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MULTIPLE SOLUTIONS FOR $(p, 2)$-EQUATIONS AT RESONANCE

Nikolaos S. Papageorgiou

Department of Mathematics, National Technical University
Zagrafou Campus, Athens, 15780, Greece

Calogero Vetro

Department of Mathematics and Computer Science, University of Palermo
Via Archirafi 34, 90123 Palermo, Italy

Francesca Vetro

Department of Energy, Information Engineering and Mathematical Models, University of Palermo
Viale delle Scienze, 90128 Palermo, Italy

Dedicated to Professor Vicentiu D. Radulescu with friendship and admiration.

Abstract. We consider a nonlinear nonhomogeneous Dirichlet problem driven by the sum of a p-Laplacian and a Laplacian and a reaction term which is $(p-1)$-linear near $\pm\infty$ and resonant with respect to any nonprincipal variational eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$. Using variational tools together with truncation and comparison techniques and Morse Theory (critical groups), we establish the existence of six nontrivial smooth solutions. For five of them we provide sign information and order them.

1. **Introduction.** Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following nonlinear nonhomogeneous Dirichlet problem

$$-\Delta_p u(z) - \Delta u(z) = f(z, u(z)) \quad \text{in } \Omega, \quad u\Big|_{\partial\Omega} = 0, \quad 2 < p. \tag{1}$$

In this problem Δ_p denotes the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}\,(|\nabla u|^{p-2}\nabla u) \quad \text{ for all } u \in W^{1,p}(\Omega).$$

When $p = 2$, we have $\Delta_2 = \Delta$ the usual Laplacian. The reaction term $f(z, x)$ is a measurable function which is C^1 in the x-variable. We assume that $f(z, \cdot)$ exhibits $(p-1)$-linear growth near $\pm\infty$ and resonance can occur at $\pm\infty$ with respect to any nonprincipal variational eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$. Near zero the reaction term $f(z, \cdot)$ exhibits a kind of oscillatory behavior. Using variational tools (critical point theory), together with truncation techniques and Morse theory (critical groups), we show that the problem has at least six nontrivial smooth solutions. For five of these solutions we provide sign information and order them. Elliptic equations driven by the sum of a p-Laplacian and a Laplacian ($(p, 2)$-equations for short), arise in problems of mathematical physics. We refer to Aris [4], Benci-D'Avenia-Fortunato-Pisani [5], Cherfils-Il'yasov [8], Fife [11] for such applications. Recently

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there have been some existence and multiplicity results for such equations. We mention the works of Aizicovici-Papageorgiou-Staicu [3], Cingolani-Degiovanni [9], Gasiński-Papageorgiou [15, 16], Liang-Han-Li [20], Papageorgiou-Rădulescu [27, 28, 29], Papageorgiou-Rădulescu-Repovš [32], Pei-Zhang [33], Sun [35], Sun-Zhang-Su [36], Yang-Bai [38]. However, none of the aforementioned works treats problems resonant at higher parts of the spectrum or produces six nontrivial smooth solutions with sign and order information.

2. **Mathematical Background - Hypotheses.** Let X be a Banach space and X^* its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X^*, X). Given $\varphi \in C^1(X,\mathbb{R})$, we say that it satisfies the "Cerami condition" (the "C-condition" for short), if the following property holds:

"Every sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that

$$\{\varphi(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R} \text{ is bounded,}$$
$$(1+\|u_n\|)\varphi'(u_n) \to 0 \text{ in } X^* \text{ as } n \to +\infty,$$

admits a strongly convergent subsequence".

This compactness-type condition on φ, leads to a deformation theorem, from which one can derive the minimax theory for the critical values of φ. A basic result in that theory is the so-called "mountain pass theorem" which we recall here.

Theorem 2.1. *If $\varphi \in C^1(X,\mathbb{R})$ satisfies the C-condition, $u_0, u_1 \in X$, $\|u_1 - u_0\| > r > 0$,*

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf\{\varphi(u) : \|u - u_0\| = r\} = m_r,$$
$$\textit{and } c = \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} \varphi(\gamma(t)) \textit{ where } \Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_0, \gamma(1) = u_1\},$$

then $c \ge m_r$ and c is a critical value of φ (that is, there exists $u \in X$ such that $\varphi(u) = c$, $\varphi'(u) = 0$).

In the analysis of problem (1) we will use the Sobolev spaces

$$W_0^{1,p}(\Omega) \quad \text{and} \quad H_0^1(\Omega)$$

and the Banach space

$$C_0^1(\overline{\Omega}) = \left\{ u \in C^1(\overline{\Omega}) \, : \, u\Big|_{\partial\Omega} = 0 \right\}.$$

By $\|\cdot\|$ we denote the norm of $W_0^{1,p}(\Omega)$. As a consequence of the Poincaré inequality, we can have

$$\|u\| = \|\nabla u\|_p \text{ for all } u \in W_0^{1,p}(\Omega).$$

The Sobolev space $H_0^1(\Omega)$ is a Hilbert space. Again thanks to the Poincaré inequality we can take as inner product of $H_0^1(\Omega)$

$$\langle u, h\rangle = (\nabla u, \nabla h)_{L^2(\Omega,\mathbb{R}^N)} = \int_\Omega (\nabla u, \nabla h)_{\mathbb{R}^N} dz \quad \text{for all } u, h \in H_0^1(\Omega).$$

Then the corresponding norm of $H_0^1(\Omega)$ is

$$\|u\|_{H_0^1(\Omega)} = \|\nabla u\|_2 \text{ for all } u \in H_0^1(\Omega).$$

The Banach space $C_0^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone given by

$$C_+ = \{u \in C_0^1(\overline{\Omega}) \, : \, u(z) \ge 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{ u \in C_+ \,:\, u(z) > 0 \text{ for all } z \in \Omega, \quad \frac{\partial u}{\partial n}\Big|_{\partial\Omega} < 0 \right\}.$$

Here $\frac{\partial u}{\partial n}\Big|_{\partial\Omega}$ denotes the normal derivative of u at $\partial\Omega$ defined by $\frac{\partial u}{\partial n} = (\nabla u, n)_{\mathbb{R}^N}$ with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. The Banach space $C_0^1(\overline{\Omega})$ is dense in both Sobolev spaces $W_0^{1,p}(\Omega)$ and $H_0^1(\Omega)$.

Consider a Carathéodory function $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ which satisfies

$$|f_0(z,x)| \le a_0(z)[1 + |x|^{r-1}] \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R},$$

with $a_0 \in L^\infty(\Omega)$, $1 < r < p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } N \le p, \end{cases}$ (the critical Sobolev exponent for p). We set $F_0(z,x) = \int_0^x f_0(z,s)ds$ and consider the C^1-functional $\varphi_0 : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega F_0(z,u)dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

The next proposition is a special case of a more general result of Aizicovici-Papageorgiou-Staicu [2]. We refer to Papageorgiou-Rădulescu [30, 31] for similar results suitable for the Neumann and Robin problems. These results are essentially an outgrowth of the nonlinear regularity theory of Lieberman [22].

Proposition 1. *If $u_0 \in W_0^{1,p}(\Omega)$ is a local $C_0^1(\overline{\Omega})$-minimizer of φ_0, that is, there exists $\rho_0 > 0$ such that $\varphi_0(u_0) \le \varphi_0(u_0 + h)$ for all $h \in C_0^1(\overline{\Omega})$, $\|h\|_{C_0^1(\overline{\Omega})} \le \rho_0$, then $u_0 \in C_0^{1,\alpha}(\overline{\Omega})$ with $\alpha \in (0,1)$ and it is also a local $W_0^{1,p}(\Omega)$-minimizer of φ_0, that is, there exists $\rho_1 > 0$ such that $\varphi_0(u_0) \le \varphi_0(u_0+h)$ for all $h \in W_0^{1,p}(\Omega)$, $\|h\| \le \rho_1$.*

Our hypotheses and arguments will involve the spectra of

$$(-\Delta_p, W_0^{1,p}(\Omega)) \quad \text{and} \quad (-\Delta, H_0^1(\Omega)).$$

We consider the following nonlinear eigenvalue problem

$$-\Delta_p u(z) = \widehat{\lambda}|u(z)|^{p-2}u(z) \quad \text{in } \Omega, \quad u\Big|_{\partial\Omega} = 0. \tag{2}$$

Actually the results for (2) which we will present are valid for $1 < p < +\infty$. We say that $\widehat{\lambda} \in \mathbb{R}$ is an "eigenvalue", if problem (2) admits a nontrivial solution $\widehat{u} \in W_0^{1,p}(\Omega)$ known as an eigenfunction corresponding to the eigenvalue $\widehat{\lambda}$. There is a smallest eigenvalue $\widehat{\lambda}_1(p)$ which has the following properties:

- $\widehat{\lambda}_1(p) > 0$ and it is isolated in the spectrum $\widehat{\sigma}(p)$ of $(-\Delta_p, W_0^{1,p}(\Omega))$ (that is, there exists $\varepsilon > 0$ such that $(\widehat{\lambda}_1(p), \widehat{\lambda}_1(p) + \varepsilon) \cap \widehat{\sigma}(p) = \emptyset$).
- $\widehat{\lambda}_1(p)$ is simple (that is, if $\widehat{u}, \widetilde{u} \in W_0^{1,p}(\Omega)$ are eigenfunctions corresponding to the eigenvalue $\widehat{\lambda}_1(p)$, then $\widehat{u} = \xi\widetilde{u}$ for some $\xi \in \mathbb{R} \setminus \{0\}$).
- $$\widehat{\lambda}_1(p) = \inf\left[\frac{\|\nabla u\|_p^p}{\|u\|_p^p} \,:\, u \in W_0^{1,p}(\Omega),\, u \ne 0\right]. \tag{3}$$

The infimum in (3) is realized on the corresponding one dimensional eigenspace. The above properties imply that the elements of this eigenspace have fixed sign. By $\widehat{u}_1(p)$ we denote the L^p-normalized (that is, $\|\widehat{u}_1(p)\|_p = 1$) positive eigenfunction corresponding to $\widehat{\lambda}_1(p)$. The nonlinear regularity theory implies that $\widehat{u}_1(p) \in C_+$.

Moreover, the nonlinear strong maximum principle (see, for example Gasiński-Papageorgiou [14], p. 738) gives that $\widehat{u}_1(p) \in \operatorname{int} C_+$. Similarly all eigenfunctions belong in $C_0^1(\overline{\Omega})$ and if $\widehat{\lambda} \neq \widehat{\lambda}_1(p)$, then the corresponding eigenfunctions are nodal (that is, sign changing). Since the spectrum $\widehat{\sigma}(p)$ is closed and $\widehat{\lambda}_1(p) > 0$ is isolated, the second eigenvalue is well-defined by

$$\widehat{\lambda}_2(p) = \min\left[\widehat{\lambda} \in \widehat{\sigma}(p) : \widehat{\lambda} > \widehat{\lambda}_1(p)\right].$$

Additional eigenvalues can be produced using the Ljusternik-Schnirelmann minimax scheme. This way we can generate a sequence $\{\widehat{\lambda}_k(p)\}_{k\in\mathbb{N}}$ of eigenvalues such that $\widehat{\lambda}_k(p) \to +\infty$ as $k \to +\infty$. These are the so-called "variational eigenvalues" and depending on the index used in the Ljusternik-Schnirelmann scheme, we can have different such sequences. They all coincide in the first two elements $\widehat{\lambda}_2(p) > \widehat{\lambda}_1(p) > 0$, but for the rest we do not know if this is the case. Here we employ the sequence generated using the Fadell-Rabinowitz cohomological index (see, Cingolani-Degiovanni [9]). Moreover, we do not know if the sequence of variational eigenvalues exhausts $\widehat{\sigma}(p)$. This is the case if $p = 2$ (linear eigenvalue problem) or if $N = 1$ (ordinary differential case).

For the linear eigenvalue problem ($p = 2$) we have

$$-\Delta u(z) = \widehat{\lambda} u(z) \quad \text{in } \Omega, \quad u\Big|_{\partial\Omega} = 0.$$

In this case we have

$$\widehat{\sigma}(2) = \{\widehat{\lambda}_k(2)\}_{k\in\mathbb{N}} \text{ with } 0 < \widehat{\lambda}_1(2) < \widehat{\lambda}_2(2) < \cdots < \widehat{\lambda}_k(2) \to +\infty \text{ as } k \to +\infty.$$

The corresponding eigenspaces $E(\widehat{\lambda}_k(2))$, $k \in \mathbb{N}$, are linear subspaces of $H_0^1(\Omega)$ and we have

$$H_0^1(\Omega) = \overline{\oplus_{k\in\mathbb{N}} E(\widehat{\lambda}_k(2))}.$$

Such a decomposition is not possible for the nonlinear problem (2), where the corresponding eigenspaces for $k \geq 2$, are cones. This makes the study of resonant problems for the p-Laplacian more difficult.

