta(\Gamma_b; \mathbb{R}^d) \to 2^{L^{\theta'}(\Gamma_b;\mathbb{R}^d)}$ given by

$$J(\boldsymbol{u}) := \{\boldsymbol{\xi} \in L^{\theta'}(\Gamma_b; \mathbb{R}^d) \mid \boldsymbol{\xi}(\boldsymbol{x}) \in \partial j(\boldsymbol{x}, \boldsymbol{u}_\tau) \text{ for a.e. } \boldsymbol{x} \in \Gamma_b\} \tag{3.5}$$

for all $\boldsymbol{u} \in L^\theta(\Gamma_b; \mathbb{R}^d)$.

Lemma 3.6. *Assume that $H(j)$ is fulfilled. Then, we have*

- (i) *$J\colon L^\theta(\Gamma_b; \mathbb{R}^d) \to 2^{L^{\theta'}(\Gamma_b;\mathbb{R}^d)}$ is well-defined, more precisely, for each $\boldsymbol{u} \in L^\theta(\Gamma_b; \mathbb{R}^d)$ the set $J(\boldsymbol{u})$ is nonempty, bounded, closed and convex;*
- (ii) *J is strongly-weakly u.s.c.*

Proof. (i) For any $\boldsymbol{u} \in L^\theta(\Gamma_b; \mathbb{R}^d)$ fixed, we utilize hypotheses $H(j)$(i), (ii), Proposition 2.2(ii) and Yankov-von Neumann-Aumann selection theorem (see Papageorgiou-Winkert [37, Theorem 2.7.25]) to conclude that there exists a measurable function $\boldsymbol{\xi}\colon \Gamma_b \to \mathbb{R}^d$ satisfying $\boldsymbol{\xi}(\boldsymbol{x}) \in \partial j(\boldsymbol{x}, \boldsymbol{u})$ for a.e. $\boldsymbol{x} \in \Gamma_b$. Whereas, from hypothesis $H(j)$(iii), we infer

$$\begin{aligned}\|\boldsymbol{\xi}\|_{L^{\theta'}(\Gamma_b;\mathbb{R}^d)} &= \left(\int_{\Gamma_b} \|\boldsymbol{\xi}\|_{\mathbb{R}^d}^{\theta'}\, d\Gamma\right)^{\frac{1}{\theta'}} \leq \left(\int_{\Gamma_b} \left(\alpha_j(\boldsymbol{x}) + \beta_j \|\boldsymbol{u}\|_{\mathbb{R}^d}^{\theta-1}\right)^{\theta'} d\Gamma\right)^{\frac{1}{\theta'}} \\ &\leq 2^{\frac{1}{\theta}} \left(\|\alpha_j\|_{L^{\theta'}(\Gamma_b)} + \beta_j \|\boldsymbol{u}\|_{L^\theta(\Gamma_b;\mathbb{R}^d)}^{\theta-1}\right),\end{aligned} \tag{3.6}$$

where we have applied the elementary inequality $(s+t)^r \leq 2^{r-1}(s^r + t^r)$ for all $s, t \geq 0$ and $r \geq 1$. This means that $\boldsymbol{\xi} \in L^{\theta'}(\Gamma_b; \mathbb{R}^d)$, i.e., J is well-defined and $J(\boldsymbol{u})$ is nonempty for each $\boldsymbol{u} \in L^\theta(\Gamma_b; \mathbb{R}^d)$. However, the convexity and closedness of $\partial j(\boldsymbol{x}, \boldsymbol{u})$ guarantee that $J(\boldsymbol{u})$ is closed and convex in $L^{\theta'}(\Gamma_b; \mathbb{R}^d)$ for each $\boldsymbol{u} \in L^\theta(\Gamma_b; \mathbb{R}^d)$.

