

# FRAMES AND WEAK FRAMES FOR UNBOUNDED OPERATORS

GIORGIA BELLOMONTE AND ROSARIO CORSO

ABSTRACT. In 2012 Găvruta introduced the notions of  $K$ -frame and of atomic system for a linear bounded operator  $K$  in a Hilbert space  $\mathcal{H}$ , in order to decompose its range  $\mathcal{R}(K)$  with a frame-like expansion. In this article we revisit these concepts for an unbounded and densely defined operator  $A : \mathcal{D}(A) \rightarrow \mathcal{H}$  in two different ways. In one case we consider a non-Bessel sequence where the coefficient sequence depends continuously on  $f \in \mathcal{D}(A)$  with respect to the norm of  $\mathcal{H}$ . In the other case we consider a Bessel sequence and the coefficient sequence depends continuously on  $f \in \mathcal{D}(A)$  with respect to the graph norm of  $A$ .

## 1. INTRODUCTION

The notion of frame in Hilbert spaces dates back to 1952 when it was introduced in the pioneering paper of J. Duffin and A.C. Schaffer [21], and was resumed in 1986 by I. Daubechies, A. Grossman and Y. Meyer in [19]. This notion is a generalization of that of orthonormal bases. Indeed, let  $\mathcal{H}$  be a Hilbert space with inner product  $\langle \cdot | \cdot \rangle$  and norm  $\| \cdot \|$ , a frame is a sequence in  $\mathcal{H}$  that allows every element of  $\mathcal{H}$  to be written as a stable, potentially infinite, linear combination of the elements of the sequence. The uniqueness of the decomposition is lost, in general, and this gives a certain freedom in the choice of the coefficients in the expansion which is in fact a good quality in applications.

L. Găvruta introduced in [23] the notion of atomic system for a linear, *bounded operator*  $K$  defined everywhere on  $\mathcal{H}$ . This notion generalizes frames and also *atomic systems for subspaces* in [22]. More precisely,  $\{g_n\}_{n \in \mathbb{N}} \subset \mathcal{H}$  is an atomic system for  $K$  if there exists  $\gamma > 0$  such that for every  $f \in \mathcal{H}$  there exists  $a_f = \{a_n(f)\}_{n \in \mathbb{N}} \in \ell^2$ , the usual Hilbert space of complex sequences, such that  $\|a_f\| \leq$

---

*Date:* January 29, 2020.

*2010 Mathematics Subject Classification.* 42C15, 47A05, 47A63, 41A65.

*Key words and phrases.*  $A$ -frames, weak  $A$ -frames, atomic systems, reconstruction formulas, unbounded operators.

$\gamma\|f\|$  and

$$Kf = \sum_{n=1}^{\infty} a_n(f)g_n.$$

This notion turns out to be equivalent to that of  $K$ -frame [23]; i.e. a sequence  $\{g_n\}_{n \in \mathbb{N}} \subset \mathcal{H}$  satisfying

$$(1.1) \quad \alpha\|K^*f\|^2 \leq \sum_{n=1}^{\infty} |\langle f|g_n \rangle|^2 \leq \beta\|f\|^2, \quad \forall f \in \mathcal{H},$$

for some constants  $\alpha, \beta > 0$ , where  $K^*$  is the adjoint of  $K$ . The main theorem in [23] states that if  $\{g_n\}_{n \in \mathbb{N}}$  is a  $K$ -frame, then there exists a Bessel sequence  $\{h_n\}_{n \in \mathbb{N}}$  in  $\mathcal{H}$ , i.e.  $\sum_{n=1}^{\infty} |\langle f|h_n \rangle|^2 \leq \gamma\|f\|^2$  for all  $f \in \mathcal{H}$  and some  $\gamma > 0$ , such that

$$Kf = \sum_{n=1}^{\infty} \langle f|h_n \rangle g_n, \quad \forall f \in \mathcal{H}.$$

This generalization of frames allows to write every element of  $\mathcal{R}(K)$ , the range of  $K$ , which need not be closed, as a superposition of the elements  $\{g_n\}_{n \in \mathbb{N}}$  which do not necessarily belong to  $\mathcal{R}(K)$ . A question can arise at this point: why develop a theory of  $K$ -frames since there already exists a well-studied theory of frames that reconstruct the entire space  $\mathcal{H}$ ? The answer is that in a specific situation we are looking for sequences with some properties, then we may not find any possible frame, but we may find a  $K$ -frame because this notion is weaker and we could want to decompose just  $\mathcal{R}(K)$ .

Let us see a concrete example: let  $\mathcal{H} = L^2(\mathbb{R})$ ,  $\phi \in L^2(\mathbb{R})$  and consider the translation system  $\{\phi_n(x)\}_{n \in \mathbb{Z}} := \{\phi(x - cn)\}_{n \in \mathbb{Z}}$  and the Gabor system  $G(\phi, a, b) = \{\phi_{m,n}(x)\}_{m,n \in \mathbb{Z}} := \{e^{2\pi imbx} \phi(x - na)\}$  with  $a, b, c > 0$ . As it is known [16], there is no hope to have  $\{\phi_n\}_{n \in \mathbb{Z}}$  (or  $\{\phi_{m,n}\}_{m,n \in \mathbb{Z}}$  with  $ab > 1$ ) as a frame, whatever  $\phi$  is in  $L^2(\mathbb{R})$ . But if  $K$  is a bounded operator on  $L^2(\mathbb{R})$  and  $\mathcal{R}(K) \neq \mathcal{H}$ , then we might find  $\phi$  such that one of the previous sequences is a  $K$ -frame.

We have taken inspiration to [31, Example 1] for the following simple example. We write  $\hat{f}$  for the Fourier transform of  $f$ , which is defined for  $f \in L^1(\mathbb{R})$  as  $\hat{f}(\gamma) := \int_{\mathbb{R}} f(x)e^{-2\pi i x \gamma} dx$ ,  $\gamma \in \mathbb{R}$ , and it is extended to  $f \in L^2(\mathbb{R})$  in a standard

way. Let  $PW_{\frac{1}{4}} = \{f \in L^2(\mathbb{R}) : \text{supp}(\widehat{f}) \subset [-\frac{1}{4}, \frac{1}{4}]\}$ . If  $\phi \in L^2(\mathbb{R})$  is such that

$$\widehat{\phi}(\gamma) = \begin{cases} 1 & \text{for } |\gamma| \leq \frac{1}{4} \\ \text{decaying to zero continuously} & \text{for } \frac{1}{4} \leq |\gamma| < \frac{1}{2} \\ 0 & \text{for } \frac{1}{2} \leq |\gamma|, \end{cases}$$

then we have for  $f \in PW_{\frac{1}{4}}$

$$\widehat{f} = \widehat{\phi}\widehat{f} = \widehat{\phi} \sum_{n \in \mathbb{Z}} \langle \widehat{f} | e_n \rangle e_n = \sum_{n \in \mathbb{Z}} \langle \widehat{f} | e_n \rangle \widehat{\phi} e_n = \sum_{n \in \mathbb{Z}} \langle \widehat{f} | f_n \rangle \widehat{\phi} e_n,$$

where

$$e_n(\gamma) = \begin{cases} e^{2\pi i n \gamma} & \text{for } |\gamma| \leq \frac{1}{2} \\ 0 & \text{for } |\gamma| > \frac{1}{2}, \end{cases} \quad \text{and} \quad f_n(\gamma) = \begin{cases} e^{2\pi i n \gamma} & \text{for } |\gamma| \leq \frac{1}{4} \\ 0 & \text{for } |\gamma| > \frac{1}{4}. \end{cases}$$

Thus  $f = \sum_{n \in \mathbb{Z}} \langle f | \psi_n \rangle \phi_n$  for  $f \in PW_{\frac{1}{4}}$  where  $\phi_n$  is the inverse Fourier transform of  $\widehat{\phi} e_{-n}$ , i.e.  $\phi_n(x) := \phi(x - n)$ , and  $\psi_n := \widetilde{f_{-n}}$  is the inverse Fourier transform of  $f_{-n}$ , i.e.

$$\psi_n(x) = \widetilde{f_{-n}}(x) = \begin{cases} 4 \frac{\sin(\frac{\pi}{2}(x-n))}{\pi(x-n)} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

If  $P$  is the orthogonal projection of  $L^2(\mathbb{R})$  onto  $PW_{\frac{1}{4}}$ , then we can write

$$Pf = \sum_{n \in \mathbb{Z}} \langle Pf | \psi_n \rangle \phi_n = \sum_{n \in \mathbb{Z}} \langle f | \psi_n \rangle \phi_n, \quad \forall f \in L^2(\mathbb{R})$$

since  $\psi_n \in PW_{\frac{1}{4}}$ . In conclusion,  $\{\phi_n\}_{n \in \mathbb{Z}}$  is a  $K$ -frame with  $K = P$  (it fulfills (1.1) as one can easily see by taking the Fourier transform) but of course  $\{\phi_n\}_{n \in \mathbb{Z}}$  is not contained in  $\mathcal{R}(P) = PW_{\frac{1}{4}}$ . Moreover, it is not even a *frame sequence*, i.e. a frame for its closed span (indeed  $\{\phi_n\}_{n \in \mathbb{Z}}$  does not satisfy [16, Theorem 9.2.5]).

In the literature there are many further studies or variations of  $K$ -frames (see for example [24, 26, 29, 32, 33, 36] and the references therein).

In this paper we deal with two different generalizations of [23] which involve a *closed densely defined operator*  $A$  on  $\mathcal{H}$ . When the operator is bounded, all definitions do coincide with those in [23]. To justify our two different approaches, let us consider a Bessel sequence  $\{g_n\}_{n \in \mathbb{N}} \subset \mathcal{H}$  and assume that, for  $f \in \mathcal{D}(A)$ , the domain of  $A$ , we have a decomposition

$$Af = \sum_{n=1}^{\infty} a_n(f) g_n,$$

for some  $a_f := \{a_n(f)\}_{n \in \mathbb{N}} \in \ell^2$ ; in particular, this situation appears when  $\{g_n\}_{n \in \mathbb{N}}$  is a frame. If  $A$  is unbounded, then the coefficients sequence  $a_f$  *can not depend continuously on  $f$* , i.e. it can not exist  $\gamma > 0$  such that  $\|a_f\| \leq \gamma \|f\|$  for every  $f \in \mathcal{D}(A)$ ; this fact may represent another issue when we want to decompose  $\mathcal{R}(A)$  by a frame.

