ON MULTIPLICITIES OF COCHARACTERS FOR ALGEBRAS WITH SUPERINVOLUTION

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Abstract. In this paper we deal with finitely generated superalgebras with superinvolution, satisfying a non-trivial identity, whose multiplicities of the cocharacters are bounded by a constant. Along the way, we prove that the codimension sequence of such algebras is polynomially bounded if and only if their colength sequence is bounded by a constant.

1. Introduction

Let F be a field of characteristic zero and let A be an F-algebra. In this paper we deal with $\mathbb{Z}_2$-graded algebras (superalgebras) endowed with a superinvolution $*$.

Algebras with superinvolution are a natural generalization of the algebras with involution. Indeed, as the set of symmetric elements of an algebra with involution with respect to the symmetrized product $a \circ b = ab + ba$ is a Jordan algebra and the set of skew elements with respect to the Lie product $[a, b] = ab - ba$ is a Lie algebra, similarly the set of symmetric elements of an algebra with superinvolution with respect to the supersymmetrized product $a \circ b = ab + (-1)^{|a||b|}ba$ forms a Jordan superalgebra and the skew elements under the graded bracket $[a, b] = ab - (-1)^{|a||b|}ba$ form a Lie superalgebra. One can find several examples of Jordan and Lie simple superalgebras for instance in [17] and [25].

Similarly to the ordinary case, one can attach to a superalgebra with superinvolution A two special numerical sequences. The first one is the $*$-codimension sequence, $c_n^*(A)$, $n = 1, 2, \ldots$, where $c_n^*(A)$ denote the dimension of the space of multilinear polynomials in n $*$-variables in the corresponding relatively free superalgebra with superinvolution of countable rank. The second numerical sequence is the $*$-colength sequence, $l_n^*(A)$, $n = 1, 2, \ldots$, that is the sum of the multiplicities in the decomposition of the $(n_1, \ldots, n_4)$-cocharacter $\chi_{n_1,\ldots,n_4}(A)$, for all $n = n_1 + \cdots + n_4 \geq 1$. Recall that the $(n_1, \ldots, n_4)$-cocharacter is the character corresponding to the action of the group $S_{n_1} \times \cdots \times S_{n_4}$ on $P_{n_1,\ldots,n_4}(A)$, the space of multilinear $*$-polynomials in n_1 even symmetric variables, n_2 even skew variables, n_3 odd symmetric variables and n_4 odd skew variables, modulo the $*$-identities of A, by permutation of the variables of the same homogeneous degree which are all symmetric or all skew at the same time with respect to the superinvolution $*$.

Given a variety of superalgebras with superinvolution $\mathcal{V}$, its growth is defined as the growth of the $*$-codimension sequence of any superalgebra with superinvolution A generating $\mathcal{V}$, i.e., $\mathcal{V} = \mathrm{var}^*(A)$. We say that $\mathcal{V}$ has polynomial growth if $c_n^*(\mathcal{V})$ is polynomially bounded and we say that $\mathcal{V}$ has almost polynomial growth if $c_n^*(\mathcal{V})$ is not polynomially bounded but every proper subvariety of $\mathcal{V}$ has polynomial growth.

Recently, superalgebras with superinvolution have been extensively studied in several papers. In [6, 7] the authors proved that, as in the ordinary case, $c_n^*(A)$ is exponentially bounded and they also classified the superalgebras with superinvolution generating varieties of almost polynomial growth. It turned out that a variety of superalgebras with superinvolution $\mathcal{V}$ has polynomial growth if and only if it does not contain a list of five suitable superalgebras with superinvolution generating the only varieties of almost polynomial growth.

In [7, 15], the authors classified the subvarieties of each variety of almost polynomial growth by giving a complete list of finite dimensional superalgebras with superinvolution generating them.

In [11, 12], Giambruno and Zaicev answered in the affirmative to a famous conjecture posed by Amitsur in the eighties: the exponential growth of the codimension sequence of a PI-algebra is an

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integer. In the setting of superalgebras with superinvolution, the analogous result was given by Ioppolo in [14].

Other results were proved in the setting of matrix superalgebras with superinvolution. In [8] the authors gave an analogue of the Amitsur-Levitzki theorem concerning the minimal degree of standard identities whereas in [16], it was proved that on the $n \times n$ upper-triangular matrix algebra UT_n, over an algebraically closed field F of characteristic zero, there are only two classes of inequivalent superinvolutions.

Finally, in 2017, Aljadeff, Giambruno and Karasik showed that any algebra with involution has the same identities of the Grassmann envelope of a finite dimensional superalgebra with superinvolution (see [1]).

The purpose of this paper is to give a characterization of superalgebras with superinvolution with multiplicities of the corresponding $(n_1, \ldots, n_4)$-cocharacters bounded by a constant.

In [21] such a characterization was given in the setting of algebras with ordinary polynomial identities. More precisely, the authors proved that the multiplicities of the S_n-cocharacter of a variety $\mathcal{V}$ are bounded by a constant if and only if $\mathcal{V}$ does not contain the algebra UT_2. A similar result was obtained by Otera in [23] for finitely generated superalgebras: in this case the variety of superalgebras $\mathcal{V}$ does not contain the superalgebra UT_2 (trivial grading) and UT_2^{sup}, i.e., the algebra UT_2 with the canonical non-trivial $\mathbb{Z}_2$-grading. The latter characterization was extended in [3] for G-graded algebras, where G is any finite abelian group, by excluding from the variety of G-graded algebras $\mathcal{V}$ the algebra UT_2 with any G-grading. Finally, in [28], Vieira studied the same problem in the setting of finitely generated algebras with involution.

Here we deal with finitely generated superalgebras with superinvolution. If A is such an algebra, we give some conditions on $\mathrm{var}^*(A)$ ensuring that the multiplicities of its $(n_1, \ldots, n_4)$-cocharacter are bounded by a constant. In particular, we prove that this happens when $\mathrm{var}^*(A)$ does not contain the algebra M, a suitable 4-dimensional subalgebra of the algebra of 4×4 upper triangular matrices endowed with trivial grading and reflection superinvolution, M^{sup}, i.e., the algebra M with a non-trivial grading and reflection superinvolution and $M_{0,2}(F)$, the algebra of 2×2 matrices with trivial grading and orthosymplectic superinvolution.

As a direct consequence, in the last section, we shall see that the $*$-codimension sequence of A grows polynomially if and only if there exists a constant k such that $l_n^*(A) \leq k$, for all $n \geq 1$.

2. Preliminaries

Let F be a field of characteristic zero and $A = A_0 \oplus A_1$ an associative superalgebra over F endowed with a superinvolution $*$. The subspaces A_0 and A_1 satisfy the conditions $A_0A_0 + A_1A_1 \subseteq A_0$ and $A_0A_1 + A_1A_0 \subseteq A_1$ and their elements are called homogeneous of degree zero (even elements) and of degree one (odd elements), respectively. A superinvolution on A is a graded linear map $* : A \to A$ such that $(x^*)^* = x$, for all $x \in A$, and $(ab)^* = (-1)^{|a||b|} b^* a^*$, for elements $a, b \in A$ of homogeneous degree $|a|$, $|b|$, respectively. Since $\mathrm{char} F = 0$, we can write $A = A_0^+ \oplus A_0^- \oplus A_1^+ \oplus A_1^-$, where, for $i = 0, 1$, $A_i^+ = \{a \in A_i : a^* = a\}$ and $A_i^- = \{a \in A_i : a^* = -a\}$ denote the sets of homogeneous symmetric and skew elements of A_i, respectively. From now on we shall refer to a superalgebra with superinvolution as a $*$-algebra.

As in the case of graded algebras or of algebras with involution, one can define a superinvolution on the free algebra $F\langle X \rangle$ in a natural way. We write the set X as the union of two disjoint infinite sets Y and Z, requiring that their elements are of homogeneous degree 0 and 1, respectively. Then each set is written as the disjoint union of two other infinite sets of symmetric and skew elements, respectively. The free superalgebra with superinvolution is denoted by $F\langle Y \cup Z, * \rangle$ and it is generated by symmetric and skew elements of even and odd degree. We write

$$F\langle Y \cup Z, * \rangle = F\langle y_1^+, y_1^-, z_1^+, z_1^-, y_2^+, y_2^-, z_2^+, z_2^-, \ldots \rangle,$$

where y_i^+ stands for a symmetric variable of even degree, y_i^- for a skew variable of even degree, z_i^+ for a symmetric variable of odd degree and z_i^- for a skew variable of odd degree.

We denote by $\mathrm{Id}^*(A) = \{f \in F\langle Y \cup Z, * \rangle \mid f \equiv 0 \text{ on } A\}$ the T_2^*-ideal of $*$-identities of A, i.e., $\mathrm{Id}^*(A)$ is an ideal of $F\langle Y \cup Z, * \rangle$ invariant under all $\mathbb{Z}_2$-graded endomorphisms of the free superalgebra commuting with the superinvolution $*$.

Given polynomials $f_1, \ldots, f_n \in F\langle Y \cup Z, *\rangle$ we shall denote by $\langle f_1, \ldots, f_n\rangle_{T_2^*}$ the T_2^*-ideal generated by $f_1, \ldots, f_n$. Moreover, in order to simplify the notation, we shall denote by y any even variable, by z any odd variable and by x an arbitrary variable.

It is well known that in characteristic zero, every $*$-identity is equivalent to a system of multilinear $*$-identities. Hence if we denote by

$$P_n^* = \operatorname{span}_F\left\{w_{\sigma(1)} \cdots w_{\sigma(n)} \mid \sigma \in S_n, \quad w_i \in \{y_i^+, y_i^-, z_i^+, z_i^-\},\ i = 1, \ldots, n\right\}$$

the space of multilinear polynomials of degree n in $y_1^+, y_1^-, z_1^+, z_1^-, \ldots, y_n^+, y_n^-, z_n^+, z_n^-$ (i.e., y_i^+ or y_i^- or z_i^+ or z_i^- appears in each monomial at degree 1) the study of $\mathrm{Id}^*(A)$ is equivalent to the study of $P_n^* \cap \mathrm{Id}^*(A)$, for all $n \geq 1$. The non-negative integer

$$c_n^*(A) = \dim_F \frac{P_n^*}{P_n^* \cap \mathrm{Id}^*(A)},\ n \geq 1,$$

is called the n-th $*$-codimension of A.