(ii) By Proposition 2.1(ii), it is enough to show that for each weakly closed set $W \subset L^{\theta'}(\Gamma_b; \mathbb{R}^d)$ with $J^-(W) = \{\boldsymbol{u} \in L^\theta(\Gamma_b; \mathbb{R}^d) \mid J(\boldsymbol{u}) \cap W \neq \emptyset\} \neq \emptyset$, the set $J^-(W)$ is closed in $L^\theta(\Gamma_b; \mathbb{R}^d)$. Let $\{\boldsymbol{u}_n\} \subset J^-(W)$ be a sequence such that $\boldsymbol{u}_n \to \boldsymbol{u}$ in $L^\theta(\Gamma_b; \mathbb{R}^d)$ for some $\boldsymbol{u} \in L^\theta(\Gamma_b; \mathbb{R}^d)$. So, without loss of generality, we may assume that $\boldsymbol{u}_n(\boldsymbol{x}) \to \boldsymbol{u}(\boldsymbol{x})$ for a.e. $\boldsymbol{x} \in \Gamma_b$. Also, there is a sequence $\{\boldsymbol{\xi}_n\} \subset L^{\theta'}(\Gamma_b; \mathbb{R}^d)$ satisfying $\boldsymbol{\xi}_n \in J(\boldsymbol{u}_n) \cap W$ for each $n \in \mathbb{N}$. But, estimates (3.6) indicate that $\{\boldsymbol{\xi}_n\}$ is bounded in $L^{\theta'}(\Gamma_b; \mathbb{R}^d)$. Passing to a subsequence if necessary, we can find a function $\boldsymbol{\xi} \in L^{\theta'}(\Gamma_b; \mathbb{R}^d)$ such that $\boldsymbol{\xi}_n \xrightarrow{w} \boldsymbol{\xi}$ in $L^{\theta'}(\Gamma_b; \mathbb{R}^d)$. Invoking Mazur's theorem, we can find a sequence $\{\boldsymbol{\eta}_n\}$ of convex combinations of $\{\boldsymbol{\xi}_n\}$ satisfying

$$\boldsymbol{\eta}_n \to \boldsymbol{\xi} \quad \text{in } L^{\theta'}(\Gamma_b; \mathbb{R}^d) \quad \text{and} \quad \boldsymbol{\eta}_n(x) \to \boldsymbol{\xi}(x) \quad \text{for a.e. } x \in \Gamma_b \quad \text{as} \quad n \to \infty. \tag{3.7}$$

The convexity of $\partial j(\boldsymbol{x}, \boldsymbol{u}_n)$ points out that $\boldsymbol{\eta}_n(\boldsymbol{x}) \in \partial j(\boldsymbol{x}, \boldsymbol{u}_n)$ for a.e. $\boldsymbol{x} \in \Gamma_b$. But, the upper semicontinuity of $\boldsymbol{u} \mapsto \partial j(\boldsymbol{x}, \boldsymbol{u})$ implies that $\boldsymbol{\xi} \in \partial j(\boldsymbol{x}, \boldsymbol{u})$ for a.e. $\boldsymbol{x} \in \Gamma_b$. This means that $\boldsymbol{\xi} \in J(\boldsymbol{u})$. Moreover, the weak closedness of W reveals that $\boldsymbol{\xi} \in W$, i.e., $\boldsymbol{\xi} \in J^-(W)$. Therefore, we utilize Proposition 2.1(ii) to conclude that J is strongly-weakly u.s.c. □

Using the lemmas above, we are now in a position to give the detailed proof of Theorem 3.1.

Proof of Theorem 3.1. **Boundedness.** We first prove that the solution set of Problem 3.3 is bounded. If the solution set of Problem 3.3 was unbounded, then there exists a solution sequence $\{\boldsymbol{u}_n\}$ such that $\|\boldsymbol{u}_n\|_E \to +\infty$ as $n \to \infty$. So, for each $n \in \mathbb{N}$, we have

$$\begin{aligned}&\int_\Omega \boldsymbol{S}(\boldsymbol{x}, \boldsymbol{D}(\boldsymbol{u}_n)) : \boldsymbol{D}(\boldsymbol{w} - \boldsymbol{u}_n)\, d\boldsymbol{x} + \int_\Omega ((\boldsymbol{u}_n \cdot \nabla)\boldsymbol{u}_n) \cdot (\boldsymbol{w} - \boldsymbol{u}_n)\, d\boldsymbol{x}\\ &+ \int_{\Gamma_b} k(\boldsymbol{u}_\tau^{(n)}) j^0(\boldsymbol{x}, \boldsymbol{u}_\tau^{(n)}; \boldsymbol{w}_\tau - \boldsymbol{u}_\tau^{(n)})\, d\Gamma + \int_{\Gamma_c} h(u_\nu^{(n)})(w_\nu - u_\nu^{(n)})\, d\Gamma\\ \geq &\int_\Omega \boldsymbol{f} \cdot (\boldsymbol{w} - \boldsymbol{u}_n)\, d\boldsymbol{x}\end{aligned}$$

for all $\boldsymbol{w} \in C(\boldsymbol{u}_n)$. Noting that $\boldsymbol{0} \in C(\boldsymbol{u}_n)$ for each $n \in \mathbb{N}$, we take $\boldsymbol{w} = \boldsymbol{0}$ into the inequality above, and hence we get

$$\begin{aligned}&\int_\Omega \boldsymbol{S}(\boldsymbol{x}, \boldsymbol{D}(\boldsymbol{u}_n)) : \boldsymbol{D}(\boldsymbol{u}_n)\, d\boldsymbol{x} + \int_\Omega ((\boldsymbol{u}_n \cdot \nabla)\boldsymbol{u}_n) \cdot (\boldsymbol{u}_n)\, d\boldsymbol{x}\\ &\leq -\int_{\Gamma_b} k(\boldsymbol{u}_\tau^{(n)}) \boldsymbol{\xi}_n \cdot \boldsymbol{u}_\tau^{(n)}\, d\Gamma - \int_{\Gamma_c} h(u_\nu^{(n)}) u_\nu^{(n)}\, d\Gamma - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u}_n\, d\boldsymbol{x},\end{aligned}$$

