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A musical reading of a contemporary installation and back: mathematical investigations of patterns in *Qwalala*

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Mathematical music theory helps us investigate musical compositions in mathematical terms. Some hints can be extended towards the visual arts. Mathematical approaches can also help formalize a "translation" from the visual domain to the auditory one and vice versa. Thus, a visual artwork can be mathematically investigated, then translated into music. The final, refined musical rendition can be compared to the initial visual idea. Can an artistic idea be preserved through these changes of media? Can a non-trivial *pattern* be envisaged in an artwork, and then still be identified after the change of medium? Here, we consider a contemporary installation and an ensemble musical piece derived from it. We first mathematically investigate the installation, finding its patterns and structure, and then we compare them with structure and patterns of the musical composition. In particular, we apply two concepts of mathematical music theory, the Quantum GestART and the gestural similarity conjecture, to the analysis of *Qwalala*, realized for the Venice Biennale by Pae White, comparing it to its musical rendition in the homonymous piece for harp and ensemble composed by Federico Favali. Some sketches of generalizations follow, with the "Souvenir Theorem" and the "Art Conjecture."

Keywords: glass; pattern; gestures; contour; category

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2012 Computing Classification Scheme: applied computing

1. Introduction: an installation and an analytic technique

1.1. *The installation* Qwalala

A giant wall made up of colored glass blocks, to represent a river, with the glass and the water being two symbols of the fragile beauty of Venice. This is the intuition the American artist Pae White (1963-) had for the *57th Esposizione Internazionale d'Arte — La Biennale di Venezia*. The installation named *Qwalala* has been displayed from 2017 to 2019 at "Le Stanze del Vetro" in the Island of San Giorgio in Venice (Fondazione Giorgio Cini 2017); see Figures 1 and 2.

The installation was named after the river "Gualala" in California, the state where Pae White lives. The word "Qwalala" comes from the language of Pomo, an Indigenous American Tribe, meaning "place where water descends."

From the top, *Qwalala* may seem a colored snake (Figure 3). In fact, the sketch of a

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curved brickwall and the sketch of a colored snake can be similar; see Mannone (2020) for a detailed discussion.

Figure 1. A side view of *Qwalala*, from the Manica Lunga Library at Fondazione Cini. Picture by F. Favali.

Figure 2. Another view of *Qwalala*'s wall. Picture by M. Mannone.

Figure 3. The *Qwalala* installation, seen from the bell tower of the San Giorgio Cathedral. Picture by M. Mannone.

1.2. *A technique: Gestural similarity and Quantum GestART*

The installation *Qwalala* can be modeled mathematically, and then it can be transposed into music. We have to see *how*. In this section, we briefly describe a possible technique to investigate creative gestures in music and in the visual arts, and how an image can be mapped to sound.

Disclaimer: this is not a compendium of the *mare magnum* of sonification techniques; this is just a summary of a particular approach based on mathematical music theory. In Section 2, we apply the proposed technique to the modeling of *Qwalala*, with an emphasis on its non-trivial patterns. Section 3 gives some hints for the musical rendition of *Qwalala*, with a discussion of how to "translate" its visual patterns and structure into music. In Section 4, we analyze the main elements of an actual musical composition inspired by this installation. This piece has been freely composed with the theoretical ideas as a guide. The structure of the installation constituted the starting point for a personal composition.

This technique is based on some hints from the mathematical theory of musical gestures. This is a fertile field of research (Mazzola and Andreatta 2007; Jedrzejewski 2019; Arias 2018; Arias-Valero and Lluis-Puebla 2020; Mannone and Turchet 2019; Clark 2020). The language of categories and some of its results and constructions (Mac Lane 1978; Fuyama, Saigo, and Takahashi 2020) are decisive to build the abstract and general tools of gesture theory. In particular, commutative diagrams and universal constructions such as limits and colimits have been applied to the orchestral gestures (Mannone 2018a). With the language of categories, we can compare gestures[1] of musicians finding their *similarity* despite their differences.

[1] A gesture can be seen as a trajectory connecting points in space and time, or in a topological space (Mazzola and Andreatta 2007), or in a parameter space (Mannone 2018a).

A similar change, e.g., an increase of pressure for the violin's bowing and of air pressure for the flute blowing, can generate a similar sound effect, i.e., *crescendo/forte* on the violin and on the flute, respectively. There will be similar changes within the described gestures and in the spectrograms of the sound output (*gestural similarity conjecture*). The gestural changes have to happen in specific parameter spaces, that are different for each instrument. As an extreme example, an increase of hammer's acceleration to perform *forte* for percussion, cannot be applied to an instrument belonging to a different parameter space, such as the flute: the hammer would just break the flute.

With this in mind, we can envisage a similarity between music and images: e.g., a detached movement of the bow can produce a *staccato* sequence of notes on the violin, and a similar detached movement with paper and pencil can produce a collection of dots (Mannone 2018a). These ideas can be extended from simple sound sequences and visual shapes to more organized and structured musical sequences and visual forms. That is, we can try to "translate" a *visual Gestalt* into a *musical Gestalt*. We can create a technique to make this translation possible, defining criteria to preserve the essential characteristics of the initial work, and making the work "invariant" — at a certain degree — under translation (Mannone et al. 2020). Such a technique makes use of some hints from quantum mechanics, and in particular, of the ideas of states as superpositions, of *destructive measure*, and of the Dirac formalism. Suggestions from quantum mechanics are often applied to music theory, as witnessed by very recent works (Blutner and beim Graben 2020). The relevance of quantum hints for this formalism is described in detail in the *Quantum GestART* paper (Mannone et al. 2020).

