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Reduction procedures for hyperbolic equations: applications to nonlinear wave problems

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Abstract

Department of Mathematics and Computer Sciences

Doctor of Philosophy

Reduction procedures for hyperbolic equations: applications to nonlinear wave problems

by ALESSANDRA RIZZO

Partial differential equations (PDEs) play a key role in the description of a wide range of complex phenomena such as the propagation of heat or sound, fluid flow, elasticity, electrostatics, electrodynamic and so on. For this reason, determining solutions of PDEs is a great challenge in applied mathematics and mechanics.
Motivated by this viewpoint, the aim of this thesis is to briefly review some of the most useful procedures for solving PDEs and to develop new approaches to construct exact solutions for this kind of models. Both the cases of partial differential equations of higher order and of systems of first order are treated.
Hyperbolic PDEs of higher order are studied in the framework of the intermediate integrals method: a reduction procedure to simplify the problem of solving a second order equation to the one of studying a first order equation is developed and an algorithm that under appropriate circustances permits to determine the general integral of linear equations is proposed.
On the other hand, hyperbolic systems are approached through the method of differential constraints: the well known Riemann problems (RP), Riemann problems with structure (RPS) and the generalised Riemann problems (GRP) are considered.
At the state of art, a general theory for solving this type of problems for non homogeneous systems does not exist: in this thesis, we provide results concerning the solution of a Riemann problem for a traffic flow model and of GRP and RPS for the non homogeneous p-system.

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Introduction

Determining exact solutions of partial differential equations (PDEs) is of great interest not only for its theoretical meaning but also for possible applications. Given a mathematical model, their exact solutions permit to describe real processes, to validate the model, to predict new behaviours. Furthermore the knowledge of exact solutions is useful to compare and to test numerical integration procedures.
Along the years many mathematical approaches have been proposed for determining exact solutions of PDEs. Most of them are concerned with degenerate hodograph and differential constraints methods.
The degenerate hodograph method requires the existence of some finite relation between the dependent field variables. The related exact solutions are called multiple waves and they are classified depending on the rank of the jacobian matrix of the dependent variables. For instance, the famous simple waves and double waves are, respectively, multiple waves of rank 1 and multiple waves of rank 2. Furthermore, within the theoretical framework of group analysis, it can be proved that multiple waves are partial invariant solutions.
Some fundamental results concerning simple and double waves are given in [44] (and references therein).
The Method of Differential Constraints was proposed in 1964 by Yanenko [65] for the gas-dynamics model. In general, exact solutions satisfy their own PDEs along with further differential relations. For instance, travelling waves $u(x-ct)$ are also solution of the transport equation $u_t + cu_x = 0$. Starting from such a remark, the main idea proposed by Yanenko is to add to the governing system under interest some further differential equations which play the role of constraints because they select classes of particular exact solutions admitted by the overderdetermined set of equations consisting of the original ones along with the additional differential constraints. The method is developed on different steps: first we have to choose the form of the differential constraints; then the compatibility of the related overdetermined system must be studied; finally exact solutions of the full set of equations must be determined. Such solutions will be given in terms of arbitrary functions whose number depends on the number of the constraints and therefore classes of initial and/or boundary conditions can be solved. Usually the presence of some constraints facilitate the procedure of solving the equations at hand. The method has a high degree of freedom and in fact it includes many of the approaches developed along the years for characterizing exact solutions of PDEs. Unfortunately, because of its generality, without any further hypotesis, it is not always useful in the applications. Therefore, in order to overcome this limitation, in [16, 55, 59], the involutiveness of the resulting overdetermined system is required (for more details see [44]). A special case of the method of differential constraints is the approach based on the use of the intermediate integrals for determining exact solutions of higher order PDEs. A first order PDE is called an intermediate integral of a given higher order PDE if all its solutions are also solutions of the higher order PDE under interest. Therefore, in order to solve the original PDE we are led to integrate a first order (eventually nonlinear) PDE. In such a case the obtained solution are given in terms of al least one arbitrary function.

Both the degenerate hodograph method and the differential constraints methods require a compatibility analysis of a resulting overdetermined system. The algebraic approaches useful for studying such a compatibility are based on the Cartan and Riquier theories. In particular, the Cartan theorem proves that after a finite number of prolongations, any analytical system becomes involutive or incompatible. Unfortunately, it is not possible to know a priori how many steps are needed to get to this result.

Among partial differential equations, the hyperbolic equations distinguish themselves for their capability of describing nonlinear waves propagation. Exact regular solutions of such models usually exist locally and after a critical time they produce discontinuity solutions like shock waves, acceleration waves, sub-shocks. Within such a theoretical framework, a famous problem is the Riemann Problem (RP).

In 1860 B. Riemann [57, 58] studied a situation in which a tube filled by a fluid is divided by a wall. On both side of the wall the fluid is at rest and it has constant but different values for mass density and pressure. At $t = 0$ the wall is broken and the problem is to study the resulting wave phenomenon. Riemann proved that the corresponding solution involves rarefaction waves, shock waves and contact discontinuities [12]. After the Riemann's paper, a RP is defined as an initial value problem characterized by two constant states separated by a discontinuity in a point. In the fundamental paper of Lax [31] it has been proved that in the case of homogeneous hyperbolic systems written in a conservative form, if the initial constant states are not "far", the general solution of a RP is given in terms of constant states separated by rarefaction waves, shock waves and/or contact discontinuities (see also [60], [12]). In particular, rarefaction waves are exact smooth solutions which are characterized by simple wave solutions. Unfortunately, nonhomogeneous systems, in general, do not admit simple wave solutions and, in turn, they do not admit rarefaction waves so that the general solutions of a RP is still an open problem for such a class of models.

An even more complex nonlinear wave problem to solve is the generalized Riemann problem (GRP), where two initial non constant states are assigned with a discontinuity in a point. The general solution of GRP is not known and usually such a class of problems is studied numerically [63], [64]. The main difficulties lie in characterising rarefaction waves capable of connecting continuously the solutions to the initial value problem with non constant states.

In the end, a more precise description of nonlineare wave propagation is given by Riemann Problems with Structure (RPS), consisting in initial value problems with continuous initial data that connect at infinity with two constant states. Also in this last case, determining exact solutions is a hard task. Quite recently, within the framework of the differential constraints method, an approach has been developed in order to solve RP, GRP and RPS also for nonhomogeneous first order hyperbolic systems.

In the following, we consider different approaches based on the degenerate hodograph method and on the method of differential constraints useful for determining exact solutions of hyperbolic PDEs. Such solutions allow to study different problems of interest in nonlinear waves propagation. In particular, the thesis is divided into two main chapters: the first one dedicated to exact solutions of partial differential equations of higher order and the second one dealing with first order hyperbolic systems.

In particular, in the first part a brief summary of the more common approaches for solving partial differential equations is presented. Among them, the Intermediate

Integral Method is widely discussed and original results are given. In fact, it is developed a systematic reduction procedure for determining intermediate integrals of second order hyperbolic equations so that exact solutions of the second order PDEs under interest can be obtained by solving first order PDEs. The conditions for which such a procedure holds are given and it is also characterized a class of linear second order hyperbolic equations for which the general solution can be found. Several examples are provided.
In the second chapter, the focus is on hyperbolic systems of first order partial differential equations. The approaches discussed to obtain exact solutions are the Differential Constraints and the Degenerate Hodograph methods and different applications are proposed. First, the Aw Rascle model for the vehicular traffic flow is considered. Within the framework of the method of differential constraints, a suitable reduction procedure is developed for solving a class of Riemann problems which are of interest in traffic flows theory: for a given source term, the general solution of the Riemann problem in terms of shock waves, contact discontinuities and generalized rarefaction waves is provided. The interaction between a shock wave and a generalized rarefaction wave is also studied and a related Generalized Riemann Problem is solved.
After that, a section is dedicated to the celebrated p-system. All the possible differential constraints compatible with the original governing system are classified. In solving the compatibility conditions between the original governing system and the appended differential constraint, several model laws for the pressure $p(v)$ are characterised. Therefore, the analysis developed is carried out in a case of physical interest and an exact solution that generalises simple waves is determined. This allows to study and to solve a class of generalised Riemann problems. In particular, a proof that the solution of the GRP can be discussed in the (p, v) plane through rarefaction-like curves and shock curves is presented. Finally, a Riemann problem with structure is studied, proving the existence of a critical time after which a GRP is solved in terms of non-constant states separated by a shock wave and a rarefaction-like wave.
As a last result, it is showed a reduction procedure aimed at determining exact solutions to a first order hyperbolic system which describes nerve pulse propagation. The main idea is to look for particular double wave solutions and to reduce the integration of the full system to that of a suitable 2×2 sub-system. The outcoming solutions are determined in terms of arbitrary functions.

Chapter 1

Reduction Methods for higher order equations

In this Chapter, we will focus our attention on higher order hyperbolic equations. As a first step, will briefly review some of the classical methods used in literature to solve this kind of equations. Among them, we will in particular discuss the Intermediate Integral Method, applying this procedure to a particular class of second order hyperbolic equations.

1.1 Riemann Method

Among the methods known in literature to solve secord order hyperbolic equations, one of the most important is surely that of Riemann [58].
In order to explain this method, we will first analyse the simple Cauchy problem

$$\begin{cases} u_{\xi\eta} = 0 \\ u|_{\gamma} = u_0 \\ \dfrac{du}{dn}_{|\gamma} = u_1, \end{cases} \tag{1.1}$$

where ξ and η are the characteristics variables, γ is a smooth curve, not tangential to the characteristics and u_0 , u_1 are the initial functions assigned for u and $\dfrac{du}{dn}$, where n is the normal to γ.
Let $P = (x, y)$ of the $\xi\eta$-plane be a point with the property that the characteristics $\xi = x$ and $\eta = y$ passing by it intersect the curve γ. Let $A = (\xi_A, y)$, $B = (x, \eta_B)$ be the two point of intersection. We will define D_P as the domain of dependence of the solution delimited by the arc $\widehat{AB}$ and by the two segments AP, BP (as in Figure 1.1).
Since $u_{\xi\eta} = 0$ for ξ, $\eta > 0$, we can integrate our equation in the domain D_P, trough Green formulas, obtaining

$$\oint_{\partial D_P} u_\xi d\xi = \oint_{\partial D_P} u_\eta d\eta = 0, \tag{1.2}$$

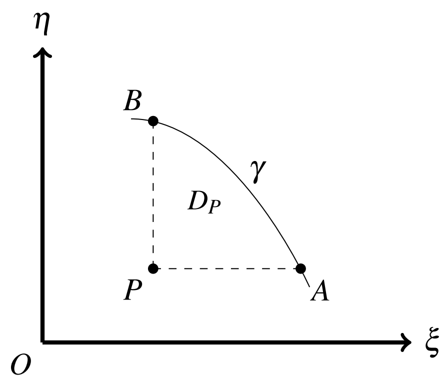

FIGURE 1.1: Domain of dependence D_P

where

$$\oint_{\partial D_P} u_\xi d\xi = \oint_{PA} u_\xi d\xi + \oint_{\widehat{AB}} u_\xi d\xi + \oint_{BP} u_\xi d\xi = [u(\xi, y)]_x^{\xi_A} + \oint_{\widehat{AB}} u_\xi d\xi =$$
$$= u(A) - u(P) + \oint_{\widehat{AB}} u_\xi d\xi,$$
$$\oint_{\partial D_P} u_\eta d\eta = \oint_{PA} u_\eta d\eta + \oint_{\widehat{AB}} u_\eta d\eta + \oint_{BP} u_\eta d\eta = \oint_{\widehat{AB}} u_\eta d\eta + [u(x, \eta)]_{\eta_B}^{y} =$$
$$= \oint_{\widehat{AB}} u_\eta d\eta + u(P) - u(B).$$

From relation (1.2), one can deduce

$$u(P) = u(A) + \oint_{\widehat{AB}} u_\xi d\xi, \qquad u(P) = u(B) - \oint_{\widehat{AB}} u_\eta d\eta \tag{1.3}$$

and summing the two expressions one obtains the formula of integral representation

$$u(P) = \frac{1}{2}[u(A) + u(B)] + \frac{1}{2}\oint_{\widehat{AB}} (u_\xi d\xi - u_\eta d\eta), \tag{1.4}$$

that represents the generalization of the famous of D'Almbert's formula. Notice that the derivative u_ξ and u_η can be evaluated along γ, because it's not a characteristic curve.

We can now analyze the more general case of a complete hyperbolic equation. We consider the following Cauchy problem

$$\begin{cases} L(u) = 0 \\ u|_\gamma = \Phi \\ \dfrac{du}{dn}|_\gamma = \Psi \end{cases}, \tag{1.5}$$

where γ is a smooth curve, not tangential to the characteristics, Φ , Ψ are the initial functions assigned for u and $\dfrac{du}{dn}$, where n is the normal to γ, while L is the operator defined as

$$L(u) = u_{\xi\eta} + M(\xi,\eta)u_\xi + N(\xi,\eta)u_\eta + Q(\xi,\eta)u - E(\xi,\eta) \tag{1.6}$$

with con $M, N \in \mathbf{C^1}$, while $E, Q \in \mathbf{C^0}$.
We introduce the adjoint operator $L^\star(\cdot)$ of $L(\cdot)$, requiring that $\forall\, u, v \in \mathbf{C^2}$ there exists a vector field $W = (W_1, W_2)$, with $W_1, W_2 \in \mathbf{C^1}$, satysfying

$$vL(u) - uL^\star(v) = \nabla \cdot W. \tag{1.7}$$

In particular, we can define

$$L^\star(v) = v_{\xi\eta} - (Mv)_\xi - (Nv)_\eta + Qv. \tag{1.8}$$

and we can compute

$$\begin{aligned} vL(u) - uL^\star(v) &= (Muv - uv_\eta)_\xi + (Nuv + vu_\xi)_\eta = \\ &= (Muv - uv_\eta + f\eta)_\xi + (Nuv + vu_\xi - f_\xi)_\eta, \end{aligned}$$

with $f \in \mathbf{C^2}$ arbitrary function, introduced in order to generalise the procedure. We integrate the above expression on the domain D_P and we get

$$\iint_{D_P} vL(u) - uL^\star(v) = \oint_{\partial D_P} [Muv - uv_\eta + f_\eta]d\eta - \oint_{\partial D_P} [Nuv + vu_\xi - f_\xi]d\xi. \tag{1.9}$$

We evaluate the integrals at the second member

$$\begin{aligned} &\oint_{\partial D_P} [Muv - uv_\eta + f_\eta]d\eta = \int_{\widehat{AB}} [Muv - uv_\eta + f_\eta]d\eta + \int_{BP} [Muv - uv_\eta + f_\eta]d\eta = \\ &= \int_{\widehat{AB}} [Muv - uv_\eta + f_\eta]d\eta + \int_{\eta_B}^{y} [(Mv - v_\eta)u + f_\eta]_{\xi=x} d\eta = \\ &= \int_{\widehat{AB}} [Muv - uv_\eta + f_\eta]d\eta + \int_{\eta_B}^{y} [(Mv - v_\eta)u]_{\xi=x} d\eta + f(P) - f(B). \\ &\oint_{\partial D_P} [Nuv + vu_\xi - f_\xi]d\xi = \int_{\widehat{AB}} [Nuv + vu_\xi - f_\xi]d\xi + \int_{PA} [Nuv + vu_\xi - f_\xi]d\xi = \\ &= \int_{\widehat{AB}} [Nuv + vu_\xi - f_\xi]d\xi + \int_{x}^{\xi_A} [(Nv - v_\xi)u + (uv - f)_\xi]_{\eta=y} d\xi. \end{aligned}$$

In order to simplify the above integrals, we introduce supplementary conditions on v. In particular, we require

$$\begin{cases} L^\star(v) = 0 \\ (Mv - v_\eta)_{\xi=x} = 0 \\ (Nv - v_\xi)_{\eta=y} = 0 \\ v(P) = 1 \end{cases} . \tag{1.10}$$

In this way, we can rewrite

$$\begin{aligned} &\oint_{\partial D_P} [Muv - uv_\eta + f_\eta]d\eta == \int_{\widehat{AB}} [Muv - uv_\eta + f_\eta]d\eta + f(P) - f(B). \\ &\oint_{\partial D_P} [Nuv + vu_\xi - f_\xi]d\xi = \int_{\widehat{AB}} [Nuv + vu_\xi - f_\xi]d\xi + \int_{x}^{\xi_A} [(uv - f)_\xi]_{\eta=y} d\xi. \end{aligned}$$

We also assume $f = \frac{1}{2}uv$, so that

$$\oint_{\partial D_P} [Muv - uv_\eta + f_\eta] d\eta = \int_{\widehat{AB}} [Muv - \frac{1}{2}uv_\eta + \frac{1}{2}vu_\eta] d\eta + f(P) - f(B).$$
$$\oint_{\partial D_P} [Nuv + vu_\xi - f_\xi] d\xi = \int_{\widehat{AB}} [Nuv + \frac{1}{2}vu_\xi - \frac{1}{2}uv_\xi] d\xi + f(A) - f(P).$$

In (1.9), we get

$$\iint_{D_P} vL(u) = \iint vE(\xi,\eta) d\xi d\eta = 2f(P) - f(A) - f(B) +$$
$$+ \int_{\widehat{AB}} [Muv - \frac{1}{2}uv_\eta + \frac{1}{2}vu_\eta] d\eta - \int_{\widehat{AB}} [Nuv + \frac{1}{2}vu_\xi - \frac{1}{2}uv_\xi] d\xi.$$

In the end, taking into account the form of f and the fact that $v(P) = 1$, we obtain

$$u(P) = \frac{1}{2}[u(A)v(A) + u(B)v(B)] + \int_{\widehat{AB}} [Nuv + \frac{1}{2}vu_\xi - \frac{1}{2}uv_\xi] d\xi +$$
$$- \int_{\widehat{AB}} [Muv - \frac{1}{2}uv_\eta + \frac{1}{2}vu_\eta] d\eta + \iint vE d\xi d\eta,$$

that is the formula for the integral representation of the solution of (1.5). The conditions (1.10) that guarantee the existence of the Riemann function v represent a Goursat Problem, that differs from the Cauchy Problem because the data are assigned along the characteristic curves.
Let us consider equation $(1.10)_2$: it is an ODE and it can be integrated taking the condition $v(P) = 1$ into account. Hence, we have

$$v(x,\eta) = \exp\left(\int_y^\eta M(x,\chi) d\chi\right). \tag{1.11}$$

Similarly, from $(1.10)_3$,

$$v(\xi,y) = \exp\left(\int_x^\xi N(\chi,y) d\chi\right). \tag{1.12}$$

Just to give an example of a solution of a Goursat Problem, we consider the particular case $M = N = 0$, $P = (0,0)$, with the further assumption $Q = \lambda \in \mathcal{R}^+$. Summarizing, we have

$$\begin{cases} u_{xy} + \lambda u = 0 \\ v(x,0) = v(0,y) = 1 \\ v(0,0) = 1 \end{cases}. \tag{1.13}$$

We notice that v assumes the same value for $x = 0$ and for $y = 0$, so we look for a solution in the form

$$v = U(\rho(\xi)) \qquad \xi = xy. \tag{1.14}$$

We try to characterize the dependence of ρ with respect to ξ in order to obtain a Bessel equation of order 0 for $U(\rho)$, knowning that a Bessel equation of order p is an equation of the form

$$x^2y'' + xy' + (x^2 - p^2)y = 0, \qquad p \in \mathcal{R}.$$

We remark that a Bessel equation is a whole family of differential equations (one for each value of p). We also notice the unfortunate terminology since for each order p the Bessel equation is always a second order ODE.
The general solutions of this kind of equations are called "Bessel functions" of order p. In particular, the first specie, identified with $J_p(x)$, expressed as series as

$$J_p(x) = \sum_{k=0}^{+\infty} (-1)^k \frac{x^{2k+p}}{k!(k+p)!}. \tag{1.15}$$

When $p = 0$

$$J_0(x) = \sum_{k=0}^{+\infty} (-1)^k \frac{x^{2k}}{(k!)^2}, \qquad J_0(0) = 1. \tag{1.16}$$

and the resolutive Bessel equation is

$$xy'' + y' + xy = 0.$$

We can compute

$$\begin{aligned} v_x &= U'\dot{\rho}y, \\ v_{xy} &= (U''\dot{\rho}x)(\dot{\rho}y) + U'(\ddot{\rho}x)y + U'\dot{\rho} = (\dot{\rho})^2 U''xy + (\dot{\rho}\xi)^{\cdot} U'. \end{aligned}$$

We require $\dot{\rho}\xi = 1$, that is $\rho = 2\sqrt{\xi}$. In this way,

$$(\dot{\rho}\xi)^{\cdot} = \frac{1}{2\sqrt{\xi}} = \frac{1}{\rho} \tag{1.17}$$

and the equation becomes

$$\rho U'' + U' + \lambda \rho U = 0. \tag{1.18}$$

In order to recover a Bessel equation of order zero, one needs to rescale the variable ρ. We define $z = a\rho$, from which $\rho = \dfrac{z}{a}$.
We compute

$$U' = a\frac{dU}{dz}, \qquad U'' = a^2 \frac{d^2U}{dz^2}. \tag{1.19}$$

Substituting (1.19) into the transformed equation (1.18), we get

$$az\frac{d^2U}{dz^2} + a\frac{dU}{dz} + \lambda \frac{z}{a} U = 0 \tag{1.20}$$

and requiring that it has the same form of (1.1), we obtain $\lambda/a = a$, that is $a = \sqrt{\lambda}$. Hence $z = 2\sqrt{\lambda xy}$ and dividing by a we have

$$z\frac{d^2U}{dz^2} + \frac{dU}{dz} + zU = 0, \tag{1.21}$$

whose solution is the Bessel function of order zero

$$v(x,y) = J_0(2\sqrt{\lambda xy}) = \sum_{k=0}^{\infty} (-1)^k \frac{(2\sqrt{\lambda xy})^{2k}}{(k!)^2}. \tag{1.22}$$

Then, the Riemann function is expressed through a series and satisfies the condition $v(x,0) = v(0,y) = J_0(0) = 1$.

1.2 Euler and Laplace Methods

In this section, we will discuss two approaches that permit to obtain exact solutions to linear hyperbolic partial differential equations: Euler and Laplace methods. In order to do that, it could be useful to review the preliminary definitions in A.1.

1.2.1 Euler Method

We consider the hyperbolic differential equation

$$u_{xt} + A(x,t)u_x + B(x,t)u_t + C(x,t)u = 0. \tag{1.23}$$

Euler proved that if the above equation can be integrated by solving two first order ODEs if one of its Laplace invariants is equal to zero (See Appendix A.1 for definitions of Laplace invariants).
We notice that equation (1.23) can be rewritten under the form.

$$\left(\frac{\partial}{\partial x} + B\right)(u_t + Au) - h_0 u = 0,$$

where $h_0 = A_x + AB - C$ is one of the Laplace invariants. If $h_0 = 0$, the equation reduces to

$$\left(\frac{\partial}{\partial x} + B\right)(u_t + Au) = 0. \tag{1.24}$$

We denote $u_1 = u_t + Au$, so that the above becomes a first order relation

$$u_{1x} + Bu_1 = 0, \tag{1.25}$$

that can be easily integrated and gives us the solution

$$u_1 = q_1(t)e^{-\int B(x,t)dx}. \tag{1.26}$$

Substituting in $u_1 = u_t + Au$ and solving the resulting linear ODE

$$u_t + Au = q_1(t)e^{-\int B(x,t)dx} \tag{1.27}$$

the solution of (1.23) when $h_0 = 0$ is

$$u = \left[p(x) + \int q(t)e^{\int Adx - Bdt}\right]e^{-\int Adt}, \tag{1.28}$$

where p and q are arbitrary functions.