where $\boldsymbol{\xi}_n \in \partial j(\boldsymbol{x}, \boldsymbol{u}_\tau^{(n)})$ is such that $\boldsymbol{\xi}_n \cdot \boldsymbol{u}_\tau^{(n)} = j^0(\boldsymbol{x}, \boldsymbol{u}_\tau^{(n)}; -\boldsymbol{u}_\tau^{(n)})$. Taking into account the inequality above, hypotheses $H(\boldsymbol{S})$(iv), $H(k)$, $H(h)$, $H(\boldsymbol{f})$ and (3.3), one has

$$\begin{aligned}a_S\|\boldsymbol{u}_n\|_E^p - m_1\|\boldsymbol{u}_n\|_E^\tau + \|c_S\|_{L^1(\Omega)} &\leq \int_\Omega \boldsymbol{S}(\boldsymbol{x}, \boldsymbol{D}(\boldsymbol{u}_n)) : \boldsymbol{D}(\boldsymbol{u}_n)\, d\boldsymbol{x} + \int_\Omega ((\boldsymbol{u}_n \cdot \nabla)\boldsymbol{u}_n) \cdot (\boldsymbol{u}_n)\, d\boldsymbol{x}\\ \leq -\int_{\Gamma_b} k(\boldsymbol{u}_\tau^{(n)}) \boldsymbol{\xi}_n \cdot \boldsymbol{u}_\tau^{(n)}\, d\Gamma &- \int_{\Gamma_c} h(u_\nu^{(n)}) u_\nu^{(n)}\, d\Gamma - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u}_n\, d\boldsymbol{x}\\ \leq k_0\|\boldsymbol{\xi}_n\|_{L^{\theta'}(\Gamma_b;\mathbb{R}^d)}\|\boldsymbol{u}_n\|_{L^\theta(\Gamma_c;\mathbb{R}^d)} &+ \|h(u_\nu^{(n)})\|_{L^{s_3'}(\Gamma_c)}\|\boldsymbol{u}_n\|_{L^{s_3}(\Gamma_c;\mathbb{R}^d)} + \|\boldsymbol{f}\|_{E^*}\|\boldsymbol{u}_n\|_E\end{aligned}$$

for some $m_1 > 0$. On the other hand, from (3.6), we obtain the estimate

$$\|\boldsymbol{\xi}_n\|_{L^{\theta'}(\Gamma_b;\mathbb{R}^d)} \leq 2^{\frac{1}{\theta}} \left(\|\alpha_j\|_{L^{\theta'}(\Gamma_b)} + \beta_j\|\boldsymbol{u}_n\|_{L^\theta(\Gamma_b;\mathbb{R}^d)}^{\theta-1}\right).$$

Hence, we deduce that

$$\begin{aligned}&a_S\|\boldsymbol{u}_n\|_E^p - m_1\|\boldsymbol{u}_n\|_E^\tau + \|c_S\|_{L^1(\Omega)}\\ \leq &k_0 2^{\frac{1}{\theta}} \left(\|\alpha_j\|_{L^{\theta'}(\Gamma_b)} + \beta_j\|\boldsymbol{u}_n\|_{L^\theta(\Gamma_b;\mathbb{R}^d)}^{\theta-1}\right)\|\boldsymbol{u}_n\|_{L^\theta(\Gamma_c;\mathbb{R}^d)} + \frac{\|h(u_\nu^{(n)})\|_{L^{s_3'}(\Gamma_c)}}{\|\boldsymbol{u}_n\|_{L^{s_3}(\Gamma_c;\mathbb{R}^d)}}\|\boldsymbol{u}_n\|_{L^{s_3}(\Gamma_c;\mathbb{R}^d)}^2\\ &+ \|\boldsymbol{f}\|_{E^*}\|\boldsymbol{u}_n\|_E.\end{aligned}$$