The sequence $c_n(A)$ of the ordinary codimensions was introduced by Regev in [26] where it was proved that, if A satisfies a non-trivial polynomial identity, then $c_n(A)$ is exponentially bounded. An analogue result holds for $*$-algebras (see [6]).

If $\mathcal{V}$ is a variety of $*$-algebras ($*$-variety) generated by A, i.e., $\mathcal{V} = \mathrm{var}^*(A)$, then we write $c_n^*(\mathcal{V}) = c_n^*(A)$. We shall say that $\mathcal{V}$ has polynomial growth if there exist k, l such that $c_n^*(\mathcal{V}) \leq kn^t$ and that $\mathcal{V}$ has almost polynomial growth if $c_n^*(\mathcal{V})$ is not polynomially bounded but every proper subvariety of $\mathcal{V}$ has polynomial growth.

Let now $n \geq 1$ and write $n = n_1 + \cdots + n_4$ as a sum of four non-negative integers. We denote by $P_{n_1,\ldots,n_4} \subseteq P_n^*$ the vector space of multilinear $*$-polynomials in which n_1 variables are symmetric of even degree, n_2 variables are skew of even degree, n_3 variables are symmetric of odd degree and n_4 variables are skew of odd degree. The group $S_{n_1} \times \cdots \times S_{n_4}$ acts on the left on the vector space $P_{n_1,\ldots,n_4}$ by permuting the variables of the same homogeneous degree which are all symmetric or all skew at the same time. Thus S_{n_1} permutes the variables $y_1^+, \ldots, y_{n_1}^+$, S_{n_2} permutes the variables $y_1^-, \ldots, y_{n_2}^-$, and so on. In this way $P_{n_1,\ldots,n_4}$ becomes an $(S_{n_1} \times \cdots \times S_{n_4})$-left module. Now, $P_{n_1,\ldots,n_4} \cap \mathrm{Id}^*(A)$ is invariant under this action and so the vector space

$$P_{n_1,\ldots,n_4}(A) = \frac{P_{n_1,\ldots,n_4}}{P_{n_1,\ldots,n_4} \cap \mathrm{Id}^*(A)}$$

is an $(S_{n_1} \times \cdots \times S_{n_4})$-left module with induced action. We denote by $\chi_{n_1,\ldots,n_4}(A)$ its character and we call it the $(n_1, \ldots, n_4)$-th cocharacter of A.

If $\lambda = (\lambda_1, \ldots, \lambda_r)$ is a partition of n, we write $\lambda \vdash n$. It is well-known that there is a one-to-one correspondence between partitions of n and irreducible S_n-characters. Hence if $\lambda \vdash n$, we denote by χ_λ the corresponding irreducible S_n-character. If $\lambda(1) \vdash n_1, \ldots, \lambda(4) \vdash n_4$ are partitions we write $\langle\lambda\rangle = (\lambda(1), \ldots, \lambda(4)) \vdash (n_1, \ldots, n_4)$ or $\langle\lambda\rangle \vdash n$ and we say that $\langle\lambda\rangle$ is a multipartition of $n = n_1 + \cdots + n_4$. Since $\mathrm{char}\, F = 0$, by complete reducibility, $\chi_{n_1,\ldots,n_4}(A)$ can be written as a sum of irreducible characters

$$\chi_{n_1,\ldots,n_4}(A) = \sum_{\langle\lambda\rangle \vdash (n_1,\ldots,n_4)} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}, \tag{1}$$

where $m_{\langle\lambda\rangle} \geq 0$ is the multiplicity of $\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}$ in $\chi_{n_1,\ldots,n_4}(A)$.

Another numerical sequence that can be attached to a $*$-algebra A is the sequence of $*$-colengths. If $\chi_{n_1,\ldots,n_4}(A) = \sum_{\langle\lambda\rangle \vdash (n_1,\ldots,n_4)} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}$ is the decomposition of the $(n_1, \ldots, n_4)$-th cocharacter of A, then the n-th colength of A is defined as

$$l_n^*(A) = \sum_{\substack{\langle\lambda\rangle \vdash (n_1,\ldots,n_4) \\ n_1 + \cdots + n_4 = n}} m_{\langle\lambda\rangle}.$$

We conclude this section by recalling some basic results concerning the sequences of cocharacters and colengths.

Remark 2.1. *Let A and B be two $*$-algebras such that*

$$\chi_{n_1,\ldots,n_4}(A) = \sum m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)} \quad \textit{and} \quad \chi_{n_1,\ldots,n_4}(B) = \sum m'_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}.$$

1. *If $B \in var^*(A)$, then $m'_{\langle\lambda\rangle} \leq m_{\langle\lambda\rangle}$, for all $\langle\lambda\rangle \vdash (n_1, \dots, n_4)$ and $l_n^*(B) \leq l_n^*(A)$, for all n.*
2. *The direct sum $A \oplus B$ is also a $*$-algebra, with superinvolution induced by the superinvolutions defined on A and B. Moreover, if*
$$\chi_{n_1,\dots,n_4}(A \oplus B) = \sum_{\langle\lambda\rangle \vdash (n_1,\dots,n_4)} \bar{m}_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}$$
is the decomposition of the $(n_1, \dots, n_4)$-th cocharacter of $A \oplus B$, then $\bar{m}_{\langle\lambda\rangle} \leq m_{\langle\lambda\rangle} + m'_{\langle\lambda\rangle}$, for all $\langle\lambda\rangle \vdash (n_1, \dots, n_4)$.

3. $*$-varieties of almost polynomial growth

In this section we shall introduce three finite dimensional $*$-algebras generating varieties of almost polynomial growth.

First we want to highlight that any finitely generated $*$-algebra has the same polynomial identities of a finite dimensional $*$-algebra in case of an algebraically closed field of characteristic zero. In fact the following result holds.

Theorem 3.1. [1]. *Let $\mathcal{V}$ be a $*$-variety generated by a finitely generated $*$-algebra B over an algebraically closed field F, satisfying an ordinary identity. Then $\mathcal{V} = var^*(C)$, for some finite dimensional $*$-algebra C over F.*

In light of the previous theorem, from now on it suffices to study only finite dimensional $*$-algebras.

Let $F \oplus F$ be the two dimensional group algebra of $\mathbb{Z}_2$. We denote by D the algebra $F \oplus F$ with trivial grading and exchange superinvolution ex given by $(a, b)^{ex} = (b, a)$, for all $(a, b) \in D$. Such a $*$-algebra was extensively studied in [6, 9]. It generates a $*$-variety of almost polynomial growth, $\mathrm{Id}^*(D) = \langle [x_1, x_2], z^+, z^- \rangle_{T_2^*}$ and, if $\chi_{n_1,\dots,n_4}(D) = \sum_{\langle\lambda\rangle \vdash (n_1,\dots,n_4)} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}$ is the $(n_1, \dots, n_4)$-th cocharacter of D, $n_1 + \cdots + n_4 = n$, then
$$m_{\langle\lambda\rangle} = \begin{cases} 1 & \text{if } \langle\lambda\rangle = ((n_1), (n_2), \emptyset, \emptyset),\ n_1 + n_2 = n \\ 0 & \text{otherwise} \end{cases}.$$

As a consequence,
$$l_n^*(D) = n + 1, \text{ for all } n \geq 1. \tag{2}$$

Let now
$$M = F(e_{11} + e_{44}) \oplus F(e_{22} + e_{33}) \oplus Fe_{12} \oplus Fe_{34},$$
be a subalgebra of UT_4, the algebra of 4×4 upper-triangular matrices, endowed with the reflection involution $\circ$, i.e., the involution obtained by reflecting a matrix along its secondary diagonal. Hence, if $a = \alpha(e_{11} + e_{44}) + \beta(e_{22} + e_{33}) + \gamma e_{12} + \delta e_{34}$ then
$$a^\circ = \alpha(e_{11} + e_{44}) + \beta(e_{22} + e_{33}) + \delta e_{12} + \gamma e_{34}.$$

If we regard M as endowed with trivial grading, then the above involution is a superinvolution. Such a $*$-algebra (see [6, 22]) generates a $*$-variety of almost polynomial growth with T_2^*-ideal of identities $\mathrm{Id}^*(M) = \langle y_1^- y_2^-, z^+, z^- \rangle_{T_2^*}$. Moreover, if $\chi_{n_1,\dots,n_4}(M) = \sum_{\langle\lambda\rangle \vdash (n_1,\dots,n_4)} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}$ is the $(n_1, \dots, n_4)$-th cocharacter of M, $n_1 + \cdots + n_4 = n$, then
$$m_{\langle\lambda\rangle} = \begin{cases} 1 & \text{if } \langle\lambda\rangle = ((n), \emptyset, \emptyset, \emptyset) \\ q+1 & \text{if } \langle\lambda\rangle = ((p+q, p), (1), \emptyset, \emptyset),\ 2p + q = n - 1 \\ q+1 & \text{if } \langle\lambda\rangle = ((p+q, p), \emptyset, \emptyset, \emptyset),\ 2p + q = n \\ q+1 & \text{if } \langle\lambda\rangle = ((p+q, p, 1), \emptyset, \emptyset, \emptyset),\ 2p + q = n - 1 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

and so
$$l_n^*(M) = \begin{cases} \frac{3n^2+4}{4} & \text{if } n \text{ is even} \\ \frac{3n^2+5}{4} & \text{if } n \text{ is odd} \end{cases}. \tag{4}$$

Next we consider a non-trivial grading on M: we denote by M^{sup} the algebra M with grading $M_0 = F(e_{11} + e_{44}) \oplus F(e_{22} + e_{33})$ and $M_1 = Fe_{12} \oplus Fe_{34}$. Notice that the reflection involution on M^{sup} is a superinvolution, since $M_1^2 = 0$. The $*$-algebra M^{sup} generates a $*$-variety of almost polynomial growth with $\mathrm{Id}^*(M^{sup}) = \langle y^-, z_1z_2\rangle_{T_2^*}$. Moreover, if $\chi_{n_1,\dots,n_4}(M^{sup}) = \sum_{\langle\lambda\rangle\vdash(n_1,\dots,n_4)} m_{\langle\lambda\rangle}\chi_{\lambda(1)}\otimes\cdots\otimes\chi_{\lambda(4)}$ is the $(n_1,\dots,n_4)$-th cocharacter of M^{sup}, $n_1+\cdots+n_4 = n$, then

$$m_{\langle\lambda\rangle} = \begin{cases} 1 & \text{if } \langle\lambda\rangle = ((n),\emptyset,\emptyset,\emptyset) \\ q+1 & \text{if } \langle\lambda\rangle = ((p+q,p),\emptyset,(1),\emptyset) \\ q+1 & \text{if } \langle\lambda\rangle = ((p+q,p),\emptyset,\emptyset,(1)) \\ 0 & \text{otherwise} \end{cases}, \tag{5}$$

where $p, q \geq 0$ and $2p + q + 1 = n$ (see [6, 10]). It follows that

$$l_n^*(M^{sup}) = \begin{cases} \frac{n^2+2n+2}{2} & \text{if } n \text{ is even} \\ \frac{n^2+2n+3}{2} & \text{if } n \text{ is odd} \end{cases}. \tag{6}$$

The above $*$-algebras characterize the $*$-varieties of polynomial growth.