1.2.2 Laplace Cascade Method

Euler method was generalised by an approach proposed in 1773 by Laplace. This method is known as Cascade Method and also in this case the invariants h_0 and k_0 play a key role. Indeed, Laplace proposed two equivalent trasformations that preserve the differential structure of equation (1.23). The two trasformations are

$$u_1 = u_t + Au \qquad u_{-1} = u_x + Bu. \tag{1.29}$$

We assume $h_0, k_0 \neq 0$ and we consider the trasformation $(1.29)_1$. If u is solution of (1.23), then it is easy to verify that u_1 satisfies

$$u_{1x} + Bu_1 = h_0 u. \tag{1.30}$$

On the other hand, if we obtain u from (1.30) and we substitute its expression in (1.23), we notice that u_1 verifies

$$u_{1xt} + A_1 u_{1x} + B_1 u_{1t} + C_1 u_1 = 0, \tag{1.31}$$

with coefficients

$$A_1 = A - (\ln h_0)_t, \qquad B_1 = B, \qquad C_1 = A_1 B + B_t - h_0 \tag{1.32}$$

and Laplace invariants

$$h_1 = 2h_0 - k_0 - (\ln h_0)_{xt}, \qquad k_1 = h_0 \tag{1.33}$$

If $h_1 = 0$, it is possible to solve (1.31), by applying the Euler method explained in the previous section.
The form of u is then obtained from (1.30) as

$$u = \frac{u_{1x} + Bu_1}{h_0}. \tag{1.34}$$

Instead, if $h_1 \neq 0$, we apply $(1.29)_1$ to the quation for u_1 and so on. The result is a chain of equations

$$u_{ixt} + A_i u_{ix} + B_i u_{it} + C_i u_i = 0, \qquad i \in \mathbb{N} \tag{1.35}$$

with coefficients

$$A_i = A_{i-1} - (\ln h_{i-1})_t, \qquad B_i = B, \qquad C_i = A_i B + B_t - h_{i-1} \tag{1.36}$$

and invariants

$$h_i = 2h_{i-1} - h_{i-2} - (\ln h_{i-1})_{xt}, \qquad k_i = h_{i-1}. \tag{1.37}$$

In the above relations, $A_0 = A$ and $C_0 = C$.
If we repeat the procedure choosing the transformation $(1.29)_2$ we get to the chain of equations

$$u_{-ixt} + A_{-i} u_{-ix} + B_{-i} u_{-it} + C_{-i} u_{-i} = 0, \qquad i \in \mathbb{N}, \tag{1.38}$$

with coefficients

$$A_{-i} = A, \qquad B_{-i} = B_{-i+1} - (\ln h_{-i+1})_x, \qquad C_{-i} = AB_{-i} + A_x - h_{-i+1} \tag{1.39}$$

and invariants

$$h_{-i} = k_{-i+1}, \qquad k_{-i} = 2h_{-i} - h_{-i+1} - (\ln h_{-i})_{xt}. \tag{1.40}$$

The set of Laplace invariants of equations (1.35) (1.38) is univocally defined by the recursive relations

$$h_i = 2h_{i-1} - h_{i-2} - (\ln h_{i-1})_{xt}, \qquad i \in \mathbb{N},$$
$$k_{-i} = 2h_{-i} - h_{-i+1} - (\ln h_{-i})_{xt}, \qquad i \in \mathbb{N}$$

and by the initial data

$$h_0 = A_x + AB - C \qquad h_{-1} = B_t + AB - C. \tag{1.41}$$

The first transformation maps the Laplace invariants as

$$(h_0, k_0) \to (h_1, h_0) \to (h_2, h_1) \to (h_3, h_2) \to \ldots,, \tag{1.42}$$

while the second follows the mapping rule

$$(h_0, k_0) \to (k_0, k_{-1}) \to (k_{-1}, k_{-2}) \to (k_{-2}, k_{-3}) \to \ldots, \tag{1.43}$$

so that in the end one obtains the series

$$\ldots; (k_{-1}, k_{-2}); (k_0, k_{-1}); (h_0, k_0); (h_1, h_0); (h_2, h_1); \ldots, \tag{1.44}$$

that is called Laplace series.
Starting from (h_0, k_0), the shift to the right is made through the first Laplace transformation, while for the shift to the left the second one is applied. This series results to be very helpful, because if there exits a value of n for which $h_n = 0$ or $k_{-n} = 0$, the general solution of the initial equation can be recovered by quadrature in terms of two arbitrary functions. In fact, if n is the minimum index for which one the two invariants vanishes (for example, $h_n = 0$), the equation with invariants $(0, h_{n-1})$ is factorizable and the general solution $u_n = u_n(x,t)$ of this equation is obtained by quadrature applying Euler method. After that, we compute

$$u_{i-1} = \frac{u_{ix} + Bu_i}{h_{i-1}} \qquad i = n, n-1, \ldots, 2, 1, \tag{1.45}$$

till the solution of the initial equation $u_0(x,t)$.
The procedure is analogue if we assume $k_{-n} = 0$ and we consider the equation with invariants $(k_{-n+1}, 0)$.

1.3 Intermediate Integral Method

The Intermediate Integral Method permits to solve partial differential equations by reducing their order. This idea can be explained by considering a second order PDE

$$F(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}) = 0. \tag{1.46}$$

Definition 1 *An intermediate integral for the second order PDE (1.46) is a first order differential equation*

$$\Phi(x, t, u, u_x, u_t) = 0 \tag{1.47}$$

such that each solution of (1.47) is also solution of (1.46).

The analysis of the conditions that guarantee the existence of an intermediate integral can be performed through the use of the Poisson Brackets ($A.2$) .
As an example, we can consider the second order hyperbolic equation

$$s + A(x,t,u)q + B(x,t,u)p + C(x,t,u) = 0, \tag{1.48}$$

where $s = u_{xt}$, $p = u_t$, $q = u_x$, $r = u_{tt}$ and $z = u_{xx}$.
We look for an intermediate integral in the form

$$\Phi(x,t,u,p,q) = 0. \tag{1.49}$$

Differentiating the (1.49) with respect to x and t and considering the equations obtained along with (1.48), we determine a linear algebraic system for the second order derivatives

$$\begin{cases} s + A(x,t,u)q + B(x,t,u)p + C(x,t,u) = 0 \\ \Phi_x + \Phi_u q + \Phi_q z + \Phi_p s = 0 \\ \Phi_t + \Phi_u p + \Phi_q s + \Phi_p r = 0 \end{cases} \tag{1.50}$$

Since the general solution of (1.49) depends on one abitrary function, equation (1.49) is an intermediate integral only if the solution of system (1.50) has the same level of arbitrariness. This implies that the rank of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & -Aq - Bp - C \\ \Phi_q & \Phi_p & 0 & -\Phi_x - \Phi_u q \\ 0 & \Phi_q & \Phi_p & -\Phi_t - \Phi_u p \end{pmatrix}$$

is less than three. As a consequence, we obtain

$$\Phi_p \Phi_q = 0. \tag{1.51}$$

We consider the case

$$\Phi_q \neq 0 \qquad \Phi_p = 0. \tag{1.52}$$

Taking into account the condition on the rank of the matrix A, the following must hold

$$\begin{vmatrix} 0 & 1 & -Aq - Bp - C \\ \Phi_q & \Phi_p & -\Phi_x - \Phi_u q \\ 0 & \Phi_q & -\Phi_t - \Phi_u p \end{vmatrix} = 0. \tag{1.53}$$

As a result

$$p(\Phi_u - B\Phi_q) - \Phi_q(Aq + C) + \Phi_t = 0. \tag{1.54}$$

Since $u(x,t)$ is an arbitrary solution of (1.49), from the previous relation it follows

$$\Phi_u - B\Phi_q = 0 \tag{1.55}$$

$$\Phi_t - \Phi_q(Aq + C) = 0 \tag{1.56}$$

We notice that if (1.52), (1.55) and (1.56) are satisfied, then the last condition of system (1.50) is a linear combination of the first and the second one. At this point, we compute the Poisson brackets of (1.52), (1.55) and (1.56). The only Poisson bracket that is not identically equal to zero is the one involving equations (1.55) , (1.56). In this last case, we obtain

$$AB + B_t - A_u q - C_u = 0. \tag{1.57}$$

This gives us only two options: the first possibility is that (1.57) is the desired intermediate integral, the second one is that (1.57) is linearly independent of (1.49).
In the first case, substituting $\Phi = AB + B_t - A_u q - C_u = 0$ into the compatibility conditions (1.55), (1.56) we have

$$\begin{aligned} &2A_u B + AB_u + B_{tu} - A_{uu} q - C_{uu} = 0 \\ &A_t B + AB_t + B_{tt} - A_{ut} q - C_{ut} + (Aq + C)A_u = 0. \end{aligned}$$

In order to conclude the procedure in this case, we just need to require that these two last conditions depend on (1.57). This leads us to the following system of two equations into the three unknown functions A, B and C

$$\begin{cases} \dfrac{AB + B_t - C_u}{A_u} = \dfrac{2A_u B + AB_u + B_{tu} - C_{uu}}{A_{uu}} \\[2ex] \dfrac{AB + B_t - C_u}{A_u} = \dfrac{A_t B + AB_t + B_{tt} + CA_u - C_{ut}}{A_{ut} - AA_u} \end{cases}.$$

On the other hand, if (1.57) is linearly independent of (1.49), the two conditions

$$A_u = 0 \qquad AB + B_t - C_u = 0 \tag{1.58}$$

are enough to prove the existence of an intermediate integral. In fact, if (1.58) are satisfied, it follows that (1.57) is an identity and then the system of compatibility conditions (1.52), (1.55) and (1.56) is complete and its solutions are defined up to an arbitrary function.
The case $\Phi_p \neq 0$ and $\Phi_q = 0$ is treated analogously.

1.3.1 Application to wave-type equations

In this subsection, we develop a procedure for characterizing intemediate integrals for a class of second order hyperbolic equations [40, 43]. After that, we present some examples of wave type hyperbolic equations for which the approach developed is helpful for characterizing their exact solutions. In the end, we apply our procedure to determine the general solution of a class of linear second order hyperbolic equations.
First of all, the hyperbolic equations under interest are of the form

$$u_{tt} - a^2(x,t,u)u_{xx} = f(x,t,u,u_x,u_t) \tag{1.59}$$

where $a(x,t,u)$ is the wave speed and $f(x,t,u,u_x,u_t)$ is a given function. We append to (1.59) the constraint

$$F(x,t,u,u_x,u_t) = 0. \tag{1.60}$$

We are able to prove the following

Theorem 1 *The relation (1.60) is an intermediate integral of the equation (1.59) if F assumes the form*

$$F = u_t - \lambda u_x - g(x,t,u) \tag{1.61}$$

and the following condition is satisfied

$$(\lambda_t + \lambda\lambda_x + 2\lambda g_u + g\lambda_u)\, u_x + 2\lambda\lambda_u u_x^2 + g_t + \lambda g_x + g g_u = f \tag{1.62}$$

where $\lambda = \mp a(x,t,u)$ and $g(x,t,u)$ is a function to be determined.

Proof 1 *Setting $u_x = q$, $u_t = p$, $u_{xx} = z$, $u_{tt} = r$ and $u_{xt} = s$, in order to study the compatibility between (1.60) and (1.59), we differentiate (1.60) so that, taking (1.59) into account, we have the following linear system in z, r and s*

$$F_p s + F_q z = -F_x - qF_u \tag{1.63}$$
$$F_p r + F_q s = -F_t - pF_u \tag{1.64}$$
$$r - a^2 z = f \tag{1.65}$$

The solution of the equation (1.61) is obtained in terms of one arbitary function. Therefore (1.61) is an intermediate integral of (1.59) only if the solution of (1.59) has the same arbitrariness so that we are led to require that

$$det \begin{Vmatrix} F_p & F_q & 0 \\ F_q & 0 & F_p \\ 0 & -a^2 & 1 \end{Vmatrix} = 0. \tag{1.66}$$

In fact, if the system (1.63)-(1.65) admits one and only one solution (i. e. if the second order derivatives of u can be calculated univocally from (1.63)-(1.65)), then the solution of (1.59) should be given in terms of arbitrary constants. Condition (1.66) gives

$$F_q - \lambda F_p = 0, \qquad \lambda = \pm a$$

whose integration leads to

$$F = p - \lambda q - g(x, t, u) \tag{1.67}$$

Finally, by substitution (1.67) into (1.63)-(1.65), we obtain the condition (1.62).

Remark 1 *Using Theorem 1, since the relation (1.60) is an intermediate integral only if F assumes the form (1.61), in order to determine exact solutions of (1.59), we have to integrate the first order equation*

$$u_t - \lambda(x, t, u)u_x = g(x, t, u) \tag{1.68}$$

where the function g must be determined according to (1.62). Therefore, the solution of (1.59) will be given in terms of one arbitrary function. We will show later that, in some cases, the procedure here developed leads to obtain the solution in terms of two arbitrary functions (i. e. we find the general solution of the equation under interest). This happens, for instance, when the second order equation is linear.

Remark 2 *As consequence of Theorem 1, using condition (1.62) and taking (1.68) into account, we notice that the function f involved in the equation (1.59) can adopt one of the following forms*

$$f = A_0 + B_0 u_x + C_0 u_x^2 \tag{1.69}$$
$$f = A_1 + B_1 u_t + C_1 u_x^2 \tag{1.70}$$
$$f = A_2 + B_2 u_t + C_t u_t^2 \tag{1.71}$$
$$f = A_3 + B_3 u_t + C_3 u_x u_t + D_3 u_x \tag{1.72}$$
$$f = A_4 + B_4 u_x + C_4 u_t + D_4 u_t^2 \tag{1.73}$$
$$f = A_5 + B_5 u_x + C_5 u_x u_t \tag{1.74}$$

where $A_i(x,t,u)$, $B_i(x,t,u)$, $C_i(x,t,u)$ and $D_i(x,t,u)$ are suitable functions given in terms of the coefficients involved in the equation at hand. Therefore, the class of second order hyperbolic equations admitting intermediate integrals is given by (1.59) supplemented by f defined according to one of the forms (1.69)-(1.74).

Remark 3 *Using Theorem 1, since $\lambda = \mp a(x,t,u)$, from (1.68) we have two intermediate integrals of (1.59). Thus, two different particular solutions of (1.59) are obtained by integrating the two reductions (1.68).*
The key point of the reduction procedure described in the previous section is the condition (1.62). In fact, once the function f is specified, we have to require that relation (1.62) is satisfied for all solutions of (1.68) (i. e. $\forall\, u_x$). Such a requirement leads to a set of compatibility conditions involving the coefficients of the equation at hand as well as the unknown function $g(x,t,u)$. Once the compatibility conditions are solved and, in turn, g is determined, we can integrate (1.68).

1. As first example we consider the equation

$$u_{tt} - a^2(u)u_{xx} = 2aa'u_x^2 + \Phi(u)u_x + h(x,t) + q(u) \tag{1.75}$$

which was widely studied in the literature. In fact when $\Phi = 0$ and $h = 0$ or when $q = 0$ and $h = 0$ equation (1.75) has been considered in [24] while when $\Phi = 0$ and $q = 0$ it was studied in [67]. In the present case the function f adopts the form

$$f = 2aa'u_x^2 + \Phi(u)u_x + h(x,t) + q(u)$$

so that from (1.62) the following compatibility conditions are obtained

$$2\lambda g_u + \lambda_u g = \Phi(u) \tag{1.76}$$
$$g_t + \lambda g_x + g g_u = h(x,t) + q(u) \tag{1.77}$$

where $\lambda = \pm a(u)$. In what follows we consider the case $\lambda = a$ because a similar reduction can be obtained when $\lambda = -a$.

Thus, integration of (1.76) leads to

$$g = \frac{1}{\sqrt{a}}\left(\varphi(u) + G(x,t)\right), \quad \varphi = \int \frac{\Phi(u)}{2\sqrt{a}}\, du \tag{1.78}$$

where $G(x,t)$ is a function to be specified. By substitution of (1.78) in (1.77), after some calculations, we get the following two cases

1.1) If $G \neq const.$, we have

$$\frac{d}{du}\left(\frac{1}{\sqrt{a}}\right) = \beta_0\sqrt{a} - \alpha_0 a - \gamma_0 \tag{1.79}$$
$$\frac{d}{du}\left(\frac{\varphi}{\sqrt{a}}\right) + \varphi\frac{d}{du}\left(\frac{1}{\sqrt{a}}\right) = \beta_1\sqrt{a} - \alpha_1 a - \gamma_1 \tag{1.80}$$
$$q\sqrt{a} = \varphi\frac{d}{du}\left(\frac{\varphi}{\sqrt{a}}\right) - \beta_2\sqrt{a} + \alpha_2 a + \gamma_2 \tag{1.81}$$
$$h = \beta_0 G^2 + \beta_1 G + \beta_2 \tag{1.82}$$

where α_i, β_i, γ_i are constants. Furthermore $G(x,t)$ must satisfy the relations

$$G_x = \alpha_0 G^2 + \alpha_1 G + \alpha_2 \tag{1.83}$$
$$G_t = \gamma_0 G^2 + \gamma_1 G + \gamma_2 \tag{1.84}$$

whose compatibility conditions require

$$\alpha_0\gamma_1 = \alpha_1\gamma_0, \quad \alpha_0\gamma_2 = \alpha_2\gamma_0, \quad \alpha_1\gamma_2 = \alpha_2\gamma_1. \tag{1.85}$$

1.2) If $G = k_0 = const.$, we find

$$h = h_0 = const., \qquad q = \frac{1}{2}\frac{d}{du}\left(\frac{\varphi + k_0}{\sqrt{a}}\right)^2 \tag{1.86}$$

while $a(u)$ is unspecified. Therefore, once $a(u)$, $q(u)$, $\Phi(u)$, $h(x,t)$ are assigned according to (1.79)-(1.82) or to (1.86), exact solutions of (1.75) can be obtained by solving equation (1.68) supplemented by (1.78) where G must be calculated from (1.83), (1.84) (in the case 1.1)) or $G = k_0$ (in the case 1.2)). In what follows we consider the three model equations arising from (1.75) when $q = \Phi = 0$ or $q = h = 0$ or $h = \Phi = 0$.

i) When $q = \Phi = 0$, it is simple matter to verify that from (1.80)-(1.82) and (1.83), (1.84) we obtain

$$g = -\frac{1}{\sqrt{a}\,(\alpha_0 x + \gamma_0 t)}, \qquad h = \frac{\beta_0}{(\alpha_0 x + \gamma_0 t)^2} \tag{1.87}$$

while $a(u)$ must be given according to (1.79). Therefore, taking (1.87) into account, integration of (1.68) will give an exact solution of (1.87) parameterized by one arbitrary function. For instance, if we assume $\alpha_0 = 0$, from (1.68) we find

$$\int_{u_0(\sigma)}^{u} \sqrt{a(u)}\, du = -\frac{1}{\gamma_0}\ln\left(\frac{t}{t_0}\right), \qquad x = -\int_{t_0}^{t} a\left(u\left(\sigma, t\right)\right)\, dt + \sigma$$

where t_0 is a constant while $u_0(\sigma)$ denotes an arbitrary function.

As far as the case 1.2) is concerned, it is simple matter to verify that it is not compatible with the assumption $q = \Phi = 0$ unless $a = const.$

ii) When $q = h = 0$, we find that the case 1.2) leads to

$$g = k_1, \qquad \Phi = k_1 a'(u) \tag{1.88}$$

where k_1 is an arbitrary constant. Therefore, integration of (1.68) gives

$$u = k_1 t + u_0(\sigma), \qquad x = -\int_0^t a\left(u\left(t, \sigma\right)\right)\, dt + \sigma \tag{1.89}$$

where $u_0(\sigma)$ is an arbitrary function. Relation (1.89) characterizes a solution of (1.75) (with $q = h = 0$, $\Phi = k_1 a'(u)$ and $a(u)$ unspecified) in terms of one arbitrary function. Moreover we notice that, using $(1.88)_2$, the equation (1.75) is equivalent to the first order system

$$u_t - v_x = 0$$
$$v_t - a^2(u) u_x = k_1 a(u)$$

which is the well known non-homogeneous p-system. Furthermore it is not difficult to ascertain that the case 1.1) is consistent with the procedure here developed only if $a = const$.

iii) When $h = \Phi = 0$, in the case 1.2) we deduce that

$$g = \frac{k_0}{\sqrt{a}}, \qquad q = \frac{k_0^2}{2}\frac{d}{du}\left(\frac{1}{a(u)}\right) \tag{1.90}$$

while $a(u)$ is unspecified. In such a case, using to (1.90)$_1$, from (1.68) we have

$$\int_{u_0(\sigma)}^{u} \sqrt{a(u)}\,du = k_0 t, \qquad x = -\int_0^t a\left(u\left(\sigma,t\right)\right)\,dt + \sigma \tag{1.91}$$

where $u_0(\sigma)$ is an arbitrary function. Therefore, once $a(u)$ is given, relation (1.91) gives a solution of (1.75) supplemented by (1.90)$_2$.

Furthermore, in the case 1.1) we obtain

$$q = \frac{\gamma_2}{\sqrt{a}} + \alpha_2\sqrt{a}, \qquad g = \frac{G(x,t)}{\sqrt{a}} \tag{1.92}$$

where, assuming $a(u) \neq const.$ (i. e. $\alpha_0^2 + \gamma_0^2 \neq 0$), using (1.83), (1.84) we find

$$G = c_0 \tan\left(c_0\sigma + c_1\right) \qquad \text{if} \quad \frac{\alpha_2}{\alpha_0} = c_0^2 \tag{1.93}$$

$$G = c_0 \frac{1 + e^{2c_0\sigma + c_1}}{1 - e^{2c_0\sigma + c_1}} \qquad \text{if} \quad \frac{\alpha_2}{\alpha_0} = -c_0^2 \tag{1.94}$$

$$G = -\frac{1}{\sigma + c_1} \qquad \text{if} \quad \alpha_2 = \gamma_2 = 0 \tag{1.95}$$

while, from (1.79) we obtain

$$\frac{1}{\sqrt{a}} - c_2 \arctan\left(\frac{1}{c_2\sqrt{a}}\right) = -\gamma_0 u \qquad \text{if} \quad \gamma_0 \neq 0 \text{ and } \frac{\alpha_0}{\gamma_0} = c_2^2 \tag{1.96}$$

$$\frac{1}{\sqrt{a}} - \frac{c_2}{2}\ln\left(\frac{1 + c_2\sqrt{a}}{1 - c_2\sqrt{a}}\right) = -\gamma_0 u \qquad \text{if } \gamma_0 \neq 0 \text{ and } \frac{\alpha_0}{\gamma_0} = -c_2^2 \tag{1.97}$$

$$a = \frac{1}{\gamma_0^2 u^2} \qquad \text{if} \quad \alpha_0 = 0 \tag{1.98}$$

$$a = \left(-3\alpha_0 u\right)^{-\frac{2}{3}} \qquad \text{if} \quad \gamma_0 = 0 \tag{1.99}$$

In (1.93)-(1.99) c_i are constants while $\sigma = \alpha_0 x + \gamma_0 t$. In such a case, integration of (1.68) can be accomplished once $a(u)$ is given according to (1.96)-(1.99) and taking (1.92)$_2$ into account supplemented by (1.93)-(1.95). For instance if $\alpha_0 = \alpha_2 = \gamma_2 = 0$ the functions a and G are given, respectively, by (1.98) and (1.95), so that from (1.68) the following solution of (1.75) is obtained

$$u = \frac{u_0(z)}{t + t_0}, \qquad x = -\frac{1}{3\gamma_0^2 u_0^2(z)}\left(\left(t + t_0\right)^3 - t_0^3\right) + z$$

where $u_0(z)$ is an arbitrary function and we set $t_0 = \frac{c_1}{\gamma_0}$.

As final case, we assume $h = \Phi = q = 0$ so that equation (1.75) specializes to

$$u_{tt} = \partial_x\left(a^2(u)u_x\right). \tag{1.100}$$

which by setting $v_x = u_t$ and $v_t = a^2 u_x$ is equivalent to the homogeneous p-system. Furthermore, if we require $g = 0$, the compatibility conditions (1.76), (1.77) are identically satisfied and from (1.68) we obtain

$$u = u_0(\xi), \qquad \xi = x + a(u_0(\xi))t \tag{1.101}$$

with $u_0(\xi)$ denoting an arbitrary function. Therefore, relation (1.101) gives a solution of (1.100) $\forall a(u)$.

Remark 4 *We notice that when $a(u) = \frac{1}{u}$, $h = 2$, $\Phi = q = 0$, then equation (1.75) specializes to the constant astigmatism equation considered in [39] where different new solutions of such equation have been obtained. It is simple to verify that the results determined in [39] can be recovered by means of the more general approach here developed starting from the compatibility conditions (1.76), (1.77). Furthermore it is also of interest to remark that a parametric solution of equation (1.100) depending of two arbitrary functions was obtained in [27] when the coefficient $a(u)$ adopts the form*

$$a = u^{\frac{4n}{1-2n}}$$

for any integers n.