Letting $n \to \infty$ in the above inequality and using smallness condition (2.3) and hypothesis $H(h)$(ii), it leads to a contradiction. Therefore, we conclude that the solution set of Problem 3.3 is bounded. This means that there exist constants $R_1, R_2, R_3 > 0$ such that for each solution $\boldsymbol{u}$ of Problem 3.3 we have

$$\|\boldsymbol{u}\|_E \leq R_1, \quad \|\gamma\boldsymbol{w}\|_{L^\theta(\Gamma_b;\mathbb{R}^d)} \leq R_2 \text{ and } \|\partial j(\cdot, \boldsymbol{w}_\tau)\|_{L^{\theta'}(\Gamma_b;\mathbb{R}^d)} \leq R_3 \text{ for all } \boldsymbol{w} \in B_E(R_1),$$

where $B_E(R_1) \subset E$ is defined by $B_E(R_1) := \{\boldsymbol{u} \in E \mid \|\boldsymbol{u}\|_E \leq R_1\}$. Also, we introduce the R_2-radial retraction $\mathcal{R}\colon L^\theta(\Gamma_b;\mathbb{R}^d) \to L^\theta(\Gamma_b;\mathbb{R}^d)$ given by

$$\mathcal{R}(\boldsymbol{u}) := \begin{cases} \boldsymbol{u} & \text{if } \|\boldsymbol{u}\|_{L^\theta(\Gamma_b;\mathbb{R}^d)} \leq R_2,\\ \frac{R_2\boldsymbol{u}}{\|\boldsymbol{u}\|_{L^\theta(\Gamma_b;\mathbb{R}^d)}} & \text{if } \|\boldsymbol{u}\|_{L^\theta(\Gamma_b;\mathbb{R}^d)} > R_2.\end{cases}$$

It is obvious that $\mathcal{R}\colon L^\theta(\Gamma_b;\mathbb{R}^d) \to L^\theta(\Gamma_b;\mathbb{R}^d)$ is bounded and continuous.

Existence. Let us consider the multivalued operator $P\colon K \times L^{\theta'}(\Gamma_b;\mathbb{R}^d) \to 2^{K\times L^{\theta'}(\Gamma_b;\mathbb{R}^d)}$ defined by

$$P(\boldsymbol{u}, \boldsymbol{\xi}) := (M(\boldsymbol{u}, \boldsymbol{\xi}), J(\mathcal{R}(\gamma\boldsymbol{u}))) \text{ for all } (\boldsymbol{u}, \boldsymbol{\xi}) \in K \times L^{\theta'}(\Gamma_b;\mathbb{R}^d).$$

The closedness and convexity of M and J (see Lemmas 3.5 and 3.6) guarantee that P has closed and convex values. Moreover, if $(\boldsymbol{u},\boldsymbol{\xi})\in K\times L^{\theta'}(\Gamma_b;\mathbb{R}^d)$ is a fixed point of P with $\|\gamma\boldsymbol{u}\|_{L^\theta(\Gamma_b;\mathbb{R}^d)}\le R_2$, i.e., $\boldsymbol{u}\in M(\boldsymbol{u},\boldsymbol{\xi})$ and $\boldsymbol{\xi}\in J(\gamma\boldsymbol{u})$, then, from the definition of J and Clarke subgradient, we have

$$\begin{aligned}\int_\Omega \boldsymbol{f}\cdot(\boldsymbol{w}-\boldsymbol{u})\,d\boldsymbol{x} &\le \int_\Omega \boldsymbol{S}(\boldsymbol{x},\boldsymbol{D}(\boldsymbol{u})):\boldsymbol{D}(\boldsymbol{w}-\boldsymbol{u})\,d\boldsymbol{x}+\int_\Omega((\boldsymbol{u}\cdot\nabla)\boldsymbol{u})\cdot(\boldsymbol{w}-\boldsymbol{u})\,d\boldsymbol{x}\\ &\quad+\int_{\Gamma_b}k(\boldsymbol{u}_\tau)\boldsymbol{\xi}\cdot(\boldsymbol{w}_\tau-\boldsymbol{u}_\tau)\,d\Gamma+\int_{\Gamma_c}h(u_\nu)(w_\nu-u_\nu)\,d\Gamma\\ &\le \int_\Omega \boldsymbol{S}(\boldsymbol{x},\boldsymbol{D}(\boldsymbol{u})):\boldsymbol{D}(\boldsymbol{w}-\boldsymbol{u})\,d\boldsymbol{x}+\int_\Omega((\boldsymbol{u}\cdot\nabla)\boldsymbol{u})\cdot(\boldsymbol{w}-\boldsymbol{u})\,d\boldsymbol{x}\\ &\quad+\int_{\Gamma_b}k(\boldsymbol{u}_\tau)j^0(\boldsymbol{x},\boldsymbol{u}_\tau;\boldsymbol{w}_\tau-\boldsymbol{u}_\tau)\,d\Gamma+\int_{\Gamma_c}h(u_\nu)(w_\nu-u_\nu)\,d\Gamma\end{aligned}$$

for all $\boldsymbol{w}\in C(\boldsymbol{u})$. This reveals that the first component of the fixed point to P, which satisfies the inequality $\|\boldsymbol{u}\|_{L^\theta(\Gamma_b;\mathbb{R}^d)}\le R_2$, is also a solution of Problem 2.4. Based on this important fact, we are going to verify that the fixed point set of P is nonempty by using Kakutani-Ky Fan theorem for multi-valued mappings.