For these reasons, we develop two approaches where either the sequence  $\{g_n\}_{n \in \mathbb{N}}$  or the coefficients sequence  $a_f$  is what represents the unboundedness of  $A$ . To go into more details, in the first case we consider a non-Bessel sequence  $\{g_n\}_{n \in \mathbb{N}}$  but the coefficients depend continuously on  $f \in \mathcal{D}(A)$ . In the second case, we take a Bessel sequence  $\{g_n\}_{n \in \mathbb{N}}$  and coefficients depending continuously on  $f \in \mathcal{D}(A)$  only in the graph topology of  $A$ , which is stronger than the one of  $\mathcal{H}$  when  $A$  is unbounded.

The paper is organized as follows. After some preliminaries, see Section 2, we introduce in Section 3, the notions of *weak  $A$ -frame* and *weak atomic system* for  $A$  (Definitions 3.1 and 3.6, respectively), where  $A$  is a, possibly unbounded, densely defined operator. The word *weak* is due to the fact that the decomposition of  $\mathcal{R}(A)$ , with  $A$  also closable, holds only in a weak sense, in general; i.e., we find a Bessel sequence  $\{t_n\}_{n \in \mathbb{N}}$  of  $\mathcal{H}$  such that

$$\langle Af|u \rangle = \sum_{n=1}^{\infty} \langle f|t_n \rangle \langle g_n|u \rangle \quad \forall f \in \mathcal{D}(A), u \in \mathcal{D}(A^*)$$

see Theorem 3.10. Like in the bounded case (see [33, Lemma 2.2]), we have also

$$A^*u = \sum_{n=1}^{\infty} \langle u|g_n \rangle t_n, \quad \forall u \in \mathcal{D}(A^*),$$

and thus we note a change of the point of view: a weak  $A$ -frame may be used to get a strong decomposition of  $A^*$  rather than  $A$ .

In Section 4 we face our second approach, giving the general notions of *atomic system* for  $A$  and  *$A$ -frame*, see Subsection 4.1, where  $A$  is a, possibly unbounded, closed densely defined operator. Denote by  $\langle \cdot | \cdot \rangle_A$  the inner product which induces the graph norm  $\|\cdot\|_A$  of  $A$ . The resulting decomposition is

$$Af = \sum_{n=1}^{\infty} \langle f|k_n \rangle_A g_n \quad \forall f \in \mathcal{D}(A),$$

for some Bessel sequence  $\{k_n\}_{n \in \mathbb{N}}$  of the Hilbert space  $\mathcal{D}(A)[\|\cdot\|_A]$ , see Corollary 4.8. Actually, this second approach is a particular case of  $K$ -frames, in the

Găvruta-like sense, where  $K \in \mathcal{B}(\mathcal{J}, \mathcal{H})$  is a bounded operator between two different Hilbert spaces  $\mathcal{J}$  and  $\mathcal{H}$ , see Section 4. Indeed, for a densely defined closed operator  $A$  on  $\mathcal{H}$  we take  $K = A$  and  $\mathcal{J} = \mathcal{D}(A)[\|\cdot\|_A]$ , see Corollary 4.8.

Throughout the paper we give some examples of weak  $A$ -frames or  $A$ -frames that can be obtained from frames or that involve Gabor or wavelets systems.

## 2. PRELIMINARIES

In the paper we consider an infinite dimensional Hilbert space  $\mathcal{H}$  with inner product  $\langle \cdot | \cdot \rangle$  and norm  $\|\cdot\|$ . The term operator is used for a linear mapping. Given an operator  $F$ , we denote its domain by  $\mathcal{D}(F)$ , its range by  $\mathcal{R}(F)$  and its adjoint by  $F^*$ , if  $F$  is densely defined. By  $\mathcal{B}(\mathcal{H})$  we denote the set of bounded operators with domain  $\mathcal{H}$  and we indicate by  $\|F\|$  the usual norm of the operator  $F \in \mathcal{B}(\mathcal{H})$ . In some examples we need the usual Hilbert spaces  $L^2(0, 1)$ ,  $L^2(\mathbb{R})$  and the Sobolev spaces, denoted with standard notations,  $H^1(0, 1)$ ,  $H_0^1(0, 1)$ ,  $H^1(\mathbb{R})$ , see [34, Section 1.3]. As usual, we will indicate by  $\ell^2$  the Hilbert space consisting of all sequences  $x := \{x_n\}_{n \in \mathbb{N}}$  satisfying  $\sum_{n=1}^{\infty} |x_n|^2 < \infty$  with norm  $\|x\|_2 = (\sum_{n=1}^{\infty} |x_n|^2)^{1/2}$ .

We will say that a series  $\sum_{n=1}^{\infty} g_n$ , with  $\{g_n\}_{n \in \mathbb{N}} \subset \mathcal{H}$ , is convergent to  $g$  in  $\mathcal{H}$  if  $\lim_{n \rightarrow \infty} \|\sum_{k=1}^n g_k - g\| = 0$ . We will write  $\{g_n\}$  to mean a sequence  $\{g_n\}_{n \in \mathbb{N}}$  of elements of  $\mathcal{H}$ . For the following definitions the reader could refer e.g. to [1, 3, 15, 16, 25, 27].

A sequence  $\{g_n\}$  of elements in  $\mathcal{H}$  is a *Bessel sequence of  $\mathcal{H}$*  if any of the following equivalent conditions are satisfied, see [16, Corollary 3.2.4]

- i) there exists a constant  $\beta > 0$  such that  $\sum_{n=1}^{\infty} |\langle f | g_n \rangle|^2 \leq \beta \|f\|^2$ , for all  $f \in \mathcal{H}$ ;
- ii) the series  $\sum_{n=1}^{\infty} c_n g_n$  converges for all  $c = \{c_n\} \in \ell^2$ .

A sequence  $\{g_n\}$  of elements in  $\mathcal{H}$  is a *lower semi-frame for  $\mathcal{H}$*  with lower bound  $\alpha > 0$  if  $\alpha \|f\|^2 \leq \sum_{n=1}^{\infty} |\langle f | g_n \rangle|^2$ , for every  $f \in \mathcal{H}$ . Note that the series on the right hand side may diverge for some  $f \in \mathcal{H}$ .

A sequence  $\{g_n\}$  of elements in  $\mathcal{H}$  is a *frame for  $\mathcal{H}$*  if there exist  $\alpha, \beta > 0$  such that

$$\alpha \|f\|^2 \leq \sum_{n=1}^{\infty} |\langle f | g_n \rangle|^2 \leq \beta \|f\|^2, \quad \forall f \in \mathcal{H}.$$

We now recall some operators which are classically used in the study of sequences, see [1, 2, 3, 15]. Let  $\{g_n\}$  be a sequence of elements of  $\mathcal{H}$ . The *analysis*

operator  $C : \mathcal{D}(C) \subseteq \mathcal{H} \rightarrow \ell^2$  of  $\{g_n\}$  is defined by

$$\mathcal{D}(C) = \left\{ f \in \mathcal{H} : \sum_{n=1}^{\infty} |\langle f | g_n \rangle|^2 < \infty \right\}$$

$$Cf = \{\langle f | g_n \rangle\}, \quad \forall f \in \mathcal{D}(C).$$

The *synthesis operator*  $D : \mathcal{D}(D) \subseteq \ell^2 \rightarrow \mathcal{H}$  of  $\{g_n\}$  is defined on the dense domain

$$\mathcal{D}(D) := \left\{ \{c_n\} \in \ell^2 : \sum_{n=1}^{\infty} c_n g_n \text{ is convergent in } \mathcal{H} \right\}$$

by

$$D\{c_n\} = \sum_{n=1}^{\infty} c_n g_n, \quad \forall \{c_n\} \in \mathcal{D}(D).$$

The *frame operator*  $S : \mathcal{D}(S) \subseteq \mathcal{H} \rightarrow \mathcal{H}$  of  $\{g_n\}$  is defined by

$$\mathcal{D}(S) := \left\{ f \in \mathcal{H} : \sum_{n=1}^{\infty} \langle f | g_n \rangle g_n \text{ is convergent in } \mathcal{H} \right\}$$

$$Sf = \sum_{n=1}^{\infty} \langle f | g_n \rangle g_n, \quad \forall f \in \mathcal{D}(S).$$

The main properties of these operators are summarized below.

**Proposition 2.1** ([3, Prop. 3.3]). *Let  $\{g_n\}$  be a sequence of  $\mathcal{H}$ . The following statements hold.*

- i)  $C = D^*$  and therefore  $C$  is closed.
- ii)  $D$  is closable if and only if  $C$  is densely defined. In this case,  $D \subseteq C^*$ .
- iii)  $D$  is closed if and only if  $C$  is densely defined and  $D = C^*$ .
- iv)  $S = DC$ .

A sequence  $\{g_n\}$  is a Bessel sequence if and only if one, and then all, the operators  $C$ ,  $D$  and  $S$  are bounded. Moreover, if  $\{g_n\}$  is a frame then  $S$  is invertible with bounded inverse and the following reconstruction formula holds

$$(2.1) \quad f = \sum_{n=1}^{\infty} \langle f | h_n \rangle g_n, \quad f \in \mathcal{H},$$

where  $\{h_n\}$  is a frame for  $\mathcal{H}$  called a dual of  $\{g_n\}$ . A choice of  $\{h_n\}$ , which is always possible, is  $\{S^{-1}g_n\}$ , called the *canonical dual* of  $\{g_n\}$ , but it can be different if  $\{g_n\}$  is overcomplete, i.e.  $\{g_n\}$  is not a basis. As a consequence of (2.1), the Hilbert space  $\mathcal{H}$  must be separable.