Theorem 3.2. [6, Theorem 5.1] *Let A be a finite dimensional $*$-algebra. Then $var^*(A)$ has polynomial growth if and only if $D, M, M^{sup} \notin var^*(A)$.*

As a consequence we have the following corollary.

Corollary 3.1. [6, Corollary 5.1] *The $*$-algebras M, M^{sup} and D are the only finite dimensional $*$-algebras generating varieties of almost polynomial growth.*

4. On the Wedderburn-Malcev decomposition

In this section we analyze the Wedderburn-Malcev decomposition of a finite dimensional $*$-algebra A, in case $M, M^{sup} \notin \mathrm{var}^*(A)$. We shall prove that the simple components of the semisimple part of such a decomposition can be chosen only in a list of five $*$-algebras.

In [6], the authors gave an analogue of the Wedderburn-Malcev decomposition in the setting of finite dimensional $*$-algebras. In order to present such a result we first recall some definitions. An ideal (subalgebra) I of an algebra A with superinvolution $*$ is a $*$-ideal (subalgebra) of A if it is a graded ideal (subalgebra) and $I^* = I$. The algebra A is a simple $*$-algebra if $A^2 \neq 0$ and A has no non-trivial $*$-ideals.

Theorem 4.1. [6, Theorem 4.1] *Let A be a finite dimensional $*$-algebra over a field F of characteristic 0. Then there exists a semisimple $*$-subalgebra B such that $A = B + J(A)$ and $J(A)$ is a $*$-ideal of A. Moreover $B = B_1 \oplus \cdots \oplus B_k$, where $B_1, \dots, B_k$ are simple $*$-algebras.*

Next we shall present the classification of the finite dimensional simple $*$-algebras over an algebraically closed field F. Recall that if A and B are two superalgebras endowed with superinvolutions $*$ and $\star$, respectively, then $(A, *)$ and $(B, \star)$ are isomorphic, as $*$-algebras, if there exists an isomorphism of superalgebras $\psi : A \to B$ such that $\psi(x^*) = \psi(x)^\star$, for all $x \in A$.

If $n = k + h$, then the matrix algebra $M_n(F)$ becomes a superalgebra, denoted by $M_{k,h}(F)$, with grading

$$(M_{k,h}(F))_0 = \left\{ \begin{pmatrix} X & 0 \\ 0 & T \end{pmatrix} \mid X \in M_k(F),\ T \in M_h(F) \right\},$$
$$(M_{k,h}(F))_1 = \left\{ \begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix} \mid Y \in M_{k\times h}(F),\ Z \in M_{h\times k}(F) \right\}.$$

In [24], Racine proved that, up to isomorphism and if the field F is algebraically closed and of characteristic different from 2, it is possible to define on $M_{k,h}(F)$ only the following superinvolutions.

1. The *transpose* superinvolution denoted *trp*, defined for $h = k$ by
$$\begin{pmatrix} X & Y \\ Z & T \end{pmatrix}^{trp} = \begin{pmatrix} T^t & -Y^t \\ Z^t & X^t \end{pmatrix},$$
where t is the usual transpose.

2. The *orthosymplectic* superinvolution osp, defined when $h = 2l$ is even by
$$\begin{pmatrix} X & Y \\ Z & T \end{pmatrix}^{osp} = \begin{pmatrix} I_k & 0 \\ 0 & Q \end{pmatrix}^{-1} \begin{pmatrix} X & -Y \\ Z & T \end{pmatrix}^{t} \begin{pmatrix} I_k & 0 \\ 0 & Q \end{pmatrix} = \begin{pmatrix} X^t & Z^tQ \\ QY^t & -QT^tQ \end{pmatrix},$$
where $Q = \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix}$ and I_k, I_l are the $k \times k$, $l \times l$ identity matrices, respectively.

Furthermore, if A is a superalgebra, we denote by A^{sop} the superalgebra with the same graded vector space structure of A and product given on homogeneous elements $a, b \in A^{sop}$ by
$$a \circ b = (-1)^{|a||b|} ba.$$
The direct sum $R = A \oplus A^{sop}$ is a superalgebra with $R_0 = A_0 \oplus A_0^{sop}$ and $R_1 = A_1 \oplus A_1^{sop}$ and it is endowed with the exchange superinvolution
$$(a, b)^* = (b, a).$$
For example, if we consider the superalgebra $Q(n) = M_n(F \oplus cF) = Q(n)_0 \oplus Q(n)_1$, where $Q(n)_0 = M_n(F)$ and $Q(n)_1 = cM_n(F)$, with $c^2 = 1$, then $Q(n) \oplus Q(n)^{sop}$ is a $*$-algebra with exchange superinvolution.

The following theorem gives the classification of the finite dimensional simple $*$-algebras over an algebraically closed field F.

Theorem 4.2 ([2, 13, 24]). *Let A be a finite dimensional simple $*$-algebra over an algebraically closed field F of characteristic different from 2. Then A is isomorphic to one of the following:*

(1) *$M_{k,h}(F)$ with the orthosymplectic or the transpose superinvolution,*
(2) *$M_{k,h}(F) \oplus M_{k,l}(F)^{sop}$ with the exchange superinvolution,*
(3) *$Q(n) \oplus Q(n)^{sop}$ with the exchange superinvolution.*

Since any finite dimensional $*$-algebra A can be naturally embedded into the $*$-algebra $A \otimes_F \bar{F}$, which is finite dimensional over the algebraic closure $\bar{F} \supseteq F$, then, without loss of generality, we may assume that a finite dimensional $*$-algebra A over a field F of characteristic zero has a Wedderburn-Malcev decomposition such that any $*$-simple component A_i, $i = 1, \dots, m$, is isomorphic to one of the algebras in the previous theorem.

Now we can focus our attention to the proof of the main result of this section, concerning finite dimensional $*$-algebras not containing M and M^{sup}. Recall that they are subalgebras of $UT_4(F)$ with trivial and natural grading, respectively, both endowed with the reflection superinvolution $\circ$. To reach our goal we have to prove first several lemmas.

Lemma 4.1. *Let A be a $*$-algebra. If $M \notin var^*(A)$, then $(M_{k,k}(F), trp) \notin var^*(A)$, for any $k \geq 2$.*

Proof. Suppose by contradiction that $(M_{k,k}(F), trp) \in \mathrm{var}^*(A)$ and let us consider its subalgebra
$$C = \mathrm{span}_F \left\{ \underbrace{e_{11} + e_{k+1,k+1}}_{a}, \underbrace{e_{22} + e_{k+2,k+2}}_{b}, \underbrace{e_{12}}_{c}, \underbrace{e_{k+2,k+1}}_{c^*} \right\} \subseteq (M_{k,k}(F), trp),$$
spanned by four elements of homogeneous degree zero and with induced superinvolution. Here the e_{ij}'s are the usual matrix units.

Now let $\{e_{11} + e_{44}, e_{22} + e_{33}, e_{12}, e_{34}\}$ be a basis of the $*$-algebra M. Then the map $\varphi : M \to C$ given by
$$\varphi(e_{11} + e_{44}) = a, \quad \varphi(e_{22} + e_{33}) = b, \quad \varphi(e_{12}) = c, \quad \varphi(e_{34}) = c^*,$$
is clearly an isomorphism of superalgebras. Moreover, for all $X \in M$, $\varphi(X^\circ) = \varphi(X)^{trp}$ and so M and C are isomorphic as $*$-algebras. This implies $M \cong C \in \mathrm{var}^*((M_{k,k}(F), trp)) \subseteq \mathrm{var}^*(A)$, a contradiction. □

Lemma 4.2. *If $M \notin var^*(A)$, then $(M_{k,2l}(F), osp) \notin var^*(A)$, for any $k \geq 2$ or $l \geq 2$.*

Proof. Suppose by contradiction that $(M_{k,2l}(F), osp) \in \mathrm{var}^*(A)$.

First let $l \geq 2$. We consider the following four even elements of $(M_{k,2l}(F), osp)$:

$$\begin{aligned} a &= e_{k+1,k+1} + e_{k+l+1,k+l+1}, \quad b = e_{k+1,k+l+2} - e_{k+l+1,k+2}, \\ c &= e_{k+l+2,k+1} - e_{k+2,k+l+1}, \quad d = e_{k+2,k+2} + e_{k+l+2,k+l+2}. \end{aligned}$$

Let $C = \mathrm{span}_F\{a, b, c, d\}$ be a subalgebra of $(M_{k,2l}(F), osp)$ with induced superinvolution. If $\{e_{11}, e_{12}, e_{21}, e_{22}\}$ is a basis of the $*$-algebra $(M_2(F), t)$, endowed with trivial grading and transpose involution, then the linear map $\varphi : C \to M_2(F)$, such that

$$\varphi(a) = e_{11},\ \varphi(b) = e_{12},\ \varphi(c) = e_{21},\ \varphi(d) = e_{22},$$

is an isomorphism of $*$-algebras. Moreover, by [28, Remark 3.2] and recalling that here the grading is trivial, we have that $M \in \mathrm{var}^*((M_2(F), t)) = \mathrm{var}^*(C) \subseteq \mathrm{var}^*((M_{k,2l}(F), osp)) \subseteq \mathrm{var}^*(A)$, a contradiction.