Standard regularity theory implies that

$$E(\widehat{\lambda}_k(2)) \subseteq C_0^1(\overline{\Omega}) \quad \text{for all } k \in \mathbb{N}.$$

Moreover, each eigenspace $E(\widehat{\lambda}_k(2))$, $k \in \mathbb{N}$, has the so-called "Unique Continuation Property" (UCP for short), that is, if $u \in E(\widehat{\lambda}_k(2))$, $k \in \mathbb{N}$, and $u(\cdot)$ vanishes on a set of positive measure, then $u \equiv 0$. For the linear eigenvalue problem, all eigenvalues admit variational characterizations:

$$\widehat{\lambda}_1(2) = \inf\left[\frac{\|\nabla u\|_2^2}{\|u\|_2^2} : u \in H_0^1(\Omega),\, u \neq 0\right], \tag{4}$$

$$\begin{aligned}\widehat{\lambda}_l(2) &= \inf\left[\frac{\|\nabla u\|_2^2}{\|u\|_2^2} : u \in \widehat{H}_l = \overline{\oplus_{k\geq l} E(\widehat{\lambda}_k(2))}\right] \\ &= \sup\left[\frac{\|\nabla u\|_2^2}{\|u\|_2^2} : u \in \overline{H}_l = \oplus_{k=1}^{l} E(\widehat{\lambda}_k(2))\right], \quad l \geq 2.\end{aligned} \tag{5}$$

In (4) the infimum is realized on $E(\widehat{\lambda}_1(2))$. In (5) both the infimum and the supremum are realized on $E(\widehat{\lambda}_l(2))$.

Using (4), (5) and the UCP of the eigenspaces, we deduce the following useful inequalities.

Proposition 2. *We have:*

(a) *If* $l \in \mathbb{N}$, $\eta \in L^\infty(\Omega)$, $\eta(z) \geq \widehat{\lambda}_l(2)$ *for a.a.* $z \in \Omega$ *and* $\eta \not\equiv \widehat{\lambda}_l(2)$, *then there exists* $c_1 > 0$ *such that*
$$\|\nabla u\|_2^2 - \int_\Omega \eta(z)u^2dz \leq -c_1\|u\|^2 \quad \text{for all } u \in \overline{H}_l.$$

(b) *If* $l \in \mathbb{N}$, $\eta \in L^\infty(\Omega)$, $\eta(z) \leq \widehat{\lambda}_l(2)$ *for a.a.* $z \in \Omega$ *and* $\eta \not\equiv \widehat{\lambda}_l(2)$, *then there exists* $c_2 > 0$ *such that*
$$\|\nabla u\|_2^2 - \int_\Omega \eta(z)u^2dz \geq c_2\|u\|^2 \quad \text{for all } u \in \overline{H}_l.$$

Consider the operator $A_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^* \left(\frac{1}{p} + \frac{1}{p'} = 1\right)$ defined by
$$\langle A_p(u), h\rangle = \int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla h)_{\mathbb{R}^N}dz.$$
This map has the following well-known properties (see, for example, Gasiński-Papageorgiou [14]).

Proposition 3. $A_p(\cdot)$ *is bounded (that is, maps bounded sets to bounded sets), continuous and strictly monotone (thus maximal monotone too) and of type* $(S)_+$ *(that is,* $u_n \xrightarrow{w} u$ *in* $W_0^{1,p}(\Omega)$ *and* $\limsup_{n\to+\infty}\langle A(u_n), u_n - u\rangle \leq 0$ *imply* $u_n \to u$ *in* $W_0^{1,p}(\Omega)$*).*

Next let us recall some basic definitions and facts from Morse theory (critical groups), which we will use in the sequel. So, let X be a Banach space, $\varphi \in C^1(X,\mathbb{R})$ and $c \in \mathbb{R}$. We introduce the following sets
$$K_\varphi = \{u \in X : \varphi'(u) = 0\},$$
$$K_\varphi^c = \{u \in K_\varphi : \varphi(u) = c\},$$
$$\varphi^c = \{u \in X : \varphi(u) \leq c\}.$$
Let (Y_1, Y_2) be a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$ and $k \in \mathbb{N}_0$. By $H_k(Y_1, Y_2)$ we denote the kth-relative singular homology group with $\mathbb{Z}$-coefficients. Suppose $u \in K_\varphi^c$ is isolated. Then the critical groups of φ at u are defined by
$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}) \quad \text{for all } k \in \mathbb{N}_0.$$
Here U is a neighborhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of singular homology implies that the above definition of critical groups is independent of the choice of the neighborhood U.

Suppose that $\varphi \in C^1(X,\mathbb{R})$ satisfies the C-condition and $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. The critical groups of φ at infinity are defined by
$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all } k \in \mathbb{N}_0.$$

This definition is independent of the choice of $c < \inf \varphi(K_\varphi)$. Indeed, let $c' < c < \inf \varphi(K_\varphi)$. From Corollary 5.35, p. 115 of Motreanu-Motreanu-Papageorgiou [25] we have that
$$\varphi^{c'} \text{ is a strong deformation retract of } \varphi^c,$$
$$\Rightarrow \quad H_k(X, \varphi^c) = H_k(X, \varphi^{c'}) \quad \text{for all } k \in \mathbb{N}_0 \text{ (see [25], Corollary 6.15(a), p. 145).}$$

If u is a local minimizer of φ, then

$$C_k(\varphi, 0) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0,$$

where $\delta_{k,m}$ is the Kronecker symbol defined by

$$\delta_{k,m} = \begin{cases} 1 & \text{if } k = m, \\ 0 & \text{if } k \neq m, \end{cases} \quad k, m \in \mathbb{N}_0.$$

If u is a critical point of mountain pass type, then

$$C_1(\varphi, u) \neq 0 \quad \text{(see [25], Corollary 6.81, p. 168).}$$

Before introducing the hypotheses on the reaction term, let us fix our notation.

For $x \in \mathbb{R}$, we set $x^{\pm} = \max\{\pm x, 0\}$. Then given $u \in W^{1,p}(\Omega)$, we define $u^{\pm}(\cdot) = u(\cdot)^{\pm}$. We know that

$$u^{\pm} \in W_0^{1,p}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.$$

By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$. If $g(z,x)$ is a measurable function (for example, a Carathéodory function), then by $N_g(\cdot)$ we denote the Nemytski (superposition) operator corresponding to g, that is,

$$N_g(u)(\cdot) = g(\cdot, u(\cdot)) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Also if $u, v \in W^{1,p}(\Omega)$, then we set

$$[u, v] = \{h \in W_0^{1,p}(\Omega) : u(z) \leq h(z) \leq v(z) \text{ for a.a. } z \in \Omega\}.$$

By $\mathrm{int}_{C_0^1(\overline{\Omega})}[u, v]$, we denote the interior in $C_0^1(\overline{\Omega})$ of $[u, v] \cap C_0^1(\overline{\Omega})$.

Now we are ready to introduce the hypotheses on the reaction term $f(z, x)$.

H(f): $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that for a.a. $z \in \Omega$ $f(z, 0) = 0$, $f(z, \cdot) \in C^1(\mathbb{R})$ and

(i) $|f'_x(z, x)| \leq a(z)[1 + |x|^{r-2}]$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)$, $p < r \leq p^*$;

(ii) there exist functions $w_{\pm} \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ such that

$$w_-(z) \leq c_- < 0 < c_+ \leq w_+(z) \quad \text{for all } z \in \overline{\Omega},$$
$$A_p(w_-) + A(w_-) \leq 0 \leq A_p(w_+) + A(w_+) \quad \text{in } W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*,$$
$$f(z, w_+(z)) \leq 0 \leq f(z, w_-(z)) \quad \text{for a.a. } z \in \Omega;$$

(iii) there exist $m \in \mathbb{N}$, $m \geq 2$ such that

$$\lim_{x \to \pm\infty} \frac{f(z, x)}{|x|^{p-2}x} = \widehat{\lambda}_m(p) \quad \text{uniformly for a.a. } z \in \Omega$$

and if $F(z, x) = \int_0^x f(z, s)ds$, then

$$f(z, x)x - pF(z, x) \to +\infty \quad \text{uniformly for a.a. } z \in \Omega, \text{ as } x \to \pm\infty;$$

(iv) there exist $l \in \mathbb{N}$, $l \geq 2$ such that $d_l \neq m$ with

$$d_l = \dim \overline{H}_l = \dim \ \oplus_{k=1}^{l} E(\widehat{\lambda}_k(2))$$

and functions $\eta_1, \eta_2 \in L^\infty(\Omega)$ such that

$$\widehat{\lambda}_l(2) \le \eta_1(z) \quad \text{for a.a. } z \in \Omega, \quad \eta_1 \not\equiv \widehat{\lambda}_l(2),$$
$$\eta_2(z) \le \widehat{\lambda}_{l+1}(2) \quad \text{for a.a. } z \in \Omega, \quad \eta_2 \not\equiv \widehat{\lambda}_{l+1}(2),$$
$$\eta_1(z) \le f'_x(z,0) = \lim_{x\to 0} \frac{f(z,x)}{x} \le \eta_2(z) \quad \text{uniformly for a.a. } z \in \Omega;$$

(v) for every $\rho > 0$, there exists $\widehat{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$, the function

$$x \to f(z,x) + \widehat{\xi}_\rho |x|^{p-2}x$$

is nondecreasing on $[-\rho, \rho]$.

Remark 1. Hypotheses $H(f)\,(ii),(iv)$ imply that near zero $f(z,\cdot)$ has a kind of oscillatory behavior. Evidently hypothesis $H(f)\,(ii)$ is satisfied if there exist $c_- < 0 < c_+$ such that $f(z,c_+) \le 0 \le f(z,c_-)$ for a.a. $z \in \Omega$. Hypothesis $H(f)\,(iii)$ implies that at $\pm\infty$ we have resonance with respect to a nonprincipal variational eigenvalue.

Example 1. The following function satisfies hypotheses $H(f)$ with $w_- \equiv -1$, $w_+ \equiv 1$. For the sake of simplicity, we drop the z-dependence:

$$f(x) = \begin{cases} \eta x - \xi |x|^{r-2}x & \text{if } |x| \le 1, \\ \widehat{\lambda}_m(p)|x|^{p-2}x + \beta |x|^{\tau-2}x + \theta & \text{if } 1 < |x|, \end{cases}$$

with $\eta \in (\widehat{\lambda}_l(2), \widehat{\lambda}_{l+1}(2))$ where $l \in \mathbb{N}$, $l \ge 2$, $d_l \ne m$, $2 < r$, $2 < \tau < p$, $\xi > \eta$ and $\theta = (p-2)\widehat{\lambda}_m(p) + (r-2)\xi$.

3. **Constant Sign Solutions.** In this section, we prove the existence of four nontrivial smooth solutions of constant sign (two positive and two negative). We also localize and order the solutions.