We assert that the graph of $P\colon K\times L^{\theta'}(\Gamma_b;\mathbb{R}^d)\to 2^{K\times L^{\theta'}(\Gamma_b;\mathbb{R}^d)}$ is sequentially closed in $E_w\times L^{\theta'}(\Gamma_b;\mathbb{R}^d)_w$ topology. Let $\{(\boldsymbol{v}_n,\boldsymbol{\xi}_n)\},\{(\boldsymbol{u}_n,\boldsymbol{\eta}_n)\}\subset K\times L^{\theta'}(\Gamma_b;\mathbb{R}^d)$ and $(\boldsymbol{v},\boldsymbol{\xi}),(\boldsymbol{u},\boldsymbol{\eta})\in K\times L^{\theta'}(\Gamma_b;\mathbb{R}^d)$ be such that

$$\begin{cases}(\boldsymbol{u}_n,\boldsymbol{\eta}_n)\in P(\boldsymbol{v}_n,\boldsymbol{\xi}_n)\text{ for every }n\in\mathbb{N},\\ (\boldsymbol{v}_n,\boldsymbol{\xi}_n)\xrightarrow{w}(\boldsymbol{v},\boldsymbol{\xi})\text{ as }n\to\infty,\\ (\boldsymbol{u}_n,\boldsymbol{\eta}_n)\xrightarrow{w}(\boldsymbol{u},\boldsymbol{\eta})\text{ as }n\to\infty.\end{cases}$$

So, for each $n\in\mathbb{N}$, we have $\boldsymbol{u}_n\in C(\boldsymbol{v}_n)$ and

$$\begin{aligned}&\int_\Omega \boldsymbol{S}(\boldsymbol{x},\boldsymbol{D}(\boldsymbol{u}_n)):\boldsymbol{D}(\boldsymbol{w}-\boldsymbol{u}_n)\,d\boldsymbol{x}+\int_\Omega((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{u}_n)\cdot(\boldsymbol{w}-\boldsymbol{u}_n)\,d\boldsymbol{x}+\int_{\Gamma_b}k(\boldsymbol{v}_\tau^{(n)})\boldsymbol{\xi}_n\cdot(\boldsymbol{w}_\tau-\boldsymbol{u}_\tau^{(n)})\,d\Gamma\\ &+\int_{\Gamma_c}h(v_\nu^{(n)})(w_\nu-u_\nu^{(n)})\,d\Gamma\ge\int_\Omega\boldsymbol{f}\cdot(\boldsymbol{w}-\boldsymbol{u}_n)\,d\boldsymbol{x}\end{aligned}\tag{3.8}$$

for all $\boldsymbol{w}\in C(\boldsymbol{v}_n)$. From the sequentially weak closedness of C and convergences $\boldsymbol{u}_n\xrightarrow{w}\boldsymbol{u}$ and $\boldsymbol{v}_n\xrightarrow{w}\boldsymbol{v}$ in E, we have $\boldsymbol{u}\in C(\boldsymbol{v})$. For any $\boldsymbol{z}\in C(\boldsymbol{v})$, Lemma 3.4(iii) permits us to find a sequence $\{\boldsymbol{w}_n\}\subset E$ with $\boldsymbol{w}_n\in C(\boldsymbol{v}_n)$ such that $\boldsymbol{w}_n\to\boldsymbol{z}$ in E. Inserting $\boldsymbol{w}=\boldsymbol{w}_n$ into (3.8), it yields

$$\begin{aligned}&\int_\Omega \boldsymbol{S}(\boldsymbol{x},\boldsymbol{D}(\boldsymbol{u}_n)):\boldsymbol{D}(\boldsymbol{w}_n-\boldsymbol{u}_n)\,d\boldsymbol{x}+\int_\Omega((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{u}_n)\cdot(\boldsymbol{w}_n-\boldsymbol{u}_n)\,d\boldsymbol{x}\\ &+\int_{\Gamma_b}k(\boldsymbol{v}_\tau^{(n)})\boldsymbol{\xi}_n\cdot(\boldsymbol{w}_\tau^{(n)}-\boldsymbol{u}_\tau^{(n)})\,d\Gamma+\int_{\Gamma_c}h(v_\nu^{(n)})(w_\nu^{(n)}-u_\nu^{(n)})\,d\Gamma\\ &\ge\int_\Omega\boldsymbol{f}\cdot(\boldsymbol{w}_n-\boldsymbol{u}_n)\,d\boldsymbol{x}.\end{aligned}\tag{3.9}$$