1.2.1. RTF method, patterns, and envelopes

We call our technique *filter, transform, and refine* (RTF). It uses the Dirac notation from quantum mechanics (Mannone et al. 2020). An image is seen as a visual state $|\psi^v\rangle$. This state can be approximated or "filtered" with respect to a collection (that we could call "basis" with some abuse of terminology) of simple shapes $|\phi_i^v\rangle$, with coefficients b_i. This is the step F, *filter(ing)*. Patterns and envelopes can be seen as the result of creators, generating gestures; thus the formalism of gestures and nested gestures can be invoked. This idea may be akin to triangular (basis) approximation of a space. However, this deals with basic forms (gestures) richer than triangles. Thus, we can relate these works to homology of gestures (Arias-Valero and Lluis-Puebla 2020).

Then, we can associate each simple visual shape $|\phi_i^v\rangle$ with a simple sound/musical sequence $|\phi_i^s\rangle$, according to gestural similarity. Also, coefficients b_i, describing characteristics of the shape (how many $|\phi_i^s\rangle$? how high/low? where? how big?) can be associated with coefficients a_i, describing characteristics of the sound sequence (how often? how high? when? how loud?), still according to gestural similarity. This means creating a translation from a vocabulary of visual forms into a vocabulary of sound forms, and a translation from an approximated form to a rough musical piece. This is the step T, *transform* — because we make a transform from the domain of visuals to the domain of sounds. In the *Quantum GestART*, T is described as semi-linear. It is not linear because the visual and sound coefficients are not the same.

Finally, such a rough musical piece can be refined, creating a musical composition $|\psi^s\rangle$. This is the step R, *refine*. The technique is thus F-T-R, or RTF, if we consider these transformations as acting on a state $|\psi^v\rangle$ from the right to the left, see Eq. (1). Mannone et al. (2020, Figure 2) give the example: photograph of a tree → sketch of the tree →

sonified sketch → musical piece.

$$RTF|\psi^v\rangle \sim RT\left(\sum_i^N b_i|\phi_i^v\rangle\right) = R\left(\sum_i^N a_i|\phi_i^s\rangle\right) = |\psi^s\rangle \tag{1}$$

We can also define *patterns* and *envelopes* (Mannone et al. 2020). *Patterns* are fine-grained approximations, distributed along with overall shapes, called *envelopes*. An envelope can be the pattern of a larger envelope, creating nested structures. E.g., while drawing a schematic leopard, the spots are the patterns, and the leopard's silhouette is the envelope. Patterns (spots) are distributed along and modified according to the envelope (leopard's profile). In a *musical leopard*, patterns could be groups of *staccato* notes representing the spots, and their repetitions and variations (e.g., with overall pitch/loudness changes) represent the scattering of spots along the envelope. If we have a set of leopards on a tree in the savannah, the single leopard is the new pattern, and the tree is the new envelope. Thus, we can investigate structures through nested patterns. Regarding notation, we represent the operation of inserting a visual element within another one through the tensor symbol[2] (Figure 7). We can separately modify envelopes or patterns, e.g., (pattern 1)⊗(envelope 1)$\xrightarrow{\text{change}\otimes 1}$(pattern 2)⊗(envelope 1). We can define monoidal categories if these transformations verify pentagonal and triangular identities.

Remark 1.1 The tensor provides a dimensional lifting: e.g., with an n-dimensional gesture, gesture⊗gesture is an $(n+1)$-dimensional hypergesture: in fact, an $(n+1)$-path connects two n-paths, as proven by Mannone (2018b, Theorem 3, page 11). Similarly, a dimensional lifting occurs when we combine a pattern within an envelope: pattern⊗envelope recalls gesture⊗gesture. In fact, here we can implicitly identify a pattern and an envelope with the gestures needed to produce them. However, when we just compose in the sense of concatenating gestures one after the other, we stay within the same n-dimension. With a little abuse of notation, we can use the tensor symbol more generally. Visually, we can use tensor to obtain, given a pattern of dots (1-dim) and a curved line (1-dim) envelope, a curved surface (2-dim) decorated with dots. (A dotted line would not be interesting in this framework). If dimensions of patterns and envelopes are different, dimensional lifting may occur in the envelope *or* in the pattern. A pattern of ellipses (2-dim) ⊗ an envelope with a curved line (1-dim) gives us a curved surface (2-dim) with ellipses. If we have blocks (3-dim) and a curved line (1-dim), we obtain a "surface" (2-dim) with blocks that in fact is a wall (3-dim) with blocks. If we have blocks (3-dim) and a curved *surface* (2-dim), we obtain again a wall (3-dim) with blocks, see Figure 7. From afar, a long and tight surface can be sketched as a line, as well as a tight 3-dim wall appears as a surface (see again Figure 7 and Figure 3). If we have a pattern of chaotic lines (1-dim) and the envelope of a transparent block (3-dim), we obtain a block (3-dim) decorated with lines distributed through it.

The idea of nested patterns and envelopes (and brick-walls with decorated glass blocks) is applied to *Qwalala* (Section 2). The RTF method allows us to envisage its possible musical rendition in Section 3. This is the result of a research project and a residency of the author at the *Fondazione Cini* in the Island of San Giorgio. After a description of how to find patterns in *Qwalala* between chaotic paths and repetition of modules, we

[2]As assessed in the Appendix, in *Quantum GestART* decompositions, some coefficients (from 1 to j) may indicate patterns and other ones (from j to N) the envelope, with the direct sum ⊕. It is like building up block-matrices to be summed together. Alternatively, we could have smaller matrices and obtain a bigger block matrix through their tensor product. The equivalence of these methods is described in the Appendix.

analyze the structure of the homonymous musical piece, finding the musical renditions of the installation's patterns and overall shape. The piece is *Qwalala* for harp and ensemble, composed by Federico Favali[3] freely *but* inspired by discussions on thinking with patterns/envelopes (Mannone 2020).