Now we spend some words on non-Bessel sequences and reconstruction formulas. In general, if  $\{g_n\}$  is a lower semi-frame, then by [14, Proposition 3.4] or [18, Sect. 4], there exists a Bessel sequence  $\{h_n\}$  such that

$$h = \sum_{n=1}^{\infty} \langle h|g_n \rangle h_n, \quad \forall h \in \mathcal{D}(C).$$

Hence a reconstruction formula holds in weak sense as

$$(2.2) \quad \langle f|h \rangle = \sum_{n=1}^{\infty} \langle f|h_n \rangle \langle g_n|h \rangle, \quad f \in \mathcal{H}, h \in \mathcal{D}(C).$$

Moreover, if  $\mathcal{D}(C)$  is dense, then one can take  $h_n = T^{-1}g_n$ , where  $T := |C|^2 = C^*C$ , a self-adjoint operator with bounded inverse on  $\mathcal{H}$ , see [17, 18]. The “weakness” of the formula (2.2) is a consequence of the fact that the synthesis operator  $D$  is not closed, in general. If  $\{g_n\}$  is a lower semi-frame,  $\mathcal{D}(C)$  is dense and the synthesis operator  $D$  of  $\{g_n\}$  is closed, then  $D = C^*$ , by Proposition 2.1. Thus  $S = C^*C$  and the strong reconstruction formula again holds

$$f = SS^{-1}f = \sum_{n=1}^{\infty} \langle f|S^{-1}g_n \rangle g_n, \quad \forall f \in \mathcal{H}.$$

**Remark 2.2.** In the light of (2.2), we compare the pair  $(\{g_n\}, \{h_n\})$  with reproducing pairs [5, 6, 10, 11], weakly dual pairs [30], also called pairs of pseudoframes for  $\mathcal{H}$ , and pairs of pseudoframes for subspaces [31]. If in (2.2) the formula holds for every  $h \in \mathcal{H}$ , then by definition  $(\{g_n\}, \{h_n\})$  is a weakly dual pair. In (2.2), if in addition  $\mathcal{D}(C)$  is dense, the pair  $(\{g_n\}, \{h_n\})$  is a reproducing pair if and only if it is a weakly dual pair. In order the pair  $(\{g_n\}, \{h_n\})$  in (2.2) to be a pseudoframe for  $\mathcal{D}(C)$ , this space has to be closed and  $\{g_n\}$  and  $\{h_n\}$  have to be Bessel sequences for  $\mathcal{D}(C)$  and  $\mathcal{H}$ , respectively, so the nature of  $\{g_n\}$  and  $\{h_n\}$  in (2.2) is very different from the setting of pseudoframe for subspace, in general.

Now we recall the two notions we will generalize in the present paper. Let  $K \in \mathcal{B}(\mathcal{H})$ . A sequence  $\{g_n\} \subset \mathcal{H}$  is an *atomic system for K* [23] if the following statements hold

- i)  $\{g_n\}$  is a Bessel sequence of  $\mathcal{H}$ ;
- ii) there exists  $C > 0$  such that for every  $f \in \mathcal{H}$  there exists  $a_f = \{a_n(f)\} \in \ell^2$  such that  $\|a_f\| \leq C\|f\|$  and  $Kf = \sum_{n=1}^{\infty} a_n(f)g_n$ .

In [23, Theorem 3], the author proves the following

**Theorem 2.3.** *Let  $K \in \mathcal{B}(\mathcal{H})$  and  $\{g_n\}$  a sequence of  $\mathcal{H}$ . The following statements are equivalent.*

- i)  $\{g_n\}$  is an atomic system for  $K$ .
- ii) there exist constants  $\alpha, \beta > 0$  such that

$$\alpha \|K^* f\|^2 \leq \sum_{n=1}^{\infty} |\langle f | g_n \rangle|^2 \leq \beta \|f\|^2, \quad \forall f \in \mathcal{H}.$$

- iii) there exists a Bessel sequence  $\{h_n\}$  of  $\mathcal{H}$  such that

$$Kf = \sum_{n=1}^{\infty} \langle f | h_n \rangle g_n, \quad \forall f \in \mathcal{H}.$$

Due to the inequalities in *ii)* above, a sequence satisfying any of the conditions in Theorem 2.3 is also called a  $K$ -frame for  $\mathcal{H}$ .

Lastly, we will use the next lemma that can be obtained by Lemma 1.1 and Corollary 1.2 in [13].

**Lemma 2.4.** *Let  $\mathcal{H}$  and  $\mathcal{K}$  be Hilbert spaces. Let  $W : \mathcal{D}(W) \subset \mathcal{K} \rightarrow \mathcal{H}$  be a closed densely defined operator with closed range  $\mathcal{R}(W)$ . Then, there exists a unique  $W^\dagger \in \mathcal{B}(\mathcal{H}, \mathcal{K})$  such that*

$$\mathcal{N}(W^\dagger) = \mathcal{R}(W)^\perp, \quad \overline{\mathcal{R}(W^\dagger)} = \mathcal{N}(W)^\perp, \quad WW^\dagger f = f, \quad f \in \mathcal{R}(W).$$

The operator  $W^\dagger$  is called the *pseudo-inverse* of  $W$ .

### 3. WEAK $A$ -FRAMES AND WEAK ATOMIC SYSTEMS FOR $A$

In this section we introduce our first generalization of the notion of  $K$ -frames to a densely defined operator on a Hilbert space  $\mathcal{H}$ .

**Definition 3.1.** Let  $A$  be a densely defined operator on  $\mathcal{H}$ . A *weak  $A$ -frame* for  $\mathcal{H}$  is a sequence  $\{g_n\} \subset \mathcal{H}$  such that

$$(3.1) \quad \alpha \|A^* f\|^2 \leq \sum_{n=1}^{\infty} |\langle f | g_n \rangle|^2 < \infty,$$

for every  $f \in \mathcal{D}(A^*)$  and some  $\alpha > 0$ .

By [27, Theorem 7.2], if  $A \in \mathcal{B}(\mathcal{H})$  then  $\{g_n\}$  is a weak  $A$ -frame if and only if it is an  $A$ -frame in the sense of [23].

**Remark 3.2.** As it is clear from (3.1), the property of being a weak  $A$ -frame does not depend on the ordering of the sequence.



**Remark 3.3.** Let  $A$  be a closable densely defined operator and  $\{g_n\}$  a weak  $A$ -frame. The domain  $\mathcal{D}(C)$  of the analysis operator  $C$  of  $\{g_n\}$  contains  $\mathcal{D}(A^*)$ . It is therefore dense and the synthesis operator  $D$  is closable. Moreover,

$$\alpha \|A^* f\|^2 \leq \sum_{n=1}^{\infty} |\langle f | g_n \rangle|^2 = \|Cf\|^2 = \|T^{\frac{1}{2}} f\|^2, \quad \forall f \in \mathcal{D}(A^*),$$

where  $T = C^*C$ . This shows that the series in (3.1) is also bounded from above by the norm of a self-adjoint operator acting on  $f \in \mathcal{D}(A^*)$ .

**Example 3.4.** Let  $A$  be a densely defined operator on a separable Hilbert space  $\mathcal{H}$ . Then a weak  $A$ -frame for  $\mathcal{H}$  always exists. Indeed, let  $\{e_n\}$  be an orthonormal basis for  $\mathcal{H}$  contained in  $\mathcal{D}(A)$  (there always exists such a one, by [37, Ch. 1, Corollary 1] and the Gram-Schmidt orthonormalization process), it suffices to take  $g_n = Ae_n$ , because for every  $f \in \mathcal{D}(A^*)$ ,  $\|A^* f\|^2 = \sum_{n=1}^{\infty} |\langle f | g_n \rangle|^2$ , by the Parseval identity.

**Example 3.5.** Let  $A$  be a densely defined operator on a separable Hilbert space  $\mathcal{H}$ . A more general example of weak  $A$ -frame is obtained by taking a frame  $\{f_n\} \subset \mathcal{D}(A)$  for  $\mathcal{H}$ . In this case, in fact, there exist  $\alpha, \beta > 0$  such that

$$\alpha \|A^* f\|^2 \leq \sum_{n=1}^{\infty} |\langle A^* f | f_n \rangle|^2 \leq \beta \|A^* f\|^2, \quad \forall f \in \mathcal{D}(A^*).$$

Therefore,  $\{Af_n\}$  is a weak  $A$ -frame for  $\mathcal{H}$ .

Now we generalize the notion of atomic system to the case of an unbounded operator.

**Definition 3.6.** Let  $A$  be a densely defined operator on  $\mathcal{H}$ . A *weak atomic system for  $A$*  is a sequence  $\{g_n\} \subset \mathcal{H}$  such that

- i)  $\sum_{n=1}^{\infty} |\langle f | g_n \rangle|^2 < \infty$  for every  $f \in \mathcal{D}(A^*)$ ;
- ii) there exists  $\gamma > 0$  such that, for every  $h \in \mathcal{D}(A)$ , there exists  $a_h = \{a_n(h)\} \in \ell^2$  satisfying  $\|a_h\| \leq \gamma \|h\|$  and

$$(3.2) \quad \langle Ah | u \rangle = \sum_{n=1}^{\infty} a_n(h) \langle g_n | u \rangle, \quad \forall u \in \mathcal{D}(A^*).$$

**Remark 3.7.** If  $\{g_n\}$  is a weak atomic system for  $A$  then the series in (3.2) is unconditionally convergent. Indeed it is *absolutely* convergent: fix any  $h \in \mathcal{D}(A)$ ,  $u \in \mathcal{D}(A^*)$ , then  $\sum_{n=1}^{\infty} |a_n(h) \langle g_n | u \rangle| \leq \|a_h\| \left( \sum_{n=1}^{\infty} |\langle g_n | u \rangle|^2 \right)^{1/2} < \infty$ .

The following lemma, which is a variation of [20, Theorem 2], will be useful in Theorem 3.10 for a characterization of weak atomic systems for  $A$  and weak  $A$ -frames.

**Lemma 3.8.** *Let  $(\mathcal{H}, \|\cdot\|)$ ,  $(\mathcal{H}_1, \|\cdot\|_1)$  and  $(\mathcal{H}_2, \|\cdot\|_2)$  be Hilbert spaces and  $T_1 : \mathcal{D}(T_1) \subseteq \mathcal{H}_1 \rightarrow \mathcal{H}$ ,  $T_2 : \mathcal{D}(T_2) \subseteq \mathcal{H} \rightarrow \mathcal{H}_2$  densely defined operators. Denote by  $T_1^* : \mathcal{D}(T_1^*) \subseteq \mathcal{H} \rightarrow \mathcal{H}_1$  and  $T_2^* : \mathcal{D}(T_2^*) \subseteq \mathcal{H}_2 \rightarrow \mathcal{H}$  the adjoint operators of  $T_1, T_2$ , respectively. Assume that*

- i)  $T_1$  is closed;
- ii)  $\mathcal{D}(T_1^*) = \mathcal{D}(T_2)$ ;
- iii)  $\|T_1^*f\|_1 \leq \lambda\|T_2f\|_2$  for all  $f \in \mathcal{D}(T_1^*)$  and some  $\lambda > 0$ .