2. A second example is given by the equation

$$u_{tt} = u^2 u_{xx} - u_t + \frac{2}{u} u_t^2 \tag{1.102}$$

which was considered in [2]. In the present case we have $\lambda = \mp u$ while

$$f = -g + \frac{2}{u} g^2 + (4g - u)\, u_x + 2u u_x^2.$$

In what follows we will consider the case $\lambda = u$. Therefore, from (1.62) we obtain

$$2u g_u - 3g = -u, \qquad g_t + u g_x + g g_u = -g + \frac{2}{u} g^2$$

whose integration leads to $g = u$, so that from (1.68) the following solution of (1.102) is given

$$u = u_0(\sigma)\, e^t, \tag{1.103}$$

where the characteristic variable σ is given by

$$x = -u_0(\sigma)\left(e^t - 1\right) + \sigma,$$

with $u_0(\sigma)$ arbitrary function.

3. Here, we consider the equation

$$u_{tt} = c^2 u_{xx} + q_1(x) u_t + q_2(x) u \tag{1.104}$$

which was studied in [34]. It results that $\lambda = \mp c$, with $c = const.$, while

$$f = q_2 u + q_1 g - \lambda q_1 u_x$$

so that from (1.62) we get

$$2g_u = q_1, \qquad g_t + \lambda g_x + g g_u = q_1 g + q_2 u. \tag{1.105}$$

Integration of (1.105) leads to

$$g = \frac{q_1(x)}{2} u + \gamma(x,t) \tag{1.106}$$

$$\gamma_t + \lambda \gamma_x = \frac{q_1(x)}{2} \gamma \tag{1.107}$$

$$q_2(x) = \frac{\lambda}{2} q_1'(x) - \frac{q_1^2(x)}{4} \tag{1.108}$$

Therefore, once $q_1(x)$ and $q_2(x)$ are given according to (1.108), taking (1.107) into account, integration of (1.68) supplemented by (1.106) leads to an exact solution of (1.104). For instance, if we assume $\gamma = 0$, by integration of (1.68) we have, in the case $\lambda = -c$,

$$u_1 = \hat{u}_0(\sigma) e^{\int \frac{q_1(x)}{2c} dx}, \qquad \sigma = x - ct \tag{1.109}$$

while, when $\lambda = c$, we obtain

$$u_2 = \tilde{u}_0(\xi) e^{-\int \frac{q_1(x)}{2c} dx}, \qquad \xi = x + ct. \tag{1.110}$$

where $\hat{u}_0(\sigma)$ and $\tilde{u}_0(\xi)$ are arbitrary functions.

In passing we notice that when $q_1 = 2k_0$ (where k_0 is a constant) so that $q_2 = -k_0^2$, equation (1.104) specializes to the linear telegraph equation studied in [61], while when $q_1 = -1$ and $q_2 = -\frac{1}{4}$, equation (1.104) is the hyperbolic Cahn-Allen equation with free energy under the form $\epsilon = \left(\frac{u^2}{8M_u} - c\right)$ where M_u denotes the mobility parameter of the order parameter u [50]. In the next paragraph, we will study such linear cases.

4. Now we put our attention to the equation

$$u_{tt} - u_{xx} = -c_0 u_t + h(x,t,u) \tag{1.111}$$

which has been considered in [1]. Since here $f = -c_0 u_t + h$, from (1.62) we find

$$2g_u = -c_0 \tag{1.112}$$

$$g_t + \lambda g_x + g g_u = -c_0 g + h \tag{1.113}$$

where $\lambda = \mp 1$. Integration of (1.112) and (1.113) gives

$$g = -\frac{c_0}{2} u + \gamma(x,t) \tag{1.114}$$

$$h = -\frac{c_0^2}{4} u + h_0(x,t) \tag{1.115}$$

where $h_0(x,t)$ is an unspecified function, while $\gamma(x,t)$ must satisfy the equation

$$\gamma_t + \lambda \gamma_x = -\frac{c_0}{2} \gamma + h_0(x,t). \tag{1.116}$$

Because of the form of the function $h(x,t)$ given in (1.115), the equation (1.111) assumes a linear form. In the next paragraph we will give the general solution of such linear equation.

5. As a last case, we consider the wave equation with non constant speed

$$u_{tt} - a^2(x,t,u)u_{xx} = 0. \tag{1.117}$$

Since $f = 0$, we obtain

$$\begin{aligned} &\lambda_u = 0 \\ &\lambda_t + \lambda\lambda_x + 2\lambda g_u = 0 \\ &g_t + \lambda g_x + g g_u = 0 \end{aligned} \tag{1.118}$$

After some simple algebra, integration of (1.118) gives

$$\lambda = \frac{1}{(A(\eta)t + B(\eta))^2} \tag{1.119}$$

$$g = \sqrt{\lambda}\,(A(\eta)u + C(\eta)) \tag{1.120}$$

where, if $A(\eta) \neq 0$, η is given by

$$x = -\frac{1}{A(\eta)\,(A(\eta)t + B(\eta))} + \eta \tag{1.121}$$

while, if $A(\eta) = 0$, η is defined by

$$x = \frac{t}{B^2(\eta)} + \eta \tag{1.122}$$

In (1.119), (1.120) $A(\eta)$, $B(\eta)$ and $C(\eta)$ are unspecified functions. Therefore, using (1.60) and (1.61), exact solutions of (1.117) are obtained by solving the first order semilinear equation

$$u_t - \lambda u_x = \sqrt{\lambda}\,(A(\eta)u + C(\eta))\,. \tag{1.123}$$

It is of relevant interest to notice that, since the integration of (1.123) is parameterized by one arbitrary function and moreover the function $C(\eta)$ is arbitrary, by solving (1.123) by the method of characteristics, an exact solution of (1.117) is obtained in terms of two arbitrary functions. Therefore we were able to prove the following theorem:

Theorem 2 *If the function $a(x,t)$ assumes the form*

$$a(x,t) = \frac{1}{(A(\eta)t + B(\eta))^2}$$

with η given by (1.121) or (1.122), then the solution of (1.117) can be obtained in terms of two arbitrary functions by integrating the first order equation (1.123).

According to Theorem 2, the wave speed $a^2(x,t)$ is defined in implicit form by (1.121) or (1.122) so that, once $a(x,t)$ is given, the general solution of (1.117) will be determined by integrating (1.123). Hereafter, in order to give some explicit forms of $a(x,t)$

which allow the solution of the wave equation (1.117) in a closed form, we will consider some different cases.

Case 1. Here, in the case where $A(\eta) \neq 0$, we assume

$$A = \frac{1}{\sqrt{c_0 \eta}}, \qquad B = \frac{t_0}{\sqrt{c_0 \eta}}$$

where t_0 and c_0 are arbitrary constants. In such a case, from (1.119) and (1.121) we obtain

$$a^2(x,t) = \frac{c_0^2 x^2}{(t+t_0)^2 (t+t_1)^2}, \quad \eta = \frac{x(t+t_0)}{t+t_1} \tag{1.124}$$

where we set $t_1 = t_0 - c_0$. Moreover, integration of (1.123) gives the following solution of the wave equation (1.117)

$$f = \sqrt{x\,(t+t_0)\,(t+t_1)}\;(\theta_1(\eta) + \theta_2(\sigma)) \tag{1.125}$$

where $\theta_1(\eta)$ and $\theta_2(\sigma)$ are arbitrary functions, while

$$\sigma = \frac{x\,(t+t_1)}{t+t_0}.$$

Case 2. We require $A(\eta) = 0$ and $B(\eta) = \frac{t_0}{\eta}$, where t_0 is an arbitrary constant. In such a case, from (1.119) and (1.122) we get

$$a^2 = \frac{x^2}{(t+t_0)^2}, \qquad \eta = \frac{t_0\,x}{t+t_0} \tag{1.126}$$

while integration of (1.123) gives

$$u = \sqrt{x(t+t_0)}\;(\theta_1(\eta) + \theta_2(\sigma)) \tag{1.127}$$

where $\theta_1(\eta)$, $\theta_2(\sigma)$ are arbitrary functions and

$$\sigma = x(t+t_0). \tag{1.128}$$

Case 3. Here, we assume $a(x,t) = p(t)\,q(x)$. Therefore, using to the analysis above, from (1.118) we find that $p(t)$ and $q(x)$ must satisfy the relations

$$2\frac{d}{dt}\left(\frac{p'}{p}\right) = p^2\left(\frac{p'}{p^2}\right)^2 + k_0 p^2 \tag{1.129}$$

$$2\frac{q''}{q'} - \frac{q'}{q} = -\frac{k_0}{qq'} \tag{1.130}$$

where k_0 is an arbitrary constant. Equations (1.129) and (1.130) can be easily integrated when $k_0 = 0$. In such a case we have

$$p(t) = \frac{c_0}{(t+t_0)^2}, \qquad q(x) = c_1(x+x_0)^2 \tag{1.131}$$

where c_0, c_1, v_0 and u_0 are constants, so that the wave speeds $a^2(x,t)$ assumes the form

$$a^2 = a_0^2 \left(\frac{x + x_0}{t + t_0} \right)^4 \tag{1.132}$$

with $a_0 = c_0 c_1$. Finally, by integrating (1.123) we obtain

$$u = (x + x_0)(t + t_0) \left(\theta_1(\eta) + \theta_2(\sigma) \right) \tag{1.133}$$

where $\theta_1(\eta)$ and $\theta_2(\sigma)$ are arbitrary functions, while

$$\eta = \frac{1}{x + x_0} + \frac{a_0}{t + t_0}, \qquad \sigma = \frac{1}{x + x_0} - \frac{a_0}{t + t_0}. \tag{1.134}$$

Finally, as particular cases, we first consider $q = 1$, so that the function $a(t)$ adopts the form

$$a^2(t) = \frac{k_0^2}{t^4} \tag{1.135}$$

with k_0 an arbitrary constant, while the solution of (1.117) is given by

$$u = t \left(\theta_1(\eta) + \theta_2(\sigma) \right) \tag{1.136}$$

where $\theta_1(\eta)$ and $\theta_2(\sigma)$ are arbitrary functions and

$$\eta = x + \frac{k_0}{t}, \qquad \sigma = x - \frac{k_0}{t}.$$

Next, if we require $p = 1$, we soon get

$$a^2(x) = k_1^4 x^4 \tag{1.137}$$

where k_1 is an arbitrary constant, while integration of (1.123) gives

$$u = x \left(\theta_1(\eta) + \theta_2(\sigma) \right) \tag{1.138}$$

with $\theta_1(\eta)$ and $\theta_2(\sigma)$ arbitrary functions, while

$$\eta = t + \frac{1}{k_1^2 x}, \qquad \sigma = t - \frac{1}{k_1^2 x}.$$

Remark 5 *Notice that the last two cases are equivalent. Indeed, using the change of dependent variables $x \leftrightarrow t$ and renaming $k_0 = 1/k_1^2$, we map the second case into the third. Indeed, after the chosen change of variables we obtain*

$$u_{xx} - \frac{1}{k_1^4 x^4} u_{tt} = 0 \longleftrightarrow u_{tt} - k_1^4 x^4 u_{xx} = 0 \tag{1.139}$$

Remark 6 *The solutions of the wave equation (1.117) given in cases 1, 2, 3, are obtained iff the coefficient $a(x,t)$ assumes one of the form (1.132), (1.135) or (1.137). If we want to determine solutions of (1.117) with arbitrary coefficient $a(x,t)$ a different strategy must be adopted. For instance, in the case $a(t)$, looking for solution under the form*

$$u(x,t) = F(x)G(t) \tag{1.140}$$

we find

$$F''(x) - \hat{k}F(x) = 0 \quad (1.141)$$
$$G''(t) + \hat{k}^2 a^2(t) G(t) = 0 \quad (1.142)$$

where $\hat{k}$ is an arbitrary constant. Therefore, once the coefficient $a(t)$ is specified, from (1.141), (1.142) different exact particular solutions of (1.117) can be determined depending if the constant $\hat{k}$ is positive or negative. Of course similar results can be obtained in the case $a(x)$.

Remark 7 *The solutions obtained for the wave equation with non constant speed can be useful also in the applications. An interesting example is given by the case of Hamiltonian systems.*
Let us focus on 2 components Hamiltonian quasilinear systems , i.e. hydrodynamic type systems of the form

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_x \begin{pmatrix} h_u \\ h_v \end{pmatrix}, \quad \begin{pmatrix} u \\ v \end{pmatrix}_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_x \begin{pmatrix} f_u \\ f_v \end{pmatrix} \quad (1.143)$$

whose Hamiltonian functionals are

$$H = \int h(u,v)\, dx \qquad F = \int f(u,v)\, dx.$$

Relevant examples of this type in physics are the shallow water equations in Eulerian and Lagrangian coordinates, the gas dynamic systems and the Hamiltonian systems describing a nonlinear elastic medium. It can be proved that this kind of systems commute if and only if their hamiltonian densities satisfy

$$h_{uu} f_{vv} - h_{vv} f_{uu} = 0, \quad (1.144)$$

that, avoiding the trivial cases, and setting

$$a^2(u,v) = \frac{h_{vv}}{h_{uu}}$$

can be rewritten under the form of the wave equation

$$f_{vv} - a^2(u,v) f_{uu} = 0. \quad (1.145)$$

Therefore, for a given wave speed $a^2(u,v)$, any pair of functions $f(u,v)$, $h(u,v)$ determined from the above relations ensure that the Hamiltonian systems under consideration commute. We remark that the idea of commuting flows is of extreme importance in the theory of hamiltonian systems: indeed, if a system has infinitely many commuting flows than it has the integrability property.
Just to give a more practical idea, as an example it is possible to consider the Hamiltonian density (inserire cit)

$$h(u,v) = -\frac{1}{2}u^2 v - s(v), \quad (1.146)$$

which generates the isoentropic gas dynamics system. In such a case u denotes the velocity, v the mass density and $s''(v) = \sigma'(v)/v$, where $\sigma(v)$ is the pressure. It is easy to verify that the Hamiltonian (1.146) is solution of the wave equation with speed $a^2(u,v)$ given by

(1.135) if the we adopt for the pressure $\rho(v)$ the Von-Kármán law

$$\rho(v) = -\frac{k_0^2}{v} + p_0,$$

where p_0 is a constant. Therefore, in such a case, the gas dynamics system associated with (1.146) admits infinitely many commuting flows given by (1.136).

1.3.2 General solution for linear equations

The main aim of this subsection is to characterize classes of linear second order hyperbolic equations for which the procedure here considered permits to determine their general solution. The idea is based on the fact that the equation (1.59) admits the intermediate integrals

$$u_t \mp a u_x = g^{\pm}(x,t,u) \tag{1.147}$$

provided that the condition (1.62) is satisfied for both reductions. Thus, when the equation (1.59) is linear, its general solution will be given by the linear combination of the solutions of (1.147).

Here we consider the equation (1.59) with $a(x,t)$ and $f = A(x,t)u_x + H(x,t)u + B(x,t) + G(x,t)u_t$. Taking (1.147) into account, from the compatibility condition (1.62) we get

$$\pm a_t + a a_x \pm 2a g_u^{\pm} = A \pm aG \tag{1.148}$$
$$g_t^{\pm} \pm a g_x^{\pm} + g^{\pm} g_u^{\pm} = Hu + B + Gg^{\pm} \tag{1.149}$$

After some algebra, from (1.148), (1.149), we obtain

$$g^{+} = \gamma(x,t)u + \alpha(x,t), \qquad g^{-} = \eta(x,t)u + \beta(x,t) \tag{1.150}$$

where

$$\gamma = \frac{1}{2a}\left(A + aG - (a_t + a a_x)\right), \qquad \eta = -\frac{1}{2a}\left(A - aG + (a_t - a a_x)\right) \tag{1.151}$$

while the functions $\alpha(x,t)$ and $\beta(x,t)$ are determined by

$$\alpha_t + a\alpha_x = B + (G - \gamma)\,\alpha \tag{1.152}$$
$$\beta_t - a\beta_x = B + (G - \eta)\,\beta \tag{1.153}$$

Furthermore, the following structural conditions must be satisfied

$$\gamma_t + a\gamma_x = H + G\gamma - \gamma^2 \tag{1.154}$$
$$\eta_t - a\eta_x = H + G\eta - \eta^2 \tag{1.155}$$

Therefore, we are able to give the following

Theorem 3 *The general solution of the equation*

$$u_{tt} - a^2(x,t)u_{xx} = A(x,t)u_x + H(x,t)u + B(x,t) + G(x,t)u_t$$

is given by the linear combination of the solutions of the first order equations (1.147) supplemented by (1.150), provided that the conditions (1.154) and (1.155) are satisfied.

In the following we will give some examples for which such an approach has been useful for determining the general solution of some linear equations.

i) As first example, we consider the equation (1.104). From (1.151) we have

$$\gamma = \eta = \frac{q_1}{2} \tag{1.156}$$

while from (1.154) and (1.155) we deduce

$$q_1 = const. \qquad \text{and} \qquad q_2 = -\frac{q_1^2}{4}. \tag{1.157}$$

Furthermore, integration of (1.152) and (1.153) gives

$$\alpha = \alpha_0(\sigma) e^{\frac{q_1}{2}t}, \qquad \beta = \beta_0(\xi) e^{\frac{q_1}{2}t} \tag{1.158}$$

where

$$\sigma = x - ct, \qquad \xi = x + ct$$

while α_0 and β_0 are arbitrary functions. Finally, by solving equations (1.147) supplemented by (1.150) along with (1.156) and (1.158), we have

$$u_1 = e^{-\frac{q_1}{4c}\sigma}\left(-e^{\frac{q_1}{4c}\xi}\int \frac{\alpha_0(\sigma)}{2c} d\sigma + u_1^0(\xi)\right), \qquad u_2 = e^{\frac{q_1}{4c}\xi}\left(e^{-\frac{q_1}{4c}\sigma}\int \frac{\beta_0(\xi)}{2c} d\xi + u_2^0(\sigma)\right)$$

where u_1^0, u_2^0 are arbitrary functions. Therefore, the general solution of (1.104) with (1.157) is

$$u = u_1 + u_2 = u_1^0(\xi)\, e^{-\frac{q_1}{4c}\sigma} + u_2^0(\sigma)\, e^{\frac{q_1}{4c}\xi}$$

where, without loss of generality, we set $\alpha_0 = \beta_0 = 0$.

ii) Now we consider the equation

$$u_{tt} - a(x)u_{xx} = a'(x)u_x - c(x)u + h(x,t) \tag{1.159}$$

which was studied in [51]. In the present case

$$\gamma = \frac{a'(x)}{4\sqrt{a}}, \qquad \eta = -\frac{a'(x)}{4\sqrt{a}} \tag{1.160}$$

while, from (1.154) and (1.155) we obtain the condition

$$c(x) = -\frac{a''}{4} + \left(\frac{a'}{4\sqrt{a}}\right)^2. \tag{1.161}$$

Integration of (1.152) and (1.153) leads to

$$\alpha = a^{-\frac{1}{4}}\left(\int h\left(x, t\left(x,\xi\right)\right) a^{-\frac{1}{4}}\, dx + \alpha_0(\xi)\right) \tag{1.162}$$

$$\beta = a^{-\frac{1}{4}}\left(-\int h\left(x, t\left(x,\sigma\right)\right) a^{-\frac{1}{4}}\, dx + \beta_0(\sigma)\right) \tag{1.163}$$

where

$$\xi = t - \int \frac{dx}{\sqrt{a(x)}}, \qquad \sigma = t + \int \frac{dx}{\sqrt{a(x)}}.$$

while α_0 and β_0 are arbitrary functions. Therefore, once $h(x,t)$ is given, the linear combination of the solutions of (1.147) gives the general solution of (1.159). For instance, if we assume

$$h = h_0(x)e^{-k_0 t} \tag{1.164}$$

where k_0 is a constant, taking (1.162) and (1.163) into account and setting $\alpha_0 = \beta_0 = 0$, by integrating (1.147) we obtain

$$u_1 = a^{-\frac{1}{4}}\left(h_1(x)e^{-k_0\sigma} + u_1^0(\sigma)\right), \qquad u_2 = a^{-\frac{1}{4}}\left(h_2(x)e^{-k_0\xi} + u_2^0(\xi)\right)$$

where u_1^0, u_2^0 are arbitrary functions, while

$$h_1(x) = -\int\left(\frac{e^{2k_0\int\frac{dx}{\sqrt{a}}}}{\sqrt{a}}\int h_0(x)a^{-\frac{1}{4}}e^{-k_0\int\frac{dx}{\sqrt{a}}}dx\right)dx,$$

$$h_2(x) = -\int\left(\frac{e^{-2k_0\int\frac{dx}{\sqrt{a}}}}{\sqrt{a}}\int h_0(x)a^{-\frac{1}{4}}e^{k_0\int\frac{dx}{\sqrt{a}}}dx\right)dx.$$

Thus, the general solution of (1.159), supplemented by (1.164) and $c(x)$ characterized by (1.161), is given by

$$u = \frac{1}{2}(u_1 + u_2) = \frac{a^{-\frac{1}{4}}}{2}\left(h_1(x)e^{-k_0\sigma} + u_1^0(\sigma) + h_2(x)e^{-k_0\xi} + u_2^0(\xi)\right) \tag{1.165}$$

whatever function $a(x)$ is given. Furthermore, if $h = 0$ and $c = 0$, equation (1.159) assumes the form

$$u_{tt} = \partial_x\left(a(x)u_x\right). \tag{1.166}$$

In such a case, from (1.161) we have

$$a(x) = c_0\, x\sqrt[3]{x} \tag{1.167}$$

where c_0 is a constant, so that, using to (1.165), the general solution of (1.166) assumes the form

$$u = \frac{1}{\sqrt[3]{x}}\left(u_1^0(\sigma) + u_2^0(\xi)\right), \qquad \sigma = t + \frac{3}{\sqrt{c_0}}\sqrt[3]{x}, \qquad \xi = t - \frac{3}{\sqrt{c_0}}\sqrt[3]{x}. \tag{1.168}$$

Remark 8 *It could be of some interest to notice that in [53] a transformation mapping the equation (1.166) to the Klein Gordon equation*

$$v_{tt} = v_{\xi\xi} + \mu(\xi)v \tag{1.169}$$

is found. Therefore, the general solution of (1.166) characterized by (1.168) can be also useful for solving equations like (1.169).

iii) The equation

$$u_{tt} - u_{xx} = A(x)u_x + B(x,t,u) \tag{1.170}$$

when $A = \frac{\alpha_0}{x}$ and $B = h(x,t)$ was considered in [46], while when $A = \frac{c_0}{x}$ and $B = -\frac{k_0}{x^2}u$ was studied in [29]. First, we point out our attention to the case $A = \frac{\alpha_0}{x}$

and $B = h(x,t)$, where α_0 denotes a constant. In the present case we have

$$\gamma = \frac{\alpha_0}{2x}, \qquad \eta = -\frac{\alpha_0}{2x} \tag{1.171}$$

while conditions (1.151) requires

$$\alpha_0 = 0 \qquad \text{or} \qquad \alpha_0 = 2.$$

Integration of (1.152) and (1.153) gives

$$\alpha = (\sigma+\xi)^{-\frac{\alpha_0}{2}} \left(\int \frac{h(\xi,\sigma)}{2} (\sigma+\xi)^{\frac{\alpha_0}{2}} d\xi + \mu(\sigma) \right) \tag{1.172}$$

$$\beta = (\sigma+\xi)^{-\frac{\alpha_0}{2}} \left(-\int \frac{h(\xi,\sigma)}{2} (\sigma+\xi)^{\frac{\alpha_0}{2}} d\sigma + \nu(\xi) \right) \tag{1.173}$$

where

$$\sigma = x - t, \qquad \xi = x + t.$$

while μ and ν are arbitrary. Next, by solving (1.147), we find

$$u_1 = (\sigma+\xi)^{-\frac{\alpha_0}{2}} \left\{ -\frac{1}{2} \left(\int \mu(\sigma) d\sigma + \frac{1}{2} \int \left(\int h(\xi,\sigma) \, (\sigma+\xi)^{\frac{\alpha_0}{2}} \, d\xi \right) d\sigma \right) + u_1^0(\xi) \right\}$$

$$u_2 = (\sigma+\xi)^{-\frac{\alpha_0}{2}} \left\{ \frac{1}{2} \left(\int \nu(\xi) d\xi - \frac{1}{2} \int \left(\int h(\xi,\sigma) \, (\sigma+\xi)^{\frac{\alpha_0}{2}} \, d\sigma \right) d\xi \right) + u_2^0(\sigma) \right\}$$

where the functions u_1^0, u_2^0 are arbitrary. Therefore, the general solution of (1.170) with $A = \frac{\alpha_0}{x}$ and $B = h(x,t)$ is given by

$$u = \frac{1}{2}(u_1+u_2) = \frac{(\sigma+\xi)^{-\frac{\alpha_0}{2}}}{2} \left(u_1^0(\xi) + u_2^0(\sigma) - \frac{1}{2} \int \left(\int h(\xi,\sigma) \, (\sigma+\xi)^{\frac{\alpha_0}{2}} d\xi \right) d\sigma \right)$$

if $\alpha_0 = 0$ or $\alpha_0 = 2$ and where we set, without loss of generality, $\mu = \nu = 0$.