Proposition 4. *If hypotheses $H(f)\,(i),(ii),(iv)$ hold, then problem (1) admits two constant sign smooth solutions*

$$u_0 \in int_{C_0^1(\overline{\Omega})}[0, w_+] \quad \text{and} \quad v_0 \in int_{C_0^1(\overline{\Omega})}[w_-, 0].$$

Proof. Consider the Carathéodory function $\widehat{f}_+ : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\widehat{f}_+(z,x) = \begin{cases} f(z,x^+) & \text{if } x \le w_+(z), \\ f(z,w_+(z)) & \text{if } w_+(z) < x. \end{cases} \tag{6}$$

We set $\widehat{F}_+(z,x) = \int_0^x \widehat{f}_+(z,s)ds$ and introduce the C^1-functional $\widehat{\varphi}_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\varphi}_+(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega \widehat{F}_+(z,u)dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

From (6) it is clear that $\widehat{\varphi}_+$ is coercive. Also using the Sobolev embedding theorem, we see that $\widehat{\varphi}_+$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\widehat{\varphi}_+(u_0) = \inf\left[\widehat{\varphi}_+(u) \,:\, u \in W_0^{1,p}(\Omega)\right]. \tag{7}$$

Hypothesis $H(f)\,(iii)$ implies that given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) \in (0, c_+)$ such that

$$f(z,x) = \widehat{f}_+(z,x) \geq [\eta_1(z) - \varepsilon]x \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \delta \text{ (see (6))},$$
$$\Rightarrow \quad \widehat{F}_+(z,x) \geq \frac{1}{2}[\eta_1(z) - \varepsilon]x^2 \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \delta. \tag{8}$$

Recall that $\widehat{u}_1(2) \in \text{int}\, C_+$. So, we can find $t \in (0,1)$ small such that

$$t\widehat{u}_1(2)(z) \in [0, \delta] \quad \text{for all } z \in \overline{\Omega}. \tag{9}$$

Then we have

$$\widehat{\varphi}_+(t\widehat{u}_1(2)) \leq \frac{t^p}{p}\|\nabla \widehat{u}_1(2)\|_p^p + \frac{t^2}{2}\widehat{\lambda}_1(2) - \frac{t^2}{2}\int_\Omega \eta_1(z)\widehat{u}_1(2)^2 dz + \frac{\varepsilon t^2}{2}$$
$$\text{(see (8) and (9) and recall that } \|\widehat{u}_1(2)\|_2 = 1)$$
$$\Rightarrow \quad \widehat{\varphi}_+(t\widehat{u}_1(2)) \leq \frac{t^p}{p}\|\nabla \widehat{u}_1(2)\|_p^p + \frac{t^2}{2}\left[\int_\Omega (\widehat{\lambda}_1(2) - \eta_1(z))\widehat{u}_1(2)^2 dz + \varepsilon\right].$$

Since $\widehat{u}_1(2) \in \text{int}\, C_+$ and $l \geq 2$, we have

$$\beta = \int_\Omega [\eta_1(z) - \widehat{\lambda}_1(2)]\widehat{u}_1(2)^2 dz > 0.$$

We have

$$\widehat{\varphi}_+(t\widehat{u}_1(2)) \leq \frac{t^p}{p}\|\nabla \widehat{u}_1(2)\|_p^p + \frac{t^2}{2}\left[\varepsilon - \beta\right].$$

Choosing $\varepsilon \in (0, \beta)$ and recalling that $p > 2$ and $t \in (0,1)$, by choosing $t \in (0,1)$ even smaller if necessary, we have

$$\widehat{\varphi}_+(t\widehat{u}_1(2)) < 0,$$
$$\Rightarrow \quad \widehat{\varphi}_+(u_0) < 0 = \widehat{\varphi}_+(0) \quad \text{(see (7)) and so } u_0 \neq 0.$$

From (7) we have

$$\widehat{\varphi}'_+(u_0) = 0,$$
$$\Rightarrow \quad \langle A_p(u_0), h\rangle + A(u_0), h\rangle = \int_\Omega \widehat{f}_+(z, u_0) h dz \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{10}$$

In (10) first we choose $h = -u_0^- \in W_0^{1,p}(\Omega)$. Then

$$\|\nabla u_0^-\|_p^p + \|\nabla u_0^-\|_2^2 = 0 \quad \text{(see (6))},$$
$$\Rightarrow \quad u_0 \geq 0,\ u_0 \neq 0.$$

Next in (10) we choose $h = (u_0 - w_+)^+ \in W_0^{1,p}(\Omega)$. Then

$$\langle A_p(u_0), (u_0 - w_+)^+\rangle + \langle A(u_0), (u_0 - w_+)^+\rangle$$
$$= \int_\Omega f(z, w_+)(u_0 - w_+)^+ dz \quad \text{(see (6))}$$
$$\leq \langle A_p(w_+), (u_0 - w_+)^+\rangle + \langle A(w_+), (u_0 - w_+)^+\rangle \quad \text{(see hypothesis } H(f)\,(ii))$$
$$\Rightarrow \quad \langle A_p(u_0) - A_p(w_+), (u_0 - w_+)^+\rangle + \langle A(u_0) - A(w_+), (u_0 - w_+)^+\rangle \leq 0,$$
$$\Rightarrow \quad u_0 \leq w_+.$$

So we have proved that

$$u_0 \in [0, w_+]. \tag{11}$$

From (6) and (11), we see that (10) becomes

$$\langle A_p(u_0),h\rangle + A(u_0),h\rangle = \int_\Omega f(z,u_0)hdz \quad \text{for all } h\in W_0^{1,p}(\Omega),$$
$$\Rightarrow \quad -\Delta_p u_0(z) - \Delta u_0(z) = f(z,u_0(z)) \quad \text{for a.a. } z\in\Omega, \quad u\Big|_{\partial\Omega} = 0. \tag{12}$$

From Ladyzhenskaya-Uralt́seva [19] (Theorem 7.1, p. 286), we have

$$u_0 \in L^\infty(\Omega).$$

Then we can use Theorem of Lieberman [22] and infer that

$$u_0 \in C_+ \setminus \{0\}.$$

Let $\rho = \|w_+\|_\infty$ and let $\widehat{\xi}_\rho > 0$ be as postulated by hypothesis $H(f)\,(v)$. Then from (12) we have

$$\Delta_p u_0(z) + \Delta u_0(z) \le \widehat{\xi}_\rho u_0(z)^{p-1} \quad \text{for a.a. } z\in\Omega.$$

Theorem 5.4.1, p. 111, of Pucci-Serrin [34] implies that

$$0 < u_0(z) \quad \text{for all } z\in\Omega$$

(alternatively one can use Harnack's inequality, see Motreanu-Motreanu-Papageorgiou [25], p. 212). Finally using the boundary point lemma of Pucci-Serrin [34], we conclude that

$$u_0 \in \operatorname{int} C_+. \tag{13}$$

Let $a:\mathbb{R}^N\to\mathbb{R}^N$ be defined by

$$a(y) = |y|^{p-2}y + y \quad \text{for all } y\in\mathbb{R}^N.$$

Evidently $a\in C^1(\mathbb{R}^N,\mathbb{R}^N)$ and

$$\operatorname{div} a(\nabla u) = \Delta_p u + \Delta u \quad \text{for all } u\in W_0^{1,p}(\Omega).$$

We have

$$\nabla a(y) = |y|^{p-2}\left[\mathrm{id}_N + (p-2)\frac{y\otimes y}{|y|^2}\right] + \mathrm{id}_N,$$
$$\Rightarrow \quad (\nabla a(y)\xi,\xi)_{\mathbb{R}^N} \ge |\xi|^2 \quad \text{for all } y,\xi\in\mathbb{R}^N. \tag{14}$$

Note that

$$A_p(u_0) + A(u_0) - N_f(u_0) = 0 \le A_p(w_+) + A(w_+) - N_f(w_+) \quad \text{in } W^{-1,p'}(\Omega) \tag{15}$$
(see hypothesis $H(f)\,(ii)$).

Then (11), (14) and (15) permit the use of the tangency principle of Pucci-Serrin [34] (Theorem 2.5.2, p. 35) and we obtain

$$u_0(z) < w_+(z) \quad \text{for all } z\in\overline{\Omega}. \tag{16}$$

From (13) and (16) we conclude that

$$u_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[0,w_+].$$

Similarly, for the negative solution, we start with the Carathéodory function

$$\widehat{f}_-(z,x) = \begin{cases} f(z,w_-(z)) & \text{if } x\le w_-(z),\\ f(z,-x^-) & \text{if } w_-(z) < x.\end{cases}$$

We set $\widehat{F}_-(z,x)=\int_0^x \widehat{f}_-(z,s)ds$ and introduce the C^1-functional $\widehat{\varphi}_-: W_0^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\widehat{\varphi}_-(u)=\frac{1}{p}\|\nabla u\|_p^p+\frac{1}{2}\|\nabla u\|_2^2-\int_\Omega \widehat{F}_-(z,u)dz \quad \text{for all } u\in W_0^{1,p}(\Omega).$$

Then reasoning as above (with $\widehat{\varphi}_-$ instead of $\widehat{\varphi}_+$), we generate a negative solution $v_0\in \mathrm{int}_{C_0^1(\overline{\Omega})}[w_-,0]$. $\square$

Next using $u_0\in \mathrm{int}\, C_+$ and $v_0\in -\mathrm{int}\, C_+$ produced in Proposition 4, we will generate two more smooth constant sign solutions $\widehat{u}\in \mathrm{int}\, C_+$ and $\widehat{v}\in -\mathrm{int}\, C_+$ such that

$$\widehat{v}(z)<v_0(z)<0<u_0(z)<\widehat{u}(z) \quad \text{for all } z\in\Omega.$$

Proposition 5. *If hypotheses $H(f)$ hold, then problem (1) has two more smooth constant sign solutions $\widehat{u}\in$ int C_+, $\widehat{v}\in -$int C_+ such that $\widehat{u}-u_0\in$ int C_+, $v_0-\widehat{v}\in$ int C_+.*

Proof. Let $u_0\in \mathrm{int}\, C_+$ be the positive solution produced in Proposition 4. Using $u_0(\cdot)$ we introduce the following truncation of the reaction term $f(z,\cdot)$:

$$k_+(z,x)=\begin{cases} f(z,u_0(z)) & \text{if } x\le u_0(z),\\ f(z,x) & \text{if } u_0(z)<x.\end{cases} \tag{17}$$

We set $K_+(z,x)=\int_0^x k_+(z,s)ds$ and introduce the C^1-functional $\widetilde{\varphi}_+: W_0^{1,p}(\Omega)\to \mathbb{R}$ defined by

$$\widetilde{\varphi}_+(u)=\frac{1}{p}\|\nabla u\|_p^p+\frac{1}{2}\|\nabla u\|_2^2-\int_\Omega K_+(z,u)dz \quad \text{for all } u\in W_0^{1,p}(\Omega).$$

Claim 1: $\widetilde{\varphi}_+$ satisfies the C-condition.

Consider a sequence $\{u_n\}_{n\in\mathbb{N}}\subseteq W_0^{1,p}(\Omega)$ such that

$$|\widetilde{\varphi}_+(u_n)|\le M_1 \quad \text{for some } M_1>0,\ \text{all } n\in\mathbb{N},$$
$$(1+\|u_n\|)\widetilde{\varphi}_+'(u_n)\to 0 \text{ in } W^{-1,p'}(\Omega) \text{ as } n\to+\infty. \tag{18}$$

From (18) we have

$$|\langle\widetilde{\varphi}_+'(u_n),h\rangle|\le \frac{\varepsilon_n\|h\|}{1+\|u_n\|}, \quad \text{for all } h\in W_0^{1,p}(\Omega), \text{ with } \varepsilon_n\to 0^+,$$
$$\Rightarrow \quad \left|\langle A_p(u_n),h\rangle+\langle A(u_n),h\rangle-\int_\Omega \widehat{f}_+(z,u_n)\,h\,dz\right|\le\frac{\varepsilon_n\|h\|}{1+\|u_n\|}, \text{ for all } n\in\mathbb{N}. \tag{19}$$

In (19) we choose $h=-u_n^-\in W_0^{1,p}(\Omega)$. Using (17), we obtain

$$\|\nabla u_n^-\|_p^p\le c_3\|u_n^-\| \quad \text{for some } c_3>0,\ \text{all } n\in\mathbb{N},$$
$$\Rightarrow \quad \{u_n^-\}_{n\in\mathbb{N}}\subseteq W_0^{1,p}(\Omega) \text{ is bounded.} \tag{20}$$

Suppose that

$$\|u_n^+\|\to+\infty \quad \text{as } n\to+\infty.$$

Let $y_n=\dfrac{u_n^+}{\|u_n^+\|}$, $n\in\mathbb{N}$. Then $\|y_n\|=1$, $y_n\ge 0$ for all $n\in\mathbb{N}$. So, we may assume that

$$y_n\xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega) \text{ and } y_n\to y \text{ in } L^p(\Omega). \tag{21}$$

From (19) and (20), we obtain

$$\Big|\langle A_p(u_n^+),h\rangle + \langle A(u_n^+),h\rangle - \int_\Omega \widehat{f}_+(z,u_n^+)\,h\,dz\Big| \le c_4\|h\|$$
$$\text{for some } c_4>0, \text{ all } n\in\mathbb{N},\ h\in W_0^{1,p}(\Omega),$$
$$\Rightarrow \Big|\langle A_p(y_n),h\rangle + \frac{1}{\|u_n^+\|^{p-2}}\langle A(y_n),h\rangle - \int_\Omega \frac{N_{\widehat{f}_+}(u_n^+)}{\|u_n^+\|^{p-1}}\,h\,dz\Big| \le \frac{c_4\|h\|}{\|u_n^+\|^{p-1}}, \text{ for all } n\in\mathbb{N}. \tag{22}$$

Hypotheses $H(f)\,(i),(iii),(iv)$ imply that

$$|f(z,x)|\le c_5[|x|+|x|^{p-1}] \quad \text{for a.a. } z\in\Omega, \text{ all } x\in\mathbb{R}, \text{ some } c_5>0. \tag{23}$$

From, (17) and (23), we infer that

$$\left\{\frac{N_{\widehat{f}_+}(u_n^+)}{\|u_n^+\|^{p-1}}\right\}_{n\in\mathbb{N}} \subseteq L^{p'}(\Omega) \text{ is bounded.} \tag{24}$$

Hence, by passing to a subsequence if necessary and using hypothesis $H(f)\,(iii)$, we have

$$\frac{N_{\widehat{f}_+}(u_n^+)}{\|u_n^+\|^{p-1}} \xrightarrow{w} \widehat{\lambda}_m(p)y^{p-1} \text{ in } L^{p'}(\Omega) \text{ as } n\to+\infty. \tag{25}$$

(see Aizicovici-Papageorgiou-Staicu [1], proof of Proposition 16).