By the monotonicity and continuity of $\boldsymbol{S}$, we have

$$\begin{aligned}\limsup_{n\to\infty}\int_\Omega \boldsymbol{S}(\boldsymbol{x},\boldsymbol{D}(\boldsymbol{u}_n)):\boldsymbol{D}(\boldsymbol{w}_n-\boldsymbol{u}_n)\,d\boldsymbol{x}&\le\limsup_{n\to\infty}\int_\Omega \boldsymbol{S}(\boldsymbol{x},\boldsymbol{D}(\boldsymbol{w}_n)):\boldsymbol{D}(\boldsymbol{w}_n-\boldsymbol{u}_n)\,d\boldsymbol{x}\\ &\le\int_\Omega \boldsymbol{S}(\boldsymbol{x},\boldsymbol{D}(\boldsymbol{z})):\boldsymbol{D}(\boldsymbol{z}-\boldsymbol{u})\,d\boldsymbol{x}.\end{aligned}\tag{3.10}$$

From divergence theorem, we deduce that

$$\begin{aligned}&\int_\Omega((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{u}_n)\cdot(\boldsymbol{w}_n-\boldsymbol{u}_n)\,d\boldsymbol{x}\\ &=\int_\Omega((\boldsymbol{v}_n\cdot\nabla)(\boldsymbol{u}_n-\boldsymbol{w}_n))\cdot(\boldsymbol{w}_n-\boldsymbol{u}_n)\,d\boldsymbol{x}+\int_\Omega((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{w}_n)\cdot(\boldsymbol{w}_n-\boldsymbol{u}_n)\,d\boldsymbol{x}\end{aligned}$$

$$= -\frac{1}{2}\int_{\Gamma_c} v_\nu^{(n)} \sum_{i=1}^d [w_i^{(n)} - u_i^{(n)}]^2\, d\boldsymbol{x} + \int_\Omega \sum_{i,j=1}^d v_i^{(n)} w_{j,i}^{(n)} (w_j^{(n)} - u_j^{(n)})\, d\boldsymbol{x}. \tag{3.11}$$

Recall that $p \geq 2$, so, E is compactly embedded into $L^1(\Gamma_c;\mathbb{R}^d)$ and $L^4(\Omega;\mathbb{R}^d)$, and continuously embedded into $L^4(\Gamma_c;\mathbb{R}^d)$. Hence, passing to a subsequence if necessary, we may assume that $v_\nu^{(n)} \to v_\nu$, $u_i^{(n)} \to u_i$ and $w_i^{(n)} \to z_i$ for a.e. $\boldsymbol{x} \in \Gamma_c$ and all $i = 1, \ldots, d$, and $v_i^{(n)} \to v_i$, $w_{j,i}^{(n)} \to z_{j,i}$ and $w_j^{(n)} - u_j^{(n)} \to z_j - u_j$ for a.e. $\boldsymbol{x} \in \Omega$ and all $i, j = 1, \ldots, d$. However, applying Hölder inequality and Sobolve embedding theorems, it is not difficult to see that $v_\nu^{(n)} \sum_{i=1}^d [w_i^{(n)} - u_i^{(n)}]^2 \in L^1(\Gamma_c)$ and $\sum_{i,j=1}^d v_i^{(n)} w_{j,i}^{(n)} (w_j^{(n)} - u_j^{(n)}) \in L^1(\Omega)$. Letting $n \to \infty$ in equation (3.11) and using Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \int_\Omega ((\boldsymbol{v}_n \cdot \nabla)\boldsymbol{u}_n) \cdot (\boldsymbol{w}_n - \boldsymbol{u}_n)\, d\boldsymbol{x} \\
&= -\lim_{n\to\infty} \frac{1}{2}\int_{\Gamma_c} v_\nu^{(n)} \sum_{i=1}^d [w_i^{(n)} - u_i^{(n)}]^2\, d\boldsymbol{x} + \lim_{n\to\infty} \int_\Omega \sum_{i,j=1}^d v_i^{(n)} w_{j,i}^{(n)} (w_j^{(n)} - u_j^{(n)})\, d\boldsymbol{x} \\
&= -\frac{1}{2}\int_{\Gamma_c} \lim_{n\to\infty} v_\nu^{(n)} \sum_{i=1}^d [w_i^{(n)} - u_i^{(n)}]^2\, d\boldsymbol{x} + \int_\Omega \lim_{n\to\infty} \sum_{i,j=1}^d v_i^{(n)} w_{j,i}^{(n)} (w_j^{(n)} - u_j^{(n)})\, d\boldsymbol{x} \\
&= -\frac{1}{2}\int_{\Gamma_c} v_\nu \sum_{i=1}^d [z_i - u_i]^2\, d\boldsymbol{x} + \int_\Omega \sum_{i,j=1}^d v_i z_{j,i} (z_j - u_j)\, d\boldsymbol{x} \\
&= \int_\Omega ((\boldsymbol{v} \cdot \nabla)\boldsymbol{u}) \cdot (\boldsymbol{z} - \boldsymbol{u})\, d\boldsymbol{x}.
\end{aligned} \tag{3.12}$$