Now we consider (1.170) with $A = \frac{c_0}{x}$ and $B = -\frac{k_0}{x^2} u$, where c_0 and k_0 are constants. In such a case equation (1.170) specializes to the Klein-Gordon-Fock (KGF) equation with central symmetry. It results that

$$\gamma = \frac{c_0}{2x}, \qquad \eta = -\frac{c_0}{2x}, \tag{1.174}$$

while from (1.151) we have

$$k_0 = \frac{c_0}{2} - \frac{c_0^2}{4}. \tag{1.175}$$

By solving (1.152) and (1.153) we obtain

$$\alpha = \mu(\sigma)\,(\sigma+\xi)^{-\frac{c_0}{2}}, \qquad \beta = \nu(\xi)\,(\sigma+\xi)^{-\frac{c_0}{2}}$$

where

$$\sigma = x - t, \qquad \xi = x + t.$$

and μ and ν are arbitrary functions. Integration of (1.147) gives

$$u_1 = (\sigma+\xi)^{-\frac{c_0}{2}} \left(-\int \frac{\mu(\sigma)}{2} d\sigma + u_1^0(\xi) \right), \qquad u_2 = (\sigma+\xi)^{-\frac{c_0}{2}} \left(\int \frac{\nu(\xi)}{2} d\xi + u_2^0(\sigma) \right)$$

with u_1^0 and u_2^0 arbitrary functions. By setting without loss of generality $\mu = \nu = 0$, the general solution of (1.170) when $A = \frac{c_0}{x}$ and $B = -\frac{k_0}{x^2}u$ is given by

$$u = u_1 + u_2 = x^{-\frac{c_0}{2}}\left(u_1^0(\xi) + u_2^0(\sigma)\right)$$

provided that condition (1.175) is satisfied.

iv) Finally we consider the equation (1.111) with $h = -\frac{c_0^2}{4}u + h_0(x,t)$. It follows that

$$\gamma = \eta = -\frac{c_0}{2}$$

while (1.154) and (1.155) are identically satisfied. Next, integration of (1.152) and (1.153) gives

$$\alpha = e^{-\frac{c_0}{4}\sigma}\left(\int \frac{h_0(\xi,\sigma)}{2}\, e^{\frac{c_0}{4}\sigma} d\sigma + \alpha_0(\xi)\right) \tag{1.176}$$

$$\beta = e^{\frac{c_0}{4}\xi}\left(-\int \frac{h_0(\xi,\sigma)}{2}\, e^{-\frac{c_0}{4}\xi} d\xi + \beta_0(\sigma)\right) \tag{1.177}$$

where

$$\sigma = x + t, \qquad \xi = x - t$$

while α_0 and β_0 arbitrary. Taking (1.176), (1.177) into account, the solution of the equations (1.147) is

$$u_1 = e^{\frac{c_0}{4}\xi}\left(-\int \frac{\alpha(\xi,\sigma)}{2}\, e^{-\frac{c_0}{4}\xi} d\xi + u_1^0(\sigma)\right), \qquad u_2 = e^{-\frac{c_0}{4}\sigma}\left(\int \frac{\beta(\xi,\sigma)}{2}\, e^{\frac{c_0}{4}\sigma} d\sigma + u_2^0(\xi)\right)$$

Finally, the general solution of the equation (1.111) with $h = -\frac{c_0^2}{4}u + h_0(x,t)$ is given by

$$u = \frac{u_1 + u_2}{2} = \frac{e^{-\frac{c_0}{4}t}}{2}\left(e^{\frac{c_0}{4}x}\, u_1^0(\sigma) + e^{-\frac{c_0}{4}x}\, u_2^0(\xi)\right) - \frac{e^{-\frac{c_0}{4}t}}{2}\int\left(\int \frac{h_0(\xi,\sigma)}{2} e^{\frac{c_0}{4}(\sigma-\xi)} d\xi\right) d\sigma$$

where $u_1^0(\sigma)$ and $u_2^0(\xi)$ are arbitrary functions, while, without loss of generality, we set $\alpha_0 = \beta_0 = 0$.

In this section, we developed a reduction procedure for determining exact solutions of second order hyperbolic equations. The approach considered permits to reduce the integration of a second order equation to that of a first order PDE called intermediate integral. The solutions obtained, apart from their theoretical value, can be also useful for testing numerical integration methods. We proved that any second order hyperbolic PDE admits two intermediate integrals so that two particular solutions given in terms of one arbitrary function can be calculated. We characterized the compatibility conditions in order that such intermediate integrals exist. The procedure here developed is particularly useful in the case of linear second order hyperbolic equations. In fact, in such a case, the linear combination of the solutions of the two intermediate integrals gives the solution of the second order governing equation in terms of two arbitrary functions. Therefore any initial value problems can be solved. In the end, characterized the class of the linear second order hyperbolic PDEs for which it is possible to obtain the general solution by means of the procedure here considered. The reduction method here developed could be applied, in principle, to any second order or higher order PDE but, as far as we know, such a procedure has been applied only for hyperbolic equations. Therefore it could be of some interest to look for intemediate integrals, for instance, for parabolic reaction-diffusion equations.

Chapter 2

Reduction Methods for hyperbolic systems

In this Chapter, we approach the study of hyperbolic systems of first order partial differential equations. In order to do so, we present two possible approaches to obtain exact solutions to this kind of models: the differential constraints and the degenerate hodograph methods. For both the procedures, we present original results. First, we show how the method of differential constraints allowed us to solve different Riemann Problems. After that, within the framework of the Degenerate Hodograph approach, we develop a reduction procedure aimed at determining exact solutions to a first order hyperbolic system which describes nerve pulse propagation.

2.1 Method of Differential Constraints

The Method of Differential Constraints has been proposed by the russian mathematician Yanenko in 1964 [65, 44] as a tool to search exact solutions of systems of PDEs. The method is non gruppal and very general, so that a large amount of the methods used to solve partial differential equations can be obtained from it as a particular case. This generality makes the method very interesting but of difficult application in the general case.
The idea of the method is to append to a system of N partial differential equations

$$F^i(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \dots) = 0 \qquad i = 1 \dots N, \tag{2.1}$$

M more differential relations

$$G^j(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \dots) = 0 \qquad j = 1 \dots M, \tag{2.2}$$

that play the role of differential constraints and select a class of exact solutions of the initial system, depending on $N - M$ arbitrary functions.

Definition 2 *Equations (2.2) are called differential constraints.*

At this point, one needs to

- choose the form and the number of the constraints;
- study the compatibility of the overdetermined system given by (2.1) and (2.2);
- solve the overdetermined system.

It is clear to see that, in order to obtain an effective application of this method, it is necessary to make some further assumption. In [16, 55, 59] the involutiveness of the resulting overdetermined system is required (for more details see [44]).
In particular, in the case of a strictly hyperbolic system, the following theorem holds:

Theorem 4 ([65], [66]) *Let us consider a strictly hyperbolic system*

$$U_t + A(U)U_x = B(U), \tag{2.3}$$

where $U, B(U) \in \mathbb{R}^N$ and $A(U) \in \mathbb{R}^{N\times N}$ denote, respectively, the vector field, the matrix coefficients and the vector source. System (2.3) is involutive if and only if the more general first order constraints that can be appended to the system are quasilinear and take the form

$$l^k(x,t,U)\cdot U_x = p^k(x,t,U) \qquad k = 1\dots M, \qquad M < N, \tag{2.4}$$

where the functions l^k are the left eigenvectors of the matrix $A(u)$, while the functions p^k need to be determined studying the compatibility of the overdetermined system.

Particularly significant is the case in which $N-1$ constraints of the type (2.4) are added to system (2.3). In this case, assuming, without loss of generality, the choice of the first N - 1 constraints ($k = 1\dots N-1$), it is easily verified that the solutions of (2.3) that also satisfy the relations (2.4) can be determined by solving the system

$$U_t + \lambda^N U_x = B + \sum_{k=1}^{N-1} p^k(\lambda^N - \lambda^k)d^k, \tag{2.5}$$

where d^k identify the right eigenvectors of the system. Furthermore, the constraints are satisfied for all $t > 0$ provided that the following condition holds:

$$l^k(U_0(x))\cdot U_0'(x) = p^k(x,0,U_0), \qquad U_0(x) = U(x,0). \tag{2.6}$$

Notice that, in this way, the constraints select the class of the initial data that admit solutions through this approach. In system (2.5), we find the derivative of the vector U along the characteristic curves associated to the eigenvalue λ^N. Therefore, in principle, the system (2.5) can be integrated using the method of characteristics. Ultimately, since the initial datum $U_0(x)$ must satisfy the $N-1$ constraint equations (2.6), the obtained solutions are determined up to an arbitrary function. These solutions are called Generalized Simple Wave solutions because, in the case where $B = 0$ and $p^k = 0$, they specialize into the well-known Simple Waves solutions, admitted by homogeneous hyperbolic systems. In (2.1.1), we show how these solutions can be useful facing wave propagation matters.

2.1.1 Application to Riemann Problems

Within the theory of wave propagation, one of the most important problems is that of Riemann. A Riemann Problem (RP) is an initial data problem with constant states, exhibiting a discontinuity at a point. This problem is an extension of the question arised by Georg Bernhard Riemann about the study of the time-evolution of a gas under the initial condition that the gas is divided into two regions by a thin diaphragm. In the two regions the gas is mantained under different values of thermodynamic quantities (temperature, density and pressure). At a certain time, the diaphragm is removed and the evolution of the gas is analysed.
It is well known that, at least in one-space dimension, the Riemann Problem has

been completely solved for systems of conservation laws ([12], [31], [30], [64], [60]) and its unique solution is expressed in terms of constant states separated by rarefaction waves, shock waves, and/or contact waves. In particular, a rarefaction wave is a simple wave of the homogeneous system. This theory breaks down in the case of non-homogeneous systems since these models, in general, do not admit simple wave solutions and, therefore, rarefaction waves.
An even more complex problem is represented by the Generalized Riemann Problem (GRP), characterized by an initial data problem with non-constant states that exhibit a discontinuity at a point. For this problem, few results are known, mostly of a numerical nature [64] , as the main difficulties lie in determining solutions to non-constant initial data problems and in characterizing rarefaction waves that can smoothly connect such solutions.
Finally, of great interest for a more detailed description of shock wave propagation is the Riemann Problem with Structure (RPS), consisting of a continuous initial data problem that smoothly connects two constant states at infinity. In some cases, it is a more realistic description of problems given by RP. In fact a RPS can take into account small thickness of initial shocks or oscillations in a narrow zone between two initial constant states. In this case as well, the main difficulty lies in determining exact solutions to this initial problem. In ([36], [37]), it has been proved that in the case of hyperbolic conservation laws a RPS converges to the corresponding RP for large time for non degenerate waves, while, in the case of exceptional waves, it tends to travelling waves. In the nonhomogeneous case there exists a conjecture for which a RPS tends to a combination of shock structures and rarefaction waves of a suitable equilibrium subsystem [4], [45]. Such a conjecture until now was verified numerically but it still needs an analytical proof.
The method of differential constraints can have an important role in solving such problems: in fact, the solutions of generalized simple waves obtained through the analytical method previously discussed can be used to obtain a generalization of the classical rarefaction waves in the non-homogeneous case.
In the following, we show an application of this idea to Riemann problems assigned for the Aw- Rascle describing traffic flow ([26]) and for the celebrated p-system [42].

2.1.2 Single and double Riemann Problem for the non-homogeneous Aw-Rascle Model

Riemann problems and generalized Riemann problems describe issues of interest in traffic flows as, for instance, situations where discontinuities in the car density must be taken into account.
In the framework of fluid mechanics-like traffic flow models, the first order models are characterized by a single hyperbolic equation for car density. The prototype of this class is the famous Lighthill-Witham-Richards equation [33], [56]. In the second order models a further equation for the car velocity is added. The prototype of this class is the Payne-Witham system [52] which, as remarked in [13], predicts some non physical effects. Therefore, Aw and Rascle proposed a new second order model [3], which led to many interesting applications in traffic flow theory [20, 18, 17, 19, 32].
The second order model proposed in [3] by Aw and Rascle is homogeneous and, as noted by the same authors, it provides an undesiderable effect consisting in the fact that the maximal velocity of the cars depends on the initial data. Therefore, in order to avoid such a weakness, in [54] and [21] a source term has been introduced so that

the following system is obtained

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial}{\partial x}(\rho v) = 0 \\ \\ \dfrac{\partial \rho S}{\partial t} + \dfrac{\partial}{\partial x}(\rho v S) = \rho G(\rho, S) \end{cases} \tag{2.7}$$

where $\rho(x,t)$ and $v(x,t)$ denote, respectively, the density and the velocity of the cars located at position x and at time t, $S = v + p(\rho)$ where the increasing function $p(\rho)$ was introduced in order to take into account driver's reactions to the state of the traffic in front of them. Moreover $p(\rho)$ must satisfy the conditions

$$p(0) = 0, \quad \lim_{\rho \to 0} \rho p'(\rho) = 0, \quad \frac{d^2}{d\rho}(\rho p(\rho)) > 0 \tag{2.8}$$

Finally $G(\rho, S)$ is a relaxation function which takes into account possibile entries or exits on the road . The prototypes of the functions $p(\rho)$ and $G(\rho, S)$ considered in [54] and [21] are :

$$p(\rho) = \rho^{\gamma}; \qquad G = \frac{S}{\tau} \tag{2.9}$$

where $\gamma > 0$ and the relaxation time τ are constants.

The system (2.7) is hyperbolic, its characteristic wave speeds are

$$\lambda^{(1)} = v - \rho p'(\rho); \qquad \lambda^{(2)} = v \tag{2.10}$$

while the corresponding left eigenvectors are

$$\mathbf{l}^{(1)} = (0;\ 1); \qquad \mathbf{l}^{(2)} = \left(p'(\rho);\ 1\right) \tag{2.11}$$

so that, according to (2.4), the possible differential constraints assume, respectively, the form

$$v_x = q^{(1)}(x, t, \rho, v) \tag{2.12}$$

or

$$p'(\rho)\rho_x + v_x = q^{(2)}(x, t, \rho, v). \tag{2.13}$$

For further convenience, we notice that the characteristic wave speed $\lambda^{(2)}$ results to be exceptional (linearly degenerate) so that the corresponding shock wave is a contact discontinuity. In [11] an exaustive analysis of system (2.7) subjected to the constraints (2.12) or (2.13) has been worked out and also particular solutions of some RP and GRP has been obtained.

Here, in order to discuss the general solution of the RP, we consider the Aw-Rascle system (2.7) along with the constraint (2.13). The resulting overdetermined system is compatible if

$$q^{(2)} = (k_0 \rho + c_0)\, g(S); \quad G = (c_0 v + c_1)\, g(S) \tag{2.14}$$

where c_0, c_1 and k_0 are constants while $g(S)$ is an unspecified function (see [11]). Furthermore the anticipation factor $p(\rho)$ is not subjected to any restrictions. We notice that if $c_0 = 0$ and $g(S) = S$, then the model law for the relaxation term $G(\rho, S)$ proposed by Rascle and Greenberg is found [54], [21]. Moreover, taking (2.5) into account, using (2.13) along with (2.14), the system (2.7) assumes the form

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \lambda^{(1)} \dfrac{\partial \rho}{\partial x} = -\rho(k_0\rho + c_0)g(S) \\[2ex] \dfrac{\partial v}{\partial t} + \lambda^{(1)} \dfrac{\partial v}{\partial x} = (c_0 v + c_1)g(S) \end{cases} \tag{2.15}$$

Therefore exact solutions of (2.7) can be obtained by solving (2.15) along with the constraint (2.13).

Let us consider the RP

$$\rho(x,0) = \begin{cases} \rho_L, \text{ for } x < 0 \\[2ex] \rho_R, \text{ for } x > 0 \end{cases} ; \qquad v(x,0) = \begin{cases} v_L \text{ for } x < 0 \\[2ex] v_R \text{ for } x > 0 \end{cases} \tag{2.16}$$

where the constants ρ_L, ρ_R, v_L, v_R characterize an equilibrium state of (2.7) so that

$$g(S_L) = 0, \qquad g(S_R) = 0 \tag{2.17}$$

with $S_L = v_L + p_L$ and $S_R = v_R + p_R$. Moreover, here and in the following, for a given function $f(\rho)$ we set $f_L = f(\rho_L)$ and $f_R = f(\rho_R)$. In order to give the general solution of (2.16), as first step we look for solutions given in terms of generalized rarefaction waves. Integration of (2.15) with the initial datum for $x < 0$ and after for $x > 0$ gives, respectively, the left state

$$\begin{cases} \rho = \rho_L \\[2ex] v = v_L \end{cases} \qquad \text{for} \quad x < x_L(t) \tag{2.18}$$

and the right state

$$\begin{cases} \rho = \rho_L \\[2ex] v = v_R \end{cases} \qquad \text{for} \quad x > x_R(t) \tag{2.19}$$

where

$$x_L(t) = \left(v_L - \rho_L p'_L\right) t, \qquad x_R(t) = \left(v_R - \rho_R p'_R\right) t. \tag{2.20}$$

denote, respectively, the left and the right limiting characteristics starting from the point $(0,0)$.

In order to calculate the central state which connects smoothly the left state with the right one, we solve the initial value problem

$$\rho(0,0) = \hat{\rho}(a), \quad v(0,0) = \hat{v}(a), \qquad a \in [0,1] \tag{2.21}$$

requiring that

$$\hat{\rho}(0) = \rho_L, \quad \hat{\rho}(1) = \rho_R, \quad \hat{v}(0) = v_L, \quad \hat{v}(1) = v_R. \tag{2.22}$$

In (2.21) a is a real parameter characterizing the family of characteristics starting from $(0,0)$. It can be easily verified that, in virtue of (2.21), using (2.6), from (2.13) and (2.14) we obtain

$$\frac{d\hat{p}}{da} + \frac{d\hat{v}}{da} = 0$$

which, in turn, gives

$$\hat{v}(a) + \hat{p}(a) = v_L + p_L = v_R + p_R. \tag{2.23}$$

Therefore, taking (2.17) into account, integration of (2.15), supplemented by (2.13), (2.21), (2.22) and (2.23), leads to

$$\begin{cases} \rho = \hat{\rho}(a) \\ v = \hat{v}(a) = v_L + p_L - \hat{p}(a) \\ x = \left(v_L + p_L - \hat{p}(a) - \hat{\rho}(a)\hat{p}'(a)\right) t \end{cases} \tag{2.24}$$

Once $p(\rho)$ is prescribed, then from $(2.24)_3$ the function $\hat{\rho}(a)$ is determined in terms of x and t so that $(2.24)_1$ and $(2.24)_2$ determine the central state which is defined in the domain

$$x_L(t) \leq x \leq x_R(t). \tag{2.25}$$

Moreover, in order to $x_L(t) < x_R(t), \ \forall t > 0$ we require

$$\rho_R < \rho_L \tag{2.26}$$

which, taking (2.23) into account, gives $v_R > v_L$. Finally from (2.23), in the (p, v) plane, using (2.26), the following generalized rarefaction curve is obtained

$$v = R(p, p_L, v_L) = -p + v_L + p_L, \quad \text{with} \quad p < p_L \tag{2.27}$$

which defines all the initial right states $(p(\rho_R), v_R)$ that can be connected smoothly with the initial left one through the generalized rarefaction wave defined in (2.24).

Next, we look for solution of the initial data (2.16) given by shock waves or contact discontinuities. The Rankine-Hugoniot conditions for (2.7) are

$$\begin{cases} -s(\rho_R - \rho_L) + \rho_R v_R - \rho_L v_L = 0 \\ -s(\rho_R S_R - \rho_L S_L) + \rho_R v_R S_R - \rho_L v_L S_L = 0 \end{cases} \tag{2.28}$$

where s denotes the shock velocity, so that two possible cases arise.

i) $s = v_L = v_R$, while the jump of the anticipation factor $p(\rho)$ is arbitrary. In such a case we have a contact discontinuity whose line in the (p, v) plane is

$$v = v_L \tag{2.29}$$

ii)

$$s = \frac{\rho_R v_R - \rho_L v_L}{\rho_R - \rho_L} \tag{2.30}$$

$$v_R + p_R = v_L + p_L \tag{2.31}$$

Here, in order to guarantee the stability of the initial shock, the Lax conditions

$$v_R - \rho_R p_R' < s < v_R, \qquad s < v_L - \rho_L p_L' \tag{2.32}$$

must be fulfilled, which is tantamount to require that

$$\rho_L < \rho_R. \tag{2.33}$$

Of course, using to (2.31), condition (2.33) leads to the relation $v_L > v_R$. Moreover, from (2.31), the corresponding shock curve in the (p, v) plane assumes the form

$$v = H(p, p_L, v_L) = -p + v_L + p_L, \qquad p_L < p_R \tag{2.34}$$

The curves (2.29) and (2.34) characterize all the initial right states which are separated with the initial left one, respectively, by a contact discontinuity or by a shock wave. It is simple matter to verify that, taking (2.30) into account, the shock wave characterized by (2.31) propagates forward or back if $v_L > \rho_R p'(\rho_\xi)$ or $v_L < \rho_R p'(\rho_\xi)$, where $\rho_L < \rho_\xi < \rho_R$. In passing we notice also that the rarefaction curve (2.27) and the shock curve (2.34) coincide so that model (2.7) is a Temple-like system [62].

Therefore, by referring to fig. 1, if the initial right state belongs to one of the curves (2.27), (2.29) or (2.103), then the RP is solved, respectively, by means of a generalized rarefaction wave, a contact discontinuity or a shock wave. Furthermore, if in the (p, v) plane we consider the curve $F_L = R(p, p_L, v_L) \cup H(p, p_L, v_L)$, it results soon that $\forall (p^\star, v^\star) \in F_L$ there exists one and only one curve $v = v^\star$ passing on it. Therefore, if the initial right state belongs to one of the regions *I*, *II* or *III*, the solution of the RP is given by three constant states separated by a shock wave and a contact discontinuity in regions *I* or *II*, by a generalized rarefaction wave and a contact discontinuity in region *III*. If the initial right state belongs to region *IV*, the solution is given by three constant states separated by a generalized rarefaction wave and a contact discontinuity if $v_R \leq v_L + p_L$, otherwise a vacuum zone between a rarefaction wave and a contact discontinuity is formed. In fact if $v_R = v^\star > v_L + p_L$, a generalized rarefaction wave connects smoothly the constant states (p_L, v_L) and $(0, v_L + p_L)$, after, a vacuum zone between the states $(0, v_L + p_L)$ and $(0, v^\star)$ is formed and finally a contact discontinuity characterized by $v = v^\star$ propagates. Furthermore we notice that when $p_L = 0$ (i. e. $\rho_L = 0$), the corresponding initial

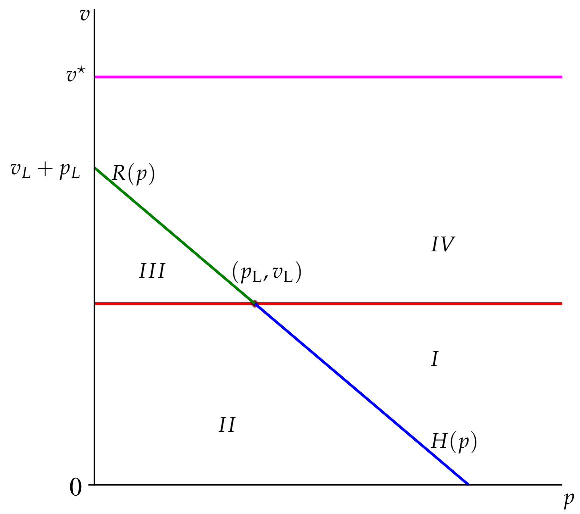

FIGURE 2.1: Generalized rarefaction curve $R(p, p_L, v_L)$ (in green), shock curve $H(p, p_L, v_L)$ (in blue) and contact discontinuity curve (in red) through the point (p_L, v_L) in the (p, v) plane. In violet the contact discontinuity $v = v^\star$.

datum v_L results to be the maximum velocity. In such a case, the RP is solved by a shock wave and a contact discontinuity if (p_R, v_R) belongs to regions I or II, by a shock if $(p_R, v_R) \in H(p, 0, v_L)$, by a contact discontinuity if $v_R = v_L$. As a final remark we notice that the general solution of the RP (2.16) here determined for the nonhomogeneous Aw-Rascle system (2.7) has the same qualitative behaviour of that admitted by the homogeneous one [3].