We are left with the case $l = 0, 1$ and $k \geq 2$. Let C be the subalgebra of $(M_{k,2l}(F), osp)$ generated by $e_{11}, e_{12}, e_{21}, e_{22}$. Clearly, C is a $*$-algebra with induced grading (trivial) isomorphic to $(M_2(F), t)$. Hence we reach a contradiction as before and we are done also in this case. $\square$

Lemma 4.3. *If $M^{sup} \notin var^*(A)$, then $(M_{1,2}(F), osp) \notin var^*(A)$.*

Proof. In order to prove the lemma we shall show that $M^{sup} \in \mathrm{var}^*(M_{1,2}(F), osp)$. If not, we would have that $\mathrm{Id}^*((M_{1,2}(F), osp)) \nsubseteq \mathrm{Id}^*(M^{sup})$ and so there would exist a non-zero multilinear polynomial f such that $f \in \mathrm{Id}^*((M_{1,2}(F), osp))$ and $f \notin \mathrm{Id}^*(M^{sup})$. In order to reach a contradiction we need only to show that f is actually the zero polynomial. Since by [10, Theorem 6.3] $\mathrm{Id}^*(M^{sup}) = \langle y^-, z_1 z_2 \rangle_{T_2^*}$, then f is either $f = \alpha z$ (when $n = 1$) or $f = \beta y_1^+ \cdots y_n^+$ or

$$f = \sum_I \alpha_I y_{i_1}^+ \cdots y_{i_k}^+ z y_{j_1}^+ \cdots y_{j_{n-k-1}}^+, \tag{7}$$

with $i_1 < \cdots < i_k$, $j_1 < \cdots < j_{n-k-1}$ and $I = \{i_1, \ldots, i_k\}$.

In the first two cases, it is easy to see, by making a suitable evaluation, that f must be the zero polynomial.

Now, consider the case in which f is as in (7). Suppose that there exists $\alpha_I \neq 0$, for some I. Then by making the evaluation $y_{i_1}^+ = \cdots = y_{i_k}^+ = e_{11}$, $y_{j_1}^+ = \cdots = y_{j_{n-k-1}}^+ = e_{22} + e_{33}$ and $z = e_{12}$, one gets $\alpha_I e_{12} = 0$. Thus $\alpha_I = 0$, a contradiction. Therefore we have proved that f is the zero polynomial and we are done. $\square$

Lemma 4.4. *If $M^{sup} \notin var^*(A)$, then $B = (M_{k,h}(F) \oplus M_{k,h}(F)^{sop}, ex) \notin var^*(A)$, for any k, h such that $k + h > 1$.*

Proof. Suppose by contradiction that $B = (M_{k,h}(F) \oplus M_{k,h}(F)^{sop}, ex) \in \mathrm{var}^*(A)$.

Let $l = k + h - 1$ and let $C = C_0 \oplus C_1$ be a subalgebra of B with

$$C_0 = \mathrm{span}_F \left\{ \underbrace{(e_{11}, e_{11})}_{a}, \underbrace{(e_{l,l}, e_{l,l})}_{b} \right\} \text{ and } C_1 = \mathrm{span}_F \left\{ \underbrace{(e_{1,l}, 0)}_{c}, \underbrace{(0, e_{1,l})}_{c^*} \right\}.$$

The map $\varphi : M^{sup} \to C$ given by

$$\varphi(e_{11} + e_{44}) = a, \quad \varphi(e_{22} + e_{33}) = b, \quad \varphi(e_{12}) = c, \quad \varphi(e_{34}) = c^*,$$

is clearly an isomorphism of superalgebras such that, for all $X \in M^{sup}$, $\varphi(X^\circ) = \varphi(X)^{ex}$. Hence M^{sup} and C are isomorphic as $*$-algebras and this implies that $M^{sup} \cong C \in \mathrm{var}^*(B) \subseteq \mathrm{var}^*(A)$, a contradiction. $\square$

With a similar argument, one can also prove the following lemma.

Lemma 4.5. *If $M \notin var^*(A)$, then $(Q(n) \oplus Q(n)^{sop}, ex) \notin var^*(A)$, for any $n > 1$.*

We say that a commutative $*$-algebra $A = A_0 \oplus A_1$ is endowed with the trivial superinvolution if $A_1 = 0$ and $*$ is the identity map.

By putting together the previous lemmas, we get the following result.

Corollary 4.1. *Let A be a finite dimensional $*$-algebra with Wedderburn-Malcev decomposition $A = A_1 \oplus \cdots \oplus A_m + J$. If $M, M^{sup} \notin var^*(A)$, then for each $i = 1, \ldots, m$, either $A_i \cong F$ with trivial superinvolution or $A_i \cong D$ or $A_i \cong (M_{1,1}(F), trp)$ or $A_i \cong (M_{0,2}(F), osp)$ or $A_i \cong (Q(1) \oplus Q(1)^{sop}, ex)$.*

Finally, we are in a position to prove the main theorem of this section.

Theorem 4.3. *Let A be a finite dimensional $*$-algebra such that $M, M^{sup} \notin var^*(A)$. Then $var^*(A) = var^*(B_1 \oplus \cdots \oplus B_m)$, where, for each $i = 1, \ldots, m$, B_i is isomorphic to one of the following algebras:*

1. *$F + J_i$, with trivial superinvolution on F,*
2. *$D + J_i$ and exchange superinvolution on D,*
3. *$M_{1,1}(F) + J_i$ and transpose superinvolution on $M_{1,1}(F)$,*
4. *$M_{0,2}(F) + J_i$ and orthosymplectic superinvolution on $M_{0,2}(F)$,*
5. *$Q(1) \oplus Q(1)^{sop} + J_i$ and exchange superinvolution on $Q(1) \oplus Q(1)^{sop}$,*

where J_i is the Jacobson radical of B_i.

Proof. By Corollary 4.1, we can decompose $A = A_1 \oplus \cdots \oplus A_m + J$, where, for each $i = 1, \ldots, m$, A_i is isomorphic either to F with trivial superinvolution or to D with exchange superinvolution or to $M_{1,1}(F)$ with transpose superinvolution or to $M_{0,2}(F)$ with orthosymplectic superinvolution or to $Q = Q(1) \oplus Q(1)^{sop}$ with exchange superinvolution.

Suppose by contradiction that there exist two $*$-simple components, say A_1 and A_2, such that $A_1 J A_2 \neq 0$. Hence $a_1 u a_2 \neq 0$, for some $a_1 \in A_1$, $u \in J$ and $a_2 \in A_2$. It clearly follows that $e_1 u e_2 \neq 0$, where e_1 and e_2 are the unit elements of A_1 and A_2, respectively.

Let $B = A_1 \oplus A_2 + J$ be an algebra with induced superinvolution $*$ and let $k \geq 1$ be such that $u \in J^k$ and $u \notin J^{k+1}$. We set $\bar{B} = B/J^{k+1}$ and we can write $\bar{B} = C_1 \oplus C_2 + \bar{J}$, where $C_i \cong A_i$, $i = 1, 2$ and $\bar{J}$ is the Jacobson radical of $\bar{B}$. Since J^{k+1} is stable under $*$, then $\bar{B}$ has induced superinvolution. Write $\bar{a} = \bar{e_1}$ and $\bar{b} = \bar{e_2}$ for the images of e_1 and e_2, respectively. If $\bar{c} = \bar{a}\bar{u}\bar{b}$, then $\bar{c}^* = \bar{b}\bar{u}^*\bar{a}$.

We now define the algebra $R = \operatorname{span}\{\bar{a}, \bar{b}, \bar{c}, \bar{c}^*\}$. It is easy to check that R has the same multiplication table of the algebra $M = F(e_{11} + e_{44}) \oplus F(e_{22} + e_{33}) \oplus F e_{12} \oplus F e_{34}$. Therefore we get that R is isomorphic to the algebra M with trivial grading and reflection superinvolution or to the algebra M^{sup}, i.e., the algebra M with natural grading and reflection superinvolution, according to the homogeneous degree of $u \in J$ (and so of $\bar{c}$). But in both cases we reach a contradiction. Thus we must have that

$$A_i J A_k = A_i A_k = 0, \text{ for all } i \neq k. \tag{8}$$

Set $B_i = A_i + J$, $i = 1, \ldots, m$. Then $A = A_1 \oplus \cdots \oplus A_m + J = (A_1 + J) + \cdots + (A_m + J) = B_1 + \cdots + B_m$. Furthermore, for each $i = 1, \ldots, m$, $J_i \subseteq B_i$ is the Jacobson radical of B_i and $B_i/J_i \cong A_i$. Hence, each B_i is isomorphic to one of the algebras 1., 2., 3., 4. or 5.. By standard arguments (see for example [28, Lemma 4.6]), we get that

$$\mathrm{Id}^*(B_1 + \cdots + B_m) = \mathrm{Id}^*(B_1) \cap \cdots \cap \mathrm{Id}^*(B_m).$$

Since $A = B_1 + \cdots + B_m$ and $\mathrm{Id}^*(B_1) \cap \cdots \cap \mathrm{Id}^*(B_m) = \mathrm{Id}^*(B_1 \oplus \cdots \oplus B_m)$, this implies that $\mathrm{Id}^*(A) = \mathrm{Id}^*(B_1 \oplus \cdots \oplus B_m)$. Hence $\mathrm{var}^*(A) = \mathrm{var}^*(B_1 \oplus \cdots \oplus B_m)$ and the proof is complete. □

5. Classifying $*$-algebras with bounded multiplicities of the cocharacter

In this section we shall prove the main theorem of this paper, dealing with finitely generated $*$-algebras with multiplicities of the $(n_1, \ldots, n_4)$-cocharacter bounded by a constant.