In (22) we choose $h=y_n-y\in W_0^{1,p}(\Omega)$, pass to the limit as $n\to+\infty$ and use (21), (24) and the fact that $p>2$. We obtain

$$\lim_{n\to+\infty}\langle A_p(y_n),y_n-y\rangle = 0,$$
$$\Rightarrow \quad y_n\to y \text{ in } W_0^{1,p}(\Omega) \text{ (see Proposition 3) and so } \|y\|=1,\ y\ge 0. \tag{26}$$

So, if in (22) we pass to limit as $n\to+\infty$ and use (25) and (26), then

$$\langle A_p(y),h\rangle = \int_\Omega \widehat{\lambda}_m(p)y^{p-1}h\,dz \quad \text{for all } h\in W_0^{1,p}(\Omega),$$
$$\Rightarrow \quad -\Delta_p y(z) = \widehat{\lambda}_m(p)y(z)^{p-1} \text{ for a.a. } z\in\Omega, \quad y\Big|_{\partial\Omega}=0. \tag{27}$$

Since $\|y\|=1$ and $m\ge 2$, from (27) we infer that y must be nodal, a contradiction to (26). Therefore

$$\{u_n^+\}_{n\in\mathbb{N}}\subseteq W_0^{1,p}(\Omega) \text{ is bounded,}$$
$$\Rightarrow \quad \{u_n\}_{n\in\mathbb{N}}\subseteq W_0^{1,p}(\Omega) \text{ is bounded (see (20)).}$$

Thus we may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega) \text{ and } u_n\to u \text{ in } L^p(\Omega). \tag{28}$$

In (19) we choose $h=(u_n-u)\in W_0^{1,p}(\Omega)$, pass to the limit as $n\to+\infty$ and use (28). Then

$$\lim_{n\to+\infty}[\langle A_p(u_n),u_n-u\rangle + \langle A(u_n),u_n-u\rangle] = 0,$$
$$\Rightarrow \quad \limsup_{n\to+\infty}[\langle A_p(u_n),u_n-u\rangle + \langle A(u),u_n-u\rangle] \le 0 \text{ (since } A(\cdot) \text{ is monotone),}$$
$$\Rightarrow \quad \limsup_{n\to+\infty}\langle A_p(u_n),u_n-u\rangle \le 0 \text{ (see (28)),}$$
$$\Rightarrow \quad u_n\to u \text{ in } W_0^{1,p}(\Omega) \text{ (see Proposition 3).}$$

This proves Claim 1.

Claim 2: We may assume that $u_0 \in \text{int } C_+$ is a local minimizer of $\widetilde{\varphi}_+$. Recall that

$$0 < (w_+ - u_0)(z) \quad \text{for all } z \in \overline{\Omega} \text{ (see Proposition 4).} \tag{29}$$

We consider the following truncation of $k_+(z,\cdot)$ (see (17)):

$$\widetilde{k}_+(z,x) = \begin{cases} k_+(z,x) & \text{if } x \leq w_+(z), \\ k_+(z,w_+(z)) & \text{if } w_+(z) < x. \end{cases} \quad \text{(see (29)).} \tag{30}$$

This is Carathéodory function. We set $\widetilde{K}_+(z,x) = \int_0^x \widetilde{k}_+(z,s)ds$ and consider the C^1-functional $\widetilde{\psi}_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widetilde{\psi}_+(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega \widetilde{K}_+(z,u)dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

From (30) it is clear that $\widetilde{\psi}_+$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, invoking the Weierstrass-Tonelli theorem, we can find $\widehat{u}_0 \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} & \widetilde{\psi}_+(\widehat{u}_0) = \inf\left[\widetilde{\psi}_+(u) \,:\, u \in W_0^{1,p}(\Omega)\right], \\ \Rightarrow \quad & \widetilde{\psi}'_+(\widehat{u}_0) = 0 \text{ in } W^{-1,p'}(\Omega). \end{aligned} \tag{31}$$

Using (31), (30), (17), as in the proof of Proposition 4, we show that

$$\widehat{u}_0 \in [u_0, w_+] \cap \text{int } C_+.$$

If $\widehat{u}_0 \neq u_0$, then this is the desired second positive smooth solution of (1). Moreover, via the tangency principle, we have that $\widehat{u}_0 - u_0 \in \text{int } C_+$. So, we are done. Therefore we may assume that $\widehat{u}_0 = u_0$. From (30) and (17), it is clear that

$$\widetilde{\psi}_+\Big|_{[u_0,w_+]} = \widetilde{\varphi}_+\Big|_{[u_0,w_+]} \tag{32}$$

From (29) and (32) it follows that

$$\begin{aligned} & u_0 \in \text{int } C_+ \text{ is a local } C_0^1(\overline{\Omega})\text{-minimizer of } \widetilde{\varphi}_+, \\ \Rightarrow \quad & u_0 \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \widetilde{\varphi}_+ \text{ (see Proposition 1).} \end{aligned}$$

This proves Claim 2.

Using (17) we can easily see that

$$K_{\widetilde{\varphi}_+} \subseteq [u_0) \cap \text{ int } C_+ = \{u \in \text{int } C_+ \,:\, u_0(z) \leq u(z) \quad \text{for all } z \in \overline{\Omega}\}.$$

So, we may assume that

$$K_{\widetilde{\varphi}_+} \text{ is finite.}$$

Otherwise we already have an infinity of positive smooth solutions of (1) all strictly bigger than u_0.

On account of Claim 2, we can find $\rho \in (0,1)$ small such that

$$\widetilde{\varphi}_+(u_0) < \inf\left[\widetilde{\varphi}_+(u) \,:\, \|u - u_0\| = \rho\right] = \widetilde{m}_+. \tag{33}$$

Hypothesis $H(f)\,(iii)$ implies that given any $\eta > 0$, we can find $\widehat{M} = \widehat{M}(\eta) > 0$ such that

$$f(z,x)x - pF(z,x) \geq \eta \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq \widehat{M}.$$

We have

$$\begin{aligned}
&\frac{d}{dx}\left[\frac{F(z,x)}{x^p}\right] = \frac{f(z,x)x^p - px^{p-1}F(z,x)}{x^{2p}}\\
&= \frac{f(z,x)x - pF(z,x)}{x^{p+1}}\\
&\geq \frac{\eta}{x^{p+1}} \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq \widehat{M},\\
\Rightarrow\ & \frac{F(z,x)}{x^p} - \frac{F(z,v)}{v^p} \geq -\frac{\eta}{p}\left[\frac{1}{x^p} - \frac{1}{v^p}\right] \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq v \geq \widehat{M}.
\end{aligned} \tag{34}$$

Hypothesis $H(f)\,(iii)$ implies that

$$\lim_{x\to\pm\infty} \frac{F(z,x)}{|x|^p} = \frac{1}{p}\widehat{\lambda}_m(p) \quad \text{uniformly for a.a. } z \in \Omega. \tag{35}$$

So, if in (34) we let $x \to +\infty$ and use (35), then

$$\begin{aligned}
&\widehat{\lambda}_m(p)v^p - pF(z,v) \geq \eta \quad \text{for a.a. } z \in \Omega, \text{ all } v \geq \widehat{M},\\
\Rightarrow\ & \widehat{\lambda}_m(p)v^p - pF(z,v) \to +\infty \text{ uniformly for a.a. } z \in \Omega \text{ as } v \to +\infty
\end{aligned} \tag{36}$$

(recall $\eta > 0$ is arbitrary).

For $t > 0$ big we have

$$\begin{aligned}
&\widetilde{\varphi}_+(t\widehat{u}_1(p)) \leq \frac{t^p}{p}\widehat{\lambda}_1(p) + \frac{t^2}{2}\|\nabla \widehat{u}_1(p)\|_2^2 + c_6 - \int_\Omega F(z, t\widehat{u}_1(p))dz\\
&\qquad\qquad \text{for some } c_6 > 0 \text{ (see (17))},\\
\Rightarrow\ & p\widetilde{\varphi}_+(t\widehat{u}_1(p)) \leq t^p[\widehat{\lambda}_1(p) - \widehat{\lambda}_m(p)] + \frac{p}{2}t^2\|\nabla \widehat{u}_1(2)\|_2^2 + pc_6\\
&\qquad + \int_\Omega [\widehat{\lambda}_m(p)(t\widehat{u}_1(p))^{p-1} - pF(z, t\widehat{u}_1(p))]dz.
\end{aligned} \tag{37}$$

Recall that $p > 2$ and $m \geq 2$. So, from (37) and using (36), we infer that

$$\widetilde{\varphi}_+(t\widehat{u}_1(p)) \to -\infty \text{ as } t \to +\infty. \tag{38}$$

Claim 1, (33) and (38) permit the use of Theorem 2.1 (the mountain pass theorem). So, we can find $\widehat{u} \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned}
&\widehat{u} \in K_{\widetilde{\varphi}_+} \subseteq [u_0) \cap \operatorname{int} C_+ \quad \text{and} \quad \widetilde{\varphi}_+(u_0) < \widetilde{m}_+ \leq \widetilde{\varphi}_+(\widehat{u}),\\
\Rightarrow\ & \widehat{u} \in \operatorname{int} C_+ \text{ is a solution of (1) (see (17)) and } \widehat{u} \neq u_0 \text{ (see (33))}.
\end{aligned}$$

Moreover, as before using the tangency principle and the boundary point lemma (see Pucci-Serrin [34], pp. 35 and 120), we have

$$\widehat{u} - u_0 \in \operatorname{int} C_+.$$

For the second negative solution, we consider the Carathéodory function

$$k_-(z,x) = \begin{cases} f(z,x) & \text{if } x \leq v_0(z),\\ f(z, v_0(z)) & \text{if } v_0(z) < x.\end{cases}$$

We set $K_-(z,x) = \int_0^x k_-(z,s)ds$ and introduce the C^1-functional $\widetilde{\varphi}_- : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widetilde{\varphi}_-(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega K_-(z,u)dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Working with $\widetilde{\varphi}_-(\cdot)$ as above, we produce a second negative smooth solution $\widehat{v}$ such that

$$\widehat{v} \in -\text{int } C_+ \quad \text{and} \quad v_0 - \widehat{v} \in \text{int } C_+.$$

□

Next we will show that problem (1) admits extremal constant sign solutions, that is, a smallest positive solution $u^* \in \text{int } C_+$ and a biggest negative solution $v^* \in -\text{int } C_+$. We will need these extremal solutions, in order to produce a nodal (sign changing) solution.