The procedure that we just showed can be applied aldo to solve the following double Riemann problem

$$\rho(x,0) = \begin{cases} \rho_1 \text{ for } x < 0 \\ \rho_2 \text{ for } 0 < x < L \\ \rho_3 \text{ for } x > L \end{cases} ; \; v(x,0) = \begin{cases} v_1 \text{ for } x < 0 \\ v_2 \text{ for } 0 < x < L \\ v_3 \text{ for } x > L \end{cases} \tag{2.35}$$

where the constants ρ_i and v_i are such that $\rho_1 < \rho_2$, $\rho_3 < \rho_2$, $v_1 > v_2$ and $v_3 > v_2$. The initial data (2.35) can model a situation where at $t = 0$ a queue of cars localized in $[0, L]$ starts to move. To this class of problems it belongs, for instance, the famous traffic light problem where in $x = L$ there is a red traffic light which a $t = 0$ switches to green.

Using the general analysis developed in section 3, the initial discontinuities in $x = 0$ and in $x = L$ can lead to different solutions through shock waves, generalized rarefaction waves and contact discontinuities. Hereafter, for the sake of simplicity and in order to show the flexibility of the approach outlined in the previous section, we limit ourselves to the case in which $(p(\rho_2), v_2) \in H(p, p_1, v_1)$ and $(p(\rho_3), v_3) \in R(p, p_2, v_2)$ so that the initial shock at $x = 0$ is stable while the shock in $x = L$ results to be unstable. Therefore, along the line of the procedure developed in the previous section, the following solution of the initial value problem (2.35) is obtained

$$\rho = \begin{cases} \rho_1 & \text{for} \quad x < x_s(t) \\ \rho_2 & \text{for} \quad x_s(t) < x < x_l(t) \\ \hat{\rho}(a) & \text{for} \quad x_l(t) \leq x \leq x_r(t) \\ \rho_3 & \text{for} \quad x > x_r(t) \end{cases} \tag{2.36}$$

and

$$v = \begin{cases} v_1 & \text{for} \quad x < x_s(t) \\ v_2 & \text{for} \quad x_s(t) < x < x_l(t) \\ v_2 + p_2 - \hat{p}(a) & \text{for} \quad x_l(t) \leq x \leq x_r(t) \\ v_3 & \text{for} \quad x > x_r(t) \end{cases} \tag{2.37}$$

provided that

$$v_1 + p_1 = v_2 + p_2 = v_3 + p_3. \tag{2.38}$$

In (2.36) and (2.37) we denote by $x = x_s(t) = s_1 t$ the shock line, while by $x = x_l(t) = (v_2 - \rho_2 p_2')\, t + L$ and $x = x_r(t) = (v_3 - \rho_3 p_3')\, t + L$, respectively, the left and the right limiting characteristics delimiting the generalized rarefaction wave. Moreover we set $\hat{p}(a) = p(\hat{\rho}(a))$, while the function $\hat{\rho}(a)$ is defined implicitely by

$$x = \left(v_2 + p_2 - \hat{p} - \hat{\rho}\hat{p}'\right) t + L \tag{2.39}$$

and the shock velocity s_1 assumes the form

$$s_1 = v_1 + \frac{\rho_2}{\rho_2 - \rho_1}\left(p_1 - p_2\right). \tag{2.40}$$

Next, , taking (2.32) into account, it is simple to verify that the shock line $x = s_1 t$ and the left limiting characteristic line $x_l(t)$ meets at the point

$$x_c = \frac{s_1 L}{s_1 + \rho_2 p_2' - v_2}, \qquad t_c = \frac{L}{s_1 + \rho_2 p_2' - v_2}. \tag{2.41}$$

Therefore the solution (2.36), (2.37), (2.39) exists for $t < t_c$ and in the point x_c, at the critical time t_c, a new discontinuity is given which characterizes the following GRP

$$\rho(x, t_c) = \begin{cases} \rho_1 & \text{for} \quad x < x_c \\ \hat{\rho}(x, t_c) & \text{for} \quad x_c < x \le \hat{x} \end{cases} \tag{2.42}$$

$$v(x, t_c) = \begin{cases} v_1 & \text{for} \quad x < x_c \\ \hat{v} = v_2 + p_2 - \hat{p}\left(\hat{\rho}\left(x, t_c\right)\right) & \text{for} \quad x_c < x \le \hat{x} \end{cases} \tag{2.43}$$

where $\hat{x} = \left(v_3 - \rho_3 p_3'\right) t_c + L$. The solution of (2.42) and (2.43) is given by a new shock. In fact, it is simple to verify that the "initial" states given in (2.42), (2.43) satisfy the relation (2.31) as well as the Lax condition (2.32). In order to calculate explicity such a new shock, here and in the following we assume $p = k_1 \rho$, where k_1 is a positive constant. In such a case from (2.39) we get

$$\hat{\rho} = \frac{1}{2k_1}\left(v_1 + k_1\rho_1 - \frac{x - L}{t}\right) \tag{2.44}$$

while the new shock velocity s_2 is given by

$$s_2 = v_1 - k_1\hat{\rho} \tag{2.45}$$

so that the corresponding shock curve starting from the point (x_c, t_c) assumes the form

$$x = \hat{x}_s(t) = \left(v_1 - k_1\rho_1\right) t - \frac{2L}{\sqrt{t_c}}\sqrt{t} + L. \tag{2.46}$$

Therefore the solution here characterized for $t \geq t_c$ is

$$\rho = \begin{cases} \rho_1 & \text{for} \quad x < \hat{x}_s(t) \\ \hat{\rho}(x,t) & \text{for} \quad \hat{x}_s(t) < x \leq x_r(t) \\ \rho_3 & \text{for} \quad x_r(t) \leq x \end{cases} \tag{2.47}$$

$$v = \begin{cases} v_1 & \text{for} \quad x < \hat{x}_s(t) \\ \hat{v}(x,t) & \text{for} \quad \hat{x}_s(t) < x \leq x_r(t) \\ v_3 & \text{for} \quad x_r(t) \leq x \end{cases} \tag{2.48}$$

where $\hat{\rho}(x,t)$ is given in (2.44), while, taking $(2.43)_2$ into account, $\hat{v}(x,t) = v_2 + k_1(\rho_2 - \hat{\rho}(x,t))$. Moreover, using (2.44) and (2.46), the shock velocity (2.45) specializes to

$$s_2 = v_1 - k_1\rho_1 - \frac{L}{\sqrt{t_c t}} \tag{2.49}$$

so that s_2 results to be upper bounded. Therefore, taking into account that the second shock propagates for $t > t_c$, the following four cases arise, depending on the initial left velocity v_1 and on the anticipation factor $p = k_1\rho$ calculated on the left or on the central initial state.

1. If $v_1 \leq k_1\rho_1$, it results $x_c < 0$ and the new shock propagates back.
2. If $v_1 = k_1\rho_2$ so that $v_1 > k_1\rho_1$, then $s_1 = 0$ and $x_c = 0$. In the present case the shock originated in $x = 0$ is stationary while the second shock formed at $t = t_c$ propagates forward. To this case it belongs the traffic light problem where $\rho_1 = \rho_3 = 0$, $\rho_2 = \rho_M$, $v_1 = v_2 = v_M$, $v_2 = 0$, with ρ_M and v_M denoting, respectively, the maximum density and the maximum velocity.
3. If $v_1 > k_1\rho_2$ so that $v_1 > k_1\rho_1$, then we find $x_c > 0$ and the second shock propagates forward.
4. If $k_1\rho_2 > v_1 > k_1\rho_1$, we have $x_c < 0$ and the shock starting at $t = t_c$ propagates back, then at $t = t^\star$ it stops and finally it moves forward, where we set

$$t^\star = \frac{k_1 L(\rho_2 - \rho_1)}{(v_1 - k_1\rho_1)^2}.$$

The behaviour in the (x,t) plane of the full solution of the double Riemann problem (2.35) in the four cases above is given in fig. (2.2).

Remark 9 *In the cases (ii)-(iv), it can be verified that the shock line $\hat{x}_s(t)$ and the right limiting characteristic $x_r(t)$ meet at a new critical time defined by*

$$\sqrt{\hat{t}_c} = \frac{L}{k_1\sqrt{t_c}\,(\rho_3 - \rho_1)} \tag{2.50}$$

if $\rho_3 > \rho_1$. In such a case, since the new discontinuity is characterized by the left state (ρ_1, v_1) and the right state $(\rho_3, v_3) \in H(\rho, \rho_1, v_1)$, the solution of the resulting new RP is determined in terms of a new third shock. Such a situation is represented in figure (2.3),

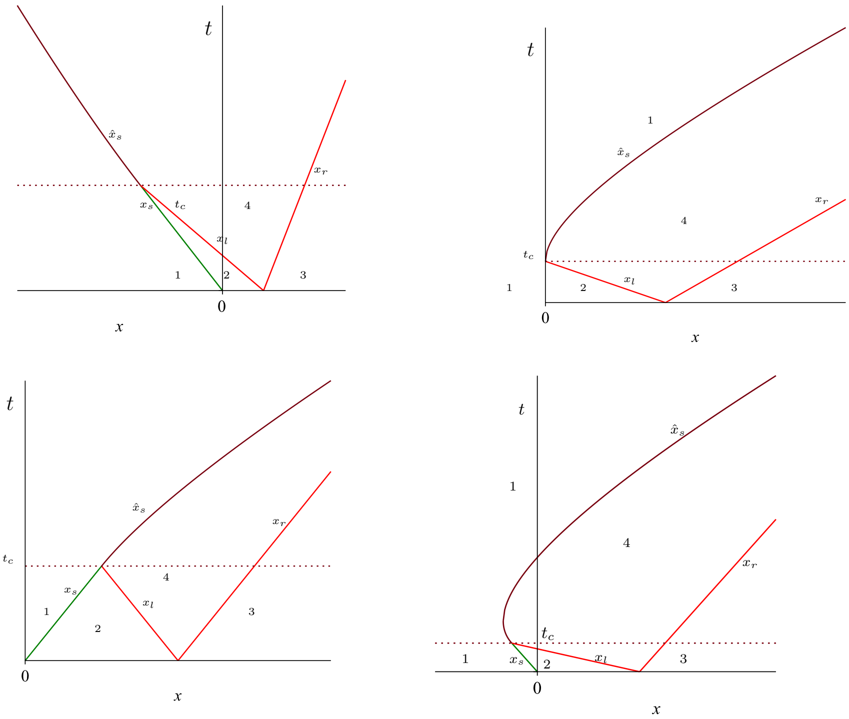

FIGURE 2.2: Behaviour in the (x, t) plane of the solution of the (2.35) in the cases (i)-(iv). In green the shock line $x_s(t)$, in red the left and right limiting characteristics $x_l(t)$ and $x_r(t)$, in black the new shock line $\hat{x}_s(t)$. Until the time t_c the solution is given by (2.36), (2.37) while for $t \geq t_c$ the solution is determined in (2.47), (2.48). In regions 1, 2 and 3 we find, respectively the constant states (ρ_1, v_1), (ρ_2, v_2) and (ρ_3, v_3). In region 4 we have the generalized rarefaction wave defined in $(2.36)_3$ and $(2.37)_3$.

where the plot of the car density versus x at different time is given in the case (iv). It can be seen that the initial shock as well as the generalized rarefaction wave propagate back until they meet. Then a new shock with non constant velocity starts propagating initially back, at $t = t^\star$ it stops and, after, it propagates forward with decreasing amplitude. Finally the new third shock is formed and it propagates forward.

Remark 10 *From (2.47), a direct inspection shows that*

$$\lim_{t \to +\infty} \left(\hat{\rho} \left(x_s(t), t \right) - \rho_1 \right) = 0 \qquad (2.51)$$

so that the amplitude of the second shock tends to zero. In particular, for large time, when $\rho_1 = \rho_3$ the car density ρ tends to the equilibrium state $\rho_1 = \rho_3$; when $\rho_1 > \rho_3$ the solution in point tends to a rarefaction wave connecting smoothly ρ_1 with ρ_3; when $\rho_1 < \rho_3$, after $\hat{t}_c$ the third shock propagates forward with constant amplitude $\rho_3 - \rho_1$.

In this section, we applied to the nonhomogeneous Aw-Rascle system describing traffic flows a strategy based on the use of the method of differential constraints for solving different Riemann problems. In particular, for a specific source term which generalizes a known source function widely adopted in the literature, we were able to give a general analysis for the RP and we proved that the corresponding solution

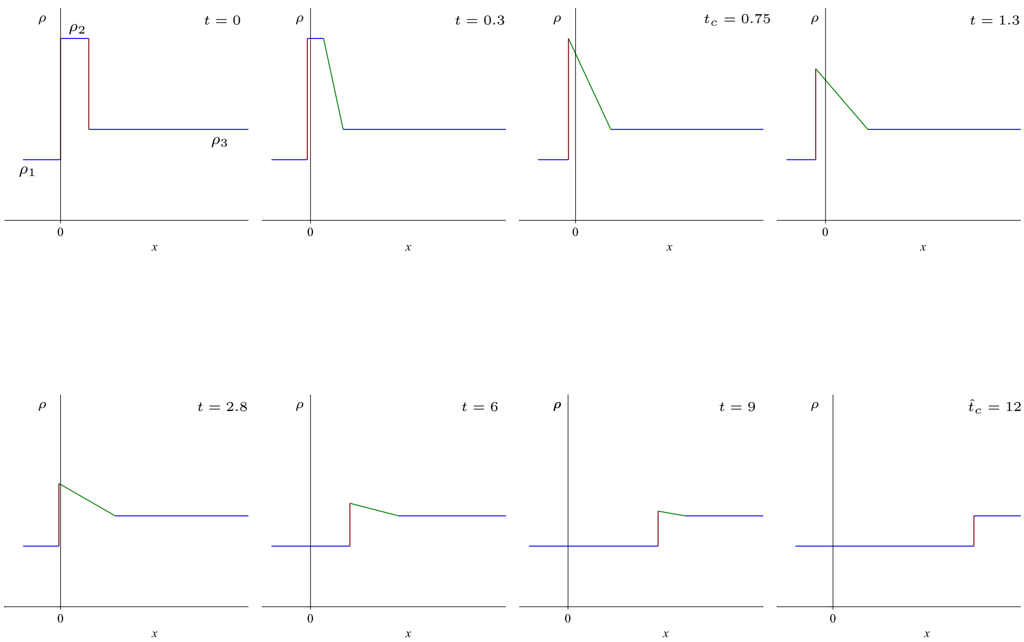

FIGURE 2.3: Plot of ρ versus x at different times given by (2.36) for $t < t_c$ and by (2.47) for $t \geq t_c$ in the case where $k_1\rho_2 > v_1 > k_1\rho_1$ and $\rho_3 > \rho_1$ (i.e. case (iv)).

is determined in terms of constant states separated by a shock or a contact discontinuity or a generalized rarefaction wave or by a combination of such wave profiles. After, we considered a double Riemann problem which models the evolution of a car's queue. In particular we considered a general case where a queue of cars is moving with a small velocity while before and after the queue a slight car's flow is on the road. Such a situation generalizes the famous traffic light problem where the queue is stopped while before and after it the road is empty. The resulting solution exists until a critical time in which a shock wave and a generalized rarefaction wave collides. Therefore a new discontinuity is formed and a GRP must be considered. In a particular case, the solution of such a GRP is given by a shock with non constant velocity whose amplitude tends to zero for large time. Such a new shock propagates back or forward depending if $v_1 < k_1\rho_1$ or $v_{\geq}k_1\rho_1$, where v_1 and $k_1\rho_1$ are, respectively, the car velocity and the anticipation factor $p = k_1\rho$ calculated in the initial left constant state. Further, we proved that if the initial right state ρ_3 of such a double RP is greater than the left one ρ_1, then the line shock meets the characteristics curve limiting on the right the generalized rarefaction wave and a new third shock can propagate. The solution obtained, for large time, evolves to an equilibrium state or to a rarefaction wave or to a shock wave. In fact, if $\rho_1 = \rho_3$, then the car's density tends to such an equilibrium state while if the initial traffic on the left is more heavy than that on the right (i.e. $\rho_1 > \rho_3$) then the car's wave density tends to a rarefaction wave. If after the queue the traffic is more heavy than that before ($\rho_1 < \rho_3$), the solution tends to a new third shock which propagates forward with constant amplitude.

2.1.3 Generalised Riemann problem for the p-system

In this section, within the framework of the Method of Differential Constraints, the celebrated p-system is studied. In particular, our aim is to outline a procedure for solving GRP and RPS for the following 2×2 homogeneous system

$$\begin{cases} v_t - u_x = 0 \\ u_t + (p(v))_x = 0 \end{cases} \tag{2.52}$$

along with the conditions

$$p'(v) < 0, \qquad p''(v) > 0. \tag{2.53}$$

In (2.53) and in the following, the prime means for ordinary differentiation. The equations (2.52) can describe an isentropic ideal gas in lagrangian coordinates. In such a case, $v = \frac{1}{\rho}$ denotes the specific volume, ρ the mass density, u the velocity, while t is the time and x the lagrangian spatial coordinate. Furthermore, for a perfect gas

$$p = p_0 v^{-\gamma}, \tag{2.54}$$

where $p(v)$ denotes the pressure, while $\gamma > 1$ is the adiabatic gas constant (i.e. the ratio between the specific heats) and $p_0 > 0$ is a constant. Under the hypotesis (2.53), system (2.52) is strictly hyperbolic. The characteristic speeds are

$$\lambda_1 = -\sqrt{-p'(v)}, \qquad \lambda_2 = \sqrt{-p'(v)}, \tag{2.55}$$

while the corresponding left eigenvectors are

$$\mathbf{l}_1 = (\lambda_1(v), 1), \qquad \mathbf{l}_2 = (\lambda_2(v), 1). \tag{2.56}$$

The pair of equations (2.52), along with (2.53), is the celebrated p-system for which a large amount of results are known in litterature, mainly for nonlinear wave propagation problems [49, 48, 35, 38]. In particular an exaustive description of the solution of the RP for the homogeneous p-system is given in [60]. Following the steps of the method of differential constraints, we look for generalised simple waves for the homogeneous p-system. Taking (2.4) into account, the possible differential constraints which can be appended to (2.52) are

$$u_x - \lambda(v) v_x = q(x, t, v, u), \tag{2.57}$$

where we set

$$\lambda = \pm\sqrt{-p'(v)}. \tag{2.58}$$

while the function $q(x, t, v, u)$ must be determined later along the reduction procedure. By requiring the differential compatibility between (2.52) and (2.57), the following consistency conditions are obtained

$$\begin{cases} 2\lambda(\lambda q_u + q_v) + \lambda' q = 0 \\ q_t + \lambda q_x + q(\lambda q_u + q_v) = 0. \end{cases} \tag{2.59}$$

After some algebra, the general solution of (2.59) is given by

$$q = \left\{ (c_0 t + c_1 x + k_0)\sqrt{|\lambda|} \right\}^{-1} \tag{2.60}$$

provided that

$$\frac{d\lambda}{dv} + 2\lambda\sqrt{|\lambda|}\,(c_0 + c_1\lambda) = 0, \tag{2.61}$$

where c_0, c_1 and k_0 are arbitrary constants. By integrating (2.61), the following cases are obtained.

i) If $c_1 = 0$, we find

$$\sqrt{|\lambda|} = \frac{1}{c_0 v}, \tag{2.62}$$

so that, taking (2.58) into account, we get

$$p = \frac{1}{3c_0^4 v^3}. \tag{2.63}$$

ii) If $c_0 = 0$, we obtain

$$|\lambda| = (\pm 3c_1 v)^{-\frac{2}{3}}, \tag{2.64}$$

which, in turn, gives

$$p = p_0 v^{-\frac{1}{3}}, \qquad p_0 = \frac{1}{\sqrt[3]{3}\,(\pm c_1)^{\frac{4}{3}}}. \tag{2.65}$$

iii) If $c_0^2 + c_1^2 \neq 0$ and $c_0 c_1 > 0$, then, in the case $\lambda = \sqrt{-p'}$ we have

$$\frac{1}{\epsilon^3}\left(\frac{\epsilon}{\Lambda} + \arctan\left(\frac{\Lambda}{\epsilon}\right)\right) = c_1 v, \tag{2.66}$$

while, if $\lambda = -\sqrt{-p'}$, we obtain

$$\frac{1}{2\epsilon^3}\left(\frac{2\epsilon}{\Lambda} + \ln\left(\frac{\Lambda - \epsilon}{\Lambda + \epsilon}\right)\right) = c_1 v, \tag{2.67}$$

where we set $\frac{c_0}{c_1} = \epsilon^2$ and $\Lambda = \sqrt{|\lambda|}$.

iv) If $c_0^2 + c_1^2 \neq 0$ and $c_0 c_1 < 0$, then in the case $\lambda = -\sqrt{-p'}$ we have

$$\frac{1}{\epsilon^3}\left(\frac{\epsilon}{\Lambda} + \arctan\left(\frac{\Lambda}{\epsilon}\right)\right) = -c_1 v, \tag{2.68}$$

while, if $\lambda = \sqrt{-p'}$ we obtain

$$\frac{1}{2\epsilon^3}\left(\frac{2\epsilon}{\Lambda} + \ln\left(\frac{\Lambda - \epsilon}{L + \epsilon}\right)\right) = -c_1 v, \tag{2.69}$$

where we set $\frac{c_0}{c_1} = -\epsilon^2$ and $\Lambda = \sqrt{|\lambda|}$.

In order to consider a case of possible physical interest, in the following we deal with the case *i)*. Therefore, using (2.60), the equations (2.5) assume the form

$$\begin{cases} v_t - \lambda v_x = \frac{v}{t+k} \\ \\ u_t - \lambda u_x = \mp \frac{1}{c_0^2 (t+k) v} \end{cases} \tag{2.70}$$

where, without loss of generality, we set $k = k_0 c_0$ and we assume $k > 0$. Integration of (2.70) by the method of characteristics leads to

$$v = \frac{t+k}{k} v_0(\sigma) \tag{2.71}$$

$$u = u_0(\sigma) \mp \frac{t}{c_0^2 (t+k) v_0(\sigma)} \tag{2.72}$$

where the functions $v_0(x) = v(x,0)$ and $u_0(x) = u(x,0)$ denote the initial data, while the characteristic variable σ is given implicitly by

$$x = \mp \frac{kt}{c_0^2(t+k)v_0^2(\sigma)} + \sigma \tag{2.73}$$

which define the family of characteristics associated, respectively, to λ_2 or to λ_1. Finally, substituting (2.71) and (2.72) in the constraint (2.57), we get

$$u_0'(x) = \pm \frac{v_0'(x)}{c_0^2 v_0^2(x)} + \frac{v_0(x)}{k} \tag{2.74}$$

which selects the class of initial value problems that can be solved by means of the present approach.

Therefore, the relations (2.71)-(2.73), along with (2.74), characterise a generalised simple wave for the p-system (2.52) supplemented by (2.63). We remark that when the constraint (2.57) is homogeneous (i. e. $k \to +\infty$), the solution (2.71)-(2.73) specializes to the classical simple wave solution. Finally it could be of a certain interest to look for a critical time t_c in which the characteristic curves meet and the corresponding solution loses its regularity. In the present case, from (2.73), by requiring that $\frac{dx}{d\sigma} = 0$, we obtain

$$t = -\frac{c_0^2 k}{c_0^2 \mp k \frac{d}{d\sigma}\left(\frac{1}{v_0^2(\sigma)}\right)}, \tag{2.75}$$

so that, if the following condition holds

$$-\frac{c_0^2}{k} \leq \frac{d}{dx}\left(\frac{1}{v_0^2(x)}\right) \leq \frac{c_0^2}{k}, \tag{2.76}$$

a critical time t_c does not exist and the solution in point exists smooth $\forall t \geq 0$.