2. A possible mathematical investigation of *Qwalala*: finding non-trivial patterns and envelopes

Qwalala is an installation made of glass brick of different colors. Each brick contains inner decorations with colored lines. These lines have probably been created with almost random movements of the glass-makers. The intricate design of lines and colors that is repeated within several bricks, can thus be identified as a "pattern" for its modularity and recognizability. This can enforce the hypothesis of gestural homology (Arias-Valero and Lluis-Puebla 2020) behind filtering. Figure 4 shows the detail of one of these decorations. These lines could be modeled as the chaotic — but recognizable — designs of a suitable strange attractor. In particular, the de Jong attractor (Bourke 1995, 2014) seems to give a good approximation of them; see Figure 5. The de Jong attractor requires the solution of the pair of recursive equations displayed in Eq. (2); see (Bourke 2014).

$$x_{n+1} = \sin(ay_n) - \cos(bx_n), \quad y_{n+1} = \sin(cx_n) - \cos(dy_n) \tag{2}$$

Figure 4. A detail of *Qwalala*'s wall. Picture taken in November 2019. Some raindrops are visibile.

Figure 5. Graphic made with Mathematica^{TM} to model a *Qwalala* brick. The inner pattern is obtained with de Jong attractors. See (Mannone 2020) for the code and more details.

By changing parameters and color functions, the numerical solution of Eq. (2) can lead to different designs and color shades. We can thus change the details of these "patterns." Each group of inner lines is contained within a glass block. Thus, we can call each block an "envelope"—using the terminology from (Mannone et al. 2020). All blocks constitute the overall form of *Qwalala*. Thus, each block is a brick of the wall.[4] Each brick is the pattern of the wall, and the wall is the envelope. In this way, we can think patterns and envelopes in a recursive way. Figures 6 and 7 visually illustrate this idea. Each single brick can be seen as the result of a variational process on an infinitesimal slice, see Figure 8, where the symbol $|\phi^v\rangle$ indicates a "state" in the formalism of (Mannone et al. 2020).

Of course, such a schematization is also a simplification: in fact, it does not take into account brick gluing and possible asymmetries or imperfections. We are schematizing a complex form in terms of simple shapes seen as patterns/envelopes recursively.

[3]F. F. is a postdoctoral researcher at the Universidad Nacional de Tres de Febrero in Buenos Aires, Argentina. He composed this piece during a residency at the *Fondazione Cini* in the Island of San Giorgio, Venice.

[4]The blocks in *Qwalala* can also remind one of ice blocks as frozen water, or of the infinitesimal volumes a fluid can be ideally divided into (as in the mechanics of fluids).

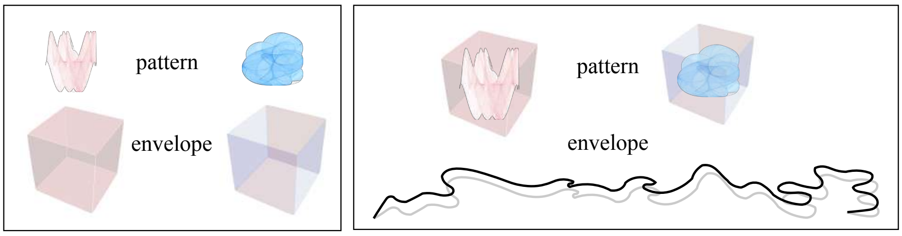

Figure 6. The majority of *Qwalala* bricks can be seen as made of inner lines (pattern) within a block (envelope). But each brick is the constitutive element of the wall: thus, each brick can also be seen as a pattern within the envelope of *Qwalala* installation. The color contrast of cube has been modified to make the figures more visible.

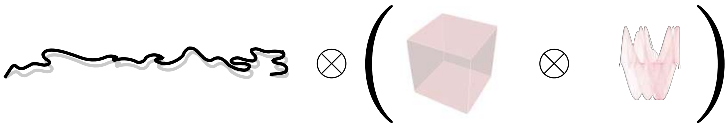

Figure 7. From the right to the left: inner lines are the patterns of blocks, and blocks are the patterns of the wall. The idea of putting a form within another one is indicated with the tensor product symbol, because we could define matrices to represent each element, and the dimensional raising is given by the tensor product operation, see (Mannone 2018b, 2020).

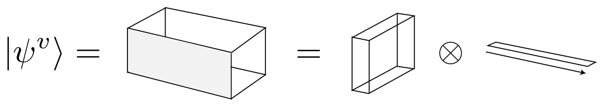

Figure 8. We can formalize a single, simple brick as a visual state $|\psi^v\rangle$ made of an infinitesimal slice expanded through space.

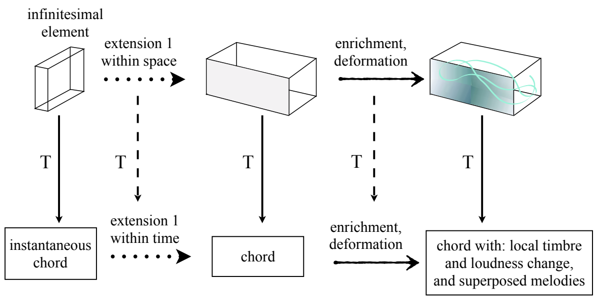

Figure 9. How the scheme of bricks and decorated bricks can become music.

3. A possible musical rendition based on these ideas

Starting from the schematization illustrated in Section 2, we can think of how to musically render these findings. Each element retrieved in the F-step can be mapped onto simple melodic fragments and chordal structures (T-step). Then, the collected musical fragments can be further elaborated to create a complete and meaningful piece of music (R-step).

Let us start from a single brick. The elements described in Figure 8 can be thought of in musical terms. For example, an infinitesimal slice can be musically rendered as an instantaneous chord; spatial extension of such a slice can be musically rendered as a time-finite chord. The visual enrichment of some bricks with inner lines may suggest a musical enrichment of a chord with melodic lines (Figures 9 and 10).