Recall that, according to the representation theory of GL_n, if a $*$-algebra A has $(n_1, \ldots, n_4)$-cocharacter

$$\chi_{n_1, \ldots, n_4}(A) = \sum_{\langle \lambda \rangle \vdash (n_1, \ldots, n_4)} m_{\langle \lambda \rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}, \tag{9}$$

where $\langle \lambda \rangle = (\lambda(1), \ldots, \lambda(4)) \vdash (n_1, \ldots, n_4)$, then $m_{\langle \lambda \rangle}$ is the maximal number of linearly independent highest weight vectors associated to the multipartition $\langle \lambda \rangle$. Moreover, a highest weight vector is obtained from the polynomial corresponding to an essential idempotent by identifying the variables whose indices lie in the same row of the corresponding Young tableau (see for instance [4, Chapter 12]).

If $\dim_F A_0^+ = d_1$, $\dim_F A_0^- = d_2$, $\dim_F A_1^+ = d_3$ and $\dim_F A_1^- = d_4$, then, in (9), we get $m_{\langle\lambda\rangle} \neq 0$ if and only if $h(\lambda(i)) \leq d_i$, for all $1 \leq i \leq 4$ (same idea of [5, Lemma 1.2]). Here $h(\lambda(i))$ stands for the high of the partition $\lambda(i)$, i.e., the number of the rows of $\lambda(i)$.

We start by proving the following lemma.

Lemma 5.1. *Let $A = C + J$ be a finite dimensional $*$-algebra, where $J = J(A)$ is its Jacobson radical and C is a $*$-simple subalgebra of A isomorphic to either $(M_{1,1}(F), trp)$ or $(Q(1) \oplus Q(1)^{sop}, ex)$. If the $(n_1, \ldots, n_4)$-cocharacter of A has decomposition as in (9), then there exist a constant N such that $m_{\langle\lambda\rangle} \leq N$, for all $\langle\lambda\rangle \vdash (n_1, \ldots, n_4)$.*

Proof. In the proof we follow the idea of [23, Lemma 7].

First, let C be isomorphic to $M_{1,1}(F)$ endowed with the transpose superinvolution and let $d = \dim_F A$. By hypothesis, one can choose

$$\{a_0, a_1, \ldots, a_{d_1-1}\}, \quad \{b_0, b_1, \ldots, b_{d_2-1}\}, \quad \{c_0, c_1, \ldots, c_{d_3-1}\}, \quad \{e_0, e_1, \ldots, e_{d_4-1}\},$$

basis of A_0^+, A_0^-, A_1^+ and A_1^-, respectively, such that $a_0 \in C_0^+$, $a_1, \ldots, a_{d_1-1} \in J_0^+$, $b_0 \in C_0^-$, $b_1, \ldots, b_{d_2-1} \in J_0^-$, $c_0 \in C_1^+$, $c_1, \ldots, c_{d_3-1} \in J_1^+$, $e_0 \in C_1^-$ and $e_1, \ldots, e_{d_4-1} \in J_1^-$. Moreover, let q be the smallest positive integer such that $J^q = 0$.

Notice that if $q = 1$, then $A \cong M_{1,1}(F)$ and, by [8, Theorem 5.1], we get that the multiplicities in (9) are bounded by a constant. So let us suppose $q \geq 2$ and prove that $m_{\langle\lambda\rangle} \leq N = d(q^d)^{d_1 d_2 d_3 d_4}$, for all $\langle\lambda\rangle \vdash (n_1, \ldots, n_4)$.

According to the considerations above, we may assume that $h(\lambda(i)) \leq d_i$, for all $1 \leq i \leq 4$. Let $\big(T_{\lambda(1)}, \ldots, T_{\lambda(4)}\big)$ be Young tableaux corresponding to $\langle\lambda\rangle$ and define, for all $1 \leq i \leq 4$,

$$R^+_{T_{\lambda(i)}} = \sum_{\sigma \in R_{T_{\lambda(i)}}} \sigma \quad \text{and} \quad C^-_{T_{\lambda(i)}} = \sum_{\sigma \in C_{T_{\lambda(i)}}} \operatorname{sgn}(\sigma)\sigma,$$

where $R_{T_{\lambda(i)}}$ and $C_{T_{\lambda(i)}}$ are the row and column stabilizers of $T_{\lambda(i)}$, respectively.

It is well-known that, for all $1 \leq i \leq 4$, the element $e_{T_{\lambda(i)}} = R^+_{T_{\lambda(i)}} C^-_{T_{\lambda(i)}}$ is an essential idempotent in the group algebra FS_{n_i}. Similarly, the element $e = e_{T_{\lambda(1)}} e_{T_{\lambda(2)}} e_{T_{\lambda(3)}} e_{T_{\lambda(4)}}$ is an essential idempotent in the group algebra $F(S_{n_1} \times S_{n_2} \times S_{n_3} \times S_{n_4})$.

Fixed $1 \leq i \leq 2$, for all $1 \leq j_i \leq d_i$ let $Y_{j_i}^{\lambda(i)}$ be the set of even variables (resp. symmetric or skew) whose indices lie in the j_i-th row of $T_{\lambda(i)}$. Similarly, fixed $3 \leq i \leq 4$, for all $1 \leq j_i \leq d_i$, let $Z_{j_i}^{\lambda(i)}$ be the set of odd variables (resp. symmetric or skew) whose indices lie in the j_i-th row of $T_{\lambda(i)}$. It turns out that, for all $f \in P_{n_1,\ldots,n_4}$, the polynomial ef is symmetric on each set $Y_1^{\lambda(1)}, \ldots, Y_{d_1}^{\lambda(1)}, Y_1^{\lambda(2)}, \ldots, Y_{d_2}^{\lambda(2)}, Z_1^{\lambda(3)}, \ldots, Z_{d_3}^{\lambda(3)}, Z_1^{\lambda(4)}, \ldots, Z_{d_4}^{\lambda(4)}$ and furthermore its variables are partitioned into the disjoint union of $d_1 + d_2 + d_3 + d_4 = d$ subsets

$$\underbrace{Y_1^{\lambda(1)} \cup \cdots \cup Y_{d_1}^{\lambda(1)}}_{\text{symmetric even variables}} \cup \underbrace{Y_1^{\lambda(2)} \cup \cdots \cup Y_{d_2}^{\lambda(2)}}_{\text{skew even variables}} \cup \underbrace{Z_1^{\lambda(3)} \cup \cdots \cup Z_{d_3}^{\lambda(3)}}_{\text{symmetric odd variables}} \cup \underbrace{Z_1^{\lambda(4)} \cup \cdots \cup Z_{d_4}^{\lambda(4)}}_{\text{skew odd variables}}.$$

In order to simplify the notation, let us denote by $X_{T_{\langle\lambda\rangle}}$ such a decomposition.

Remark that, for all $\sigma_i \in S_{n_i}$, $1 \leq i \leq 4$, $\sigma_i e_{T_{\lambda(i)}} \neq 0$. Hence, by setting $\eta = (\sigma_1, \ldots, \sigma_4)$, we get $\eta e \neq 0$. This implies that if $ef \neq 0$, where f is a multilinear $*$-polynomial, then ef and ηef generate the same irreducible $(S_{n_1} \times \cdots \times S_{n_4})$-module.

Take any $f_1, \ldots, f_m$ multilinear $*$-polynomials generating different but isomorphic irreducible $(S_{n_1} \times \cdots \times S_{n_4})$-modules corresponding to the same multipartition $\langle\lambda\rangle$. By the above remark, we can choose permutations $\eta_1, \ldots, \eta_m \in S_{n_1} \times \cdots \times S_{n_4}$ and a decomposition $X_{T_{\langle\lambda\rangle}}$ such that $\eta_1 f_1, \ldots, \eta_m f_m$ are simultaneously symmetric on $Y_{j_i}^{\lambda(i)}$, $1 \leq i \leq 2$, and $Z_{j_i}^{\lambda(i)}$, $3 \leq i \leq 4$. Thus, without loss of generality, we may assume that $f_1, \ldots, f_m$ satisfy this condition.

Let us now assume by contradiction that $m = m_{\langle\lambda\rangle} > N = d(q^d)^{d_1 d_2 d_3 d_4}$. If we prove that A satisfies a $*$-identity of the type

$$f = \mu_1 f_1 + \cdots + \mu_m f_m, \tag{10}$$

where $\mu_1, \dots, \mu_m \in F$ are not all zero, then we reach the contradiction, since this would imply that $f_1, \dots, f_m$ are linearly dependent modulo $\mathrm{Id}^*(A)$.