Remark 11 *In [10] some exact solutions for the nonhomogeneous p-system have been obtained through the method of differential constraints. In particular it was proved that when the relaxation term goes to zero as well as the source involved in the constraint, the exact solution there obtained tends to the corresponding solution of the homogeneous p-system. Such a solution is different from that given in (2.71)-(2.73). In fact the differential constraint (2.57) here considered characterizes a class of initial value problems different from those taken into account in [10]. Of course when in (2.71)-(2.74) we take $k \to +\infty$ the solutions of the homogeneous system and of the nonhomogeneous one coincide because they specialize to simple waves.*

Using the analysis developed in the *i)* case,we first approach the following GRP

$$v(x,0) = \begin{cases} v_l(x) & \text{for} \quad x < 0 \\ v_r(x) & \text{for} \quad x > 0 \end{cases}, \qquad u(x,0) = \begin{cases} u_l(x) & \text{for} \quad x < 0 \\ u_r(x) & \text{for} \quad x > 0, \end{cases} \tag{2.77}$$

where $v_l(x)$, $v_r(x)$, $u_l(x)$ and $u_r(x)$ are smooth functions such that

$$v_L = \lim_{x\to 0^-} v_l(x), \qquad v_R = \lim_{x\to 0^+} v_r(x), \tag{2.78}$$

$$u_L = \lim_{x\to 0^-} u_l(x), \qquad u_R = \lim_{x\to 0^+} u_r(x) \tag{2.79}$$

with $v_L \neq v_R$ and $u_L \neq u_R$. According to the approach outlined in section 2, since the initial data must satisfy the constraint (2.57), by substitution of (2.77) into (2.57) and by a further integration we find

$$u_l(x) = u_L \mp \frac{1}{c_0^2}\left(\frac{1}{v_l(x)} - \frac{1}{v_L}\right) + \frac{1}{k}\int_0^x v_l(z)\,dz, \tag{2.80}$$

$$u_r(x) = u_R \mp \frac{1}{c_0^2}\left(\frac{1}{v_r(x)} - \frac{1}{v_R}\right) + \frac{1}{k}\int_0^x v_r(z)\,dz. \tag{2.81}$$

Therefore, if we consider the initial data assigned for $x < 0$, from (2.71) and (2.72), we obtain

$$v = \frac{t+k}{k} v_l(\sigma_l), \tag{2.82}$$

$$u = u_l(\sigma_l) \mp \frac{t}{c_0^2 (t+k)\, v_l(\sigma_l)}, \tag{2.83}$$

where

$$x = \mp \frac{kt}{c_0^2 (t+k) v_l^2(\sigma_l)} + \sigma_l, \quad \text{with} \quad \sigma_l < 0, \tag{2.84}$$

while, taking the initial data for $x > 0$ into account, we have

$$v = \frac{t+k}{k} v_r(\sigma_r), \tag{2.85}$$

$$u = u_r(\sigma_r) \mp \frac{t}{c_0^2 (t+k)\, v_r(\sigma_r)}, \tag{2.86}$$

where

$$x = \mp \frac{kt}{c_0^2 (t+k) v_r^2(\sigma_r)} + \sigma_r, \quad \text{with} \quad \sigma_r > 0. \tag{2.87}$$

Because the characteristic parameters $\sigma_l < 0$ and $\sigma_r > 0$, the solution given by (2.82), (2.83) is defined in the region $x < x_L(t)$, while the solution (2.85), (2.86) exists in $x > x_R(t)$, where

$$x_L(t) = \lim_{\sigma_l\to 0^-}\left(\mp \frac{kt}{c_0^2 (t+k) v_l^2(\sigma_l)}\right) = \mp \frac{kt}{c_0^2 (t+k) v_L^2}, \tag{2.88}$$

$$x_R(t) = \lim_{\sigma_r\to 0^+}\left(\mp \frac{kt}{c_0^2 (t+k) v_r^2(\sigma_r)}\right) = \mp \frac{kt}{c_0^2 (t+k) v_R^2}. \tag{2.89}$$

Then, in (2.88), (2.89) $x = x_L(t)$ and $x = x_R(t)$ denote, respectively, the left and the right limiting characteristics starting from the discontinuity point $(0,0)$ of the (x,t)

plane.

In order to determine a central state solution which connects smoothly the left state (2.82), (2.83) with the right one (2.85), (2.86), we integrate the system (2.70), along with the constraint (2.57), with the initial data

$$v(0,0) = \bar{v}(a), \quad u(0,0) = \bar{u}(a), \quad \text{with} \quad a \in [0,1]\,, \tag{2.90}$$

subjected to the conditions

$$\bar{v}(0) = v_L, \quad \bar{v}(1) = v_R, \quad \bar{u}(0) = u_L, \quad \bar{u}(1) = u_R. \tag{2.91}$$

In (2.90) we indicate with a the parameter characterizing the fan of characteristics starting from the origin of the (x,t) plane. Then, after some algebra, in the central region $x_L(t) \leq x \leq x_R(t)$, the following solution is obtained

$$v = \sqrt{\mp \frac{t(t+k)}{k c_0^2 x}}, \tag{2.92}$$

$$u = u_L \pm \frac{1}{c_0^2 v_L} \mp \frac{2t+k}{c_0} \sqrt{\mp \frac{x}{kt(t+k)}}, \tag{2.93}$$

along with the condition

$$u_R \pm \frac{1}{c_0^2 v_R} = u_L \pm \frac{1}{c_0^2 v_L}. \tag{2.94}$$

Finally, by requiring $x_L(t) < x_R(t)$ we find the further conditions

$$v_L < v_R \quad \text{in the case where} \quad \lambda = \sqrt{-p'}, \tag{2.95}$$

$$v_L > v_R \quad \text{in the case where} \quad \lambda = -\sqrt{-p'}. \tag{2.96}$$

Therefore, provided that conditions (2.94) and (2.95) or (2.96) are sastisfied, the central state (2.92), (2.93) connects smoothly the left state (2.82), (2.83) with the right one (2.85), (2.86) and it characterises a generalised rarefaction wave which solves the GRP (2.77). More in general, from (2.94), we find the generalised rarefaction curves

$$u = R^{(1)}(v, v_L, u_L) = u_L - \frac{1}{c_0^2}\left(\frac{1}{v} - \frac{1}{v_L}\right), \quad \text{with} \quad v_L < v, \tag{2.97}$$

$$u = R^{(2)}(v, v_L, u_L) = u_L + \frac{1}{c_0^2}\left(\frac{1}{v} - \frac{1}{v_L}\right), \quad \text{with} \quad v_L > v, \tag{2.98}$$

which in the (v,u) plane characterize all the initial right states whose limiting values (v_R, u_R) defined in (2.78) and (2.79), by belonging to the curve $u = R^{(1)}(v, v_L, u_L)$ or $u = R^{(2)}(v, v_L, u_L)$, permit to solve the GRP (2.77) by means of the generalised rarefaction wave determined in (2.92) and (2.93).

The smooth exact solution of (2.52), supplemented by (2.77), here obtained is given in implicit form depending on the initial data $v(x,0)$ considered. In order to obtain an explicit solution which will be useful in the following, we chose the initial condition (2.77) as

$$v(x,0) = \begin{cases} v_L & \text{for} \quad x < 0 \\ v_R & \text{for} \quad x > 0, \end{cases} \tag{2.99}$$

where the constants v_L and v_R are such that $v_L \neq v_R$, so that from (2.80), (2.81) we obtain

$$u(x,0) = \begin{cases} u_l(x) = u_L + \frac{v_L}{k}x & \text{for} \quad x < 0 \\ u_r(x) = u_R + \frac{v_R}{k}x & \text{for} \quad x > 0. \end{cases} \tag{2.100}$$

In such a case the corresponding left state solution (2.82)-(2.84) assumes the form

$$v = \tilde{v}_l(t) = \frac{t+k}{k}v_L, \quad u = u_l(x) \quad \text{in} \quad x < x_L(t), \tag{2.101}$$

while the right state solution (2.85)-(2.87) specializes to

$$v = \tilde{v}_r(t) = \frac{t+k}{k}v_R, \quad u = u_r(x) \quad \text{in} \quad x > x_R(t). \tag{2.102}$$

Furthermore, the central state connecting smoothly (2.101) with (2.102) is still characterised by (2.92), (2.93). Next, in order to discuss the general solution of the GRP here considered, we look for shock wave solutions for the p-system (2.52) with (2.99) and (2.100). Such an analysis is well known for the p-system, so that we refer to [60] for more details. Therefore, by solving the Rankine-Hugoniot conditions for (2.52), we find

$$s = -\frac{u_r - u_l}{\tilde{v}_r - \tilde{v}_l}, \quad u_r = u_l \mp \sqrt{(\tilde{v}_r - \tilde{v}_l)\left(p\left(\tilde{v}_l\right) - p(\tilde{v}_r)\right)}, \tag{2.103}$$

where $(\tilde{v}_l, u_l)$ is the state on the left of the shock determined by (2.101), $(\tilde{v}_r, u_r)$ is the state on the right of the shock characterized by (2.102) and s is the shock velocity. Now, by requiring that the Lax conditions are satisfied, after some algebra, we find two shock families. The $1-$shocks in which $v_R < v_L$ and $s < 0$ and the $2-$ shock family where $v_R > v_L$ and $s > 0$. In both cases, using (2.101) and (2.102), we easily find the shock curve

$$x = x_s(t) = -\frac{u_R - u_L}{v_R - v_L}\frac{kt}{t+k} \tag{2.104}$$

so that, from $(2.103)_1$ with (2.101) and (2.102), the shock speed specializes to

$$s = s(t) = -\frac{u_R - u_L}{v_R - v_L}\left(\frac{k}{t+k}\right)^2. \tag{2.105}$$

Finally, using $(2.103)_2$ and taking (2.63), (2.101), (2.102) and (2.104) into account, for $1-$shocks the following shock curve is obtained

$$u = S^{(1)}(v, v_L, u_L) = u_L - \frac{1}{\sqrt{3c_0^2}}\sqrt{(v - v_L)\left(\frac{1}{v_L^3} - \frac{1}{v^3}\right)} \quad \text{with} \quad v < v_L, \tag{2.106}$$

while for $2-$shocks we find

$$u = S^{(2)}(v, v_L, u_L) = u_L - \frac{1}{\sqrt{3c_0^2}}\sqrt{(v - v_L)\left(\frac{1}{v_L^3} - \frac{1}{v^3}\right)} \quad \text{with} \quad v > v_L. \tag{2.107}$$

The curves (2.106), (2.107) characterize in the (v, u) plane all the right initial states which allow to solve the GRP (2.99), (2.100) by a $1-$shock or by a $2-$shock. It is relevant to notice that both the generalised rarefaction curves (2.97) and (2.98) as well the shock curves (2.106) and (2.107) involve the limiting values of the initial data for $x \to 0$ (i. e. v_L, v_R, u_L, u_R). Therefore, such a curves coincide with those of the RP for the p-system (see [60]). It follows that the general discussion of the solution of the RP for the p-system is useful here for characterizing the general solution of the GRP given in (2.99), (2.100). In fact, by referring to Figure 2.4, if (v_R, u_R) belongs to one of the curves $R^{(1,2)}$, $S^{(1,2)}$, then the solution is given in terms of the non constant states (2.101) and (2.102) separated, respectively, by the generalized rarefaction wave (2.92), (2.93) or by a shock wave ($1-$shock or $2-$shock). If, on the other hand, (v_R, u_R) belongs to one the regions I, II, III or IV, then, as in the case of the RP, the solution of (2.99), (2.100) is determined in terms of three non constant states separated by generalised rarefaction waves and/or shock waves (the interested reader can find the detailed discussion corresponding to the RP in [60]). Finally, taking (2.105) into account, we notice that, for large t, both the $1-$shocks or the $2-$shocks tend to a stationary shock.

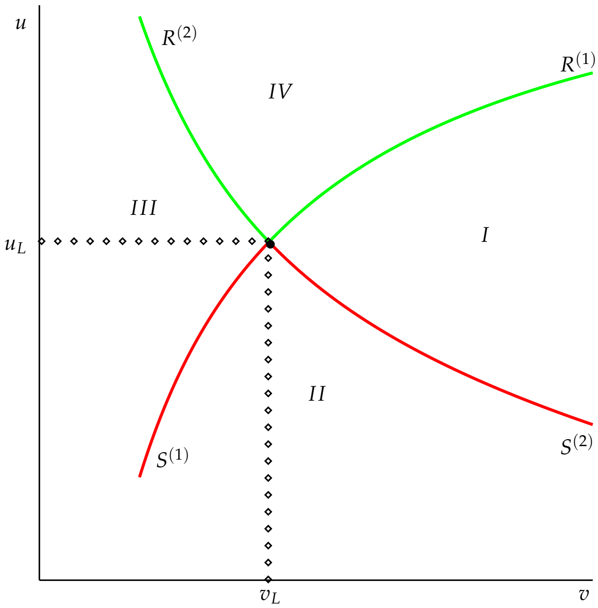

FIGURE 2.4: Generalised rarefaction curves and shock curves passing through (v_L, u_L) in the (v, u) plane. In red the shock curves $S^{(1,2)}$ given, respectively by (2.106) and (2.107). In green the generalised rarefaction curves $R^{(1,2)}$ characterised by (2.97) and (2.98).

2.1.4 Riemann problem with structure for the p-system

The analysis conducted above can be useful also to study Riemann problems with structure. In particular, we consider the following RPS

$$v(x,0) = \begin{cases} v_L & \text{for} \quad x < 0 \\ v_0(x) & \text{for} \quad 0 \leq x \leq L \\ v_R & \text{for} \quad x > L \end{cases} \tag{2.108}$$

where

$$v_0(x) = \frac{v_L}{\sqrt{1+\alpha x}}, \quad \text{with} \quad \alpha = \frac{v_L^2 - v_R^2}{L v_R^2}. \tag{2.109}$$

Taking (2.163) into account, from (2.108) we find

$$u(x,0) = \begin{cases} u_L + \frac{v_L}{k} x & \text{for} \quad x < 0 \\ u_0(x) & \text{for} \quad 0 \leq x \leq L \\ u_R + \frac{v_R}{k}(x - L) & \text{for} \quad x > L \end{cases} \tag{2.110}$$

with

$$u_0(x) = u_L + \frac{1}{c_0^2 v_L}\left(\frac{2c_0^2 v_L^2}{k\alpha} \mp 1\right)\left(\sqrt{1+\alpha x} - 1\right). \tag{2.111}$$

Moreover, in order that the initial condition (2.110) is smooth, we require

$$u_R = u_L + \frac{v_L - v_R}{c_0^2 v_L v_R}\left(\frac{2c_0^2 v_L^2}{k\alpha} \mp 1\right). \tag{2.112}$$

In passing we notice that in both cases where $\alpha > 0$ or $\alpha < 0$, it results $1 + \alpha x > 0$. Taking (2.71)-(2.73) into account, after some algebra, the solution of the initial value problem (2.108), (2.110) is given by the left state

$$v = \frac{t+k}{k} v_L, \quad u = u_L + \frac{v_L}{k} x, \quad \text{for} \quad x < x_l(t) = \mp \frac{kt}{c_0^2 (t+k) v_L^2}, \tag{2.113}$$

by the central state ($x_l(t) \leq x \leq x_r(t)$)

$$\begin{cases} v = \dfrac{t+k}{k} \dfrac{v_L}{\sqrt{1+\alpha\eta}} \\ u = u_L - \dfrac{1}{c_0^2 v_L}\left(\dfrac{2c_0^2 v_L^2}{k\alpha} \mp 1\right) + \dfrac{1}{c_0^2 v_L}\left(\dfrac{2c_0^2 v_L^2}{k\alpha} \mp \frac{2t+k}{t+k}\right)\sqrt{1+\alpha\eta} \end{cases} \tag{2.114}$$

where

$$\eta = \frac{v_L^2 c_0^2 x (t+k) \pm kt}{v_L^2 c_0^2 (t+k) \mp \alpha k t}, \tag{2.115}$$

and by the right state

$$v = \frac{t+k}{k} v_R, \quad u = u_R + \frac{v_R}{k}(x - L), \quad \text{for} \quad x > x_r(t) = \mp \frac{kt}{c_0^2 (t+k) v_R^2} + L. \tag{2.116}$$

It is simple to verify that in the case $\lambda = \sqrt{-p'}$, if

$$v_L > v_R \quad \text{and} \quad \frac{c_0^2}{k} < \frac{\alpha}{v_L^2}, \tag{2.117}$$

the limiting characteristics $x_l(t)$ and $x_r(t)$ meet at the critical time t_c given by

$$t_c = -\frac{c_0^2 k v_L^2}{c_0^2 v_L^2 - \alpha k}, \tag{2.118}$$

while, in the case $\lambda = -\sqrt{-p'}$, under the conditions

$$v_L < v_R \quad \text{and} \quad \frac{c_0^2}{k} < -\frac{\alpha}{v_L^2} \tag{2.119}$$

the curves $x_l(t)$ and $x_r(t)$ meet at the critical time

$$t_c = -\frac{c_0^2 k v_L^2}{c_0^2 v_L^2 + \alpha k}. \tag{2.120}$$

In both cases the the left and right characteristics $x_l(t)$, $x_r(t)$ meet in the point

$$x_c = -\frac{1}{\alpha}. \tag{2.121}$$

Of course, if the condition (2.117) or (2.119) is not satisfied, then the solution (2.113)-(2.116) exists smooth $\forall t \geq 0$. In passing we notice that the initial data $(2.108)_1$ and $(2.108)_3$ satisfy the relation (2.76) while, if condition (2.117) or (2.119) is satisfied, then the initial datum $(2.108)_2$ fulfills (2.76) until t_c where all the characteristics of the central region (2.114) meet in x_c.

In the following we consider the case characterised by (2.117). Of course similar results can be obtained in the remaining case. Since the solution (2.113)-(2.116) exists regular until t_c, we have now to study the following GRP

$$(v(x,t_c), u(x,t_c)) = \begin{cases} \left(\dfrac{t_c + k}{k} v_L, u_L + \dfrac{v_L}{k} x\right) & \text{for} \quad x < x_c \\ \left(\dfrac{t_c + k}{k} v_R, u_R + \frac{v_R}{k}(x - L)\right) & \text{for} \quad x > x_c \end{cases}. \tag{2.122}$$

In order to solve (2.122), it is convenient to set

$$\tau = t - t_c, \qquad \xi = x - x_c, \tag{2.123}$$

along with

$$\hat{k} = t_c + k, \quad \hat{v}_L = \frac{t_c + k}{k} v_L, \quad \hat{v}_R = \frac{t_c + k}{k} v_R, \tag{2.124}$$

$$\hat{u}_L = u_L + \frac{v_L}{k} x_c, \quad \hat{u}_R = u_R + \frac{v_R}{k}(x_c - L), \tag{2.125}$$

so that, in the variables (ξ, τ), the initial data (2.122) assume the form (2.99), (2.100). As consequence, the results obtained in section 4 can be useful for solving the present GRP. In particular, by referring to Figure 2.5, it is possible to prove that the point $(\hat{v}_R, \hat{u}_R)$, which is obtained from the initial right state (2.122) when $x \to x_c$ (or $\xi \to 0$), belongs to region III so that the solution of (2.122) is given by means of a back-shock and a forward generalised rarefaction wave. In fact, because of (2.117), we have $\hat{v}_R < \hat{v}_L$. Therefore, let $(\hat{v}_R, \hat{u}_1)$ and $(\hat{v}_R, \hat{u}_2)$ the points of abscissa $\hat{v}_R$ which belong, respectively to $S^{(1)}$ and $R^{(2)}$, since $\hat{u}_1 = S^{(1)}(\hat{v}_R, \hat{v}_L, \hat{u}_L)$ and $\hat{u}_2 = R^{(2)}(\hat{v}_R, \hat{v}_L, \hat{u}_L)$, taking (2.112), (2.124) and (2.125) into account, after some algebra, it results $\hat{u}_1 < \hat{u}_R < \hat{u}_2$, so that $(\hat{v}_R, \hat{u}_R)$ belongs to region III.

The resulting solution after t_c is given by three non constant state separated by a shock and by a generalised rarefaction wave. In particular, by referring to Figure

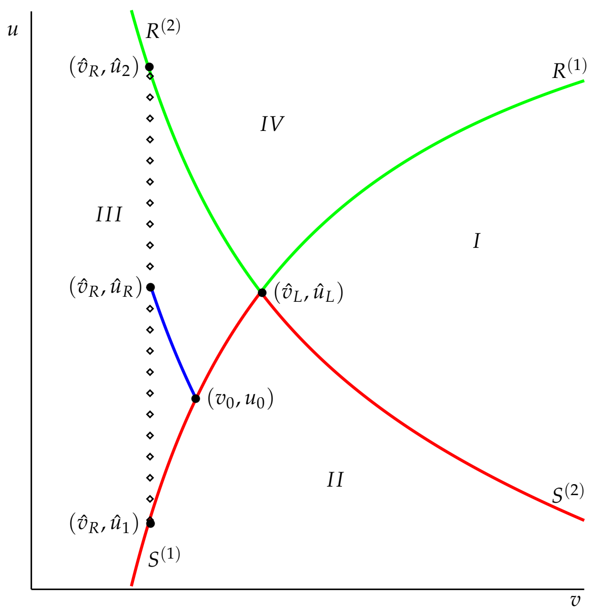

FIGURE 2.5: Generalised rarefaction curves and shock curves passing through $(\hat{v}_L, \hat{u}_L)$ in the (v, u) plane. In red the shock curves $S^{(1,2)}$ and in green the generalised rarefaction curves $R^{(1,2)}$. The point $(v_0, u_0) \in S^{(1)}(v, \hat{v}_L, \hat{u_L})$ characterizes the central state (2.126).

2.6 where, in the (x, t) plane, the solution of (2.108), (2.110) is given $\forall t \geq 0$, after the critical time t_c we find the left state (2.113) which is separated by the central state

$$v = \tilde{v}_c(t) = \frac{t+k}{t_c+k} v_0, \quad u = \tilde{u}_c(x) = u_0 + \frac{v_0}{t_c+k}(x - x_c) \tag{2.126}$$

by a back shock whose line, taking (2.123) into account, is given by

$$x = \hat{x}_s(t) = -\frac{u_0 - \hat{u}_L}{v_0 - \hat{v}_L} \frac{\hat{k}\,(t - t_c)}{k+t} + x_c. \tag{2.127}$$

The state (2.126) is connected smoothly to the right state (2.116) by a forward generalised rarefaction wave which, using (2.92) and (2.93), in terms of the (ξ, τ) variables, assumes the form

$$v = \sqrt{\frac{\tau(\tau+\hat{k})}{\hat{k} c_0^2 \xi}}, \tag{2.128}$$

$$u = u_0 - \frac{1}{c_0^2 v_0} + \frac{2\tau + \hat{k}}{c_0} \sqrt{\frac{\xi}{\hat{k}\tau(\tau+\bar{k})}}. \tag{2.129}$$

Using (2.123), the left and the right characteristics limiting the generalised rarefaction wave (2.128), (2.129) are

$$x = \hat{x}_l(t) = \frac{\hat{k}\,(t - t_c)}{c_0^2 v_0^2\,(t + k)} + x_c, \qquad x = \hat{x}_r(t) = \frac{\hat{k}\,(t - t_c)}{c_0^2 \hat{v}_R^2\,(t + k)} + x_c. \tag{2.130}$$

Furthermore, since the point $(v_0, u_0) \in S^{(1)}(v, \hat{v}_L, \hat{u}_L)$ and $(\hat{v}_R, \hat{u}_R) \in R^{(2)}(v, v_0, u_0)$ (see Figure 2.6), we have

$$u_0 = \hat{u}_L - \frac{1}{\sqrt{3}c_0^2}\sqrt{(v_0 - \hat{v}_L)\left(\frac{1}{\hat{v}_L^3} - \frac{1}{v_0^3}\right)}, \tag{2.131}$$

$$\hat{u}_R = u_0 - \frac{1}{c_0^2}\left(\frac{1}{v_0} - \frac{1}{\hat{v}_R}\right). \tag{2.132}$$

Therefore the values (v_0, u_0) which are involved in the central state (2.126) are defined by (2.131) and (2.132).

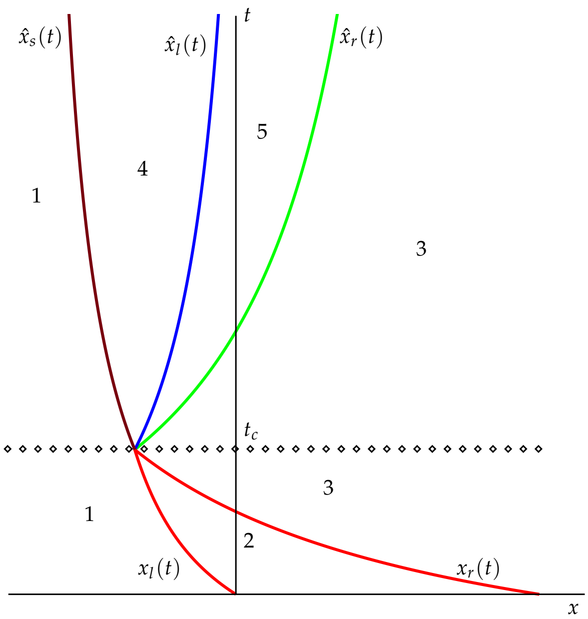

FIGURE 2.6: Behaviour in the (x, t) plane of the solution of (2.108), (2.110). In red the characteristics $x_l(t)$ and $x_r(t)$ given in (2.113) and (2.116), respectively. In black the shock line $\hat{x}_s(t)$ characterized in (2.127). In blue the characteristic $\hat{x}_l(t)$ and in green the characteristic $\hat{x}_r(t)$ determined in (2.130). In regions 1 and 3 the solution is given, respectively, by (2.113) and (2.116), while in region 2 by (2.114). In region 4 the solution is determined by (2.126), while in the region 5 we find the generalised rarefaction wave characterized in (2.128), (2.129).