The chaotic visual lines, approximated in Section 2 with strange attractors, can be rendered with apparently improvised melodic lines, giving an effect of perceived shape similarity (Figure 11). The recognizability of colors is rendered with precise timbral and harmonic references. Henceforth, the visual appearances of colored bricks leads to recognizable instrumental entries, with recurring harmonies and identifiable thematic-cell references.

In Figure 10, a specific example of lines is broken down into simple shapes to build up one of the themes used in the piece. However, the idea described in Figure 10 is just illustrative. The composer discussed the possible application of RTF to *Qwalala* with the author during an artistic residency at Fondazione Cini. Then, he freely composed the piece, and the trace of the method and the references to patterns can be found after a compositional analysis.

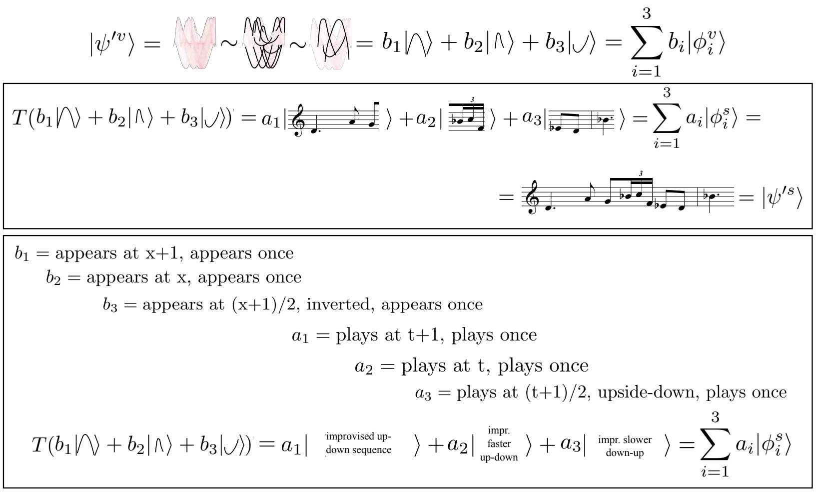

Figure 10. A visual state $|\psi'^{v}\rangle$ can be filtered (F) and musicalized (T). The example on top shows the creation of the red theme in the piece titled *Qwalala*. For the sake of simplicity, we are considering just three lines. Bottom: coefficients seen as "instructions." Another possible musical realization can involve improvised musical sequences (and, another one, algorithmically generated music). All these versions can be compared, and the technique to obtain them can be compared with natural transformations.

We can see the b_i as "instructions" on how to put the visual sketches within the visual space, and the a_i as how to put simple musical sequences in the musical space. This is the space of musical parameters. Prior to any instrument specification, this space must be constituted by pitch, time, and loudness as a minimum. The information about color can tell one about timbre. The choice of pitch contour is suggested by gestural similarity; the precise choice of pitch can be either evaluated computationally as a mapping, either up to the composer or even the improviser, but it should always respect the (perceptive criterion of) gestural similarity; see bottom of Figure 10. The scope of the remainder of this paper is finding precise patterns and schemes within the final piece, and to connect them with the installation's structure and details. The identification of patterns in the original installation is not trivial; the same for the musical piece. The overall work shows some coherence of "translation" from the visual domain to the musical domain. Of course, there is not *only one* way to translate an artwork from a domain to an artistic one, but, chosen an approximative technique, there is an entire equivalence

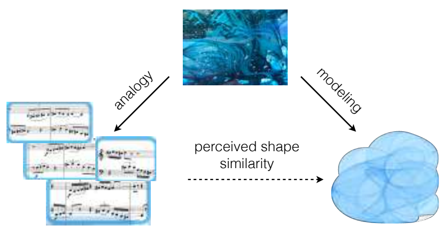

Figure 11. Chaos-like melodic lines and attractors: from decoration of blocks to perceptual shape similarity.

class of possible outcomes. This will be discussed
in Conjecture 6.1.

4. Finding patterns and comparing them with mathematical findings

Let us now consider the piece, *Qwalala*, composed for harp and ensemble by Federico Favali. Figure 14 shows the detail of the RTF method with a score excerpt. This method implies a progressive transformation: installation → scheme → musical sketch → actual score. Figure 15 shows another example from the score. Beauty codes of visual arts and of music are different, as Hanslick (2007) would remark. Thus, the R step (refine) is essential for the final and very personal choice of music details. Figure 12 shows the overall scheme of the musicalization process, from the original image, to the hand-drawn scheme, to the main elements of the score. Time durations and metronome marks are accurately defined to imitate the space development of the installation within the temporal development of music.