Keeping in mind that E is compactly embedded into $L^{s_1}(\Gamma_b;\mathbb{R}^d)$ and $L^{s_2}(\Gamma_c;\mathbb{R}^d)$ (due to $s_1, s_2 \in [1, p_*)$), we utilize the continuity of k and h to infer

$$\begin{cases}
\lim\limits_{n\to\infty} \int_{\Gamma_b} k(\boldsymbol{v}_\tau^{(n)})\boldsymbol{\xi}_n \cdot (\boldsymbol{w}_\tau^{(n)} - \boldsymbol{u}_\tau^{(n)})\, d\Gamma = \int_{\Gamma_b} k(\boldsymbol{v}_\tau)\boldsymbol{\xi} \cdot (\boldsymbol{z}_\tau - \boldsymbol{u}_\tau)\, d\Gamma, \\
\lim\limits_{n\to\infty} \int_{\Gamma_c} h(v_\nu^{(n)})(w_\nu^{(n)} - u_\nu^{(n)})\, d\Gamma = \int_{\Gamma_c} h(v_\nu)(z_\nu - u_\nu)\, d\Gamma, \\
\lim\limits_{n\to\infty} \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{w}_n - \boldsymbol{u}_n)\, d\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{z} - \boldsymbol{u})\, d\boldsymbol{x}.
\end{cases} \tag{3.13}$$

Passing to the limit as $n \to \infty$ in inequality (3.9) and employing (3.10), (3.12) and (3.13), one has

$$\begin{aligned}
&\int_\Omega \boldsymbol{S}(\boldsymbol{x}, \boldsymbol{D}(\boldsymbol{z})) : \boldsymbol{D}(\boldsymbol{z} - \boldsymbol{u})\, d\boldsymbol{x} + \int_\Omega ((\boldsymbol{v} \cdot \nabla)\boldsymbol{u}) \cdot (\boldsymbol{z} - \boldsymbol{u})\, d\boldsymbol{x} \\
&\quad + \int_{\Gamma_b} k(\boldsymbol{v}_\tau)\boldsymbol{\xi} \cdot (\boldsymbol{z}_\tau - \boldsymbol{u}_\tau)\, d\Gamma + \int_{\Gamma_c} h(v_\nu)(z_\nu - u_\nu)\, d\Gamma \\
&\geq \lim_{n\to\infty} \Big[\int_\Omega \boldsymbol{S}(\boldsymbol{x}, \boldsymbol{D}(\boldsymbol{u}_n)) : \boldsymbol{D}(\boldsymbol{w}_n - \boldsymbol{u}_n)\, d\boldsymbol{x} + \int_\Omega ((\boldsymbol{v}_n \cdot \nabla)\boldsymbol{u}_n) \cdot (\boldsymbol{w}_n - \boldsymbol{u}_n)\, d\boldsymbol{x} \\
&\quad + \int_{\Gamma_b} k(\boldsymbol{v}_\tau^{(n)})\boldsymbol{\xi}_n \cdot (\boldsymbol{w}_\tau^{(n)} - \boldsymbol{u}_\tau^{(n)})\, d\Gamma + \int_{\Gamma_c} h(v_\nu^{(n)})(w_\nu^{(n)} - u_\nu^{(n)})\, d\Gamma \Big] \\
&\geq \lim_{n\to\infty} \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{w}_n - \boldsymbol{u}_n)\, d\boldsymbol{x} \\
&= \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{z} - \boldsymbol{u})\, d\boldsymbol{x}.
\end{aligned}$$

Let $\boldsymbol{w} \in C(\boldsymbol{v})$ be arbitrary and $t \in (0, 1)$. We take $\boldsymbol{z}_t = t\boldsymbol{w} + (1 - t)\boldsymbol{u}$ into the above inequality to get

$$\int_\Omega \boldsymbol{S}(\boldsymbol{x}, \boldsymbol{D}(\boldsymbol{z}_t)) : \boldsymbol{D}(\boldsymbol{w} - \boldsymbol{u})\, d\boldsymbol{x} + \int_\Omega ((\boldsymbol{v} \cdot \nabla)\boldsymbol{u}) \cdot (\boldsymbol{w} - \boldsymbol{u})\, d\boldsymbol{x}$$

$$
\begin{aligned}
&+ \int_{\Gamma_b} k(\boldsymbol{v}_\tau)\boldsymbol{\xi} \cdot (\boldsymbol{w}_\tau - \boldsymbol{u}_\tau)\, d\Gamma + \int_{\Gamma_c} h(v_\nu)(w_\nu - u_\nu)\, d\Gamma \\
&\geq \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{w} - \boldsymbol{u})\, d\boldsymbol{x}.
\end{aligned}
$$