Since $f_1, \dots, f_m$ are multilinear, in order to prove that $f \equiv 0$, we can evaluate it only for elements of a basis of A. First, let us define substitutions of a special kind. We consider non-negative integers

$$\alpha_{j_1 0}^{\lambda(1)}, \dots, \alpha_{j_1(d_1-1)}^{\lambda(1)}, \beta_{j_2 0}^{\lambda(2)}, \dots, \beta_{j_2(d_2-1)}^{\lambda(2)}, \gamma_{j_3 0}^{\lambda(3)}, \dots, \gamma_{j_3(d_3-1)}^{\lambda(3)}, \delta_{j_4 0}^{\lambda(4)}, \dots, \delta_{j_4(d_4-1)}^{\lambda(4)}$$

such that, for all $1 \le j_i \le d_i$ and $1 \le i \le 4$,

$$\sum_{i=0}^{d_1-1} \alpha_{j_1 i}^{\lambda(1)} = |Y_{j_1}^{\lambda(1)}|, \qquad \sum_{i=0}^{d_2-1} \beta_{j_2 i}^{\lambda(2)} = |Y_{j_2}^{\lambda(2)}|, \qquad \sum_{i=0}^{d_3-1} \gamma_{j_3 i}^{\lambda(3)} = |Z_{j_3}^{\lambda(3)}|, \qquad \sum_{i=0}^{d_4-1} \delta_{j_4 i}^{\lambda(4)} = |Z_{j_4}^{\lambda(4)}|.$$

For $j_i = 1, \dots d_i$, $1 \le i \le 4$, we set $X_{j_1,\dots,j_4}^{\langle\lambda\rangle} = Y_{j_1}^{\lambda(1)} \cup Y_{j_2}^{\lambda(2)} \cup Z_{j_3}^{\lambda(3)} \cup Z_{j_4}^{\lambda(4)}$. We say that an evaluation φ has type

$$\left(\alpha_{j_1 0}^{\lambda(1)}, \dots, \alpha_{j_1(d_1-1)}^{\lambda(1)}, \beta_{j_2 0}^{\lambda(2)}, \dots, \beta_{j_2(d_2-1)}^{\lambda(2)}, \gamma_{j_3 0}^{\lambda(3)}, \dots, \gamma_{j_3(d_3-1)}^{\lambda(3)}, \delta_{j_4 0}^{\lambda(4)}, \dots, \delta_{j_4(d_4-1)}^{\lambda(4)}\right),$$

if we replace the variables in the following way: for any fixed j_1, j_2, j_3 and j_4, we evaluate the first $\alpha_{j_1 0}^{\lambda(1)}$ symmetric even variables from $X_{j_1,\dots,j_4}^{\langle\lambda\rangle}$ for a_0, the next $\alpha_{j_1 1}^{\lambda(1)}$ symmetric even variables from $X_{j_1,\dots,j_4}^{\langle\lambda\rangle}$ for a_1, and so on up to the last $\alpha_{j_1(d_1-1)}^{\lambda(1)}$ symmetric even variables for a_{d_1-1}. Similarly, we replace the first $\beta_{j_2 0}^{\lambda(2)}$ skew even variables from $X_{j_1,\dots,j_4}^{\langle\lambda\rangle}$ for b_0, and so on up to the last $\beta_{j_2(d_2-1)}^{\lambda(2)}$ symmetric even variables for b_{d_2-1}. An analogous evaluation will be made by taking into account the symmetric and skew odd variables and the basis $\{c_0, c_1, \dots, c_{d_3-1}\}$ and $\{e_0, e_1, \dots, e_{d_4-1}\}$ of A_1^+ and A_1^-, respectively.

In order to get a non-zero value of f in (10), we have to consider the nilpotency of J. Thus we get the following conditions:

1) $\displaystyle\sum_{i=1}^{d_1-1} \alpha_{j_1 i}^{\lambda(1)} \le q-1,$
2) $\displaystyle\sum_{i=1}^{d_2-1} \beta_{j_2 i}^{\lambda(2)} \le q-1,$
3) $\displaystyle\sum_{i=1}^{d_3-1} \gamma_{j_3 i}^{\lambda(3)} \le q-1,$
4) $\displaystyle\sum_{i=1}^{d_4-1} \delta_{j_4 i}^{\lambda(4)} \le q-1.$

Besides them, by definition we have some additional restrictions:

5) $\displaystyle\alpha_{j_1 0}^{\lambda(1)} = |Y_{j_1}^{\lambda(1)}| - \sum_{i=1}^{d_1-1} \alpha_{j_1 i}^{\lambda(1)},$
6) $\displaystyle\beta_{j_2 0}^{\lambda(2)} = |Y_{j_2}^{\lambda(2)}| - \sum_{i=1}^{d_2-1} \beta_{j_2 i}^{\lambda(2)},$
7) $\displaystyle\gamma_{j_3 0}^{\lambda(3)} = |Z_{j_3}^{\lambda(3)}| - \sum_{i=1}^{d_3-1} \gamma_{j_3 i}^{\lambda(3)},$
8) $\displaystyle\delta_{j_4 0}^{\lambda(4)} = |Z_{j_4}^{\lambda(4)}| - \sum_{i=1}^{d_4-1} \delta_{j_4 i}^{\lambda(4)}.$

By taking into account conditions 1) - 8), it is clear that the number of distinct d_i-tuples $\left(\varepsilon_{j_i 0}^{\lambda(i)}, \dots, \varepsilon_{j_i(d_i-1)}^{\lambda(i)}\right)$, $\varepsilon \in \{\alpha, \beta, \gamma, \delta\}$, is less than d_i, $1 \le i \le 4$.

It follows that the overall number of distinct special substitutions is at most $q^{d_1} q^{d_2} q^{d_3} q^{d_4} = q^{d_1+d_2+d_3+d_4} = q^d$, for given $1 \le j_i \le d_i$ and $1 \le i \le 4$. Since the number of 4-tuples $(j_1, \dots, j_4)$ is $d_1 d_2 d_3 d_4$, it follows that the number $\widetilde{N}$ of distinct types of substitutions is less than $N_0 = (q^d)^{d_1 d_2 d_3 d_4}$.

Let us consider now all these $\widetilde{N}$ special substitutions $\varphi_1, \ldots, \varphi_{\widetilde{N}}$ and construct the matrix (u_{ij}), where, for all $1 \leq i \leq m$ and $1 \leq j \leq \widetilde{N}$,

$$\varphi_j(f_i) = u_{ij}.$$

This matrix has m rows and $\widetilde{N}$ columns of elements of A. Since we are assuming that $m > N = dN_0 > \widetilde{N}$, we have that the rows of (u_{ij}) are linearly dependent. Hence there exist $\mu_1, \ldots, \mu_m \in F$ not all zero such that

$$\sum_{i=1}^{m} \mu_i u_{ij} = 0, \text{ for all } 1 \leq j \leq \widetilde{N}.$$

Thus

$$0 = \sum_{i=1}^{m} \mu_i(\varphi_j(f_i)) = \varphi_j \left(\sum_{i=1}^{m} \mu_i f_i \right),$$

for all $1 \leq j \leq \widetilde{N}$. This means that the polynomial $f = \sum_{i=1}^{m} \mu_i f_i$ is zero under all special substitutions $\varphi_1, \ldots, \varphi_{\widetilde{N}}$. Now it suffices to show that this implies that $f \in \mathrm{Id}^*(A)$.

To this end, let ρ be any substitutions of the variables of f in the elements of the basis of A_0^+, A_0^-, A_1^+ and A_1^-. Let $l_{j_1 0}^{\lambda(1)}$ be the number of variables in $Y_{j_1}^{\lambda(1)}$ mapped by ρ to a_0, $l_{j_1 1}^{\lambda(1)}$ be the number of variables in $Y_{j_1}^{\lambda(1)}$ mapped by ρ to a_1 and so on. Similarly, let $l_{j_2 0}^{\lambda(2)}$ be the number of variables in $Y_{j_2}^{\lambda(2)}$ mapped by ρ to b_0, $l_{j_2 1}^{\lambda(2)}$ be the number of variables in $Y_{j_2}^{\lambda(2)}$ mapped by ρ to b_1 and so on. Analogously, let $l_{j_3 k}^{\lambda(3)}$ and $l_{j_4 h}^{\lambda(4)}$, $1 \leq k \leq d_3 - 1$, $1 \leq h \leq d_4 - 1$, be the number of variables in $Z_{j_3}^{\lambda(3)}$ and $Z_{j_4}^{\lambda(4)}$ mapped by ρ to c_k and e_h, respectively. Since f is symmetric on each $Y_1^{\lambda(1)}, \ldots, Y_{d_1}^{\lambda(1)}, Y_1^{\lambda(2)}, \ldots, Y_{d_2}^{\lambda(2)}, Z_1^{\lambda(3)}, \ldots, Z_{d_3}^{\lambda(3)}, Z_1^{\lambda(4)}, \ldots, Z_{d_4}^{\lambda(4)}$, we get that, for all $\eta \in S_{n_1} \times \cdots \times S_{n_4}$ such that $\eta(Y_{j_1}^{\lambda(1)}) = Y_{j_1}^{\lambda(1)}$, $\eta(Y_{j_2}^{\lambda(2)}) = Y_{j_2}^{\lambda(2)}$, $\eta(Z_{j_3}^{\lambda(3)}) = Z_{j_3}^{\lambda(3)}$ and $\eta(Z_{j_4}^{\lambda(4)}) = Z_{j_4}^{\lambda(4)}$, for all $1 \leq j_i \leq d_i$ and $1 \leq i \leq 4$, we have

$$\rho(f) = \rho(\eta f) = (\rho\eta) f.$$

In particular, we can choose η such that $\rho\eta$ is the special substitution of the type

$$\left(l_{j_1 0}^{\lambda(1)}, \ldots, l_{j_1 (d_1-1)}^{\lambda(1)}, l_{j_2 0}^{\lambda(2)}, \ldots, l_{j_2 (d_2-1)}^{\lambda(2)}, l_{j_3 0}^{\lambda(3)}, \ldots, l_{j_3 (d_3-1)}^{\lambda(3)}, l_{j_4 0}^{\lambda(4)}, \ldots, l_{j_4 (d_4-1)}^{\lambda(4)} \right).$$

According to what was proved above, $\rho(f) = (\rho\eta)f = 0$ and $f \in \mathrm{Id}^*(A)$, a contradiction. Hence we must have $m_{\langle\lambda\rangle} \leq N$, for all $\langle\lambda\rangle \vdash (n_1, \ldots, n_4)$ and we get the desired conclusion.

A similar proof holds also in the case $C \cong Q(1) \oplus Q(1)^{sop}$, so we omit it. □

Lemma 5.2. [28, Lemma 5.2] *Let $A = C + J$ be a finite dimensional $*$-algebra, where $J = J(A)$ is its Jacobson radical and C is a $*$-simple subalgebra of A isomorphic to either F with trivial superinvolution or D with trivial grading and exchange superinvolution. If the $(n_1, \ldots, n_4)$-cocharacter of A has decomposition as in* (9)*, then there exist a constant N such that $m_{\langle\lambda\rangle} \leq N$, for all $\langle\lambda\rangle \vdash (n_1, \ldots, n_4)$.*

We highlight that in case $C \cong (M_{0,2}(F), osp)$, some computational difficulties arise. In fact, it seems that the technique from Lemma 5.1 does not work anymore. Notice that in the proof of [28, Lemma 5.1] there is some gap, hence we can only conjecture the following.