Summarizing, we have showed how the method of differential constraints has been useful for solving nonlinear wave problems for the celebrated p-system. Such a model, apart from its physical meaning, has been considered as prototype of more general hyperbolic systems for describing nonlinear wave propagation.

After classifying all the possible differential constraints which can be appended to the governing model under interest, we solved the consistency conditions (2.59) arising from the differential compatibility between (2.52) and (2.57). As a consequence, the material law for the pressure $p(v)$ must obey to one of the relations for $\lambda(v)$ characterized in cases *i)–iv)*. Since for a perfect gas we have $p = p_0 v^{-\gamma}$ with $\gamma > 1$, the unique case which has a physical meaning is determined by (2.62). Therefore, we developed our analysis in the case *i)* and an exact solution which generalises the classical simple wave admitted by homogeneous systems is obtained. In fact, when the source term q involved in the constraint (2.163) is zero, then the solution (2.71)-(2.73) specializes to a simple wave. Such a solution was useful for solving a class of generalised Riemann problems as well as of Riemann problems with structure.

In fact we have been able to obtain the general solution of the GRP (2.99), (2.100) in terms of non constant states separated by generalised rarefaction waves and/or by shock waves. In characterizing such a solution the generalised rarefaction curves $u = R^{(1)}(v, v_L, u_L)$, $u = R^{(2)}(v, v_L, u_L)$ play a prominent role as well as the shock curves $u = S^{(1)}(v, v_L, u_L)$, $u = S^{(2)}(v, v_L, u_L)$. Indeed, the analysis in the (v, u) plane which can be carried on for the Riemann problem can be developed also for the GRP under interest and it involves the point (v_R, u_R) determined by the limiting values for $x \to 0$ of the initial data. Furthermore, using (2.63), we have

$$\int_{v_L}^{\infty} \sqrt{-p'(v)}dv = \frac{1}{c_0^2 v_L}, \tag{2.133}$$

so that if the point (v_R, u_R) belongs to region IV of the (v, u) plane (see Figure 2.4), then a vacuum zone can be formed as it happens for the RP.

A Riemann problem with structure was also solved by means of the generalised simple wave determined in section 3. In such a case, it was interesting to notice that, if the condition (2.117) or (2.119) is satisfied, then at a critical time t_c a shock is formed and, in turn, a new GRP must be solved. By means of the analysis of the (v, u) plane (see Figure (2.5)), we have proved that such a GRP is solved by three non constant states separated by a back shock and a forward generalised rarefaction wave. The corresponding full solution is given in Figure (2.6).

2.2 Degenerate Hodograph Method

The Degenerate Hodograph Method deals with solutions characterized by finite relations among the dependent variables and it has provided the majority of solutions to systems of PDEs.
Solutions obtained through this approach are called "multiple waves".
Let us consider the following systems of N quasilinear PDEs involving m independent variables $(x_1, ..., x_m)$ and n dependent variables $\mathbf{U} = (u_1, ..., u_n)$:

$$A_\alpha(\mathbf{U}) \frac{\partial u_k}{\partial x_\alpha} = f_k(\mathbf{U}), \quad \alpha = 1, .., m; \quad k = 1, .., n \tag{2.134}$$

where $x = (x_1, \ldots, x^m)$ are the independent variables, A_α are $N \times n$ matrices with elements with the elements a_{ij}^α, f_k denote source terms, that from now on we assume equal to 0 for sake of simplicity. A particular exact solution $u_k(x_\alpha)$ of (2.134) is called multiple wave of rank r if the rank of the jacobian matrix

$$\left\| \frac{\partial\left(u_1, ..., u_n\right)}{\partial\left(x_1, ..., x_m\right)} \right\|$$

is equal to r. In such a case the corresponding solutions of (2.134) assume the form

$$u_k = u_k\left(\Phi_1, ..., \Phi_r\right), \quad k = 1, \ldots, n \tag{2.135}$$

where the functions $\Phi_j(x_\alpha)$ are wave parameters. If r is equal to 1, the solution is called "simple wave", if it is equal to 2 we have a "double wave" and so on. Notice that if $r = n$, we recover the case of nondegenerate solutions, while multiple waves of rank $r \leq n - 1$ give a class of solutions with a degenerate hodograph. These solutions generalize the classical traveling wave solutions. The difference is that in the case of an r-multiple travelling wave the wave parameters are linear forms of the independent variables, while the wave parameters for an r-multiple wave are unknown functions.

The main problem for this approach is that substituting the ansatz (2.135) into (2.134) one gets an overdetermined system in the unknown $\Phi_j(x_\alpha)$.

In the case of simple waves, one looks for solutions of the form

$$u_k = u_k(\Phi_1), \quad k = i, \ldots, n$$

which substituted in the original system give

$$c_{k\alpha}(\Phi)\frac{\partial\Phi}{x_\alpha} = 0, \quad k = i, \ldots, n, \text{ where } c_{k\alpha} = a^{\alpha}_{k\beta}\frac{\partial u_\beta}{\partial\Phi_1}. \tag{2.136}$$

At this point, the structure of the solutions of (2.136) depends on the matrix C given by the coefficients $c_{k\alpha}$. In fact, one is able to produce nontrivial solutions if the rank r of the matrix C, satisfies

$$r < min(n.m). \tag{2.137}$$

If (2.137) holds, then we can rewrite (2.136) as

$$\frac{\partial\Phi_1}{\partial x_\alpha} = b_{\alpha\beta}\frac{\partial\Phi_1}{\partial x_\beta} \quad \alpha = 1, \ldots, r, \;\; \beta = r+1, \ldots, m. \tag{2.138}$$

The solutions of (2.138) is defined implictly by

$$\Phi^1 = \varphi^1(x_{r+1} + \sum_{\beta=1}^{r} x_\beta b_{\beta m} + \sum_{\beta=1}^{r} x_\beta b_{\beta r+1, ..., } x_m), \tag{2.139}$$

where $\varphi^1 : \mathbb{R}^{m-r} \to \mathbb{R}$ is an arbitrary mapping. While there is a large body of litterature dedicated to the simple wave theory, very few results have been obtained for double waves because the analysis of the overdetermined system is a very hard task to carry on. In fact, substituting the ansatz

$$u^k = u^k(\Phi_1, \Phi_2), \quad k = 1, \ldots, n$$

in the initial system one gets

$$A_\alpha\left(u_{\Phi_1}\frac{\partial\Phi_1}{\partial x_\alpha} - u_{\Phi_2}\frac{\partial\Phi_2}{\partial x_\alpha}\right) = 0.$$

This last system needs to be studied for compatibility and it is easy to notice that this is almost impossible in the general case.
In order to overcome this issues, quite recently in [7] and [9], in the case of quasilinear first order nonhomogeneous system, a possible strategy for determining double waves was proposed within the theoretical framewok of the Differential Constraints Method. We now sketch the idea of this procedure and, in the next section, we show an application to a model describing nerve fiber propagation.
In the system (2.3), without loss of generality, we can set

$$\mathbf{U}=\begin{bmatrix}\mathbf{V}\\ \mathbf{W}\end{bmatrix},\quad \mathbf{B}=\begin{bmatrix}\mathbf{B}_1\\ \mathbf{B}_2\end{bmatrix},\quad \mathrm{A}=\begin{bmatrix}\mathrm{P} & \mathrm{Q}\\ \mathrm{R} & \mathrm{S}\end{bmatrix},\quad \mathbf{l}^{(\lambda)}=\begin{bmatrix}\widehat{\mathbf{l}}^{(\lambda)}, & \bar{\mathbf{l}}^{(\lambda)}\end{bmatrix} \tag{2.140}$$

where $\mathbf{V},\mathbf{B}_1,\widehat{\mathbf{l}}^{(\lambda)}\in R^2$, $\mathbf{W},\mathbf{B}_2,\bar{\mathbf{l}}^{(\lambda)}\in R^{N-2}$ and $\mathrm{P},\mathrm{Q},\mathrm{R},\mathrm{S}$ are suitable matrix coefficients

$$\begin{aligned}&\mathrm{P}=\|P_{hk}\|,\quad \mathrm{Q}=\|Q_{hs}\|,\quad \mathrm{R}=\|R_{rk}\|,\quad \mathrm{S}=\|S_{rs}\|,\\ &h,k=1,2;\quad r,s=3,..,N.\end{aligned} \tag{2.141}$$

We look for double wave solutions of (2.3) under the form

$$\mathbf{U}=\mathbf{U}(\mathbf{V})=\begin{bmatrix}\mathbf{V}\\ \mathbf{W}(\mathbf{V})\end{bmatrix} \tag{2.142}$$

with $\mathbf{W}(\mathbf{V})$ smooth functions of $\mathbf{V}$. By sustituting the ansatz (2.142) in the equations (2.3), we get the following overdetermined system in the unknown $\mathbf{V}$

$$\frac{\partial\mathbf{V}}{\partial t}+(\mathrm{P}+\mathrm{Q}\,\nabla\mathbf{W})\,\frac{\partial\mathbf{V}}{\partial x}=\mathbf{B}_1 \tag{2.143}$$

$$(\nabla\mathbf{W})\frac{\partial\mathbf{V}}{\partial t}+(\mathrm{R}+\mathrm{S}\,\nabla\mathbf{W})\,\frac{\partial\mathbf{V}}{\partial x}=\mathbf{B}_2 \tag{2.144}$$

where

$$\nabla\mathbf{W}=\left\|\frac{\partial W_r}{\partial V_k}\right\|,\quad r=3,...,N;\quad k=1,2.$$

Of course depending on the choice of the variable $\mathbf{V}$, $\frac{N(N-1)}{2}$ reduced systems (2.143) can be characterized but the following theorem holds [9]

Theorem 5 *Let* $\mathbf{U}=\mathbf{U}(\mathbf{V})$ *a class of solutions of* (2.3), *then the hyperbolicity of* (2.3) *induces the hyperbolicity of at least a* 2×2 *reduced system in the new field variable* $\mathbf{V}$ *.*

Therefore, without loss of generality, we can assume the system (2.143) to be strictly hyperbolic so that the matrix coefficients $\mathrm{P}+\mathrm{Q}\,\nabla\mathbf{W}$ admits two real eigenvalues $\widetilde{\lambda}_1\neq\widetilde{\lambda}_2$ with left $\widetilde{\mathbf{l}}^{(\widetilde{\lambda}_k)}$ and right $\widetilde{\mathbf{d}}^{(\widetilde{\lambda}_k)}$ $(k=1,2)$ eigenvectors. The remaining $N-2$ equations (2.144) can be rewritten [9] under the form

$$\omega_{r1}\left(\widetilde{\mathbf{l}}^{(\widetilde{\lambda}_1)}\cdot\frac{\partial\mathbf{V}}{\partial x}\right)+\omega_{r2}\left(\widetilde{\mathbf{l}}^{(\widetilde{\lambda}_2)}\cdot\frac{\partial\mathbf{V}}{\partial x}\right)=B_r-\sum_{h=1}^{2}\frac{\partial W_r}{\partial V_h}B_h\quad (r=3,\ldots,N) \tag{2.145}$$

where $\omega_{rh}=\omega_{rh}(\mathbf{V},\mathbf{W}(\mathbf{V}))$ are suitable functions of $\mathbf{V}$. Therefore, an equation belonging to (2.145) is satisfied by any solutions of the reduced model (2.143) (i. e. it is a supplementary law of (2.143)) iff $\omega_{\bar{r}1}=\omega_{\bar{r}2}=B_{\bar{r}}-\sum_{h=1}^{2}\frac{\partial W_{\bar{r}}}{\partial V_h}B_h=0$, where $\bar{r}$ is the index characterizing the fixed equation in point. Furthermore, such a fixed equation is a differential constraint for the hyperbolic system (2.143) iff $\omega_{\overline{r}1} = 0$ or alternatively $\omega_{\overline{r}2} = 0$. In fact in such a case the equation under consideration assumes the form

$$\widetilde{\mathbf{l}}^{(\widetilde{\lambda}_2)} \cdot \frac{\partial \mathbf{V}}{\partial x} = \frac{1}{\omega_{\overline{r}2}} \left(B_{\overline{r}} - \sum_{h=1}^{2} \frac{\partial W_{\overline{r}}}{\partial V_h} B_h \right) = q_2 \left(\mathbf{V} \right) \tag{2.146}$$

or

$$\widetilde{\mathbf{l}}^{(\widetilde{\lambda}_1)} \cdot \frac{\partial \mathbf{V}}{\partial x} = \frac{1}{\omega_{\overline{r}1}} \left(B_{\overline{r}} - \sum_{h=1}^{2} \frac{\partial W_{\overline{r}}}{\partial V_h} B_h \right) = q_1 \left(\mathbf{V} \right). \tag{2.147}$$

and, using the procedure sketched in the sub-section 2.1, it results to be a possible differential constraint for the 2×2 reduced system (2.143). The main idea of the procedure here considered is based on the requirement that one of the equations (2.145) is a differential constraint of (2.143) while the remaining $N - 3$ relations are identically satisfied for all solutions of (2.143). In such a case exact double wave solutions of (2.3) can be obtained by solving the 2×2 reduced system (2.143) along with the differential constraint (2.146) or (2.147). According to the Method of Differential Constraints, the resulting solutions be determined in terms of one arbitrary function so that class of initial value problems can be solved. Of course since the procedure here considered reduces the problem of characterizing exact solution of (2.3) to that of the reduced sub-system (2.143), all the results concerning 2×2 hyperbolic systems supplemented by a differential constraint can be applied to the present case (see [5], [6]). Finally it could be of interest to recall that in order to characterize a possible differential constraint which can be appended to (2.143) the following proposition can be proved (see [8]).

Proposition 1 *Let $\widetilde{\lambda}(\mathbf{V})$ be a characteristic velocity associated to (2.143), and*

$$H_j(\widetilde{\lambda}) \neq 0 \quad \forall \widetilde{\lambda} \quad (3 \leq j \leq N) \tag{2.148}$$

where $H_j(\widetilde{\lambda})$ is the determinant of the matrix of order $N - 2$ obtained from $\left(\mathrm{S} - \nabla \mathbf{W}\,\mathrm{Q} - \widetilde{\lambda}\,\mathrm{I}\right)$ when the j−th row is replaced by $\widetilde{\mathbf{l}}^{(\widetilde{\lambda})}\,\mathrm{Q}$. Then, under assumption

$$\omega_{r1} = \omega_{r2} = B_r - \sum_{h=1}^{2} \frac{\partial W_r}{\partial V_h} B_h = 0 \quad (r = 3, \ldots, N, \quad r \neq j) \tag{2.149}$$

the $j -$ th condition (2.145) reduces to a first order differential constraint associated to $\widetilde{\lambda}_k$ iff $\widetilde{\lambda}_k$ is not a characteristic velocity of the hyperbolic system (2.3) whereas the remaing eigenvalue belongs to the spectrum of λ's.

Remark 12 *It can be proved [8] that in the case*

$$H_j(\widetilde{\lambda}_1) = H_j(\widetilde{\lambda}_2) = 0 \tag{2.150}$$

both characteristic velocities of the hyperbolic reduced system (2.143) belong to the spectrum of λ's and the $j -$ th condition (2.145) may reduce to a first order differential constraint associated both to $\widetilde{\lambda}_1$ or $\widetilde{\lambda}_2$.

2.2.1 Application to a model describing nerve fiber propagation

In this section, we show an application of the Degenerate Hodograph Method for determining a particular class of double wave solutions to a model describing nerve fiber pulse propagation [41]. A nerve cell (neuron) receives and send output messages from other neurons through an axon which is a channel permeable to different kinds of ions, mainly potassium (in the interior of the axon) and sodium (in the exterior) [22], [28]. In order to describe nerve pulse transmission, along the years, many mathematical models have been proposed. The most famous is due to Hodgkin and Huxley (HH) [23], in which the nerve pulse propagatation is described in terms of the membrane voltage and of three recovery variables that take into account of the activation and inactivation of sodium and potassium ions. Unfortunately, the mathematical analysis of the HH equations is of great complexity so that, later, different simplified systems have been proposed. One of the most studied was considered first by FitzHugh [15] and by Nagumo et al. [47]. The FitzHugh-Nagumo (FN) system involves the membrane voltage and only one recovery variable. Since it is based on a diffusion-like equation, it results to be parabolic so that nerve pulses propagate with an infinite speed. To avoid such a paradox and in order to model the nerve pulse transmission within a well-posed wave theory, in [14] the following hyperbolic system has been proposed

$$u_t + \mu_1 v_x = f(u, w) \tag{2.151}$$

$$v_t + \mu_3 u_x = \nu_0 v \tag{2.152}$$

$$w_t + \phi(u, w) v_x = \psi(u, w) \tag{2.153}$$

where x and t denotes, respectively, the distance along the axon and the time, u the potential difference across the membrane, v the axon current, w a recovery variable which takes into account the sodium inactivation and the potassium activation. Moreover $\mu_1 = \frac{1}{\pi a^2 C}$, $\mu_3 = \frac{\pi a^2}{L}$, $\nu_0 = -\frac{R}{L}$, $\phi = \bar{\Phi}(u, w)$, $\psi = \bar{\Psi}(u, w) - f\bar{\Phi}(u, w)$, $f = -\frac{2}{aC} I(u, w)$ where I is the ion current, $\bar{\Phi}$ and $\bar{\Psi}$ denote material response functions, a the axon radius, C the self-capacitance, L the specific silf-inductance and R the specific resistance. In passing we notice that the equations (2.151)-(2.153) specialize to the FH model when $L = \bar{\Phi} = 0$, $\bar{\Psi} = c_0 + c_1 u + c_2 w$ (c_0, c_1 and c_2 are constants), $I = w + k_1 u + k_3 u^3$ (k_1 and k_3 are constants). Furthermore the equation (2.153) can be rewritten under the form

$$w_t + \bar{\Phi}(u, w) u_t = \bar{\Psi}(u, w)$$

which characterizes the so-called rate-type materials where a non-istantaneous response of the material are taken into account because of short memory effects.

System (2.151)-(2.153) is hyperbolic and the characteristic wave speeds (eigenvalues of the matrix coefficients) are

$$\lambda_1 = -\sqrt{\mu_1 \mu_3}, \qquad \lambda_2 = 0, \qquad \lambda_3 = \sqrt{\mu_1 \mu_3}. \tag{2.154}$$

Our aim, here, is to develop the reduction procedure outlined in the sub-section 2.2 in order to find exact solutions of (2.151)-(2.153). Therefore, let us look for solutions of (2.151)-(2.153) under the form

$$w = w(u, v) \tag{2.155}$$

Substituting the ansatz (2.155) into (2.151)-(2.153), we obtain

$$u_t + \mu_1 v_x = \hat{f}(u, v) \tag{2.156}$$
$$v_t + \mu_3 u_x = \nu_0 v \tag{2.157}$$
$$\left(\hat{\phi} - \mu_1 w_u\right) v_x - \mu_3 w_v u_x = \hat{\psi} - \hat{f} w_u - \nu_0 v w_v \tag{2.158}$$

where $\hat{f} = f\left(u, w\left(u, v\right)\right)$, $\hat{\phi} = \phi\left(u, w\left(u, v\right)\right)$ and $\hat{\psi} = \psi\left(u, w\left(u, v\right)\right)$. The 2×2 sub-system (2.156), (2.157) results to be hyperbolic. Its characteristics speeds are

$$\tilde{\lambda}_1 = -\sqrt{\mu_1 \mu_3}, \quad \tilde{\lambda}_2 = \sqrt{\mu_1 \mu_3} \tag{2.159}$$

with left eigenvectors given by

$$\tilde{\mathbf{l}}^{(1)} = \left[1, -\sqrt{\frac{\mu_1}{\mu_3}}\right], \quad \tilde{\mathbf{l}}^{(2)} = \left[1, \sqrt{\frac{\mu_1}{\mu_3}}\right]$$

so that the possible differential contraints which can be appended to (2.156), (2.157) assume the form

$$u_x - \sqrt{\frac{\mu_1}{\mu_3}} v_x = \tilde{q}(x, t, u, v) \quad \text{or} \quad u_x + \sqrt{\frac{\mu_1}{\mu_3}} v_x = \hat{q}(x, t, u, v). \tag{2.160}$$

Since in the present case both the characteristic speeds of the reduced 2×2 system (2.156), (2.157) belong to the spectrum of the characteristic speeds of the full system (2.151)-(2.153), the equation (2.158) can be reduced either to $(2.160)_1$ or to $(2.160)_2$ (see the remark 12). Here we point out our attention to the case where (2.158) specializes to $(2.160)_2$ (of course a similar analysis could be developed in the remaining case). After some simple algebra, it turns out that (2.158) assumes the form $(2.160)_2$ if

$$\hat{\psi} = \hat{f} w_u + \nu_0 v w_v - \mu_3 w_v \hat{q} \tag{2.161}$$
$$\hat{\phi} = \mu_1 w_u - \sqrt{\mu_1 \mu_3} w_v \tag{2.162}$$

Next, by requiring the compatibility between (2.156), (2.157) and $(2.160)_2$ we are led to the consistency conditions

$$2\sqrt{\mu_1 \mu_3}\left(\hat{q}_v - \sqrt{\frac{\mu_1}{\mu_3}} \hat{q}_u\right) - \left(\hat{f}_v - \sqrt{\frac{\mu_1}{\mu_3}} \hat{f}_u\right) = \nu_0 \sqrt{\frac{\mu_1}{\mu_3}} \tag{2.163}$$
$$\hat{f} \hat{q}_u - \hat{q} \hat{f}_u + \left(\nu_0 v - \mu_3 \hat{q}\right) \hat{q}_v + \sqrt{\mu_1 \mu_3}\, \hat{q}\, \hat{q}_u = 0 \tag{2.164}$$

while, taking $(2.160)_2$ into account, the equations (2.156), (2.157) reduce to

$$u_t - \sqrt{\mu_1 \mu_3} u_x = \hat{f} - \sqrt{\mu_1 \mu_3} \hat{q} \tag{2.165}$$
$$v_t - \sqrt{\mu_1 \mu_3} v_x = \nu_0 v - \mu_3 \hat{q} \tag{2.166}$$

Therefore, once conditions (2.163), (2.164) are satisfied, the solution of (2.156), (2.157) along with (2.158) can be obtained by solving the equations (2.165), (2.166) supplemented by $(2.160)_2$.

To this end we first integrate (2.163) and we obtain

$$\hat{f} = 2\sqrt{\mu_1 \mu_3} \hat{q} + \nu_0 u + F(\sigma) \tag{2.167}$$

where $F(\sigma)$ is an arbitrary function of the variable

$$\sigma = u + \sqrt{\frac{\mu_1}{\mu_3}} v. \tag{2.168}$$

Next, substituting (2.167) in (2.164), after some algebra, we find the following two cases.