Hypotheses $H(f)\,(i),(iii)$ imply that given $\varepsilon > 0$ and $\tau \in (2, p^*)$, we can find $c_7 > 0$ such that

$$f(z,x)x \geq (\eta_1(z) - \varepsilon)x^2 - c_7|x|^\tau \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}.$$

We consider the following auxiliary Dirichlet problem:

$$-\Delta_p u(z) - \Delta u(z) = (\eta_1(z) - \varepsilon)u(z) - c_7|u(z)|^{\tau-2}u(z) \text{ in } \Omega, \quad u\Big|_{\partial\Omega} = 0. \tag{39}$$

Proposition 6. *For all $\varepsilon > 0$ small problem (39) has a unique positive solution $\widetilde{u} \in$ int C_+ and the oddness of (39) implies that $\widetilde{v} = -\widetilde{u} \in -$int C_+ is the unique negative solution of (39).*

Proof. First we prove the existence of a positive solution for problem (39).

Let $\sigma_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\sigma_+(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 + \frac{c_7}{\tau}\|u^+\|_\tau^\tau - \frac{1}{2}\int_\Omega [\eta_1(z) - \varepsilon](u^+)^2 dz$$

for all $u \in W_0^{1,p}(\Omega)$. Since $\tau > 2$, it is clear that $\sigma_+(\cdot)$ is coercive. Also, by the Sobolev embedding theorem $\sigma_+(\cdot)$ is sequentially weakly lower semicontinuous. Therefore we can find $\widetilde{u} \in W_0^{1,p}(\Omega)$ such that

$$\sigma_+(\widetilde{u}) = \inf[\sigma_+(u) \,:\, u \in W_0^{1,p}(\Omega)]. \tag{40}$$

Note that for $t \in (0,1)$ small we have

$$\sigma_+(t\widehat{u}_1(2)) = \frac{t^p}{p}\|\nabla \widehat{u}_1(2)\|_p^p + \frac{t^2}{2}\left[\int_\Omega (\widehat{\lambda}_1(2) - \eta_1(z))\widehat{u}_1(u)^2 dz + \varepsilon\right] + \frac{t^\tau}{\tau}c_7\|\widehat{u}_1(2)\|_\tau^\tau.$$

We have

$$\beta = \int_\Omega (\eta_1(z) - \widehat{\lambda}_1)\widehat{u}_1(2)dz > 0.$$

Choosing $\varepsilon \in (0, \beta)$, we have

$$\sigma_+(t\widehat{u}_1(2)) \leq \frac{t^p}{p}\|\nabla \widehat{u}_1(2)\|_p^p - \frac{t^2}{2}c_8 + \frac{t^\tau}{\tau}c_7\|\nabla \widehat{u}_1(2)\|_\tau^\tau \quad \text{for some } c_8 > 0.$$

Since $2 < \tau, p$, by taking $t \in (0,1)$ even smaller if necessary, we have

$$\begin{aligned} & \sigma_+(t\widehat{u}_1(2)) < 0 = \sigma_+(0), \\ \Rightarrow\ & \sigma_+(\widetilde{u}) < 0 = \sigma_+(0) \quad (\text{see } (40)), \\ \Rightarrow\ & \widetilde{u} \neq 0. \end{aligned}$$

From (40) we have

$$\begin{aligned} & \sigma'_+(\widetilde{u}) = 0, \\ \Rightarrow\ & \langle A_p(\widetilde{u}), h\rangle + \langle A(\widetilde{u}), h\rangle = \int_\Omega [\eta_1(z) - \varepsilon](\widetilde{u}^+)hdz - c_7\int_\Omega (\widetilde{u}^+)^{\tau-1}hdz \end{aligned} \tag{41}$$

for all $h \in W_0^{1,p}(\Omega)$.

In (41) we choose $h = -\widetilde{u}^- \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned} &\|\nabla \widetilde{u}^-\|_p^p + \|\nabla \widetilde{u}^-\|_2^2 = 0, \\ \Rightarrow\quad &\widetilde{u} \geq 0,\ \widetilde{u} \neq 0. \end{aligned}$$

From (41) we have

$$-\Delta_p \widetilde{u}(z) - \Delta \widetilde{u}(z) = (\eta_1(z) - \varepsilon)\widetilde{u}(z) - c_7 \widetilde{u}(z)^{\tau-1} \quad \text{for a.a. } z \in \Omega, \quad \widetilde{u}\Big|_{\partial\Omega} = 0.$$

As before, the nonlinear regularity theory implies that

$$\widetilde{u} \in C_+ \setminus \{0\}$$

and by the tangency principle we have

$$\begin{aligned} &\Delta_p \widetilde{u}(z) + \Delta \widetilde{u}(z) \leq c_7 \|\widetilde{u}\|_\infty^{\tau-p} \widetilde{u}(z)^{p-1} \quad \text{for a.a. } z \in \Omega, \\ \Rightarrow\quad &\widetilde{u} \in \text{int } C_+ \quad \text{(see Pucci-Serrin [34] and Montenegro [24]).} \end{aligned}$$

Next we show that this solution of (39) is unique. To this end let

$$G_0(t) = \frac{t^p}{p} + \frac{t^2}{2} \quad \text{for all } t \geq 0.$$

If $G(y) = G_0(|y|)$ for all $y \in \mathbb{R}^N$, then $\nabla G(y) = a(y) = |y|^{p-2}y + y$ and

$$\text{div } a(\nabla u) = \Delta_p u + \Delta u \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Note that $t \to G_0(t^{1/2})$ is convex.

We introduce the integral functional

$$j(u) = \begin{cases} \int_\Omega G(\nabla u^{\frac{1}{2}})dz & \text{if } u \geq 0,\ u^{\frac{1}{2}} \in W_0^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases} \tag{42}$$

Let $u_1, u_2 \in \text{dom } j = \{u \in L^1(\Omega) : j(u) < +\infty\}$ (the effective domain of $j(\cdot)$). Let $v_1 = u_1^{\frac{1}{2}}$, $v_2 = u_2^{\frac{1}{2}}$. From (42) we have $v_1, v_2 \in W_0^{1,p}(\Omega)$. Let

$$v = ((1-t)v_1 + tv_2)^{\frac{1}{2}},\ t \in [0,1].$$

From Lemma 1 of Diaz-Saá [10], we have

$$\begin{aligned} &|\nabla v(z)| \leq [(1-t)|\nabla v_1(z)|^2 + t|\nabla v_2(z)|^2]^{\frac{1}{2}} \\ \Rightarrow\quad &G_0(|\nabla v(z)|) \leq G_0\left([(1-t)|\nabla v_1(z)|^2 + t|\nabla v_2(z)|^2]^{\frac{1}{2}}\right) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\text{(since } G_0(\cdot) \text{ is increasing)} \\ &\leq (1-t)G_0(|\nabla v_1(z)|) + tG_0(|\nabla v_2(z)|) \text{ (since } t \to G_0(t^{\frac{1}{2}}) \text{ is convex)} \\ \Rightarrow\quad &G(\nabla v(z)) \leq (1-t)G(\nabla u_1(z)^{\frac{1}{2}}) + tG(\nabla u_2(z)^{\frac{1}{2}}) \quad \text{for a.a. } z \in \Omega, \\ \Rightarrow\quad &j(\cdot) \text{ is convex.} \end{aligned}$$

Suppose that $\widetilde{u}, \widetilde{u}_0$ are positive solutions of (39). From the first part of proof we have

$$\widetilde{u}, \widetilde{u}_0 \in \text{int } C_+.$$

Then given $h \in C_0^1(\overline{\Omega})$ and for $|t|$ small, we have

$$\widetilde{u} + th,\ \widetilde{u}_0 + th \in \text{dom } j.$$

We see that $j(\cdot)$ is Gâteaux differentiable at $\widetilde{u}$ and at $\widetilde{u}_0$ in the direction h. Using the chain rule, we have

$$j'(\widetilde{u})(h) = \frac{1}{2}\int_\Omega \frac{-\Delta_p \widetilde{u} - \Delta \widetilde{u}}{\widetilde{u}^2} h\, dz \quad \text{and} \quad j'(\widetilde{u}_0)(h) = \frac{1}{2}\int_\Omega \frac{-\Delta_p \widetilde{u}_0 - \Delta \widetilde{u}_0}{\widetilde{u}_0^2} h\, dz$$

for all $h \in C_0^1(\overline{\Omega})$.

The convexity of $j(\cdot)$ implies the monotonicity of $j'(\cdot)$. Hence

$$\begin{aligned} 0 &\leq \int_\Omega \left[\frac{-\Delta_p \widetilde{u} - \Delta \widetilde{u}}{\widetilde{u}^2} - \frac{-\Delta_p \widetilde{u}_0 - \Delta \widetilde{u}_0}{\widetilde{u}_0^2}\right](\widetilde{u}^2 - \widetilde{u}_0^2)dz \\ &= \int_\Omega \left[(\eta_1(z) - \varepsilon)\left(\frac{1}{\widetilde{u}} - \frac{1}{\widetilde{u}_0}\right) - c_7(\widetilde{u}^{\tau-2} - \widetilde{u}_0^{\tau-2})\right](\widetilde{u}^2 - \widetilde{u}_0^2)dz \leq 0 \\ \Rightarrow \quad & \widetilde{u} = \widetilde{u}_0 \quad (\text{since } 2 < \tau). \end{aligned}$$

Since problem (39) is odd, we infer that

$$\widetilde{v} = -\widetilde{u} \in -\text{int}\, C_+$$

is the unique negative solution of (39). □

Consider the sets

$$\begin{aligned} S_+ &= \{u \in W_0^{1,p}(\Omega) : u \text{ is a positive solution of problem (1)}\}, \\ S_- &= \{v \in W_0^{1,p}(\Omega) : v \text{ is a negative solution of problem (1)}\}. \end{aligned}$$

From Proposition 4 we have that

$$\begin{aligned} S_+ \neq \emptyset \quad &\text{and} \quad S_+ \subseteq \text{int}\, C_+, \\ S_- \neq \emptyset \quad &\text{and} \quad S_- \subseteq -\text{int}\, C_-. \end{aligned}$$

From Filippakis-Papageorgiou [13] we know that

S_+ is downward directed

(that is, if $u_1, u_2 \in S_+$, then there is $u \in S_+$ such that $u \leq u_1$, $u \leq u_2$),

S_- is upward directed

(that is, if $v_1, v_2 \in S_-$, then there is $v \in S_-$ such that $v_1 \leq v$, $v_2 \leq v$).

Proposition 7. *If hypotheses $H(f)$ hold, then $\widetilde{u} \leq u$ for all $u \in S_+$ and $v \leq \widetilde{v}$ for all $v \in S_-$.*

Proof. Let $u \in S_+$ and consider the following Carathéodory function

$$e_+(z,x) = \begin{cases} 0 & \text{if } x < 0, \\ (\eta_1(z) - \varepsilon)x - c_7 x^{\tau-1} & 0 \leq x \leq u(z), \\ (\eta_1(z) - \varepsilon)u(z) - c_7 u(z)^{\tau-1} & u(z) < x. \end{cases} \tag{43}$$

We set $E_+(z,x) = \int_0^x e_+(z,s)ds$ and consider the C^1-functional $\gamma_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\gamma_+(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega E_+(z,u)dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

From (43) it is clear that γ_+ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\widetilde{u}_0 \in W_0^{1,p}(\Omega)$ such that

$$\gamma_+(\widetilde{u}_0) = \inf[\gamma_+(u) : u \in W_0^{1,p}(\Omega)].$$

Moreover, as in the proof of Proposition 4 since $2 < \tau$, we have

$$\begin{aligned} & \gamma_+(\widetilde{u}_0) < 0 = \gamma_+(0) \\ \Rightarrow \quad & \widetilde{u}_0 \neq 0. \end{aligned}$$

We have

$$\begin{aligned} & \gamma_+'(\widetilde{u}_0) = 0, \\ \Rightarrow \quad & \langle A_p(\widetilde{u}_0), h\rangle + \langle A(\widetilde{u}_0), h\rangle = \int_\Omega e_+(z, \widetilde{u}_0) h \, dz \quad \text{for all } h \in W_0^{1,p}(\Omega). \end{aligned} \tag{44}$$

If in (44) we choose $h = -\widetilde{u}_0^- \in W_0^{1,p}(\Omega)$ and $h = (\widetilde{u}_0 - u)^+ \in W_0^{1,p}(\Omega)$, then we show that

$$\begin{aligned} & \widetilde{u}_0 \in [0, u] \cap C_+ \setminus \{0\} \quad \text{(nonlinear regularity)}, \\ \Rightarrow \quad & \widetilde{u}_0 \text{ is a positive solution of (39) (see (43))}, \\ \Rightarrow \quad & \widetilde{u}_0 = \widetilde{u} \in \operatorname{int} C_+ \text{ (see Proposition 6)}, \\ \Rightarrow \quad & \widetilde{u} \leq u \text{ for all } u \in S_+. \end{aligned}$$

Similarly we show that

$$v \leq \widetilde{v} \text{ for all } v \in S_-.$$

□

Using this proposition we can now show the existence of extremal constant sign solutions for problem (1).