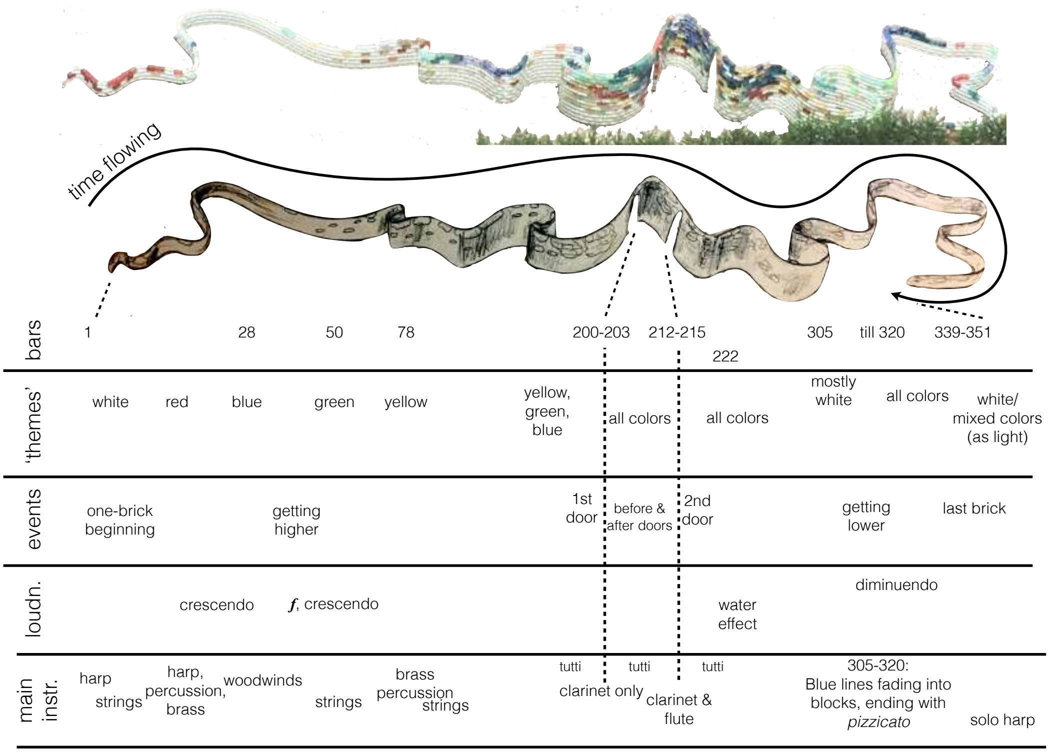

Figure 12. From *Qwalala*'s picture (top), to its schematic drawing made by the author (center), to the structure of its musical rendition (bottom). If there were a left-to-right scanning as in a score, in certain points we would have had multiple elements of the image. Instead, the overall music flows as the gesture to draw the snake, and this justifies the time flowing curve.

The sequence of bricks and the appearances of inner lines and colors in the installation is read as a score, with instrumental sections entrances and thematic cells appearances. The piece includes overall pitch and loudness variations according to the overall variations of the height of the installation, based on the conjecture of gestural similarity and on the idea of musical tridimensionality (Mannone 2018a). Two openings in the installation suggest two moments of almost sudden silence. In the first case, the sound of a solo clarinet represents the brick at the top of the door; in the second case, the sound is only sustained by a clarinet and a flute (Figure 12).

As an extension of gestural similarity to colors, we can compare colors and timbres according to our perceptive reactions to them, e.g., we could be more reactive to reddish colors and percussion/brass sounds, and we can respond in a more soft way to blue-ish colors and flute/high woodwinds sounds. These associations are not one-to-one; nevertheless, they can be investigated in terms of our response to color-bands and timbre-bands. A recent experiment investigates these associations (Mannone and Santini 2020). This is the starting point of the "chromo-gestural similarity." Kandinsky discussed choices of colors in terms of their emotional power (Seyler 2019). According to these general suggestions and to his own musical sensibility, the composer proposed the following association of colors with instruments:

- blue → woodwinds and horn;
- green → strings;
- yellow → horn, brass, high violins;
- red → brass, percussions;
- white → harp.

Figure 13. Color themes.

A different thematic element and a tonal center is associated to each color (Figure 13). The piece begins and ends with the harp. There is no melodic theme associated with this instrument. The harp's timbre is meant as being a more "neutral" element, as the uncolored[5] blocks in *Qwalala*. At the end of the piece, the fading solo harp plays thematic references to the other colors. This is a reference to the nature of white light, containing all colors.

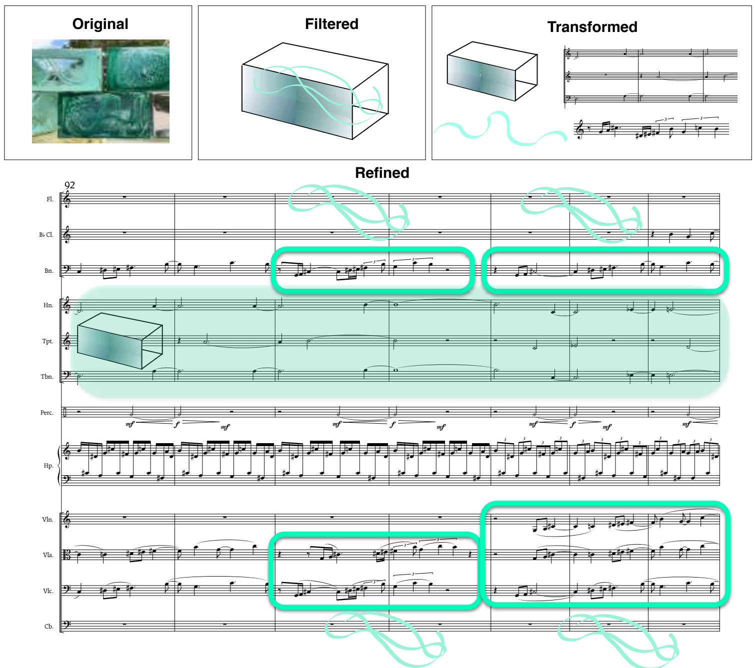

Figure 14. An example of an intuitive application of the RTF method, from some original bricks, to their schematization, to the association with musical elements, to a portion of the final score (bars 92–98; green theme). The score is published by Donemus.

[5]Completely transparent or decorated with white lines.