Conjecture 5.1. *Let $A = C + J$ be a finite dimensional $*$-algebra, where $J = J(A)$ is its Jacobson radical and C is a $*$-simple subalgebra of A isomorphic to $(M_{0,2}(F), osp)$. If the $(n_1, \ldots, n_4)$-cocharacter of A has decomposition as in* (9)*, then there exist a constant N such that $m_{\langle\lambda\rangle} \leq N$, for all $\langle\lambda\rangle \vdash (n_1, \ldots, n_4)$.*

Now we are ready to prove one of the main results of this paper.

Theorem 5.1. *Let A be a finitely generated F-algebra with superinvolution $*$ satisfying an ordinary polynomial identity and let its $(n_1, \ldots, n_4)$-cocharacter be as in* (9)*. If $M, M^{sup}, (M_{0,2}(F), osp) \notin var^*(A)$ then there exists a constant N such that, for all $\langle\lambda\rangle \vdash (n_1, \ldots, n_4)$, we have the inequality*

$$m_{\langle\lambda\rangle} \leq N.$$

Proof. By Theorem 3.1 we may assume that A is finite dimensional. Now, by using item 2 of Remark 2.1, Theorem 4.3 and Lemmas 5.1, 5.2, we get the desired conclusion. □

It is remarkable to notice that if Conjecture 5.1 is true, then one can improve the results of Theorem 5.1 by proving the following.

Conjecture 5.2. *Let A be a finitely generated F-algebra with superinvolution $*$ satisfying an ordinary polynomial identity and let its $(n_1,\dots,n_4)$-cocharacter be as in (9). Then $M, M^{sup} \notin$ var$^*(A)$ if and only if there exists a constant N such that, for all $\langle\lambda\rangle \vdash (n_1,\dots,n_4)$, we have the inequality*

$$m_{\langle\lambda\rangle} \leq N.$$

In fact, the first implication can be proved as Theorem 5.1 by considering also Conjecture 5.1.

On the other hand, if $M \in \mathrm{var}^*(A)$ or $M^{sup} \in \mathrm{var}^*(A)$, then by (3) or (5) and Remark 2.1 we get a contradiction.

In the next lemma we find a condition ensuring that the multiplicities in (9) are equal to zero. Here $\lambda(i)_j$ stands for the number of boxes in the j-th row of the partition $\lambda(i)$. Moreover, if $\lambda(i) \vdash n_i$, then $|\lambda(n_i)| = n_i$, $1 \leq i \leq 4$.

Lemma 5.3. *Let A be a finite dimensional $*$-algebra such that $(M,\circ), (M^{sup},\circ) \notin$ var$^*(A)$. Then there exists a constant q such that in (9) we have $m_{\langle\lambda\rangle} = 0$ whenever*

$$(|\lambda(1)| - \lambda(1)_1) + (|\lambda(2)| - (\lambda(2)_1 + \lambda(2)_2 + \lambda(2)_3)) + (|\lambda(3)| - \lambda(3)_1) + (|\lambda(4)| - \lambda(4)_1) \geq q.$$

Proof. Let q be the smallest positive integer such that $J^q = 0$, where J is the Jacobson radical of A. By contradiction, let us suppose that there exists $\langle\lambda\rangle \vdash (n_1,\dots,n_4)$ such that $m_{\langle\lambda\rangle} \neq 0$ and

$$(|\lambda(1)| - \lambda(1)_1) + (|\lambda(2)| - (\lambda(2)_1 + \lambda(2)_2 + \lambda(2)_3)) + (|\lambda(3)| - \lambda(3)_1) + (|\lambda(4)| - \lambda(4)_1) \geq q. \tag{11}$$

Then there exist four Young tableaux $T_{\lambda(1)},\dots,T_{\lambda(4)}$ and $f \in P_{n_1,\dots,n_4}$ such that $ef \notin \mathrm{Id}^*(A)$, where $e = e_{T_{\lambda(1)}} e_{T_{\lambda(2)}} e_{T_{\lambda(3)}} e_{T_{\lambda(4)}}$, and $F(S_{n_1} \times \cdots \times S_{n_4})ef$ is a minimal left ideal of the group algebra $F(S_{n_1} \times \cdots \times S_{n_4})$. Set now $e' = C^-_{T_{\lambda(1)}} e_{T_{\lambda(1)}} \cdots C^-_{T_{\lambda(4)}} e_{T_{\lambda(4)}}$. Since, in general, $0 \neq R^+_{T_\lambda} C^-_{T_\lambda} h$ implies $C^-_{T_\lambda} h \neq 0$, where h is a multilinear polynomial, we immediately get that $e'f$ is not a $*$-identity of A.

Moreover, it is clear that $e'f$ is alternating on each $\lambda(i)_1$ sets of variables corresponding to the columns of $T_{\lambda(i)}$, for all $1 \leq i \leq 4$. In order to get a contradiction, we shall prove that $g = e'f \in \mathrm{Id}^*(A)$.

To this end, since $M, M^{sup} \notin \mathrm{var}^*(A)$, Corollary 4.1 applies and we have that $A = A_1 \oplus \cdots \oplus A_m + J$, where, for each $1 \leq i \leq m$, either $A_i \cong F$ with trivial superinvolution or $A_i \cong D$ or $A_i \cong (M_{1,1}(F), trp)$ or $A_i \cong (M_{0,2}(F), osp)$ or $A_i \cong (Q(1) \oplus Q(1)^{sop}, ex)$. Moreover, by (8) we have that

$$A_i J A_k = 0 \text{ and } A_i A_k = 0, \text{ for all } i \neq k.$$

Thus, in order to get a non-zero value of g, we must evaluate its variables with elements of J and elements of just a single $*$-simple component of A, say A_i.

In each case, $\dim_F (A_i)_0^+ = 1$ and so we can substitute at most one element of $(A_i)_0^+$ in each alternating set of even symmetric variables. A similar argument holds also for the odd part, since $\dim_F (A_i)_1^+ \leq 1$ and $\dim_F (A_i)_1^- \leq 1$.

Finally, $\dim_F (A_i)_0^- \leq 1$ or $\dim_F (A_i)_0^- = 3$, in case $A_i \cong (M_{0,2}(F), osp)$. In the latter case, in order to have a non-zero value of g, we can substitute at most $\lambda(1)_1$ elements from $(A_i)_0^+$ and at most $\lambda(2)_1 + \lambda(2)_2 + \lambda(2)_3$ elements from $(A_i)_0^-$ (here we recall that $(A_i)_1 = 0$, so $\lambda(3) = \lambda(4) = \emptyset$). This means that we have to evaluate at least $(|\lambda(1)| - \lambda(1)_1) + (|\lambda(2)| - (\lambda(2)_1 + \lambda(2)_2 + \lambda(2)_3))$ elements from J. Since we are assuming that $J^q = 0$ and, by hypothesis, that condition (11) holds, we get $g \in \mathrm{Id}^*(A)$, a contradiction.

In case $\dim_F (A_i)_0^- \leq 1$, we can substitute at most one element of $(A_i)_0^-$ in each alternating set of even skew variables. Thus we evaluate at most $\lambda(1)_1$ elements from $(A_i)_0^+$, $\lambda(2)_1$ elements from $(A_i)_0^-$ and eventually $\lambda(3)_1$ elements from $(A_i)_1^+$ and $\lambda(4)_1$ elements from $(A_i)_1^-$, according to $\dim_F (A_i)_1^+$ and $\dim_F (A_i)_1^-$, respectively. By the considerations above, we have that at least

$$(|\lambda(1)| - \lambda(1)_1) + (|\lambda(2)| - \lambda(2)_1) + (|\lambda(3)| - \lambda(3)_1) + (|\lambda(4)| - \lambda(4)_1)$$

variables must be evaluated in elements of J. Since

$$(|\lambda(1)| - \lambda(1)_1) + (|\lambda(2)| - \lambda(2)_1) + (|\lambda(3)| - \lambda(3)_1) + (|\lambda(4)| - \lambda(4)_1) \geq$$
$$(|\lambda(1)| - \lambda(1)_1) + (|\lambda(2)| - (\lambda(2)_1 + \lambda(2)_2 + \lambda(2)_3)) + (|\lambda(3)| - \lambda(3)_1) + (|\lambda(4)| - \lambda(4)_1) \geq q,$$

we obtain that also in this case $g \in \mathrm{Id}^*(A)$, a contradiction. This concludes the proof. $\square$

We are now in a position to prove the following theorem.

Theorem 5.2. *Let A be a finitely generated F-algebra with superinvolution $*$ satisfying an ordinary polynomial identity. If its $(n_1, \ldots, n_4)$-cocharacter is as in (9), then the following conditions are equivalent:*

1. *$M, M^{sup} \notin var^*(A)$.*
2. *There exists a constant q such that, for all $\langle\lambda\rangle \vdash (n_1, \ldots, n_4)$, $m_{\langle\lambda\rangle} = 0$ whenever*

$$(|\lambda(1)| - \lambda(1)_1) + (|\lambda(2)| - (\lambda(2)_1 + \lambda(2)_2 + \lambda(2)_3)) + (|\lambda(3)| - \lambda(3)_1) + (|\lambda(4)| - \lambda(4)_1) \geq q.$$

Proof. Since we may assume that A is finite dimensional, by Lemma 5.3 condition 1. implies condition 2.

Conversely, suppose by contradiction that $M \in \mathrm{var}^*(A)$ or $M^{sup} \in \mathrm{var}^*(A)$. In the first case, according to (3), if $\langle\lambda\rangle = ((p+q,p),(1),\emptyset,\emptyset)$, then $m_{\langle\lambda\rangle} = q+1 > 0$. Thus $m_{\langle\lambda\rangle} \neq 0$, for any multipartition $\langle\lambda\rangle$ such that $\lambda(2) = (1)$, $\lambda(3) = \lambda(4) = \emptyset$ and $|\lambda(1)| - \lambda(1)_1 = q$ arbitrary large. Hence A does not satisfy condition 2. A similar argument holds also in case $M^{sup} \in \mathrm{var}^*(A)$, by considering in (5) $\langle\lambda\rangle = ((p+q,p),\emptyset,(1),\emptyset)$ and thus $m_{\langle\lambda\rangle} = q+1$.