Proposition 8. *If hypotheses $H(f)$ hold, then problem (1) admits a smallest positive solution $u^* \in int\, C_+$ and a biggest negative solution $v^* \in -int\, C_+$.*

Proof. Invoking Lemma 3.10, p. 178, of Hu-Papageorgiou [18], we can find a decreasing sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq S_+ \subseteq \operatorname{int} C_+$ such that

$$\inf S_+ = \inf_{n\in\mathbb{N}} u_n.$$

We have

$$\langle A_p(u_n), h\rangle + \langle A(u_n), h\rangle = \int_\Omega f(z, u_n) h \, dz \quad \text{for all } h \in W_0^{1,p}(\Omega), \text{ all } n \in \mathbb{N}. \tag{45}$$

In (45) we use $h = u_n \in W_0^{1,p}(\Omega)$. Using (23) and recalling that $0 \leq u \leq u_1 \in \operatorname{int} C_+$ for all $n \in \mathbb{N}$, we see that

$$\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.}$$

So, we may assume that

$$u_n \xrightarrow{w} u^* \text{ in } W_0^{1,p}(\Omega) \text{ and } u_n \to u^* \text{ in } L^p(\Omega).$$

In (45) we choose $h = (u_n - u^*) \in W_0^{1,p}(\Omega)$ and pass to the limit as $n \to +\infty$. As in the proof of Proposition 5 (see Claim 1), we have

$$\begin{aligned} & \lim_{n\to+\infty} \langle A_p(u_n), u_n - u^*\rangle \leq 0, \\ \Rightarrow \quad & u_n \to u^* \text{ in } W_0^{1,p}(\Omega) \text{ (see Proposition 3)}. \end{aligned} \tag{46}$$

So, if in (45) we pass to the limt as $n \to +\infty$ and use (46), then

$$\langle A_p(u^*), h\rangle + \langle A(u^*), h\rangle = \int_\Omega f(z, u^*) h \, dz \quad \text{for all } h \in W_0^{1,p}(\Omega).$$

Also, from Proposition 7 we have

$$\begin{aligned} & \widetilde{u} \leq u_n \quad \text{for all } n \in \mathbb{N}, \\ \Rightarrow \quad & \widetilde{u} \leq u^* \quad (\text{see } (46)), \\ \Rightarrow \quad & u^* \in S_+ \quad \text{and} \quad u^* = \inf S_+. \end{aligned}$$

Similarly we produce $v^* \in S_-$ such that $v^* = \sup S_-$. ☐

4. **Nodal solutions.** In this section we prove the existence of a nodal (that is, sign changing) solution. The strategy is the following. Let $u^* \in \operatorname{int} C_+$ and $v^* \in -\operatorname{int} C_+$ be the two extremal constant sign solutions (see Proposition 8). Using truncations of $f(z,\cdot)$ at $u^*(z)$ and $v^*(z)$, we focus on the order interval $[v^*, u^*]$. Employing variational tools (in particular using Theorem 2.1), we produce $y_0 \in [v^*, u^*]$ a solution of (1) distinct from $\{0, u^*, v^*\}$. The extremality of u^* and v^* guarantees that y_0 is nodal.

We start by considering the energy (Euler) functional $\varphi : W_0^{1,p}(\Omega) \to \mathbb{R}$ of problem (1) defined by

$$\varphi(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega F(z,u)dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

We have that $\varphi \in C^2(W_0^{1,p}(\Omega), \mathbb{R})$.

The next proposition will allow us to distinguish the trivial solution from the other solutions of problem (1).

Proposition 9. *If hypotheses $H(f)\,(i),(iv)$ hold, then $C_k(\varphi,0) = \delta_{k,d_l}\mathbb{Z}$ for all $k \in \mathbb{N}_0$ with $d_l = \dim \oplus_{i=1}^l E(\widehat{\lambda}_i(2))$.*

Proof. Consider the C^2-functional $\widehat{\mu} : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\mu}(u) = \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega F(z,u)dz \quad \text{for all } u \in H_0^1(\Omega).$$

Hypotheses $H(f)\,(i),(iv)$ imply that given $\varepsilon > 0$, we can find $c_9 > 0$ such that

$$\frac{1}{2}[f'_x(z,0) - \varepsilon]x^2 - c_9|x|^r \leq F(z,x) \leq \frac{1}{2}[f'_x(z,0) + \varepsilon]x^2 + c_9|x|^r \tag{47}$$

for a.a. $z \in \Omega$, all $x \in \mathbb{R}$.

Let $u \in \overline{H}_l = \oplus_{i=1}^l E(\widehat{\lambda}_i(2))$. Then

$$\begin{aligned} \widehat{\mu}(u) &\leq \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2}\int_\Omega [f'_x(z,0) - \varepsilon]u^2 dz + c_{10}\|u\|^r \text{ for some } c_{10} > 0 \text{ (see (47))} \\ &\leq -c_{11}\|u\|^2 + c_{10}\|u\|^r \text{ for some } c_{11} > 0. \end{aligned}$$

(see Proposition 2 and choose $\varepsilon > 0$ small).

Since $r > 2$, we can find $\rho \in (0,1)$ small such that

$$\widehat{\mu}(u) \leq 0 \quad \text{for all } u \in \overline{H}_l = \oplus_{i=1}^l E(\widehat{\lambda}_i(2)),\ \|u\|_{H_0^1(\Omega)} \leq \rho. \tag{48}$$

On the other hand, if $u \in \widehat{H}_{l+1} = \overline{H}_l^\perp = \overline{\oplus_{i \geq l+1} E(\widehat{\lambda}_i(2))}$, then

$$\begin{aligned} \widehat{\mu}(u) &\geq \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2}\int_\Omega [f'_x(z,0) + \varepsilon]u^2 dz - c_{12}\|u\|^r \text{ (for some } c_{12} > 0) \\ &\geq c_{13}\|u\|^2 - c_{12}\|u\|^r \text{ for some } c_{13} > 0 \text{ (see Proposition 2 and choose } \varepsilon > 0 \text{ small).} \end{aligned}$$

Since $r > 2$, by choosing $\rho \in (0,1)$ even smaller if necessary, we can also have

$$\widehat{\mu}(u) > 0 \text{ for all } u \in \widehat{H}_{l+1} = \overline{\oplus_{i\geq l+1} E(\widehat{\lambda}_i(2))},\ 0 < \|u\| \leq \rho. \tag{49}$$

From (48) and (49) it follows that μ has local linking at 0. Using Proposition 2.3 of Su [37], we infer that

$$C_k(\widehat{\mu}, 0) = \delta_{k,d_l}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{50}$$

Let $\mu = \widehat{\mu}\big|_{W_0^{1,p}(\Omega)}$. Since $W_0^{1,p}(\Omega)$ is dense in $H_0^1(\Omega)$, we have

$$C_k(\mu, 0) = C_k(\widehat{\mu}, 0) \quad \text{for all } k \in \mathbb{N}_0$$

(see Palais [26] and Chang [6] (p. 14)). Hence

$$C_k(\mu, 0) = \delta_{k,d_l}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0 \quad \text{(see (50))}. \tag{51}$$

We have

$$|\varphi(u) - \mu(u)| \leq \frac{1}{p}\|u\|^p \text{ for all } u \in W_0^{1,p}(\Omega). \tag{52}$$

Also, for all $h \in W_0^{1,p}(\Omega)$ we have

$$\begin{aligned} & |\langle \varphi'(u) - \mu'(u), h\rangle| = |\langle A_p(u), h\rangle \leq c_{14}\|u\|^{p-1}\|h\| \text{ for some } c_{14} > 0, \\ \Rightarrow\quad & \|\varphi'(u) - \mu'(u)\|_* \leq c_{14}\|u\|^{p-1} \text{ for all } u \in W_0^{1,p}(\Omega). \end{aligned} \tag{53}$$

From (52), (53) and the invariance of the critical groups in the C^1-topology (see Chang [7] (Corollary 5.1.25, p. 336) and Gasiński-Papageorgiou [17] (Theorem 5.126, p. 836)), we have

$$\begin{aligned} & C_k(\varphi, 0) = C_k(\mu, 0) \text{ for all } k \in \mathbb{N}_0, \\ \Rightarrow\quad & C_k(\varphi, 0) = \delta_{k,d_l}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0 \quad \text{(see (51))}. \end{aligned}$$

□

Remark 2. In this remark, we present an alternative proof of the above result. This new proof illustrates the power of the nonlinear regularity theory and uses the other basic invariance property of critical groups, namely the homotopy invariance property.

So, let $\lambda \in (\widehat{\lambda}_l(2), \widehat{\lambda}_{l+1}(2))$ and consider the C^2-functional $\widehat{\mu}_0 : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\mu}_0(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{\lambda}{2}\|u\|_2^2 \quad \text{for all } u \in H_0^1(\Omega).$$

The choice of λ implies

$$C_k(\widehat{\mu}_0, 0) = \delta_{k,d_l}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0 \tag{54}$$

(see Motreanu-Motreanu-Papageorgiou [25], Theorem 6.51, p. 155). Let $\mu_0 = \widehat{\mu}_0\big|_{W_0^{1,p}(\Omega)}$. Since $W_0^{1,p}(\Omega)$ is dense in $H_0^1(\Omega)$, we have $C_k(\mu_0, 0) = C_k(\widehat{\mu}_0, 0)$ for all $k \in \mathbb{N}_0$ (see Palais [26] and Chang [6] (p. 14)). Hence

$$C_k(\mu_0, 0) = \delta_{k,d_l}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0 \quad \text{(see (54))}. \tag{55}$$

We consider the homotopy

$$h_t(u) = (1-t)\varphi(u) + t\mu(u) \quad \text{for all } (t, u) \in [0,1] \times W_0^{1,p}(\Omega).$$

Suppose we can find $\{t_n\}_{n\in\mathbb{N}} \subseteq [0,1]$ and $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ such that

$$t_n \to t,\ u_n \to u \text{ in } W_0^{1,p}(\Omega) \text{ and } (h_{t_n})'(u_n) = 0 \text{ for all } n \in \mathbb{N}. \tag{56}$$

From (56) we have

$$\langle A_p(u_n), h\rangle + \langle A(u_n), h\rangle = \int_\Omega f(z, u_n) h dz \text{ for all } h \in W_0^{1,p}(\Omega), \text{ all } n \in \mathbb{N}, \quad (57)$$

$$\Rightarrow -\Delta_p u_n(z) - \Delta u_n(z) = f(z, u_n(z)) \text{ for a.a. } z \in \Omega, \quad u_n\big|_{\partial\Omega} = 0, \text{ all } n \in \mathbb{N}.$$

The nonlinear regularity theory (see Lieberman [22]) implies that there exist $\theta \in (0, 1)$ and $c_{15} > 0$ such that

$$u_n \in C_0^{1,\theta}(\overline{\Omega}), \quad \|u_n\|_{C_0^{1,\theta}(\overline{\Omega})} \le c_{15} \quad \text{for all } n \in \mathbb{N}. \quad (58)$$

Let $y_n = \dfrac{u_n}{\|u_n\|}$, $n \in \mathbb{N}$. Then $\|y_n\| = 1$ for all $n \in \mathbb{N}$ and so we may assume that

$$y_n \to y \text{ in } W_0^{1,p}(\Omega) \text{ and } y_n \to y \text{ in } L^p(\Omega). \quad (59)$$

From (57) we have

$$\|u_n\|^{p-2}\langle A_p(y_n), h\rangle + \langle A(y_n), h\rangle = \int_\Omega \frac{N_f(u_n)}{\|u_n\|} h dz \text{ for all } h \in W_0^{1,p}, \text{ all } n \in \mathbb{N} \quad (60)$$

$$\Rightarrow -\|u_n\|^{p-2}\Delta_p y_n(z) - \Delta y_n(z) = \frac{f(z, u_n(z))}{\|u_n\|} \text{ for a.a. } z \in \Omega, \ y_n\big|_{\partial\Omega} = 0. \quad (61)$$