Then there exists an operator  $U \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$  such that  $T_1 = T_2^*U$ .

*Proof.* Define an operator  $J$  on  $R(T_2) \subseteq \mathcal{H}_2$  as  $JT_2f = T_1^*f \in \mathcal{H}_1$ . Then  $J$  is a well-defined bounded operator by *iii*). Now we extend  $J$  to the closure of  $R(T_2)$  by continuity and define it to be zero on  $R(T_2)^\perp$ . Therefore  $J \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$  and  $JT_2 = T_1^*$ , i.e.  $T_1 = T_2^*J^*$  and the statement is proved by taking  $U = J^*$ .  $\square$

For the characterization in Theorem 3.10 we need the following definition.

**Definition 3.9.** Let  $A$  be a densely defined operator and  $\{g_n\}$  a sequence on  $\mathcal{H}$ , then a sequence  $\{t_n\}$  of  $\mathcal{H}$  is called a *weak  $A$ -dual of  $\{g_n\}$*  if

$$(3.3) \quad \langle Ah|u \rangle = \sum_{n=1}^{\infty} \langle h|t_n \rangle \langle g_n|u \rangle \quad \forall h \in \mathcal{D}(A), u \in \mathcal{D}(A^*).$$

**Theorem 3.10.** *Let  $\{g_n\} \subset \mathcal{H}$  and  $A$  a closable densely defined operator on  $\mathcal{H}$ . Then the following statements are equivalent.*

- i)  $\{g_n\}$  is a weak atomic system for  $A$ ;
- ii)  $\{g_n\}$  is a weak  $A$ -frame;
- iii)  $\sum_{n=1}^{\infty} |\langle f|g_n \rangle|^2 < \infty$  for every  $f \in \mathcal{D}(A^*)$  and there exists a Bessel weak  $A$ -dual  $\{t_n\}$ .

*Proof.*  $i) \Rightarrow ii)$  Let  $f \in \mathcal{D}(A^*)$ . Then  $\|A^*f\| = \sup_{h \in \mathcal{H}, \|h\|=1} |\langle A^*f|h \rangle|$  and by the density of  $\mathcal{D}(A)$  in  $\mathcal{H}$

$$\begin{aligned} \|A^*f\| &= \sup_{h \in \mathcal{D}(A), \|h\|=1} |\langle A^*f|h \rangle| = \sup_{h \in \mathcal{D}(A), \|h\|=1} |\langle f|Ah \rangle| \\ &= \sup_{h \in \mathcal{D}(A), \|h\|=1} \left| \sum_{n=1}^{\infty} \overline{a_n(h)} \langle f|g_n \rangle \right| \\ &\leq \sup_{h \in \mathcal{D}(A), \|h\|=1} \left( \sum_{n=1}^{\infty} |a_n(h)|^2 \right)^{1/2} \left( \sum_{n=1}^{\infty} |\langle f|g_n \rangle|^2 \right)^{1/2} \\ &\leq \gamma_A \left( \sum_{n=1}^{\infty} |\langle f|g_n \rangle|^2 \right)^{1/2}, \end{aligned}$$

taking into account that  $\|a_h\| \leq \gamma_A \|h\|$  for some  $\gamma_A > 0$  and every  $h \in \mathcal{D}(A)$ .

$ii) \Rightarrow iii)$  Let  $\{e_n\}$  be an orthonormal basis of  $\ell^2$ . Consider the densely defined operator  $B : \mathcal{D}(A^*) \rightarrow \ell^2$  given by  $Bf = \{\langle f|g_n \rangle\}$  which is a restriction of the analysis operator  $C : \mathcal{D}(C) \rightarrow \ell^2$ . Since  $C$  is closed,  $B$  is closable.

We apply Lemma 3.8 to  $T_1 := \bar{A}$  and  $T_2 := B$  noting that  $\|Bf\|^2 = \sum_{n=1}^{\infty} |\langle f|g_n \rangle|^2$ . There exists  $M \in \mathcal{B}(\mathcal{H}, \ell^2)$  such that  $\bar{A} = B^*M$ . This implies that for  $h \in \mathcal{D}(A)$ ,  $u \in \mathcal{D}(A^*) = \mathcal{D}(B)$

$$\begin{aligned} \langle Ah|u \rangle &= \langle B^*Mh|u \rangle = \langle Mh|Bu \rangle = \sum_{n=1}^{\infty} \langle Mh|e_n \rangle \langle g_n|u \rangle \\ &= \sum_{n=1}^{\infty} \langle h|t_n \rangle \langle g_n|u \rangle, \end{aligned}$$

taking  $\{t_n\} = \{M^*e_n\}$  which is a Bessel sequence by [3, Proposition 4.6].

$iii) \Rightarrow i)$  It suffices to take  $a_h = \{a_n(h)\} = \{\langle h|t_n \rangle\}$  for all  $h \in \mathcal{D}(A)$ . Indeed for some  $\gamma_A > 0$  we have  $\sum_{n=1}^{\infty} |a_n(h)|^2 = \sum_{n=1}^{\infty} |\langle h|t_n \rangle|^2 \leq \gamma_A \|h\|^2$  since  $\{t_n\}$  is a Bessel sequence and  $\langle Ah|u \rangle = \sum_{n=1}^{\infty} a_n(h) \langle g_n|u \rangle$ , for  $u \in \mathcal{D}(A^*)$ .  $\square$

The term “weak” of weak  $A$ -frame and of weak atomic system, is due to the fact that (3.3) holds whereas, in general, the same decomposition in strong sense  $Ah = \sum_{n=1}^{\infty} \langle h|t_n \rangle g_n$  may fail, unlike the case of  $A$ -frame where  $A \in \mathcal{B}(\mathcal{H})$ , see [23, Theorem 3]. We show this with the following example.

**Example 3.11.** Suppose that  $\mathcal{H}$  is separable. Let  $\{e_n\}$  be an orthonormal basis for  $\mathcal{H}$  and  $\{g_n\}$  the sequence defined by  $g_1 = e_1$  and  $g_n = n(e_n - e_{n-1})$  for  $n \geq 2$ . We denote by  $C, D$  the analysis and synthesis operators of  $\{g_n\}$ , respectively.

As it is shown in [15],  $C$  is densely defined and  $D$  is a proper restriction of  $C^*$ . In particular,  $\{\frac{1}{n}\}_{n \in \mathbb{N}} \in \mathcal{D}(C^*) \setminus \mathcal{D}(D)$ . Let  $\mathcal{I}$  be the analysis operator of  $\{e_n\}$ . Obviously it is a bijection in  $\mathcal{B}(\mathcal{H}, \ell^2)$ . Now consider the sesquilinear form

$$\Omega(f, u) = \sum_{n=1}^{\infty} \langle f | e_n \rangle \langle g_n | u \rangle,$$

which is defined on  $\mathcal{H} \times \mathcal{D}(C)$ . Moreover  $\Omega(f, u) = \langle \mathcal{I}f | Cu \rangle$  for all  $f \in \mathcal{H}, u \in \mathcal{D}(C)$ . Therefore  $\Omega(f, u) = \langle C^* \mathcal{I}f | u \rangle$  for all  $f \in \mathcal{D}(C^* \mathcal{I}), u \in \mathcal{D}(C)$ .

This suggests to define  $A := C^* \mathcal{I}$  which is a densely defined closed operator. The adjoint  $A^*$  is equal to  $\mathcal{I}^* C$  and then it has  $\mathcal{D}(C)$  as domain. Thus

$$\langle Af | u \rangle = \sum_{n=1}^{\infty} \langle f | e_n \rangle \langle g_n | u \rangle, \quad \forall f \in \mathcal{D}(A), u \in \mathcal{D}(A^*),$$

i.e.  $\{g_n\}$  is a weak  $A$ -frame by Theorem 3.10. But the relation

$$Af = \sum_{n=1}^{\infty} \langle f | e_n \rangle g_n, \quad \forall f \in \mathcal{D}(A)$$

does not hold. Indeed, the element  $f := \sum_{n=1}^{\infty} \frac{1}{n} e_n$  belongs to  $\mathcal{D}(A)$  and the sum  $\sum_{k=1}^n \langle f | e_k \rangle g_k = e_n$  for  $n \in \mathbb{N}$ , does not converge in  $\mathcal{H}$ .

**Example 3.12.** In general, for a weak  $A$ -frame  $\{g_n\}$  for  $\mathcal{H}$  a Bessel weak  $A$ -dual  $\{t_n\}$  is not unique. For all examples we have considered we give here a possible choice of  $\{t_n\}$ .

- i) If  $\{g_n\} := \{Ae_n\}$ , where  $\{e_n\} \subset \mathcal{D}(A)$  is an orthonormal basis for  $\mathcal{H}$ , then one can take  $\{t_n\} = \{e_n\}$ .
- ii) If  $\{g_n\} := \{Af_n\}$ , where  $\{f_n\} \subset \mathcal{D}(A)$  is a frame for  $\mathcal{H}$ , then one can take for  $\{t_n\}$  any dual frame of  $\{f_n\}$ .

**Remark 3.13.** Let  $A$  be a densely defined operator,  $\{g_n\}$  a weak  $A$ -frame and  $\{t_n\}$  a Bessel weak  $A$ -dual of  $\{g_n\}$ , then for  $h \in \mathcal{D}(A)$  and  $u \in \mathcal{D}(A^*)$

$$\langle A^* u | h \rangle = \langle u | Ah \rangle = \sum_{n=1}^{\infty} \overline{\langle h | t_n \rangle} \langle g_n | u \rangle = \sum_{n=1}^{\infty} \langle u | g_n \rangle \langle t_n | h \rangle.$$

Since the sequence  $\{t_n\}$  is Bessel, the series  $\sum_{n=1}^{\infty} \langle u | g_n \rangle t_n$  is convergent. Therefore

$$\langle A^* u | h \rangle = \left\langle \sum_{n=1}^{\infty} \langle u | g_n \rangle t_n \middle| h \right\rangle, \quad \forall h \in \mathcal{D}(A), u \in \mathcal{D}(A^*)$$

and by the density of  $\mathcal{D}(A)$  we obtain

$$(3.4) \quad A^*u = \sum_{n=1}^{\infty} \langle u|g_n \rangle t_n, \quad \forall u \in \mathcal{D}(A^*).$$

In conclusion, it is worth noting that in this setting, surprisingly, from condition (3.1) the *strong* decomposition of  $A^*$  follows, whereas for  $A$  we have just a *weak* decomposition, in general. If  $A$  is symmetric, i.e.  $A \subset A^*$ , then clearly from (3.4) we have a decomposition of  $A$  in strong sense. If  $\{g_n\}$  is also a Bessel sequence, then  $A$  is bounded on its domain, thus closable, and condition (3.1) gives us decompositions in strong sense for both the closure  $\overline{A}$  and  $A^*$  (see [23, Theorem 3] and [33, Lemma 2.2]).