Letting $t \to 0^+$ and applying the continuity of $\boldsymbol{S}$, we obtain that

$$
\begin{aligned}
&\int_\Omega \boldsymbol{S}(\boldsymbol{x}, \boldsymbol{D}(\boldsymbol{u})) : \boldsymbol{D}(\boldsymbol{w} - \boldsymbol{u})\, d\boldsymbol{x} + \int_\Omega ((\boldsymbol{v} \cdot \nabla)\boldsymbol{u}) \cdot (\boldsymbol{w} - \boldsymbol{u})\, d\boldsymbol{x} \\
&\quad + \int_{\Gamma_b} k(\boldsymbol{v}_\tau)\boldsymbol{\xi} \cdot (\boldsymbol{w}_\tau - \boldsymbol{u}_\tau)\, d\Gamma + \int_{\Gamma_c} h(v_\nu)(w_\nu - u_\nu)\, d\Gamma \\
&= \lim_{t \to 0^+} \int_\Omega \boldsymbol{S}(\boldsymbol{x}, \boldsymbol{D}(\boldsymbol{z}_t)) : \boldsymbol{D}(\boldsymbol{w} - \boldsymbol{u})\, d\boldsymbol{x} + \int_\Omega ((\boldsymbol{v} \cdot \nabla)\boldsymbol{u}) \cdot (\boldsymbol{w} - \boldsymbol{u})\, d\boldsymbol{x} \\
&\quad + \int_{\Gamma_b} k(\boldsymbol{v}_\tau)\boldsymbol{\xi} \cdot (\boldsymbol{w}_\tau - \boldsymbol{u}_\tau)\, d\Gamma + \int_{\Gamma_c} h(v_\nu)(w_\nu - u_\nu)\, d\Gamma \\
&\geq \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{w} - \boldsymbol{u})\, d\boldsymbol{x}
\end{aligned}
$$

for all $\boldsymbol{w} \in C(\boldsymbol{v})$. This means that $\boldsymbol{u} \in M(\boldsymbol{v}, \boldsymbol{\xi})$. Moreover, it follows from the strong-weak upper semicontinuity of J, the continuity of $\mathcal{R}$, the fact that J has closed and convex values (see Lemma 3.6), and Theorem 1.1.4 of [21] that $J \circ \mathcal{R}$ is strongly-weakly closed. The latter combined with the compactness of embedding of E into $L^p(\Gamma_b; \mathbb{R}^d)$ and $\theta \leq p$ implies that $\boldsymbol{\eta} \in J(\mathcal{R}(\boldsymbol{v}))$. Therefore, we conclude that $(\boldsymbol{u}, \boldsymbol{\eta}) \in P(\boldsymbol{u}, \boldsymbol{\xi})$, namely, P is sequentially closed in $E_w \times L^{\theta'}(\Gamma_b; \mathbb{R}^d)_w$ topology.

Furthermore, we consider the bounded, closed and convex set D defined by

$$
D := \{(\boldsymbol{u}, \boldsymbol{\xi}) \in K \times L^\theta(\Gamma_b; \mathbb{R}^d) \mid \|\boldsymbol{u}\|_E \leq R_1 \text{ and } \|\boldsymbol{\xi}\|_{L^\theta(\Gamma_b; \mathbb{R}^d)} \leq R_3\}.
$$

Arguing as in the proof of Boundedness part, it is not hard to verify that P maps D into itself, i.e., $P\colon D \to 2^D$. Therefore, all conditions of Kakutani-Ky Fan theorem, namely Theorem 2.3, have been verified. We are now in a position to invoke this theorem for deriving that P has a fixed point $(\boldsymbol{u}, \boldsymbol{\xi}) \in D$, i.e., $(\boldsymbol{u}, \boldsymbol{\xi}) \in P(\boldsymbol{u}, \boldsymbol{\xi})$. This means that $\boldsymbol{u} \in M(\boldsymbol{u}, \boldsymbol{\xi})$ and $\boldsymbol{\xi} \in J(\mathcal{R}(\boldsymbol{u}))$. Whereas, applying the same arguments as in the proof of Boundedness part, it is obvious that $\|\boldsymbol{u}\|_E \leq R_1$ and $\|\gamma \boldsymbol{u}\|_{L^\theta(\Gamma_b; \mathbb{R}^d)} \leq R_2$. So, we have $\boldsymbol{\xi} \in J(\boldsymbol{u})$. Consequently, we conclude that the solution set of Problem 3.3 is nonempty. □

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Conflict of interests

The authors declare no conflict of interests.

Data availability statement

Data sharing not applicable to this article as no data sets were generated or analysed during the current study.

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