Using (23) we have

$$\begin{aligned} \left|\frac{f(z, u_n(z))}{\|u_n\|}\right| &\le c_5\left[|y_n(z)| + |u_n(z)|^{p-2}|y_n(z)|\right] \\ &\le c_{16}|y_n(z)| \text{ for a.a. } z \in \Omega, \text{ all } n \in \mathbb{N}, \text{ some } c_{16} > 0 \text{ (see (58))}, \\ &\Rightarrow \frac{N_f(u_n)}{\|u_n\|} \in L^\infty(\Omega) \text{ for every } n \in \mathbb{N}. \end{aligned}$$

From (61) and Corollary 8.6, p. 208 of Motreanu-Motreanu-Papageorgiou [25], we can find $c_{17} > 0$ such that

$$\|y_n\|_\infty \le c_{17} \quad \text{for all } n \in \mathbb{N}.$$

The Theorem 1 of Lieberman [22] implies the existence of $\tau \in (0, 1)$ and $c_{18} > 0$ such that

$$y_n \in C_0^{1,\tau}(\overline{\Omega}), \quad \|y_n\|_{C_0^{1,\tau}(\overline{\Omega})} \le c_{18} \text{ for all } n \in \mathbb{N}.$$

Exploiting the compact embedding of $C_0^{1,\tau}(\overline{\Omega})$ into $C_0^1(\overline{\Omega})$ and using (59), we have

$$y_n \to y \text{ in } C_0^1(\overline{\Omega}) \text{ and } \|y\| = 1. \quad (62)$$

Passing to the limit as $n \to +\infty$ in (60) and using (62), we obtain

$$\langle A(y), h\rangle = \int_\Omega f'_x(z, 0) y h dz \quad \text{for all } y \in W_0^{1,p}(\Omega). \quad (63)$$

From the density of $W_0^{1,p}(\Omega)$ in $H_0^1(\Omega)$, we infer that (63) is valid for all $h \in H_0^1(\Omega)$. Hence

$$-\Delta y(z) = f'_x(z, 0) y(z) \quad \text{for a.a. } z \in \Omega, \quad y\big|_{\partial\Omega} = 0. \quad (64)$$

From the strict monotonicity of the eigenvalues on the weights (a consequence of the UCP, see de Figueiredo-Gossez [12], Proposition 1), from (64) and hypothesis $H(f)\,(iv)$ we infer that $y = 0$, which contradicts (62). Therefore (56) can not occur and we can use the homotopy invariance property of critical groups (see [17], p. 836) and have

$$\begin{aligned} & C_k(\varphi,0) = C_k(\mu_0,0) \quad \text{for all } k\in\mathbb{N}_0, \\ \Rightarrow \quad & C_k(\varphi,0) = \delta_{k,d_l}\mathbb{Z} \quad \text{for all } k\in\mathbb{N}_0 \quad \text{(see (55))}. \end{aligned}$$

This complete the alternative proof of Proposition 9.

Now we can produce a nodal solution for problem (1).

Proposition 10. *If hypotheses $H(f)$ hold, then problem (1) admits a nodal solution $y_0 \in int_{C_0^1(\overline{\Omega})}[v^*,u^*]$.*

Proof. Let $u^* \in \text{int}\, C_+$ and $v^* \in -\text{int}\, C_+$ be the two extremal constant sign solutions of problem (1) produced in Proposition 8. We introduce the following Carathéodory function

$$\gamma(z,x) = \begin{cases} f(z,v^*(z)) & \text{if } x < v^*(z), \\ f(z,x) & \text{if } v^*(z)\le x\le u^*(z), \\ f(z,u^*(z)) & \text{if } u^*(z) < x. \end{cases} \tag{65}$$

We set $\Gamma(z,x) = \int_0^x \gamma(z,s)ds$ and consider the C^1-functional $\chi : W_0^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\chi(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega \Gamma(z,u)dz \quad \text{for all } u\in W_0^{1,p}(\Omega).$$

We also introduce the positive and negative truncations of $\gamma(z,\cdot)$, namely the Carathéodory functions $\gamma_\pm(z,x) = \gamma(z,\pm x^\pm)$. We set $\Gamma_\pm(z,x) = \int_0^x \gamma_\pm(z,s)ds$ and consider the C^1-functional $\chi_\pm : W_0^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\chi_\pm(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega \Gamma_\pm(z,u)dz \quad \text{for all } u\in W_0^{1,p}(\Omega).$$

Using (65), as in the proof of Proposition 4, we see that

$$K_\chi \subseteq [v^*,u^*]\cap C_0^1(\overline{\Omega}), \quad K_{\chi_+}\subseteq [0,u^*]\cap C_+, \quad K_{\chi_-}\subseteq [v^*,0]\cap(-C_+).$$

The extremality of u^* and v^* implies that

$$K_\chi \subseteq [v^*,u^*]\cap C_0^1(\overline{\Omega}), \quad K_{\chi_+} = \{0,u^*\}, \quad K_{\chi_-} = \{0,v^*\}. \tag{66}$$

Claim: $u^*\in \text{int}\, C_+$ and $v^*\in -\text{int}\, C_+$ are local minimizers of χ.

Evidently χ_+ is coercive (see (65)). Also, it is sequentially weakly lower semicontinuous. So, we can find $\widehat{u}^*\in W_0^{1,p}(\Omega)$ such that

$$\chi_+(\widehat{u}^*) = \inf\left[\chi_+(u) \,:\, u\in W_0^{1,p}(\Omega)\right]. \tag{67}$$

Since $u^*\in \text{int}\, C_+$, using Proposition 2.1 of Marano-Papageorgiou [23], we can find $t\in(0,1)$ small such that

$$0\le t\widehat{u}_1(2)(z)\le u^*(z) \quad \text{for all } z\in\overline{\Omega}.$$

Using hypothesis $H(f)\,(iv)$, as in the proof of Proposition 4, we have

$$\begin{aligned} & \chi_+(t\widehat{u}_1(2)) < 0, \\ \Rightarrow \quad & \chi_+(\widehat{u}^*) < 0 = \chi_+(0) \quad \text{and so} \quad \widehat{u}^*\neq 0. \end{aligned} \tag{68}$$

From (66), (67), (68) we infer that

$$\widehat{u}^* = u^* \in \text{int}\, C_+.$$

Clearly we have

$$\begin{aligned}
& \chi_+\Big|_{[0,u^*]} = \chi\Big|_{[0,u^*]} \\
\Rightarrow \quad & u^* \text{ is a local } C_0^1(\overline{\Omega})\text{-minimizer of } \chi, \\
\Rightarrow \quad & u^* \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \chi \text{ (see Proposition 1).}
\end{aligned}$$

Similarly for $v^* \in -\text{int}\, C_+$ using this time the functional χ_-. This proves Claim.

We can always assume that $K_\chi \subseteq C_0^1(\overline{\Omega})$ is finite. Otherwise on account of (66), we already have an infinity of smooth nodal solutions of (1) and so we are done. Using the Claim and Proposition 5.42, p. 119, of Motreanu-Motreanu-Papageorgiou [25], we can find $y_0 \in K_\chi \subseteq [v^*, u^*] \cap C_0^1(\overline{\Omega})$ of mountain pass type. So, we have

$$C_1(\chi, y_0) \neq 0 \tag{69}$$

(see Corollary 6.81, p. 168, of Motreanu-Motreanu-Papageorgiou [25]). From Proposition 9, we have

$$C_k(\varphi, 0) = \delta_{k,d_l}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0.$$

Since $\varphi'\Big|_{[v^*,u^*]} = \chi'\Big|_{[v^*,u^*]}$ (see (65)), we have

$$\begin{aligned}
& C_k(\varphi, 0) = C_k(\chi, 0) \quad \text{for all } k \in \mathbb{N}_0, \\
\Rightarrow \quad & C_k(\chi, 0) = \delta_{k,d_l}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0.
\end{aligned} \tag{70}$$

Note that $d_l \geq 2$ (recall $l \geq 2$). Hence from (69) and (70) we infer that

$$\begin{aligned}
& y_0 \neq 0, \\
\Rightarrow \quad & y_0 \in [v^*, u^*] \cap C_0^1(\overline{\Omega}) \text{ is nodal.}
\end{aligned}$$

Moreover, as in the proof of Proposition 4, via the tangency principle, we have

$$y_0 \in \text{int}_{C_0^1(\overline{\Omega})}[v^*, u^*].$$

□

5. **Multiplicity Theorem.** In this section, we produce a sixth nontrivial smooth solution and we have the complete multiplicity theorem for problem (1) (six solutions theorem).

To this end we will need the following fact about the critical groups of φ at infinity.

Proposition 11. *If hypotheses $H(f)$ hold, then $C_m(\varphi, \infty) \neq 0$.*

Proof. Let $\lambda \in (\widehat{\lambda}_m(p), \widehat{\lambda}_{m+1}(p))$, $\lambda \notin \widehat{\sigma}(p)$ and consider the C^2-functional $\psi : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi(u) = \frac{1}{p}\|\nabla u\|_p^p - \frac{\lambda}{p}\|u\|_p^p \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

We consider the following homotopy

$$h_t(u) = (1-t)\varphi(u) + t\psi(u) \quad \text{for all } (t, u) \in [0,1] \times W_0^{1,p}(\Omega).$$

Claim: There exist $\beta \in \mathbb{R}$ and $\varepsilon_0 > 0$ such that

$$h_t(u) \leq \beta \Rightarrow (1 + \|u\|)\|(h_t)'(u)\|_* \geq \varepsilon_0 \quad \text{for all } t \in [0,1].$$

We argue indirectly. So, suppose that the Claim is not true. Since $(t,u) \to h_t(u)$ is bounded (that is, it maps bounded sets to bounded sets), we can find $\{t_n\}_{n\in\mathbb{N}} \subseteq [0,1]$ and $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ such that

$$t_n \to t, \|u_n\| \to +\infty, h_{t_n}(u_n) \to -\infty, (1+\|u_n\|)(h_{t_n})'(u_n) \to 0 \text{ in } W^{-1,p'}(\Omega). \quad (71)$$

For all $n \in \mathbb{N}$, we have

$$\left| \langle A_p(u_n), h\rangle + (1-t_n)\langle A(u_n), h\rangle - (1-t_n)\int_\Omega f(z,u_n)\, h\, dz - t_n\lambda \int_\Omega |u_n|^{p-2}u_n h\, dz \right|$$
$$\leq \frac{\varepsilon_n \|h\|}{1+\|u_n\|}, \text{ for all } h \in W_0^{1,p}(\Omega), \text{ with } \varepsilon_n \to 0^+. \quad (72)$$

Let $y_n = \dfrac{u_n}{\|u_n\|}$, $n \in \mathbb{N}$. Then $\|y_n\| = 1$ for all $n \in \mathbb{N}$. So, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega) \text{ and } y_n \to y \text{ in } L^p(\Omega). \quad (73)$$

From (72) we obtain

$$\left| \langle A_p(y_n), h\rangle + \frac{1-t_n}{\|u_n\|^{p-2}}\langle A(y_n), h\rangle - (1-t_n)\int_\Omega \frac{N_f(u_n)}{\|u_n\|^{p-1}}\, h\, dz - t_n\lambda \int_\Omega |y_n|^{p-2}y_n h\, dz \right|$$
$$\leq \frac{\varepsilon_n \|h\|}{(1+\|u_n\|)\|u_n\|^{p-1}}, \text{ for all } n \in \mathbb{N}. \quad (74)$$

Evidently

$$\left\{ \frac{N_f(u_n)}{\|u_n\|^{p-1}} \right\}_{n\in\mathbb{N}} \subseteq L^{p'}(\Omega) \text{ is bounded (see (23))}.$$

So, by passing to a subsequence if necessary and using hypothesis $H(f)\,(iii)$, we have

$$\frac{N_f(u_n)}{\|u_n\|^{p-1}} \xrightarrow{w} \widehat{\lambda}_m(p)|y|^{p-2}y \text{ in } L^{p'}(\Omega) \text{ as } n \to +\infty \text{ (see [1])}. \quad (75)$$

If in (74) we choose $h = y_n - y \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (71), (73), (75) and the fact that $p > 2$, then

$$\lim_{n\to+\infty} \langle A_p(y_n), y_n - y\rangle = 0,$$
$$\Rightarrow \quad y_n \to y \text{ in } W_0^{1,p}(\Omega) \text{ (see Proposition 3) and so } \|y\| = 1. \quad (76)$$