**Remark 3.14.** One could ask whether a weak  $A$ -dual  $\{t_n\}$  of a weak  $A$ -frame  $\{g_n\}$  is a weak  $A^*$ -frame, with  $A$  a closable densely defined operator. The answer is negative, in general. Indeed, if  $\{t_n\}$  is a Bessel sequence, an inequality as

$$\alpha \|Af\|^2 \leq \sum_{n=1}^{\infty} |\langle f|t_n \rangle|^2, \quad \forall f \in \mathcal{D}(A)$$

with  $\alpha > 0$ , implies that  $A$  is bounded on its domain.

Under further assumption of  $A$ , weak  $A$ -frames can be used to decompose the domain of  $A^*$ .

**Theorem 3.15.** *Let  $A$  be a densely defined closed operator with  $\mathcal{R}(A) = \mathcal{H}$  and  $(A^\dagger)^* \in \mathcal{B}(\mathcal{H})$  the adjoint of the pseudo-inverse  $A^\dagger$  of  $A$ . Let  $\{g_n\}$  be a weak  $A$ -frame and  $\{t_n\}$  a Bessel weak  $A$ -dual of  $\{g_n\}$ . Then, the sequence  $\{h_n\}$ , with  $h_n := (A^\dagger)^* t_n \in \mathcal{H}$  for every  $n \in \mathbb{N}$ , is Bessel and*

$$u = \sum_{n=1}^{\infty} \langle u|g_n \rangle h_n, \quad u \in \mathcal{D}(A^*).$$

*Proof.* First observe that, since  $A$  is onto,  $f = AA^\dagger f$ , for every  $f \in \mathcal{H}$ . Let  $\{g_n\}$ ,  $\{t_n\}$  and  $\{h_n\}$  be as in the statement. Then, by (3.3), we have that for  $f \in \mathcal{H}, u \in \mathcal{D}(A^*)$

$$\langle f|u \rangle = \langle AA^\dagger f|u \rangle = \sum_{n=1}^{\infty} \langle A^\dagger f|t_n \rangle \langle g_n|u \rangle = \sum_{n=1}^{\infty} \langle f|h_n \rangle \langle g_n|u \rangle$$

and for some  $\gamma > 0$

$$\sum_{n=1}^{\infty} |\langle f|h_n \rangle|^2 = \sum_{n=1}^{\infty} |\langle A^\dagger f|t_n \rangle|^2 \leq \gamma \|A^\dagger f\|^2 \leq \gamma \|A^\dagger\|^2 \|f\|^2$$

since  $\{t_n\}$  is Bessel for  $\mathcal{H}$  and  $A^\dagger$  is bounded. Hence,  $\{h_n\}$  is a Bessel sequence of  $\mathcal{H}$ . Finally, for any  $f \in \mathcal{H}$ ,  $u \in \mathcal{D}(A^*)$ , we have  $\langle u|f \rangle = \sum_{n=1}^{\infty} \langle \langle u|g_n \rangle h_n | f \rangle$ . Since the sequence  $\{h_n\}$  is Bessel, the series  $\sum_{n=1}^{\infty} \langle u|g_n \rangle h_n$  is convergent and we conclude that  $u = \sum_{n=1}^{\infty} \langle u|g_n \rangle h_n$ , for all  $u \in \mathcal{D}(A^*)$ .  $\square$

Now we give another theorem of characterization for weak  $A$ -frames involving the synthesis operator.

**Theorem 3.16.** *Let  $A$  be a closed densely defined operator,  $\{g_n\} \subset \mathcal{H}$  and  $D : \mathcal{D}(D) \subset \ell^2 \rightarrow \mathcal{H}$  the synthesis operator of  $\{g_n\}$ . The following statements are equivalent.*

- i) *The sequence  $\{g_n\}$  is a weak  $A$ -frame for  $\mathcal{H}$ ;*
- ii) *there exists a densely defined, closed extension  $R$  of  $D$  such that  $A = RQ$  with some  $Q \in \mathcal{B}(\mathcal{H}, \ell^2)$ ;*
- iii) *there exists a closed densely defined operator  $L : \mathcal{D}(L) \subset \ell^2 \rightarrow \mathcal{H}$  such that and  $\mathcal{D}(A^*) \subset \mathcal{D}(L^*)$ ,  $g_n = Le'_n$  where  $\{e'_n\} \subset \mathcal{D}(L)$  is an orthonormal basis for  $\ell^2$  and  $A = LU$  for some  $U \in \mathcal{B}(\mathcal{H}, \ell^2)$ .*

*Proof.* i)  $\Rightarrow$  ii) Following the proof of Theorem 3.10,  $A = B^*M$ . Then the statement is proved taking  $Q = M$  and  $R = B^*$ , since  $B^* \supseteq C^* \supseteq D$ .

ii)  $\Rightarrow$  iii) Since  $R$  is an extension of the synthesis operator  $D$ , it suffices to take  $L = R, U = M$  and  $\{e'_n\}$  the canonical orthonormal basis of  $\ell^2$ .

iii)  $\Rightarrow$  i) For every  $f \in \mathcal{D}(A^*)$  the adjoint of  $L$  is given by

$$L^*f = \sum_{n=1}^{\infty} \langle f|g_n \rangle e'_n.$$

Indeed, for  $c \in \ell^2$

$$\begin{aligned} \langle L^*f|c \rangle &= \langle L^*f | \sum_{n=1}^{\infty} c_n e'_n \rangle = \sum_{n=1}^{\infty} \overline{c_n} \langle f|Le'_n \rangle \\ &= \sum_{n=1}^{\infty} \langle e'_n|c \rangle \langle f|g_n \rangle = \langle \sum_{n=1}^{\infty} \langle f|g_n \rangle e'_n | c \rangle. \end{aligned}$$

Moreover,  $\{g_n\}$  is a weak  $A$ -frame because for every  $f \in \mathcal{D}(A^*)$  we have  $\sum_{n=1}^{\infty} |\langle f|g_n \rangle|^2 = \|L^*f\|^2 < \infty$  and  $\|A^*f\|^2 \leq \|U^*L^*f\|^2 \leq \|U\|^2 \|L^*f\|^2$ .  $\square$

We conclude this section with some concrete examples.

**Example 3.17.** Let us consider the differential operator  $Af = -if'$  with domain  $H^1(0,1)$  which is a densely defined closed operator on  $\mathcal{H} = L^2(0,1)$ , see [34,

Section 1.3]. The sequence  $\{g_n\}_{n \in \mathbb{Z}} = \{e_{nb}\}_{n \in \mathbb{Z}}$ , where  $0 < b \leq 1$  and  $e_{nb}(x) = e^{2\pi i n b x}$  for  $x \in (0, 1)$ , is a frame for  $L^2(0, 1)$ , see [16, Section 9.8]. Therefore  $\{Ag_n\} = \{2\pi n b e_{bn}\}$  is a weak  $A$ -frame for  $L^2(0, 1)$  by Example 3.5. The canonical dual frame of  $\{e_{nb}\}$  is  $\{\frac{1}{b}e_{nb}\}$ , then according to Example 3.12 we can take  $\{\frac{1}{b}e_{nb}\}$  as weak  $A$ -dual of  $\{g_n\}$ . The adjoint  $A^*$  is the operator  $A^*f = -if'$  with  $\mathcal{D}(A^*) = H_0^1(0, 1)$ , see again [34, Section 1.3]. Note that  $A^* \subset A$ . Hence the decomposition in weak sense of Theorem 3.10 reads as

$$\langle -if'|h \rangle = \langle Af|h \rangle = \sum_{n \in \mathbb{Z}} 2\pi n \langle f|e_{nb} \rangle \langle e_{nb}|h \rangle, \quad \forall f \in H^1(0, 1), h \in H_0^1(0, 1).$$

Finally, we have also a strong decomposition of  $A^*$  by (3.4):

$$-if' = A^*f = \sum_{n \in \mathbb{Z}} 2\pi n \langle f|e_{nb} \rangle e_{nb}, \quad \forall f \in H_0^1(0, 1).$$

**Example 3.18.** Let  $\mathcal{H} := L^2(\mathbb{R})$  and denote by  $A$  the selfadjoint operator  $Af = -if'$  with domain  $\mathcal{D}(A) = H^1(\mathbb{R})$ . Let  $g : \mathbb{R} \rightarrow \mathbb{C}$  be a continuous and differentiable function with support  $[0, L]$ , more generally, one can take a function  $g \in H^1(\mathbb{R})$  such that  $g \in W$  where  $W$  is the Wiener space, see e.g. [16, Section 11.5] for the definition of  $W$ .

Let  $y \in \mathbb{R}$ ,  $\omega \in \mathbb{R}$  and  $T_y, M_\omega : \mathcal{H} \rightarrow \mathcal{H}$  be the *translation* and *modulation* operators defined, for  $f \in \mathcal{H}$ , by  $(T_y f)(x) = f(x - y)$  and  $(M_\omega f)(x) = e^{2\pi i \omega x} f(x)$ , respectively. Consider the Gabor system  $G(g, a, b)$ . By the hypothesis,  $\{g_{m,n}\}_{m,n \in \mathbb{Z}} \subseteq \mathcal{D}(A)$ . Assume in particular that  $\{g_{m,n}\}_{m,n \in \mathbb{Z}}$  is a frame for  $L^2(\mathbb{R})$ , a necessary and sufficient condition is given in [25, Theorem 6.4.1]. Then, by Example 3.5,  $\{Ag_{m,n}\}_{m,n \in \mathbb{Z}}$  is a weak  $A$ -frame; i.e., for some  $\gamma > 0$

$$\gamma \|A^*f\|^2 \leq \sum_{m,n \in \mathbb{Z}} |\langle f|Ag_{m,n} \rangle|^2 < \infty \quad \forall f \in \mathcal{D}(A^*) = \mathcal{D}(A) = H^1(\mathbb{R}).$$

Explicitly,

$$\begin{aligned} Ag_{m,n}(x) &= 2\pi b n e^{2\pi i b n x} g(x - a m) - i e^{2\pi i b n x} g'(x - a m) \\ &= 2\pi b n (M_{bn} T_{am} g)(x) - i (M_{bn} T_{am} g')(x). \end{aligned}$$

For the decomposition of  $A$  we can use the canonical dual of the Gabor frame  $\{g_{m,n}\}_{m,n \in \mathbb{Z}}$  which is a Gabor frame  $\{h_{m,n}\}_{m,n \in \mathbb{Z}}$  with some window  $h \in L^2(0, 1)$ . Since  $A$  is selfadjoint we can write directly a decomposition in strong sense of  $A$  according to (3.4)

$$-if' = Af = \sum_{m,n \in \mathbb{Z}} \langle f|M_{bn} T_{am} (2\pi b n g - ig') \rangle M_{bn} T_{am} h, \quad \forall f \in H^1(\mathbb{R}).$$

Once more we point out that the property of being a weak  $A$ -frame does not depend on the ordering of the sequence  $\{M_{bn}T_{am}(2\pi bng - ig')\}_{m,n \in \mathbb{Z}}$ , see Remark 3.2.