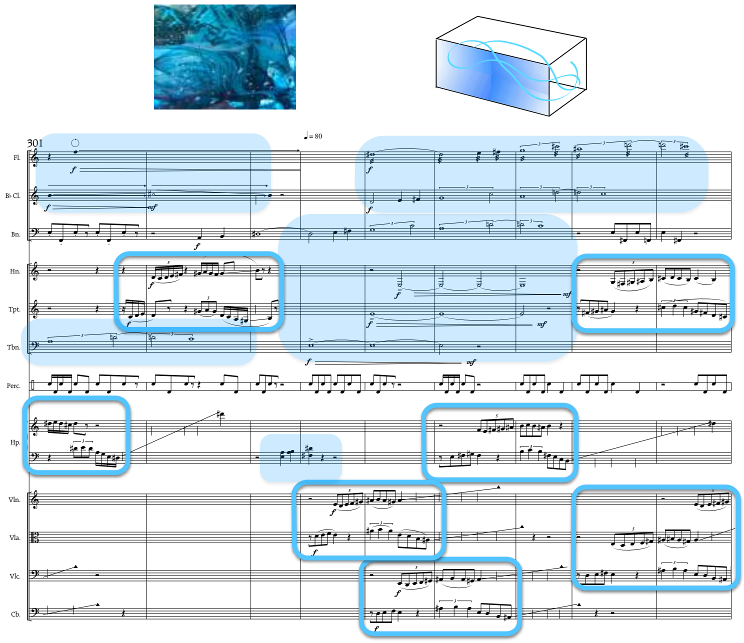

Figure 15. An example with the blue theme: a detail from a brick's decoration (top left; the small spots in the picture are rain droplets); the symbolic structure of a decorated block (top right); an excerpt from bars 301–309 with highlighted renditions of blocks and lines (bottom).

The piece does make explicit references to the water flow. There is a structural reference, with the overall ongoing flow of the rhythm. In some passages, a percussionist uses an instrument filled with water (Figure 12).

5. Discussion

We can compare different visual patterns and compare their musical renditions. The formal structure of patterns and envelopes can lead to some theoretical developments; see the Appendix.

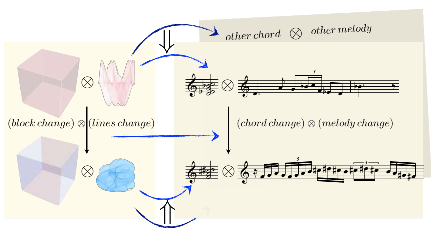

Figure 16. Natural transformations can describe cross-comparisons between different musical renditions of visual elements.

If we compose another piece based on the same installation, we will be able to cross-compare a first musical rendition with a second musical rendition of the same elements. This idea is captured by the concept of natural transformations in category theory (Mac Lane 1978), see Figure 16 for a visual example. We can open up a discussion about the "identity" and the essence of an artwork. Starting from the same visual artwork, we can obtain different musical renditions.

If all these renditions keep some essential features of the initial artwork, i.e., if they are all realized according to the gestural similarity conjecture, the main idea of the initial work could be preserved through these translations (Mannone et al. 2020), and some main ideas can be still recognizable. In the case of *Qwalala*, there would be the ideas of "flowing," of variety within blocks, and of nested structure made by smaller and identifiable units. The possible musical renditions of *Qwalala* are in principle infinite. Let us consider some of them: they constitute a small category (with morphisms given by small variations to their scores). We can define a functor (a morphism between categories) from the category of visuals, that contains *Qwalala*, to the category of its musical renditions. The comparisons between these renditions are described by natural transformations. More precisely: A *category* is constituted by objects and morphisms between them, which verify associativity and identity properties. The composition of two morphisms of a category gives a morphism of the same category. Here we define:

- A category A of visuals constituted by one object only. Object: "Qwalala," morphisms: loop arrows from and into Qwalala. Alternatively, we could extend the idea to small variations (non-homotopic in general) which do not compromise its recognizability (e.g. taking away a block or putting another one). Identity: no block changes; composition: two block variations are still block variations; associativity: verified;
- A category B of Qwalala's musical renditions. Objects: musical pieces which verify gestural similarity condition; morphisms: (classes of) local and global transformations which morph a piece into another one. We can compose different transformations to get another musical piece. Identity is given by the zero-change in a piece. Associativity: verified;
- A *sonification functor 1*, $S_1 : A \to B$, which maps Qwalala into a musical piece, and small block variations into small musical variations leading to a slightly different piece. If A contains the loops as the only arrows, thus S_1 maps loops into small musical variations. A *sonification functor 2*, $S_2 : A \to B$, which maps Qwalala into another musical piece. If the RTF-method is applied, S_i mapping images into pieces acts as T;
- A natural transformation $\alpha_{1,2} : S_1 \to S_2$.

Each of these transformations contains the information about how the single patterns are musically rendered and how these renditions change from a piece to another one. We can think of homotopy equivalence[6] between effective renditions, because of the analogy rendition/functor (path) and comparison/natural transformation (path homotopy). More generally, we can start from an "ideal gesture" that produces visual or musical artworks with similarities. The produced artworks can be analyzed, and they can remind of the same general "artistic idea." Using the language of categories, we can think of these objects as limits or colimits, see Figure 17. An artistic idea can be the starting point for another ideal gesture to create other artworks, and so on, as infinite chains. This is the "Art Conjecture" (see the Appendix). Its underlying aesthetic hypothesis is supported by philosophical references such as the studies by Michel Henry (2009, 2004) on music/art as expression of external forces and thus *abstract* (Seyler 2019), with references to Kandinsky's painting and Briesen's musical drawings. The "artistic idea" and the concept of recognizability can lead to the rather metaphorical "Souvenir Theorem" (Theorem 5.1): a souvenir is seen as an an optimal representative of a homotopy class (Mannone 2020). A musical rendition of the souvenir could constitute a "musical souvenir." The "proof" of this theorem is extra-mathematical, being up to some statistical validation. A souvenir is more likely to be sold if a similarity condition is verified, and if some "practical" features, such as a balance of price and quality, are advantageous. As a corollary, we can define *musical souvenirs* using sonification functors, verifying similarity condition, whose support (CD, file) is strong enough, and with an enough small price.