Case i) If $F \neq -\nu_0 \sigma$, integration of (2.164) leads to

$$\hat{q} = -\nu_0 q_0(\xi) \frac{\eta(\sigma)}{\eta'(\sigma)} \tag{2.169}$$

where $q_0(\xi)$ is an arbitrary function, the variable ξ is defined implicitely in terms of u and σ by

$$u\eta(\sigma) = \sqrt{\mu_1 \mu_3}\, q_0(\xi) \int \eta(\sigma)\, d\sigma + \sigma \eta(\sigma) + \xi \tag{2.170}$$

while, for further convenience, we set

$$F(\sigma) + \nu_0 \sigma = -\nu_0 \frac{\eta(\sigma)}{\eta'(\sigma)}$$

with $\eta'(\sigma) \neq 0$. As far as the integration of (2.165) and (2.166) is concerned, it is simply to verify that under the change of variables $(u, v) \leftrightarrow (\sigma, \xi)$, the equations (2.165), (2.166) assume the form

$$\sigma_t - \sqrt{\mu_1 \mu_3} \sigma_x = -\nu_0 \frac{\eta(\sigma)}{\eta'(\sigma)} \tag{2.171}$$

$$\xi_t - \sqrt{\mu_1 \mu_3} \xi_x = 0 \tag{2.172}$$

whose integration gives

$$\eta(\sigma) = \eta_0 \left(\sigma_0 \left(z \right) \right) e^{-\nu_0 t}, \qquad \xi = \xi_0 \left(z \right) \tag{2.173}$$

where

$$z = x + \sqrt{\mu_1 \mu_3} t.$$

Moreover $\sigma_0(z)$ and $\xi_0(z)$ are unspecified functions which, according to the method of differential constraints, must satisfy the constraint $(2.160)_2$. In fact, substituting (2.173) in $(2.160)_2$, we obtain

$$\frac{d}{dx} \left(\ln \left(\eta_0 \left(\sigma_0 \left(x \right) \right) \right) \right) = -\nu_0 q_0 \left(\xi_0 \right). \tag{2.174}$$

Therefore, once $q_0(\xi)$ and $\eta(\sigma)$ are given, the functions $u(x, t)$, $v(x, t)$ solutions of (2.151)-(2.152) can be obtained through (2.173) and (2.174), taking (2.168) and (2.170) into account, while, using (2.167), $w(x, t)$ can be determined from

$$f\left(u\left(x,t \right), w\left(x,t \right) \right) = -\nu_0 \left(2\sqrt{\mu_1 \mu_3} q_0 \left(\xi_0 \right) + 1 \right) \frac{\eta\left(\sigma \right)}{\eta'\left(\sigma \right)} - \nu_0 \sqrt{\frac{\mu_1}{\mu_3}} v. \tag{2.175}$$

We notice that, because of the constraint (2.174) such a solution is given in terms of one arbitrary function.

As an example, we make the simpler choice $q_0(\xi) = \xi$. In such a case, after some calculations, the function $\eta(\sigma)$ must necessarily assumes the form

$$\eta = \begin{cases} \dfrac{c_2}{(1-c_0)\sqrt{\mu_1\mu_3}}\,\sigma^{\frac{c_0}{1-c_0}} & \text{if } c_0 \neq 1 \\[2ex] \dfrac{c_2}{\sqrt{\mu_1\mu_3}}\,e^{c_2\sigma} & \text{if } c_0 = 1 \end{cases}$$

where $c_0 \neq 0$ and $c_2 \neq 0$ are constants, so that the following solution of (2.151)–(2.153) is obtained

$$v = \frac{\mu_3 c_0}{\nu_0}\sigma_0'(z)\, e^{\frac{\nu_0(c_0-1)}{c_0}t} \tag{2.176}$$

$$u = \begin{cases} \left(\sigma_0(z) - \frac{c_0}{\nu_0}\sqrt{\mu_1\mu_3}\,\sigma_0'(z)\right) e^{\frac{\nu_0(c_0-1)}{c_0}t} & \text{if } c_0 \neq 1 \\[2ex] \sigma_0(z) - \frac{\sqrt{\mu_1\mu_3}}{\nu_0}\sigma_0'(z) - \frac{\nu_0}{c_2}t & \text{if } c_0 = 1 \end{cases} \tag{2.177}$$

$$f\left(u\left(x,t\right), w\left(x,t\right)\right) = \begin{cases} \frac{\nu_0}{c_0}\left(\left(c_0-1\right)u + \sqrt{\frac{\mu_1}{\mu_3}}v\right) & \text{if } c_0 \neq 1 \\[2ex] -\frac{\nu_0}{c_2} + \nu_0\sqrt{\frac{\mu_1}{\mu_3}}v & \text{if } c_0 = 1 \end{cases} \tag{2.178}$$

where

$$\sigma_0(x) = \frac{\nu_0}{\mu_3 c_0}\int v_0(x)\,dx \tag{2.179}$$

with $v_0(x) = v(x,0)$.

Finally, as far as the constitutive functions $\Phi(u,w)$ and $\Psi(u,w)$ are concerned, from (2.161), (2.162) we find

$$\Phi = \frac{1}{f_w}\left(\nu_0\left(\mu_1 - f_u\right) - \frac{2\nu_0\mu_1}{c_0}\right) \tag{2.180}$$

$$\Psi = \frac{1}{f_w}\left(2\frac{\nu_0}{c_0}\left(c_0-1\right)f - ff_u - \frac{\nu_0^2}{c_0^2}\left(c_0-1\right)^2 u\right) \tag{2.181}$$

while $f(u,w)$ is still unspecified. Therefore, once $f(u,w)$ is assigned, then the solution of (2.151)-(2.153) is determined by (2.176)-(2.178) provided that Φ and Ψ assume, respectively, the form (2.180) and (2.181). Such a solution is given in terms of one arbitrary functions so that it is consistency with the arbitrary initial datum $v(x,0) = v_0(x)$ or, equivalently, $u(x,0) = u_0(x)$.

Case ii) If $F(\sigma) = -\nu_0\sigma$, by solving (2.164) we find $\hat{q} = q_0(\sigma)$. In such a case, from (2.156), (2.157) we have

$$\sigma_t - \sqrt{\mu_1\mu_3}\,\sigma_x = 0 \tag{2.182}$$

whose integration, along with that of (2.157), gives

$$v = v_0(z)e^{\nu_0 t} + \frac{\mu_3}{\nu_0}\left(u_0'(z) + \sqrt{\frac{\mu_1}{\mu_3}}v_0'(z)\right)\left(1 - e^{\nu_0 t}\right) \tag{2.183}$$

$$u = u_0(z) - \sqrt{\frac{\mu_1}{\mu_3}}\left[v_0(z) - \frac{\mu_3}{\nu_0}\left(u_0'(z) + \sqrt{\frac{\mu_1}{\mu_3}}v_0'(z)\right)\right]\left(e^{\nu_0 t} - 1\right) \tag{2.184}$$

where $z = x + \sqrt{\mu_1 \mu_3} t$, $u_0(x) = u(x,0)$, $v_0(x) = v(x,0)$. Furthermore, from the constraint (2.160)$_2$ we find

$$\frac{du_0}{dx} + \sqrt{\frac{\mu_1}{\mu_3}} \frac{dv_0}{dx} = q_0\left(\sigma_0(x)\right) \tag{2.185}$$

with $\sigma_0(x) = u_0(x) + \sqrt{\frac{\mu_1}{\mu_3}} v_0(x)$. Finally, taking (2.167) into account, once $f(u,w)$ is given, the function $w(x,t)$ can be calculated from

$$f\left(u\left(x,t\right), w\left(x,t\right)\right) = 2\sqrt{\mu_1 \mu_3}\, q_0\left(\sigma_0(z)\right) - \nu_0 \sqrt{\frac{\mu_1}{\mu_3}} v \tag{2.186}$$

Since $q_0(\sigma)$ is arbitrary, from (2.161), (2.162) a class of model laws for the functions $\Phi(u,w)$ and $\Psi(u,w)$ allowing the existence of the solution (2.183), (2.184), (2.186) are characterized. As an example, if we choose $q_0 = c_3 \sigma$, with c_3 constant, we find

$$\Phi = \frac{\mu_1\left(\nu_0 - f_u\right)}{f_w} \tag{2.187}$$

and

$$\Psi = \begin{cases} \dfrac{1}{f_w}\left[\left(\nu_0 - f_u\right) f + c_3 \sqrt{\mu_1 \mu_3}\left(f - \nu_0 u\right)\right] & \text{if } \; c_3 \neq \dfrac{\nu_0}{2\sqrt{\mu_1 \mu_3}} \\ \dfrac{\left(\nu_0 - f_u\right) f}{f_w} \quad \text{if } \; c_3 = \dfrac{\nu_0}{2\sqrt{\mu_1 \mu_3}} & \end{cases} \tag{2.188}$$

Remark 13 *In the case ii) the function $q_0(\sigma)$ involved in the constraint (2.185) is arbitrary. Therefore, once $q_0(\sigma)$ is given, because of (2.185) we are dealing with only one arbitrary initial datum ($v_0(x)$ or $u_0(x)$) so that the solution in point is determined in terms of one arbitrary function. Viceversa, if $u_0(x)$ and $v_0(x)$ are both arbitrary and if it is possible to calculate the inverse function of $\sigma_0(x)$, then from (2.185) we can determine $q_0(\sigma_0)$ and the corresponding solution is obtained in terms of two arbitrary functions (i.e. $u_0(x)$ and $v_0(x)$). As an example, if we choose $u_0(x) = \hat{u}_0 e^{-kx}$ and $v_0 = \hat{v}_0$ where $\hat{u}_0$, $\hat{v}_0$ and k are constants, from (2.185) we find $q_0 = -k\left(\sigma_0 - \sqrt{\frac{\mu_1}{\mu_3}} \hat{v}_0\right)$. Therefore, in such a case, the solution given by (2.183), (2.184) and (2.186) is obtained in terms of two arbitrary functions. In fact the initial data $u_0(x)$ and $v_0(x)$ are arbitrary while, taking (2.186) into account, the initial datum $w_0(x) = w(x,0)$ must obey*

$$f\left(u_0(x), w_0(x)\right) = -2k\hat{u}_0 \sqrt{\mu_1 \mu_3}\, e^{-kx} - \nu_0\, \hat{v}_0 \sqrt{\frac{\mu_1}{\mu_3}}. \tag{2.189}$$

In this section, we developed a reduction procedure for determining a particular class of double wave solutions of the first order quasilinear hyperbolic system (2.151)-(2.153) which describes nerve pulses propagation. In particular, following the idea proposed in [9], we reduced the integration of the full system (2.151)-(2.153) to that of a suitable 2×2 reduced system supplemented by a differential constraint. Therefore, within the theoretical framework of the method of differential constraints, all the results known for 2×2 first order quasilinear hyperbolic systems with a first order differential constraint can be applied to the nerve pulse model under interest.

Since we are considering a system in the one-dimensional case, of course all the solutions belong to the class of double wave solutions. However, the general analysis of the overdetermined system arising from the double wave ansatz is a very hard task to accomplish. Therefore, our procedure is aimed at determing a class of particular double wave solutions.

From (2.151)-(2.153) three possible reduced systems can be extracted depending if we look for double wave solutions under the form $u = u(v, w)$ or $v = v(u, w)$ or $w = w(u, v)$. Of course a suitable change of variables can be connected all the three cases. Here we chose the ansatz $w = w(u, v)$ and we solved, in general, the compatibility conditions arising from the reduced system (2.156), (2.157) and the constraint $(2.160)_2$. From such an analysis two cases arise. In the first we obtained a solution in terms of one arbitrary function. In the second case, under a suitable hypotesis, it is possible to determine a solution where the initial data for the potential $u(x, t)$ and the axon current $v(x, t)$ are arbitary.

A far as the model laws for the functions $f(u, w)$, $\Phi(u, w)$ and $\Psi(u, w)$ are concerned, the solutions characterized exist only if Φ and Ψ adopt some special form while f is unspeficied. Furthermore, the recovery variable $w(x, t)$ is determined once $f(u, w)$ is assigned. Therefore our procedure permitted to find double wave solutions to a large class of models described by (2.151)-(2.153).

Appendix A

Appendix

A.1 Properties of Laplace invariants of a second order hyperbolic equation

Let us consider a second order linear hyperbolic PDE.

$$u_{xt} + A(x,t)u_x + B(x,t)u_t + C(x,t)u = 0 \tag{A.1}$$

Definition 3 *We define Laplace Invariants of (A.1) the quantities*

$$h_0 = A_x + AB - C \qquad k_0 = B_t + AB - C \tag{A.2}$$

Definition 4 *We define equivalent transformations the more general trasformations that do not affect the differential structure of (A.1). Their form is the following*

$$\bar{x} = \alpha(x)x \qquad \bar{t} = \beta(t)t \qquad \bar{u} = w(x,t)u, \tag{A.3}$$

where α, β and $w(x,t) \neq 0$.

Definition 5 *We define equivalent equations two equations of form (A.1) that can be mapped one into the other with a trasformation of the class (A.3).*

At this point, we focus on equivalent trasformations with $\alpha(x) = \beta(t) = 1$, so that the independent variables remain untouched, while linear trasformations are possible for the dependent variable u:

$$\bar{x} = x \qquad \bar{t} = t \qquad \bar{u} = w(x,t)u. \tag{A.4}$$

We can now give the following theorem.

Theorem 6 *Two equations of form (A.1) with Laplace invariants (h_0, k_0) and $(\bar{h}_0, \bar{k}_0)$ can be mapped one into the other with a trasformation belonging to the class (A.4) if and only if*

$$h_0 = \bar{h}_0 \qquad k_0 = \bar{k}_0. \tag{A.5}$$

Proof 2 *Let us consider equation (A.1) and the trasformation $u = w(x,t)\bar{u}$. We compute the derivatives of u that appear in the equation:*

$$u_x = w_x\bar{u} + w\bar{u}_x \tag{A.6}$$

$$u_t = w_t\bar{u} + w\bar{u}_t \tag{A.7}$$

$$u_{xt} = w_{tx}\bar{u} + w_x\bar{u}_t + w_t\bar{u}_x + w\bar{u}_{xt}. \tag{A.8}$$

Substituting the above relations into ($A.1$)*, we get that* $\bar{u}$ *must satisfy*

$$\bar{u}_{xt} + \bar{A}\bar{u}_x + \bar{B}\bar{u}_t + \bar{C}\bar{u} = 0, \tag{A.9}$$

where

$$\bar{A} = A + (\ln w)_t, \quad \bar{B} = B + (\ln w)_x, \quad \bar{C} = C + w^{-1}(Aw_x + Bw_t + w_{xt}). \tag{A.10}$$

Performing an easy computation, it is possible to verify that the Laplace invariants of ($A.1$) *and* ($A.9$) *coincide.*
Viceversa, if $\bar{h}_0 = h_0$ *and* $\bar{k}_0 = k_0$*, then*

$$A_x + AB - C = \bar{A}_x + \bar{A}\bar{B} - \bar{C} \qquad B_t + AB - C = \bar{B}_t + \bar{A}\bar{B} - \bar{C}. \tag{A.11}$$

Hence, we can deduce

$$(A - \bar{A})_x = (B - \bar{B})_t \tag{A.12}$$

that implied the existence of a function $w(x,t)$ *such that*

$$\bar{A} - A = (\ln w)_t \qquad \bar{B} - B = (\ln w)_x. \tag{A.13}$$

Substituting ($A.13$) *into the equation* $\bar{h}_0 = h_0$*, we recover the form of* $\bar{C}$*,that is*

$$\bar{C} = C + w^{-1}(Aw_x + Bw_t + w_{xt}). \tag{A.14}$$

Relations ($A.13$) *and* ($A.14$) *prove that the transformation* $u = w(x,t)\bar{u}$ *map the differential equation with coefficients* A*,* B *and* C *into the differential relation of coefficients* $\bar{A}$*,* $\bar{B}$ *and* $\bar{C}$*.*

Remark 14 ([25]) *We notice that equation* ($A.1$) *can be mapped in*

- $u_{tx} = 0$ *if and only if* $h_0 = k_0 = 0$;
- $u_{tx} + C(x,t)u = 0$ *if and only if* $h_0 = k_0$;
- $u_{tx} + cu = 0$*,* $c =$ *constant if and only if* $h_0 = k_0 = f(x)g(t)$.

Furthermore, equation ($A.1$) *is factorable, i.e. the differential operator of the second order* $L = \partial_x\partial_t + A(x,t)\partial_x + B(x,t)\partial_t + C(x,t)$ *can be expressed as a product of two operators of the first order, if and only if one of the Laplace invariants vanishes. In particular, if* $h = 0$*, it is possible to obtain*

$$L = \partial_x\partial_t + A\partial_x + B\partial_t + A_x + AB = [\partial_x + \alpha(x,t)][\partial_t + \beta(x,t)],$$

while, if $k = 0$*, we get*

$$L = [\partial_t + \beta(x,t)][\partial_x + \alpha(x,t)].$$

A.2 Poisson Brackets' algorithm

One of the key points of the methods for solving partial differential equations is the analysis of compatibility. Poisson brackets' algorithm can be really helpful for the study of compatibility of systems of first order linear homogeneous differential equations as we have seen in the section of this thesis dedicated to the method of intermediate integrals.

For this reason, in this appendix we briefly sketch the steps of the algorithm.
We fix a system of partial differential equations in the unknown function $u(x)$

$$X_i(u) = a_{i\alpha}(x)\frac{\partial u}{\partial x_\alpha} = 0, \quad i = 1, \dots m, \alpha = 1, \dots n \tag{A.15}$$

Definition 6 *The first order differential operator*

$$\left[X_i, X_j\right] = b_{ij\alpha}\frac{\partial}{\partial x_\alpha} = \left(a_{i\beta}(x)\frac{\partial a_{j\alpha}}{\partial x_\beta} - a_{j\beta}(x)\frac{\partial a_{i\alpha}}{\partial x_\beta}\right)\frac{\partial}{\partial x_\alpha} \tag{A.16}$$

is defined Poisson bracket of the operators X_i and X_j.

Each solution u of $X_i(u) = 0$ and $X_j(u) = 0$ is also solution of $\left[X_i, X_j\right](u) = 0$. Hence, new linear homogeneous equations can be produced by means of Poisson brackets and if they are linearly independent of the equations of system (A.15), then it becomes necessary to append them to the initial system. On the other hand,

Definition 7 *System (A.15) is complete if all of its Poisson Brackets are linearly dependent of the initial system.*

Remark 15 *A complete system is compatible since it is not possible to obtain from the system new first order equations independent of it.*

If we consider system (A.15), in order to get a non constant solution, we need $m < n$. Then the solution of the system depends on $n - m$ arbitrary functions. Now,

- If from the study of the Poisson brackets we get $m' - m$ new independent conditions, we must append them to the system.
- In this way, we obtain an extendend system. The study of compatibility is reduced to the study of the Poisson brackets of this new system.

This procedure is repeated until one gets to the complete system.

A.3 Involutive systems

The study of compatibility conditions is strictly related to the notion of involutive systems. The main contributions in this field were given by E. Cartan and C. H. Riquier.
The Cartan approach is a strictly geometric one and it relies on the calculus of exterior differential forms. On the other hand, Riquier approach is a more algebric one and it is based on the prolongations of a system od PDEs and on the study of rankes of some matrices, so that the calculations turn out to be easier than the previous case. For this reason, we will discuss only the Riquier approach.
Let us consider a system of s differential relations of order q

$$\Phi^i(\mathbf{x}, \mathbf{u}, \mathbf{p}) = 0, \qquad i = 1, \dots, s \tag{A.17}$$

where $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^{\mathbb{N}}$ are the independent variables, $\mathbf{u} = (u_1(\mathbf{x}), \dots, u_m(\mathbf{x}))$ are the dependent variables and $\mathbf{p} = (p^j_\alpha)$ identify the set of partial derivatives

$$p^j_\alpha = \frac{\partial^{|\alpha|}u^j}{\partial x^\alpha}, \qquad j = 1, \dots, m, \ |\alpha| \leq q, \ |\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n.$$

All the discussed properties are analysed locally, in fact all the constructions are valid in the neighbourhood of a point $X_0 = (x_0, u_0, p_0)$.
We define the following symbol

Definition 8 *The simbol G_q of system (A.17) is defined as the vector space of the vectors of coordinates (ξ^j_α), where $j = 1, 2, \ldots, m$, $|\alpha| = q$ and the ξ^j_α satisfy*

$$\sum_{j=1}^{m} \sum_{|\alpha|=q} \xi^j_\alpha \frac{\partial \Phi^i}{\partial p^j_\alpha}(X_0) = 0 \qquad i = 1, 2, \ldots, s. \tag{A.18}$$

The subspace $(G_q)^k$ ($k = 1, 2 \ldots n-1$), contains the vectors with

$$\xi^j_{\beta,l} = 0 \qquad \beta = 1, 2, \ldots, q-1,\ l = 1, 2, \ldots, k,\ j = 1, 2, \ldots, m,$$

where $\beta, l = (\beta_1 \beta_2, \ldots, \beta_{l-1}, \beta_{l+1}, \ldots \beta_n)$

.
It is clear that $(G_q)^0 = G_q$ and $(G_q)^n = 0$. We will call τ_k the dimension of the subspace $(G_q)^k$. For example

$$\tau_0 = m \binom{n+q-1}{q} - rank \left(\frac{\partial \Phi^i}{\partial p^j_\alpha}(X_0) \right), \qquad \tau_n = 0.$$

In general, it holds

$$\tau_{i+1} \leq \tau_i \qquad i = 0, 1, \ldots, n-1.$$

The sum

$$\sum_{k=0}^{n-1} \tau_k$$

is called Cartan number, while we define the Cartan characters as follows:

$$\sigma_{k+1} = \tau_k - \tau_{k+1}, \qquad k = 1, 2, \ldots, n-1.$$

We remark that $\tau_0 = \sum_{k=1}^{n} \sigma_k$ and that the Cartan number can be expressed through the Cartan characters. In fact,

$$\sum_{k=0}^{n-1} \tau_k = \sum_{k=1}^{n} k\sigma_k.$$

Furthermore, the Cartan characters are linked to the order of the variables $(x_1, x_2, \ldots, x_n)$: hence, each change in the order of the variables can modify the Cartan characters.
We now consider the prolonged system of order $q + 1$

$$D_l \Phi^i(\mathbf{x}, \mathbf{u}, \mathbf{p}) = 0, \qquad l = 1, 2, \ldots, n\ i = 1, 2, \ldots, s, \tag{A.19}$$

where the operator D_l is defined as the total derivative with respect to x_l

$$D_l = \frac{\partial}{\partial x_l} + \sum_{|\alpha|} \sum_{j=1}^{m} p^j_{\alpha,l} \frac{\partial}{\partial p^j_\alpha}$$

Definition 9 *The system given by (A.17) and (A.19) is called a first prolongation of (A.17).*

Let G_{q+1} be the symbol of system ($A.19$), the it follows

$$dim(G_{q+1}) = m\binom{n+q}{q+1} - rank\left(\frac{\partial D_l \Phi^i}{\partial p^j_\alpha}\right) \qquad |\alpha = q+1|.$$

It can be proved that

$$dim(G_{q+1}) \leq \sum_{k=0}^{n-1} \tau_k$$

Definition 10 *A coordinate system for which*

$$dim(G_{q+1}) = \sum_{k=0}^{n-1} \tau_k$$

is called a quasiregular coordinate system. If there exists a quasiregular coordinate system, symbol G_q is defined involutive.

After studying the algebraic properties of ($A.17$), one has to analyse the differential structure of the manifold defined by ($A.19$). From this system, one can recover $N = dim(G_{q+1})$ derivatives of order $q+1$, that are called main derivatives of order $q+1$ of the system ($A.19$).
At this point, we can finally give the notion of involutive systems:

Definition 11 *System ($A.17$) is involutive if*

- *it has an involutive symbol G_q;*
- *it has the property that after substituting the main derivatives of the prolonged system ($A.19$) of order $q+1$, the remaining equations of ($A.19$) are identities because of system ($A.17$).*

The idea of involutive system is extremely important in the theory of compatibility. In fact, the following theorem holds:

Theorem 7 *(Cartan) Any analytic system of partial differential equations after a finite number of prolongations becomes either involutive or incompatible.*

Hence, in order to clarify the compatibility of a system one can apply the following alghorithm:

- the system is prolonged until its symbol becomes involutive;
- the system is prolonged again to verify if the new differential consequences are independent of the equations of the system;
- if the new differential consequences are identities, the system is involutive. If they are new independent relations, one has to go back to step one.

After a finite number of steps, one obtains the involutiveness or a contradiction.
Using Riquier's approach, it is also possible to solve the inverse problem of the theory of compatibility, that is the problem of determining the form of the functions $\Phi^i(\mathbf{x}, \mathbf{u}, \mathbf{p})$, with the requirement that system ($A.17$) has a certain arbitrariness in the solution.
We can distinguish three different inverse problems: in fact, referring to definition

(11, there are cases in which only the first condition is violated, cases in which only the second condition is not satisfied and cases in which both the conditions are not fullfilled.
In order to solve, this kind of problems, one has to require that the system becomes involutive. For example, if the second condition of definition (11 is satisfied, while the first is violated it is necessary to increase the dimension of G_{q+1}, reducing the rank of the matrix associated to system (A.19).
For what it concerns the solution of an analytic system, the following theorem holds:

Theorem 8 *(Cartan-Khäler) If a system* (A.17) *of order q is analytic and involutive, then there exists one and only one analytic solution of the Cauchy problem with assigned σ_k functions depending on k arguments ($k = 1, 2, \dots, n-1$).*

Notice that the analyticity property is not a necessary condition for the existence of a solution. In fact, theorems of existence for involutive systems of class $\mathbf{C}^1$ can be given.

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