**Example 3.19.** Let us consider the same space  $\mathcal{H} := L^2(\mathbb{R})$  and the operator  $Af = f'$  with domain  $\mathcal{D}(A) = H^1(\mathbb{R})$ . Let  $\phi \in H^1(\mathbb{R})$  and the *shift-invariant system*  $\{\phi_k(x)\}_{k \in \mathbb{Z}} := \{\phi(x - ck)\}_{k \in \mathbb{Z}}$ , with  $c > 0$ . Then  $\{(A\phi_k)(x)\}_{k \in \mathbb{Z}} = \{\phi'(x - ck)\}_{k \in \mathbb{Z}}$ . However, we cannot apply Example 3.5 to say that  $\{A\phi_k\}$  is a weak  $A$ -frame. Indeed, as it is known [16],  $\{\phi_k\}$  is never a frame for  $L^2(\mathbb{R})$ .

Consider instead the *wavelet system*  $\{\phi_{m,n}\}_{m,n \in \mathbb{Z}} := \{a^{-\frac{m}{2}}\phi(a^{-m}x - nb)\}_{m,n \in \mathbb{Z}}$  with  $a, b > 0$ . We have  $\{\phi_{m,n}\}_{m,n \in \mathbb{Z}} \subset H^1(\mathbb{R})$  and

$$\{(A\phi_{m,n})(x)\}_{m,n \in \mathbb{Z}} = \{a^{-\frac{3m}{2}}\phi'(a^{-m}x - nb)\}_{m,n \in \mathbb{Z}}.$$

The sequence we obtained is nothing but the wavelet system  $\{\phi'_{m,n}\}_{m,n \in \mathbb{Z}}$  generated by the derivative  $\phi'$  multiplied by the scalars  $\{a^{-m}\}_{m \in \mathbb{Z}}$ .

When  $\{\phi_{m,n}\}_{m,n \in \mathbb{Z}}$  is a frame for  $\mathcal{H}$ ,  $\{A\phi_{m,n}\}_{m,n \in \mathbb{Z}}$  is a weak  $A$ -frame. In particular, by [25, Theorem 10.6 (c)], for any  $k \in \mathbb{N}$ , there exists a function  $\phi$  with compact support and continuous derivatives up to order  $k$  such that  $\{\phi_{m,n}\}_{m,n \in \mathbb{Z}} := \{2^{-\frac{m}{2}}\phi(2^{-m}x - n)\}_{m,n \in \mathbb{Z}}$  is an orthonormal basis for  $L^2(\mathbb{R})$  and hence  $\{A\phi_{m,n}\}_{m,n \in \mathbb{Z}}$  is a weak  $A$ -frame.

**Example 3.20.** Let  $A$  be a closed and densely defined on  $\mathcal{H}$ . The domain  $\mathcal{D}(A)$  of  $A$  can be turned into a Hilbert space if endowed with the graph norm  $\|\cdot\|_A$ . Denote it by  $\mathcal{H}_A$  and by  $\mathcal{H}_A^\times$  its conjugate dual and construct the rigged Hilbert space  $\mathcal{H}_A \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}_A^\times$ , where  $\hookrightarrow$  means that the embeddings  $\mathcal{H}_A \subset \mathcal{H} \subset \mathcal{H}_A^\times$  are continuous with dense range, see e.g. [4, Chapter 10]. Since the sesquilinear form  $B(\cdot, \cdot)$  that puts  $\mathcal{H}_A$  and  $\mathcal{H}_A^\times$  in duality is an extension of the inner product of  $\mathcal{H}$  we write  $B(\xi, f) = \langle \xi | f \rangle$  for the action of  $\xi \in \mathcal{H}_A^\times$  on  $f \in \mathcal{H}_A$ .

Now let  $\{g_n\} \subset \mathcal{H}$ . Then  $\{g_n\}$  can be regarded as a sequence in  $\mathcal{H}_A^\times$ . Assume that it is a *Bessel-like sequence* in the sense of [12, Definition 2.10], i.e. for every bounded subset  $\mathcal{M} \subset \mathcal{H}_A$ ,

$$\sup_{f \in \mathcal{M}} \sum_{n=1}^{\infty} |\langle f | g_n \rangle|^2 < \infty.$$

Then, by [12, Proposition 2.11],  $\sum_{n=1}^{\infty} |\langle f | g_n \rangle|^2 < \infty$  and the operator  $F : \mathcal{H}_A \rightarrow \ell^2$  given by  $Ff := \{\langle f | g_n \rangle\}$  is bounded. If  $F$  is also injective, e.g. if  $\{g_n\}$  is dense



in  $\mathcal{H}$ , and has closed range, then  $\{g_n\}$  is a weak  $A^*$ -frame since

$$c\|Af\|^2 \leq c\|f\|_A^2 \leq \sum_{n=1}^{\infty} |\langle f|g_n\rangle|^2 = \|Ff\|^2 < \infty, \quad \forall f \in \mathcal{D}(A)$$

and for some  $c > 0$ .

#### 4. ATOMIC SYSTEMS FOR BOUNDED OPERATORS BETWEEN DIFFERENT HILBERT SPACES

In this section we will give another generalization of the notions and results in [23] to unbounded closed densely defined operators in a Hilbert space. If  $A : \mathcal{D}(A) \rightarrow \mathcal{H}$  is a closed and densely defined operator, then it can be seen as a bounded operator  $A : \mathcal{H}_A \rightarrow \mathcal{H}$  between two different Hilbert spaces, where by  $\mathcal{H}_A$  we indicate the Hilbert space  $\mathcal{D}(A)[\|\cdot\|_A]$  with  $\|\cdot\|_A$  the graph norm.

Thus, before going forth, we reproduce the main definitions and results in [23] for bounded operators from a Hilbert space  $\mathcal{J}$  into another, say  $\mathcal{H}$ , omitting the proofs since they are very similar to the standard ones where  $\mathcal{J} = \mathcal{H}$ , [23, 33]. We will come back to the operator  $A : \mathcal{H}_A \rightarrow \mathcal{H}$  in Section 4.1.

Let  $\langle \cdot | \cdot \rangle_{\mathcal{H}}, \langle \cdot | \cdot \rangle_{\mathcal{J}}$  be the inner products and  $\|\cdot\|_{\mathcal{H}}, \|\cdot\|_{\mathcal{J}}$  the norms of  $\mathcal{H}$  and  $\mathcal{J}$ , respectively. We denote by  $\mathcal{B}(\mathcal{J}, \mathcal{H})$  the set of bounded linear operators from  $\mathcal{J}$  into  $\mathcal{H}$ .

**Definition 4.1.** Let  $K \in \mathcal{B}(\mathcal{J}, \mathcal{H})$ . An *atomic system* for  $K$  is a sequence  $\{g_n\} \subset \mathcal{H}$  such that

- (i)  $\{g_n\}$  is a Bessel sequence,
- (ii) there exists  $\gamma > 0$  such that for all  $f \in \mathcal{J}$  there exists  $a_f = \{a_n(f)\} \in \ell^2$ , with  $\|a_f\| \leq \gamma\|f\|_{\mathcal{J}}$  and  $Kf = \sum_{n=1}^{\infty} a_n(f)g_n$ .

Clearly the previous notion reduces to that of atomic system in [23] when  $\mathcal{J} = \mathcal{H}$ .

**Example 4.2.** Let  $\mathcal{H}$  be separable and  $K \in \mathcal{B}(\mathcal{J}, \mathcal{H})$ . Every frame  $\{g_n\}$  for  $\mathcal{H}$  is an atomic system for  $K$ . Indeed, if  $\{v_n\}$  is a dual frame of  $\{g_n\}$ , then

$$Kf = \sum_{n=1}^{\infty} \langle Kf|v_n\rangle_{\mathcal{H}} g_n, \quad \forall f \in \mathcal{J}$$

and the definition is satisfied by taking  $a_f = \{\langle Kf|v_n\rangle_{\mathcal{H}}\}$  for  $f \in \mathcal{J}$ .

**Example 4.3.** Let  $\mathcal{J}$  be separable,  $K \in \mathcal{B}(\mathcal{J}, \mathcal{H})$  and  $\{f_n\}$  a frame for  $\mathcal{J}$  with dual frame  $\{h_n\} \subset \mathcal{J}$ , then for all  $f \in \mathcal{J}$

$$f = \sum_{n=1}^{\infty} \langle f | h_n \rangle_{\mathcal{J}} f_n, \text{ hence } Kf = \sum_{n=1}^{\infty} \langle f | h_n \rangle_{\mathcal{J}} Kf_n.$$

Thus the sequence  $\{g_n\} = \{Kf_n\}$  is an atomic system for  $K$ , taking  $a_f = \{a_n(f)\} := \{\langle f | h_n \rangle_{\mathcal{J}}\}$ .

For  $L \in \mathcal{B}(\mathcal{J}, \mathcal{H})$  we denote by  $L^* \in \mathcal{B}(\mathcal{H}, \mathcal{J})$  its adjoint. We now give a characterization of the atomic systems for operators in  $\mathcal{B}(\mathcal{J}, \mathcal{H})$  similar to that obtained by Găvruta in [23, Theorem 3].