Theorem 5.1 The "Souvenir Theorem." *Given an object, we can simplify it, but there is a minimum number of lines or simple shapes necessary to make its form recognizable.*

[6]Homotopy does not verify associativity, but homotopy classes do (Hatcher 2018). Transformations between gestures are often called *hypergestures* (Mazzola and Andreatta 2007).

A successful souvenir should be composed by these minimum number of parts, should have a price well below the tourist's budget, and should be made with a sufficient quality material.

6. Conclusion

We used the formalism of patterns and envelopes to analyze an installation, to move from the visual domain to the sound domain, and to investigate a musical rendition of this installation, created freely but knowing these ideas. The identification of patterns in the original installation is not trivial, but once they are found and their concept is extracted, we can find them within the musical composition, and we can imagine further musical applications. The mathematical formalism, thus, reveals itself not only as an analytical tool, but also as a creative instrument to generate new artworks.

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Appendix: some theoretical advancements and the Art Conjecture

In Quantum GestART decompositions, some coefficients (from 1 to j) can indicate patterns and other ones (from j to N) the envelope, with the direct sum $\oplus$. Patterns and envelopes can be the result of "creating gestures." In (Mannone 2018b, 2020), tensor products $\otimes$ are used while describing nested gestures. We can reconcile these notations considering matrix representations of states. Matrix sum requires the same dimensions of matrices; the result is another matrix of the same dimension. The vector product of two $n \times n$ matrices gives a $(2n \times 2n)$-matrix. We can specify the dimension of our objects, and we can use suitably-defined matrices M_1, M_2 to "lift up" matrix dimensions by adding zeroes, see Eq. (3).

$$\begin{aligned}
&\sum_{i=1}^{j} b_i|\phi_i^v\rangle^{n\times n} \oplus \sum_{i=j+1}^{N} b_i|\phi_i^v\rangle^{n\times n} = \\
&= \left(\sum_{i=1}^{j} b_i|\phi_i^v\rangle^{(n-2)\times(n-2)} \otimes M_1^{(n-2)\times(n-2)}\right) \bigoplus \left(\sum_{i=j+1}^{N} b_i|\phi_i^v\rangle^{(n-2)\times(n-2)} \otimes M_2^{(n-2)\times(n-2)}\right) = \\
&= \sum_{i=1}^{j} b_i|\phi_i^v\rangle^{(n-2)\times(n-2)} \otimes \sum_{i=j+1}^{N} b_i|\phi_i^v\rangle^{(n-2)\times(n-2)} = \left(\bigoplus_{i=1}^{j} b_i|\phi_i^v\rangle^{(n-2)\times(n-2)}\right) \bigotimes \left(\bigoplus_{i=j+1}^{N} b_i|\phi_i^v\rangle^{(n-2)\times(n-2)}\right)
\end{aligned} \tag{3}$$

Thus, we can conjecture the relationship $\otimes\bigoplus\otimes = \oplus\bigotimes\oplus$. Let us now discuss the Conjecture 6.1.

Conjecture 6.1 The "Art Conjecture." *Let A be an artwork belonging to the artistic domain represented by the category $\mathcal{A}$. Let $B_1, ..., B_n$ a set of n artworks belonging to the artistic domain represented by the category $\mathcal{B}$. The artworks B_i, with $i \in \{1, ..., n\}$, are musical renditions of A in the domain $\mathcal{B}$, verifying gestural similarity (but we could have infinite theoretical musical renditions). The existence of $B_1, ..., B_n$ and the analysis of their natural transformations contributes to the understanding of the* artistic idea *underlying A. Formally, all arrows are converging toward it.*

Discussion. Look at Figure 17 as a reference. It presents different ontological levels. It is constituted by two cones joined at their bases. The vertices of both cones contain abstract entities. The middle parts of the cones contain less abstract entities. The (coincident) bases of the cones contain "concrete" entities, such as tangible or audible artworks. Arrows from the top to the base represent a progressive *embodiment*, while the arrows from the base to the bottom represent progressive *abstraction*. These are "dual" transformations. Here, we are sketching a parallelism between a particular duality in the arts, and a particular duality in mathematics. In fact, the upper cone represents the art *creation*, while the lower cone represents the art *fruition*. From the point of view of categories, the upper cone contains a *limit*, while the lower cone contains a *colimit*: limits and colimits are dual. The limit is a generalization of the product, and the colimit is the generalization of the sum. In a nutshell, all arrows are converging to the colimit and all arrows are departing from the limit. In Figure 17, the *ideal gesture* is the start, and the *artistic idea* is the end. The vertex of the upper cone contains an *ideal gesture*, and its middle part contains a *generator gesture*. From gestural similarity (Mannone 2018a), a gestural generator is a gesture that "creates" and characterizes a simple musical articulation, or a simple line or collection of dots while drawing.Generator gestures can also be envisaged in the work of painters, glassblowers, or sculptors. The *category of gestural generators* has classes of gestural generators as objects, and classes of homotopies between them as morphisms. A *gestural generator* is the entity modifying different gestures in a similar way; see (Mannone 2018a) for a detailed discussion. *Ideal gestures* are abstractions of these creation-movements. For example, the movements necessary to draw a waved brick-wall can be oversimplified as movements to draw a waved-line with some points here and there as (oversimplified) bricks. The *category of ideal gestures*, or *conceptual gestures*, have, as objects, classes of ideal gestures, and, as morphisms, classes of homotopies between them. Summarizing, a generator gesture is a movement (or a collection of basic movements) that produces essential features of artworks. In Figure 17, if there exists a morphism f^i_{sim} such that $f^i_{sim} : A \to B_i$, thus there exists a gestural generator producing both A and B_i. Thus, the f^i_{sim} are morphisms between the category of visual artworks and the category of musical artworks (as the S_i described before). An ideal gesture is a more abstract element, taking only some essential features of gestures (of generator gestures, in our case). Let f_A : (generator gesture) $\to A$, f_{B_i} : (generator gesture) $\to B$, and $f^i_{sim} : A \to B_i$, showing an embodiment of gestures within specific artworks. For the hypothesis of gestural similarity between $A \in \mathcal{A}$ and $B_i \in \mathcal{B}$, their simple components are similar, that is, if $f^i_{sim} \circ f_A = f_{B_i}$ (i.e., if f_A, f_{B_i}, f^i_{sim} constitute a commutative triangle).