Therefore, if in (74) we pass to the limit as $n \to +\infty$ and use (75), (76), then

$$\langle A_p(y), h\rangle = \int_\Omega \left[(1-t)\widehat{\lambda}_m(p) + t\lambda\right] |y|^{p-2}yh\, dz \quad \text{for all } h \in W_0^{1,p}(\Omega),$$
$$\Rightarrow \quad -\Delta_p y(z) = \widehat{\lambda}_t |y(z)|^{p-2}y(z) \text{ for a.a. } z \in \Omega, \quad y\Big|_{\partial\Omega} = 0, \quad (77)$$
$$\text{with } \widehat{\lambda}_t = (1-t)\widehat{\lambda}_m(p) + t\lambda.$$

If $\widehat{\lambda}_t \notin \widehat{\sigma}(p)$, then from (77) it follows that $y \equiv 0$, which contradicts (76). Suppose that $\widehat{\lambda}_t \in \widehat{\sigma}(p)$. From (76) we have $y \neq 0$. Hence there exists $E \subseteq \Omega$ measurable with $|E|_N > 0$ such that

$$|u_n(z)| \to +\infty \quad \text{for all } z \in E.$$

Then (74), hypothesis $H(f)\,(iii)$ and Fatou's lemma imply

$$\begin{aligned} &\int_E [f(z,u_n)u_n - pF(z,u_n)]dz \to +\infty \quad \text{as } n\to+\infty, \\ \Rightarrow\ &\int_\Omega [f(z,u_n)u_n - pF(z,u_n)]dz \to +\infty \quad \text{as } n\to+\infty \quad (\text{see (23)}). \end{aligned} \tag{78}$$

From (71) we see that we can find $n_0 \in \mathbb{N}$ such that

$$\|\nabla u_n\|_p^p + (1-t_n)\frac{p}{2}\|\nabla u_n\|_2^2 - (1-t_n)\int_\Omega pF(z,u_n)dz - t_n\lambda\|u_n\|_p^p \le 0 \tag{79}$$

for all $n \ge n_0$. On the other hand from (72) with $h = u_n \in W_0^{1,p}(\Omega)$, we have

$$-\|\nabla u_n\|_p^p - (1-t_n)\|\nabla u_n\|_2^2 + (1-t_n)\int_\Omega f(z,u_n)u_n dz + t_n\lambda\|u_n\|_p^p \le \varepsilon_n \tag{80}$$

for all $n \in \mathbb{N}$. Adding (79) and (80) and recalling that $p > 2$, we have

$$(1-t_n)\int_\Omega [f(z,u_n)u_n - pF(z,u_n)]dz \le c_{19} \quad \text{for some } c_{19} > 0, \text{ all } n \ge n_0. \tag{81}$$

We show that $t < 1$. Indeed, if $t = 1$, then (77) becomes

$$\begin{aligned} &-\Delta_p y(z) = \lambda|y(z)|^{p-2}y(z) \quad \text{for a.a. } z\in\Omega, \quad y|_{\partial\Omega} = 0, \\ \Rightarrow\quad & y = 0 \ (\text{since } \lambda \notin \widehat{\sigma}(p)), \text{ a contradiction to (76)}. \end{aligned}$$

Then from (81) it follows that

$$\int_\Omega [f(z,u_n)u_n - pF(z,u_n)]dz \le c_{20} \quad \text{for some } c_{20} > 0, \text{ all } n \ge n_0. \tag{82}$$

Comparing (78) and (82), we have a contradiction. This proves the Claim.

From the Claim and Theorem 5.1.21, p. 334 of Chang [7] (see also Liang-Su [21]), we have

$$\begin{aligned} & C_k(h_0(\cdot),\infty) = C_k(h_1(\cdot),\infty) \quad \text{for all } k \in \mathbb{N}_0, \\ \Rightarrow\quad & C_k(\varphi,\infty) = C_k(\psi,\infty) \quad \text{for all } k\in\mathbb{N}_0. \end{aligned} \tag{83}$$

For $\rho > 0$, we consider the following two sets

$$\begin{aligned} C_\rho &= \{u \in W_0^{1,p}(\Omega) \,:\, \|u\|^p < \lambda\|u\|_p^p,\ \|u\| = \rho\}, \\ D &= \{u \in W_0^{1,p}(\Omega) \,:\, \|u\|^p \ge \lambda\|u\|_p^p\}. \end{aligned}$$

Since $\lambda \in (\widehat{\lambda}_m(p), \widehat{\lambda}_{m+1}(p))$, we have

$$\operatorname{ind} C_\rho = \operatorname{ind} D = m$$

with $\operatorname{ind}(\cdot)$ being the Fadell-Rabinowitz cohomological index (see [9]). Then, by Theorems 3.2 and 3.6 of Cingolani-Degiovanni [9], we have

$$C_m(\psi, 0) \neq 0. \tag{84}$$

But $K_\psi = \{0\}$ (since $\lambda \notin \widehat{\sigma}(p)$). Hence

$$C_k(\psi,0) = C_k(\psi,\infty) \quad \text{for all } k \in \mathbb{N}_0,$$

(see Proposition 6.61(c), p. 160, of Motreanu-Motreanu-Papageorgiou [25]), implies

$$C_m(\varphi,\infty) \neq 0 \quad (\text{see (83), (84)}).$$

□

Proposition 12. *If hypotheses $H(f)$ hold and K_φ is finite, then $C_k(\varphi, y_0) = C_k(\varphi, \widehat{u}) = C_k(\varphi, \widehat{v}) = \delta_{k,1}\mathbb{Z}$ for all $k \in \mathbb{N}_0$.*

Proof. Let $\chi \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$ be as in the proof of Proposition 10 and consider the homotopy

$$h_t(u) = (1-t)\varphi(u) + t\chi(u) \quad \text{for all } (t,u) \in [0,1] \times W_0^{1,p}(\Omega).$$

Suppose we could find $\{t_n\}_{n\in\mathbb{N}} \subseteq [0,1]$ and $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ such that

$$t_n \to t,\ u_n \to y_0 \text{ in } W_0^{1,p}(\Omega) \text{ and } (h_{t_n})'(u_n) = 0 \quad \text{for all } n \in \mathbb{N}. \tag{85}$$

From the equality in (85), we have

$$\begin{aligned} &\langle A_p(u_n), h\rangle + \langle A(u_n), h\rangle = (1-t_n)\int_\Omega f(z,u_n)\, h\, dz + t_n \int_\Omega \gamma(z,u_n) dz \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } h \in W_0^{1,p}(\Omega), \\ \Rightarrow\quad & -\Delta_p u_n(z) - \Delta u_n(z) = (1-t_n) f(z,u_n(z)) + t_n \gamma(z,u_n(z)) \text{ for a.a. } z \in \Omega, \\ &\qquad\qquad\qquad\qquad\qquad\qquad u_n\Big|_{\partial\Omega} = 0 \text{ for all } n \in \mathbb{N}. \end{aligned} \tag{86}$$

From (86) and Theorem 7.1, p. 286 of Ladyzhenskaya-Uraltseva [19], we have that

$$\|u_n\|_\infty \le c_{21} \quad \text{for some } c_{21} > 0, \text{ all } n \in \mathbb{N}.$$

Invoking Theorem 1 of Lieberman [22], we find $\tau \in (0,1)$ and $c_{22} > 0$ such that

$$u_n \in C_0^{1,\tau}(\overline{\Omega}), \quad \|u_n\|_{C_0^{1,\tau}(\overline{\Omega})} \le c_{22} \quad \text{for all } n \in \mathbb{N}. \tag{87}$$

From (85), (87) and the compact embedding of $C_0^{1,\tau}(\overline{\Omega})$ into $C^1(\overline{\Omega})$, we infer that

$$\begin{aligned} & u_n \to y_0 \text{ in } C_0^1(\overline{\Omega}), \\ \Rightarrow\quad & u_n \in [v^*, u^*] \quad \text{for all } n \ge n_0 \text{ (see Proposition 10)}. \end{aligned}$$

Since $\varphi'\Big|_{[v^*,u^*]} = \chi'\Big|_{[v^*,u^*]}$ (see (65)), it follows that

$$\{u_n\}_{n\ge n_0} \subseteq K_\varphi,$$

which contradicts the hypothesis that K_φ is finite.

Hence (85) can not hold and so the homotopy invariance of critical groups (see Gasiński-Papageorgiou [17], Theorem 5.125, p. 836), implies that

$$C_k(\varphi, y_0) = C_k(\chi, y_0) \quad \text{for all } k \in \mathbb{N}_0. \tag{88}$$

But from (69) we know that

$$\begin{aligned} & C_1(\chi, y_0) \ne 0, \\ \Rightarrow\quad & C_1(\varphi, y_0) \ne 0 \quad \text{(see (88))}. \end{aligned} \tag{89}$$

We know that $\varphi \in C^2(W_0^{1,p}(\Omega), \mathbb{R})$. Then from (89) and Papageorgiou-Radulešcu [27] (see proof of Proposition 3.5, Claim 3), we have

$$C_k(\varphi, y_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0.$$

Similarly for $\widehat{u} \in \text{int } C_+$ and $\widehat{v} \in -\text{int } C_+$, using the homotopies

$$(\widetilde{h}_+)_t(u) = (1-t)\varphi(u) + t\widetilde{\varphi}_+(u),$$
$$(\widetilde{h}_-)_t(u) = (1-t)\varphi(u) + t\widetilde{\varphi}_-(u) \quad \text{for all } (t,u) \in [0,1] \times W_0^{1,p}(\Omega).$$

$\square$

Now we are ready for the multiplicity theorem producing six nontrivial smooth solutions for problem (1). Note that we provide sign information for five of these solutions and we also order them.

Theorem 5.1. *If hypotheses $H(f)$ hold, then problem (1) admits at least six nontrivial smooth solutions*

$$u_0, \widehat{u} \in \textit{int } C_+ \quad \textit{with } \widehat{u} - u_0 \in \textit{int } C_+,$$
$$v_0, \widehat{v} \in -\textit{int } C_+ \quad \textit{with } v_0 - \widehat{v} \in \textit{int } C_+,$$
$$y_0 \in \textit{int}_{C_0^1(\overline{\Omega})}[v_0, u_0] \textit{ nodal},$$
$$\widehat{y} \in C_0^1(\overline{\Omega}) \setminus \{0\}.$$

Proof. From Proposition 4, we know that $u_0 \in \text{int } C_+$

- is a minimizer of $\widehat{\varphi}_+$;
- $u_0 \in \text{int}_{C_0^1(\overline{\Omega})}[0, w_+]$.

Since $\varphi\Big|_{[0,w_+]} = \widehat{\varphi}_+\Big|_{[0,w_+]}$ (see (6)), it follows that

$$u_0 \text{ is a local } C_0^1(\overline{\Omega})\text{-minimizer of } \varphi,$$
$$\Rightarrow \quad u_0 \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \varphi \text{ (see Proposition 1).}$$

Similarly we have that

$$v_0 \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \varphi.$$

Therefore we have

$$C_k(\varphi, u_0) = C_k(\varphi, v_0) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{90}$$

From Proposition 12, we have

$$C_k(\varphi, y_0) = C_k(\varphi, \widehat{u}) = C_k(\varphi, \widehat{v}) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{91}$$

Also, from Proposition 9, we know that

$$C_k(\varphi, 0) = \delta_{k,d_l}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{92}$$

Since $C_m(\varphi, \infty) \neq 0$ (see Proposition 11), we can find $\widehat{y} \in K_\varphi \subseteq C_0^1(\overline{\Omega})$ such that

$$C_m(\varphi, \widehat{y}) \neq 0. \tag{93}$$

(see Motreanu-Motreanu-Papageorgiou [25], Proposition 6.61, p. 160). Since $m \neq d_l$ (see hypothesis $H(f)\,(iv)$), from (92), (93) we infer that

$$\widehat{y} \neq 0.$$

Also, from (90), (91) and (93), we see that

$$\widehat{y} \notin \{u_0, v_0, y_0, \widehat{u}, \widehat{v}\}.$$

We conclude that $\widehat{y} \in C_0^1(\overline{\Omega})$ is the sixth nontrivial smooth solution of problem (1).

$\square$

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E-mail address: npapg@math.ntua.gr
E-mail address: calogero.vetro@unipa.it
E-mail address: francesca.vetro@unipa.it