**Theorem 4.4.** *Let  $\{g_n\} \subset \mathcal{H}$  and  $K \in \mathcal{B}(\mathcal{J}, \mathcal{H})$ . Then the following are equivalent.*

- i)  $\{g_n\}$  is an atomic system for  $K$ ;
- ii) there exist  $\alpha, \beta > 0$  such that for every  $f \in \mathcal{H}$

$$(4.1) \quad \alpha \|K^*f\|_{\mathcal{J}}^2 \leq \sum_{n=1}^{\infty} |\langle f | g_n \rangle_{\mathcal{H}}|^2 \leq \beta \|f\|_{\mathcal{H}}^2;$$

- iii)  $\{g_n\}$  is a Bessel sequence of  $\mathcal{H}$  and there exists a Bessel sequence  $\{k_n\}$  of  $\mathcal{J}$  such that

$$(4.2) \quad Kf = \sum_{n=1}^{\infty} \langle f | k_n \rangle_{\mathcal{J}} g_n, \quad \forall f \in \mathcal{J}.$$

**Definition 4.5.** Let  $K \in \mathcal{B}(\mathcal{J}, \mathcal{H})$ . A sequence  $\{g_n\} \subset \mathcal{H}$  is called a  $K$ -frame for  $\mathcal{H}$  if the chain of inequalities (4.1) holds true for all  $f \in \mathcal{H}$  and some  $\alpha, \beta > 0$ .

By (4.2) the range  $\mathcal{R}(K)$  must be a separable subspace of  $\mathcal{H}$ , which may be non separable. As in [33, Definition 2.1] a sequence  $\{k_n\} \subset \mathcal{J}$  as in (4.2) is called a  $K$ -dual of the  $K$ -frame  $\{g_n\} \subset \mathcal{H}$ .

**Example 4.6.** As in Section 3, we remark that, in general, a  $K$ -dual  $\{k_n\} \subset \mathcal{J}$  of a  $K$ -frame  $\{g_n\} \subset \mathcal{H}$  is not unique. Then, for the  $K$ -frames  $\{g_n\}$  considered in Examples 4.2 and 4.3 we give possible  $K$ -duals.

- i) If  $\{g_n\} := \{f_n\}$ , with  $\{f_n\} \subset \mathcal{H}$  a frame for  $\mathcal{H}$ , then one can take  $\{k_n\} = \{K^*v_n\}$  where  $\{v_n\}$  is any dual frame of  $\{f_n\}$ .
- ii) If  $\{g_n\} := \{Kf'_n\}$ , with  $\{f'_n\} \subset \mathcal{J}$  a frame for  $\mathcal{J}$ , then one can take for  $\{k_n\}$  any dual frame of  $\{f'_n\}$ .

Once at hand a  $K$ -frame  $\{g_n\}$ , the Bessel sequence  $\{k_n\} \subset \mathcal{J}$  in Theorem 4.4 is a  $K^*$ -frame, see [33, Lemma 2.2] for the case  $\mathcal{J} = \mathcal{H}$ .

We now give a characterization of  $K$ -frames involving the synthesis operator. The equivalence of the first two sentences is an easy generalization of [23, Theorem 4] and the other ones are straightforward.

**Theorem 4.7.** *Let  $K \in \mathcal{B}(\mathcal{J}, \mathcal{H})$ ,  $\{g_n\} \subset \mathcal{H}$  and  $D : \mathcal{D}(D) \subseteq \ell^2 \rightarrow \mathcal{H}$  the synthesis operator of  $\{g_n\}$ . The following statements are equivalent.*

- i)  $\{g_n\}$  is a  $K$ -frame for  $\mathcal{H}$ ;
- ii) there exists  $L \in \mathcal{B}(\ell^2, \mathcal{H})$  such that  $g_n = Le'_n$  where  $\{e'_n\}$  is an orthonormal basis for  $\ell^2$  and  $\mathcal{R}(K) \subset \mathcal{R}(L)$ ;
- iii)  $D \in \mathcal{B}(\ell^2, \mathcal{H})$  and  $\mathcal{R}(K) \subset \mathcal{R}(D)$ ;
- iv)  $D \in \mathcal{B}(\ell^2, \mathcal{H})$  and there exists  $M \in \mathcal{B}(\mathcal{J}, \ell^2)$  such that  $K = DM$ .

From Theorem 4.7 *iii*) it follows that a  $K$ -frame is not necessarily a frame sequence, indeed the range of the synthesis operator may be not closed, see [16, Corollary 5.5.2].

**4.1. Atomic systems for unbounded operators  $A$  and  $A$ -frames.** As announced at the beginning of this section, we come back to our original aim to generalize  $K$ -frames, with  $K \in \mathcal{B}(\mathcal{H})$ , in the context of unbounded closed and densely defined operator  $A$  on a Hilbert space  $\mathcal{H}$ . Here, for simplicity, we denote again by  $\langle \cdot | \cdot \rangle$  and  $\| \cdot \|$  the inner product and the norm of  $\mathcal{H}$ , respectively.

From now on we will consider  $A$  as a bounded operator in  $\mathcal{B}(\mathcal{H}_A, \mathcal{H})$ , where  $\mathcal{H}_A$  is the Hilbert space obtained endowing the domain  $\mathcal{D}(A)$  with the graph norm  $\| \cdot \|_A$ , induced by the graph inner product  $\langle \cdot | \cdot \rangle_A$ . Let  $A^\sharp : \mathcal{H} \rightarrow \mathcal{H}_A$  be the adjoint of  $A : \mathcal{H}_A \rightarrow \mathcal{H}$ , different from  $A^*$  the adjoint of the unbounded operator  $A$ .

For the reader's convenience we rewrite the definitions of atomic system for  $A \in \mathcal{B}(\mathcal{H}_A, \mathcal{H})$  and of  $A$ -frame. A sequence  $\{g_n\} \subset \mathcal{H}$  is said to be

- i) an *atomic system* for  $A$  if  $\{g_n\}$  is a Bessel sequence and there exists  $\gamma > 0$  such that for all  $f \in \mathcal{D}(A)$  there exists  $a_f = \{a_n(f)\} \in \ell^2$ , with  $\|a_f\| \leq \gamma \|f\|_A$  and  $Af = \sum_{n=1}^{\infty} a_n(f)g_n$ , with respect to the norm of  $\mathcal{H}$ ;
- ii) an  *$A$ -frame* if there exist  $\alpha, \beta > 0$  such that for every  $f \in \mathcal{H}$

$$\alpha \|A^\sharp f\|_A^2 \leq \sum_{n=1}^{\infty} |\langle f | g_n \rangle|^2 \leq \beta \|f\|^2.$$

Hence, Theorem 4.7 can be rewritten as follows.

**Corollary 4.8.** *Let  $\{g_n\} \subset \mathcal{H}$  and  $A$  a closed densely defined operator on  $\mathcal{H}$ . Then the following are equivalent.*

- i)  $\{g_n\}$  is an atomic system for  $A$ ;
- ii)  $\{g_n\}$  is an  $A$ -frame;
- iii)  $\{g_n\}$  is a Bessel sequence of  $\mathcal{H}$  and there exists a Bessel sequence  $\{k_n\}$  of  $\mathcal{H}_A$  such that

$$(4.3) \quad Af = \sum_{n=1}^{\infty} \langle f | k_n \rangle_A g_n, \quad \forall f \in \mathcal{D}(A)$$

with respect to the norm of  $\mathcal{H}$ .

- iv) the synthesis operator  $D$  of  $\{g_n\}$  is bounded and everywhere defined on  $\ell^2$  and  $\mathcal{R}(A) \subset \mathcal{R}(D)$ ;
- v) the synthesis operator  $D$  of  $\{g_n\}$  is bounded and everywhere defined on  $\ell^2$  and there exists  $M \in \mathcal{B}(\mathcal{H}_A, \ell^2)$  such that  $A = DM$ .

Note also that if  $A \in \mathcal{B}(\mathcal{H})$ , then the graph norm of  $A$  is defined on  $\mathcal{H}$  and it is equivalent to  $\|\cdot\|$ , thus our notion reduces to that of [23].

**Remark 4.9.** The expansion in (4.3) of  $Af$  in terms of  $\{g_n\}$  involves the inner product  $\langle \cdot | \cdot \rangle_A$ . One might ask if there exists also a sequence  $\{t_n\} \subset \mathcal{H}$  such that

$$Af = \sum_{n=1}^{\infty} \langle f | t_n \rangle g_n, \quad \forall f \in \mathcal{D}(A)$$

like for atomic systems for  $A \in \mathcal{B}(\mathcal{H})$ , see [23, Theorem 3]. The answer, in general, is negative if  $A$  is unbounded. Indeed, let  $\{e_n\}$  be an orthonormal basis for a separable Hilbert space  $\mathcal{H}$  and  $A$  an unbounded closed and densely defined operator in  $\mathcal{H}$ . Assume in particular that  $\{e_n\} \not\subset \mathcal{D}(A^*)$ , such an orthonormal basis for  $\mathcal{H}$  can always be found. Clearly,  $\{e_n\}$  is an  $A$ -frame. Suppose that there exists a sequence  $\{t_n\} \subset \mathcal{H}$  such that  $Af = \sum_{n=1}^{\infty} \langle f | t_n \rangle e_n$ , for all  $f \in \mathcal{D}(A)$ . Then  $\langle Af | e_n \rangle = \langle f | t_n \rangle$  for all  $f \in \mathcal{D}(A)$  and  $n \in \mathbb{N}$ . But this leads to the contradiction that  $\{e_n\} \subset \mathcal{D}(A^*)$ .

We conclude by showing an example of an  $A$ -frame which is not a frame.

**Example 4.10.** Let  $\mathcal{H} = L^2(\mathbb{R})$ ,  $\{\alpha_k\}_{k \in \mathbb{Z}}$  be a complex sequence and  $A$  the closed and densely defined operator on  $L^2(\mathbb{R})$  defined as

$$(Af)(x) = \begin{cases} \alpha_k f(x) & x \in [2k, 2k+1[ \\ \alpha_k f(x-1) & x \in [2k+1, 2k+2[ \end{cases}$$

where  $k$  varies in  $\mathbb{Z}$ , with natural domain

$$\mathcal{D}(A) = \left\{ f \in L^2(\mathbb{R}) : \sum_{k \in \mathbb{Z}} |\alpha_k|^2 \int_{2k}^{2k+1} |f(x)|^2 dx < \infty \right\}.$$

The operator  $A \in \mathcal{B}(L^2(\mathbb{R}))$  if and only if  $\{\alpha_k\}_{k \in \mathbb{Z}}$  is bounded.