If A and B_i are similar, they can be seen as embodiments within different media of the same generator gesture. We can also compare different B_i, that is, different renditions of A in the medium $\mathcal{B}$. In our example, this would correspond to different musical pieces composed from *Qwalala*'s visual structure: B_1, B_2, ..., B_n can be different musical transpositions of the same visual artwork A.

For a visual artwork, there is an entire class of musical artworks that can be derived from it according to the conjecture of gestural similarity; here, we consider the selection $B_1, ..., B_n$ of these musical artworks. Let $g_{B_1,B_2} : B_1 \to B_2$ a (non invertible) morphism in $\mathcal{B}$ verifying associativity and identity; let $f^1_{sim} : A \to B_1$, $f^2_{sim} : A \to B_2$, and let $\alpha_{1,2} : f^1_{sim} \to f^2_{sim}$ be a natural transformation comparing the rendition B_1 of A with the other rendition B_2 of A. Here, f^i_{sim} can be seen as a functor mapping objects of $\mathcal{A}$ to objects of $\mathcal{B}$, and morphisms of $\mathcal{A}$ (not indicated in Figure 17; they would correspond to variations of A in $\mathcal{A}$) into morphisms of $\mathcal{B}$. Again, commutativity is verified if $f^1_{sim} \circ f_A = f_{B_1}$ and $f^2_{sim} \circ f_A = f_{B_2}$. Let us now consider an ideal gesture. We can define f^*_A : (ideal gesture) $\to A$ and $f^*_{B_i}$: (ideal gesture) $\to B_i$. Because there is only one way to make the ideal gesture more 'specific' in the creative generator gesture, there exists one and only one h : (ideal gesture) $\to$ (generator gesture) such as the upper cone of Figure 17 commutes; i.e., $f_A \circ h = f^*_A$ and $f_{B_i} \circ h = f^*_{B_i}$. Actually, there exists one and only one h' : (generator gesture) $\to$ (ideal gesture) about information recovery s.t. $f^*_A \circ h' = f_A$ commutes as well. This cone is the *terminal* object; it is *universal*, and the ideal gesture is a limit.

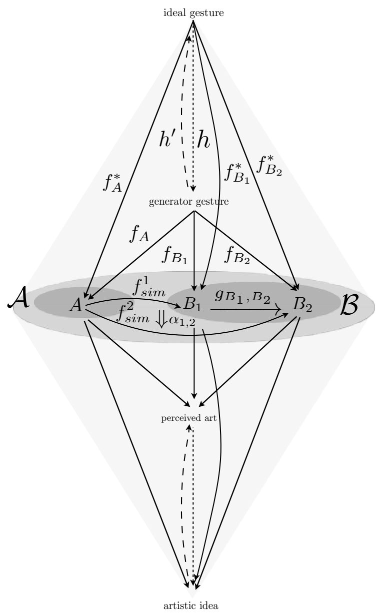

Figure 17. A diagrammatic representation of artworks (A, B_i) from different artistic domains $\mathcal{A}$, $\mathcal{B}$, with B_i derived from A, and their connection in terms of ideal (creative) gestures, generator gestures regarding the creation of the artwork, and perceived artwork and artistic idea in terms of art appreciation. The arrows in the upper cone represent *embodiment*, while the arrows in the second cone represent *abstraction*.

We can reverse all arrows and obtain the artistic evaluation idea, with the *perceived artwork* and its abstraction with the *artistic idea*. The construction process of the lower cone proceeds in terms of progressive abstraction, i.e., neglecting information from the "concrete" artworks presented within the base of the two cones. Morphisms of the lower cone represent the extraction of (gestural) information operated by listeners/audience/watchers: they are the opposite of embodiments (except the corresponding of h'). The "artistic idea" can be conceived as the result of an abstract gesture or as a purely abstract idea which can be embodied within a new ideal gesture, producing another double cone.

Different realizations of an artwork lead to a deeper understanding of it, and they all contribute toward the comprehension of its artistic idea. By duality, we may define the artistic idea as a *colimit* of the ideal gesture. When there is a particular coincidence between the ideal gesture and the artistic idea, we can imagine that, through a continuous deformation of the double cones, the two vertices touch each other.

Corollary 6.2 *We can imagine chains of cones, where the artistic idea of one of them can constitute the starting point for another creation. The intermediate passage makes us think of the structure of the light cone used in physics, with a distinction between past, present, and future. Light cones are used to represent the light particles trajectories within spacetime. In our case, we can freely consider a light cone as the space of existence (and the time of creation and fruition) of artworks connected by the same artistic concept. In physics, the middle point of the light cone indicates the present. Here, there is a temporal dimension flowing from up to down, with a "shifting present" from the artistic creation, to the conceptualization, to another creation, and so on. If, at some point, the ideal gesture goes back to a former artistic idea, then the chain of cones may close onto themselves, as creating tori or rings.*

Remark 6.3 Kubota et al. (2017) proposed a categorical model of aesthetics. It represents the process of (a single) artwork observation and critique, with the reaching of an aesthetic state; graphically, it uses points and arrows organized as an octahedron. Here, we consider (multiple) artwork creation. Also, artworks may belong to different domains, but they are built to preserve a certain degree of similarity, "converging" toward a unique concept. The octahedron in Kubota et al. (2017) can be included within the structure proposed in Figure 17.