Now let  $g \in L^2(\mathbb{R})$  be bounded with support  $[0, 2]$  and let the essential infimum of  $|g|$  on  $[0, 2]$  be positive,  $\text{ess\,inf}_{x \in [0, 2]} |g(x)| > 0$ . Consider the Gabor system  $\mathcal{G}(g, a, b) := \mathcal{G}(g, 2, 1) = \{e^{2\pi i m x} g(x - 2n)\}_{m, n \in \mathbb{Z}}$ ; it is Bessel because  $g$  is bounded and compactly supported, but it is not a frame since  $ab = 2 > 1$ . However, we show that it is an  $A$ -frame. Indeed, the range of the synthesis operator of  $\mathcal{G}(g, 1, 2)$  is

$$\mathcal{R}(D) = \{f \in L^2(\mathbb{R}) : f(x) = f(x-1), \forall x \in [2k+1, 2k+2[, \forall k \in \mathbb{Z}\}$$

and contains  $\mathcal{R}(A)$ . Therefore, by Corollary 4.8,  $\mathcal{G}(g, 2, 1)$  is an  $A$ -frame.

## 5. CONCLUSIONS

In conclusion, we make some remarks to highlight the novelty and potential applications of the notion of weak  $A$ -frame. If  $\{f_n\} \subset \mathcal{H}$  is a frame for  $\mathcal{H}$  and  $\{h_n\} \subset \mathcal{H}$  is a dual frame of  $\{f_n\}$ , then a closable densely defined operator  $A$  in  $\mathcal{H}$  can be decomposed as follows:

$$Af = \sum_{n=1}^{\infty} \langle Af | h_n \rangle f_n, \quad \forall f \in \mathcal{D}(A).$$

However, in this decomposition the action of the operator  $A$  still appears. On the contrary, if  $\{g_n\} \subset \mathcal{H}$  is a weak  $A$ -frame, then by Theorem 3.10 there exists a Bessel sequence  $\{t_n\} \subset \mathcal{H}$  such that

$$\langle Ah | u \rangle = \sum_{n=1}^{\infty} \langle h | t_n \rangle \langle g_n | u \rangle, \quad \forall h \in \mathcal{D}(A), u \in \mathcal{D}(A^*)$$

and the action of the operator  $A$  does not appear in the decomposition. Since we have also

$$A^*u = \sum_{n=1}^{\infty} \langle u | g_n \rangle t_n, \quad \forall u \in \mathcal{D}(A^*)$$

weak  $A$ -frames are clearly connected to multipliers that have been recently object of many studies, refer e.g. to the survey [35]. However, few works were directed to unbounded multipliers, so our study could give a contribution in this direction, actually it is what we did in Examples 3.17 and 3.18 for some specific operators.

We want to mention [7, 8, 9, 28] where some unbounded multipliers have been defined as model of non-selfadjoint Hamiltonians. Let us focus on [8] for a connection with weak  $A$ -frames. Fixed a complex sequence  $\alpha = \{\alpha_n\}$  and a Riesz basis  $\phi = \{\phi_n\}$  with dual  $\psi = \{\psi_n\}$ , one can construct the operator

$$(5.1) \quad H_{\phi, \psi}^{\alpha} f = \sum_{n=1}^{\infty} \alpha_n \langle f | \psi_n \rangle \phi_n$$

with  $\mathcal{D}(H_{\phi, \psi}^{\alpha})$  being the greatest subspace where (5.1) converges. Then  $\{\alpha_n \phi_n\}$  is a weak  $H_{\phi, \psi}^{\alpha}$ -frame, indeed by [8, Proposition 2.1]

$$\mathcal{D}(H_{\phi, \psi}^{\alpha}) = \left\{ f \in \mathcal{H} : \sum_{n=1}^{\infty} |\langle f | \alpha_n \phi_n \rangle|^2 < \infty \right\}$$

and thus Theorem 3.10 iii) is satisfied.

#### ACKNOWLEDGEMENTS

The authors warmly thank Prof. C. Trapani and the referees for their fruitful comments and remarks. This work has been supported by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

#### REFERENCES

1. J.-P. Antoine, P. Balazs, *Frames and semi-frames*, J. Phys. A: Math. Theor. **44**, 205201 (2011); Corrigendum **44**, (2011) 479501.
2. J.-P. Antoine, P. Balazs, *Frames, semi-frames, and Hilbert scales*, Numer. Funct. Anal. Optim. **33**, (2012), 736-769.
3. J.-P. Antoine, P. Balazs, D. T. Stoeva, *Classification of general sequences by frame related operators*, Sampling Theory Signal Image Proc. **10**, (2011), 151-170.
4. J.-P. Antoine, A. Inoue, C. Trapani, *Partial \*-Algebras and their Operator Realizations*, Dordrecht: Kluwer 2002.
5. J.-P. Antoine, M. Speckbacher, C. Trapani, *Reproducing Pairs of Measurable Functions*, Acta Appl. Math. **150**, (2017), 81-101.
6. J.-P. Antoine, C. Trapani, *Reproducing pairs of measurable functions and partial inner product spaces*, Adv. Operator Th. **2**, (2017), 126-146.

7. F. Bagarello, G. Bellomonte, *Hamiltonians defined by biorthogonal sets*, J. Phys. A: Math. Theor. 50(14), 145203, (2017).
8. F. Bagarello, A. Inoue, C. Trapani, *Non-self-adjoint hamiltonians defined by Riesz bases*, J. Math. Phys. 55, (2014), 033501.
9. F. Bagarello, H. Inoue, C. Trapani, *Biorthogonal vectors, sesquilinear forms, and some physical operators*, J. Math. Phys. 59, (2018), 033506.
10. P. Balazs, M. Speckbacher, *Reproducing pairs and Gabor systems at critical density*, J. Math. Anal. Appl. 455(2), (2017), 1072-1087.
11. P. Balazs, M. Speckbacher, *Reproducing pairs and the continuous nonstationary Gabor transform on LCA groups*, J. Phys. A: Math. Theor. 48, (2015), 395201.
12. G. Bellomonte, C. Trapani, *Riesz-like bases in rigged Hilbert spaces*, Z. Anal. Anwend. 35, (2016), 243-265.
13. F.J. Beutler, W.L. Root, *The operator pseudoinverse in control and systems identification*, in Generalized Inverses and Applications (M. Zuhair Nashed. Ed.), Academic Press, New York, 1976
14. P. Casazza, O. Christensen, S. Li, A. Lindner, *Riesz-Fischer sequences and lower frame bounds*, Z. Anal. Anwend. 21(2), (2002), 305-314.
15. O. Christensen, *Frames and Pseudo-inverses*, J. Math. Anal. Appl. 195, (1995), 401-414.
16. O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhäuser, Boston, 2003.
17. R. Corso, *Sesquilinear forms associated to sequences on Hilbert spaces*, Monatsh. Math., 189(4), 625-650, (2019).
18. R. Corso, *Generalized frame operator, lower semi-frames and sequences of translates*, arXiv:1912.03261, (2019)
19. I. Daubechies, A. Grossmann, Y. Meyer, *Painless nonorthogonal expansions*, J. Math. Phys. 27 (1986), 1271-1283.
20. R.G. Douglas, *On majorization, factorization, and range inclusion of operators on Hilbert space*, Proc. Amer. Math. Soc. 17 (1) (1966), 413-415.
21. J. Duffin, A.C. Schaeffer *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc. 72 (1952), 341-366.
22. H.G. Feichtinger, T. Werther, *Atomic Systems for Subspaces*, in: L. Zayed (Ed.), Proceedings SampTA 2001, Orlando, FL, (2001), 163-165.
23. L. Găvruta, *Frames for operators*, Appl. Comp. Harmon. Anal. 32 (2012), 139-144.
24. R. Geddavalasa, P.S. Johnson, *Frames for operators in Banach spaces*, Acta Math. Vietnam. 42, 4, (2017), 665-673.
25. K. Gröchenig, *Foundations of time-frequency analysis*, Birkhäuser, Boston, 2001.
26. X. Guo, *Canonical dual K-Bessel sequences and dual K-Bessel generators for unitary systems of Hilbert spaces*, J. Math. Anal. Appl. 444, (2016), 598-609.
27. C. Heil, *A Basis Theory Primer: Expanded Edition*, Birkhäuser, Boston, 2011.
28. H. Inoue, M. Takakura, *Non-self-adjoint Hamiltonians defined by generalized Riesz bases*, J. Math. Phys. 57, 083505 (2016)
29. H. Javanshiri, A.-M. Fattahi, *Continuous Atomic Systems for Subspaces*, Mediterranean Journal of Mathematics, 13 (4), (2016), 1871-1884.

30. S. Li, H. Ogawa, *Pseudo-duals of frames with applications*, Appl. Comput. Harm. Anal., 11, 289-304, (2001).
31. S. Li, H. Ogawa, *Pseudoframes for subspaces with applications*, J. Fourier Anal. Appl., 10(4), 409-431, (2004).
32. A. Najati, M. Mohammadi Saem, P. Găvruta, *Frames and operators in Hilbert  $C^*$ -modules*, Operators and Matrices, **10** (1), (2016), 73-81.
33. F.A. Neyshaburi, A.A. Arefijamaal, *Some constructions of  $K$ -frames and their duals*, Rocky Mountain J. Math. **47** (6), (2017), 1749-1764.
34. K. Schmüdgen, *Unbounded Self-adjoint Operators on Hilbert Space*, Springer, Dordrecht, (2012).
35. D.T. Stoeva, P. Balazs, *A survey on the unconditional convergence and the invertibility of multipliers with implementation*, In: Sampling - Theory and Applications (A Centennial Celebration of Claude Shannon), S. D. Casey, K. Okoudjou, M. Robinson, B. Sadler (Ed.), Applied and Numerical Harmonic Analysis Series, Springer, (accepted) 2018.
36. X. Xiao, Y. Zhu, L. Găvruta, *Some properties of  $K$ -frames in Hilbert spaces*, Results. Math. **63**, (2013), 1243-1255.
37. R. Young, *An Introduction to Nonharmonic Fourier Series*, Academic, New York (1980) (revised first edition 2001).

DIPARTIMENTO DI MATEMATICA E INFORMATICA, UNIVERSITÀ DEGLI STUDI DI PALERMO,  
I-90123 PALERMO, ITALY

*E-mail address:* `giorgia.bellomonte@unipa.it`

*E-mail address:* `rosario.corso@studium.unict.it`