The proof is now complete. $\square$

We conclude this paper by proving the following corollary that relates the growth of the $*$-codimension sequence of a finite dimensional $*$-algebra A with its $*$-colength. Recall that, for all $n \geq 1$ and for all $n_1, \ldots, n_4$ such that $n_1 + \cdots + n_4 = n$, the n-th $*$-colength is defined as

$$l_n^*(A) = \sum_{\substack{\langle\lambda\rangle \vdash (n_1,\ldots,n_4) \\ n_1+\cdots+n_4=n}} m_{\langle\lambda\rangle},$$

where $m_{\langle\lambda\rangle}$ are the multiplicities of the irreducibles appearing in the $(n_1, ..., n_4)$-cocharacter of A.

We want to highlight that a similar result concerning algebras with ordinary polynomial identities was proved by Kemer in [18], by Vieira in [27] in the setting of superalgebras and in [20] for algebras with involution.

In order to prove the last result of this paper we first need the following lemma.

Lemma 5.4. *The $*$-algebra D belongs to the $*$-variety generated by the $*$-algebra $(M_{0,2}(F), osp)$.*

Proof. Let $D' = \mathrm{span}_F\{e_{11} + e_{22}, e_{11} - e_{22}\}$ be a subalgebra of $M_{0,2}(F)$ spanned by two elements of homogeneous degree zero and with induced superinvolution. Now let $\{(1,1),(1,-1)\}$ be a basis of the $*$-algebra D. The linear map $\varphi : D \to D'$ given by

$$\varphi((1,1)) = e_{11} + e_{22}, \qquad \varphi((1,-1)) = e_{11} - e_{22},$$

is clearly an isomorphism of superalgebras. Moreover,

$$\varphi\left((1,1)^{ex}\right) = \varphi(1,1) = e_{11} + e_{22} = \left(\varphi(1,1)\right)^{osp},$$
$$\varphi\left((1,-1)^{ex}\right) = \varphi(-1,1) = -e_{11} + e_{22} = \left(\varphi(1,-1)\right)^{osp},$$

and so φ is an isomorphism of $*$-algebras and the proof is complete. $\square$

Corollary 5.1. *Let A be a finite dimensional algebra with superinvolution $*$ over a field of characteristic zero. Then $c_n^*(A)$ is polynomially bounded if and only if $l_n^*(A) \leq L$, for some constant L and for all $n \geq 1$.*

Proof. First, let us suppose that $c_n^*(A)$ is polynomially bounded, say $c_n^*(A) \approx an^k$, for some integer k and $a > 0$. Then, by Theorem 3.2, it follows that $D, M, M^{sup} \notin \mathrm{var}^*(A)$. Let

$$\chi_{n_1,\ldots,n_4}(A) = \sum_{\langle\lambda\rangle \vdash (n_1,\ldots,n_4)} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}$$

be the $(n_1, \ldots, n_4)$-cocharacter of A. By Lemma 5.4 it follows that also $(M_{0,2}(F), osp) \notin \mathrm{var}^*(A)$. Hence Theorem 5.1 applies and so there exists a constant M such that $m_{\langle\lambda\rangle} \leq M$, for all multipartition $\langle\lambda\rangle$.

Furthermore, by Lemma 5.3, there exists q such that $m_{\langle\lambda\rangle} = 0$ whenever

$$(|\lambda(1)| - \lambda(1)_1) + (|\lambda(2)| - (\lambda(2)_1 + \lambda(2)_2 + \lambda(2)_3)) + (|\lambda(3)| - \lambda(3)_1) + (|\lambda(4)| - \lambda(4)_1) \geq q. \tag{12}$$

On the other hand, since $D \notin \mathrm{var}^*(A)$, by [7, Theorem 2.1], there exists $s \geq 1$ such that

$$y_1^- w_1 y_2^- w_2 \cdots y_s^- w_s \equiv 0 \text{ on } A, \tag{13}$$

where the $w_i's$ are (eventually empty) words in even variables (symmetric or skew).

Moreover, since $c_n^*(A) \approx an^k$, we get that $z_1 z_2 \cdots z_{k+1} \equiv 0$ on A, where the $z_i's$ are any odd variables (consequence of [19, Theorem 5.1]). It trivially follows that

$$z_1 w_1 z_2 w_2 \cdots z_{k+1} w_{k+1} \equiv 0 \text{ on } A, \tag{14}$$

where the $w_i's$ are (eventually empty) words in any kind of variables.

From (12) we trivially get that $m_{\langle\lambda\rangle} = 0$ for any multipartition $\langle\lambda\rangle$ such that $|\lambda(1)| - \lambda(1)_1 \geq q$. Similarly, due to to identity (14), if $|\lambda(3)| + |\lambda(4)| \geq k+1$ than $m_{\langle\lambda\rangle} = 0$.

We are left to find a condition on $|\lambda(2)|$. We claim that, as soon as $m_{\langle\lambda\rangle} \neq 0$, it has to be $|\lambda(2)| < s(k+1)$. In fact, suppose by contradiction, that $m_{\langle\lambda\rangle} \neq 0$ and $|\lambda(2)| \geq s(k+1)$. Fix $|\lambda(3)| + |\lambda(4)| = t$, where $0 \leq t \leq k$. Then in each monomial of the corresponding highest weight vectors there are at most $t+1$ groups of even variables. Since $t+1$ is at most $k+1$ and we are assuming $|\lambda(2)| \geq s(k+1)$, it follows that there exists a group of even variables containing at least s even skew variables. Thus by (13), we get that each highest weight vector is an identity and $m_{\langle\lambda\rangle} = 0$, a contradiction. The claim is proved.

Thus $m_{\langle\lambda\rangle} = 0$, $\langle\lambda\rangle \vdash (n_1, \ldots, n_4)$, for $n = n_1 + \cdots + n_4$ large enough.

Hence only a finite number of multipartitions $\langle\lambda\rangle$ satisfies the conditions $|\lambda(1)| - \lambda(1)_1 < q$, $|\lambda(2)| < s(k+1)$ and $|\lambda(3)| + |\lambda(4)| < k+1$ and since $m_{\langle\lambda\rangle} \leq M$, for all $\langle\lambda\rangle$, if follows that, for any $(n_1, \ldots, n_4)$,

$$l_n^*(A) \leq L, \text{ for some constant } L.$$

Conversely, let us now assume that $l_n^*(A) \leq L$, for some L. By using Remark 2.1 and equations (2), (4) and (6), we get that $D, M, M^{sup} \notin \mathrm{var}^*(A)$. Thus by Theorem 3.2, $c_n^*(A)$ must be polynomially bounded and this complete the proof. □

Remark that, due to Theorem 3.1, the previous corollary holds also in case of finitely generated $*$-algebras.

References

[1] E. Aljadeff, A. Giambruno, Y. Karasik, *Polynomial identities with involution, superinvolutions and the Grassmann envelope*, Proc. Amer. Math. Soc. **145** (2017), no. 5, 1843–1857.

[2] Yu Bahturin, M. Tvalavadze, T. Tvalavadze, *Group gradings on superinvolution simple superalgebras*, Linear Algebra Appl. **431** (2009), no. 5-7, 1054–1069.

[3] A. Cirrito, A. Giambruno, *Group graded algebras and multiplicities bounded by a constant*, J. Pure Appl. Algebra **217** (2013), no. 2, 259–268.

[4] V. Drensky, *Free algebras and PI-algebras. Graduate course in algebra*, Springer-Verlag, Singapore, Singapore, 2000.

[5] V. Drensky, A. Giambruno, *Cocharacters, codimensions and Hilbert series of the polynomial identities for* 2×2 *matrices with involution*, Canad. J. Math. **46** (1994), 718–733.

[6] A. Giambruno, A. Ioppolo, D. La Mattina, *Varieties of algebras with superinvolution of almost polynomial growth*, Algebr. Represent. Theory **19** (2016), no. 3, 599–611.

[7] A. Giambruno, A. Ioppolo, D. La Mattina, *Superalgebras with involution or superinvolution and almost polynomial growth of the codimensions*, Algebr. Represent. Theory **22** (2019), no. 4, 961–976.

[8] A. Giambruno, A. Ioppolo, F. Martino, *Standard polynomials and matrices with superinvolutions*, Linear Algebra Appl. **504** (2016), 272–291.

[9] A. Giambruno, S. Mishchenko, *Polynomial growth of the $*$-codimensions and Young diagrams*, Comm. Algebra **29** (2001), no. 1, 277–284.

[10] A. Giambruno, R. B. dos Santos, A. C. Vieira, *Identities of $*$-superalgebras and almost polynomial growth*, Linear Multilinear Algebra **64** (2016), no. 3, 484-501.

[11] A. Giambruno, M. Zaicev, *On codimension growth of finitely generated associative algebras*, Adv. Math. **140** (1998), 145–155.

[12] A. Giambruno, M. Zaicev, *Exponential codimension growth of PI-algebras: an exact estimate*, Adv. Math. **142** (1999), 221–243.

[13] C. Gomez-Ambrosi, I. P. Shestakov, *On the Lie structure of the skew-elements of a simple superalgebra with involution*, J. Algebra **208** (1998), 43–71.

[14] A. Ioppolo, *The exponent for superalgebras with superinvolution*, Linear Algebra Appl. **555** (2018), 1–20.

[15] A. Ioppolo, D. La Mattina, *Polynomial codimension growth of algebras with involutions and superinvolutions*, J. Algebra **472** (2017), 519–545.

[16] A. Ioppolo, F. Martino, *Superinvolutions on upper-triangular matrix algebras*, J. Pure Appl. Algebra **222** (2018), no. 8, 2022–2039.

[17] V. G. Kac, *Lie superalgebras*, Adv. Math. **26** (1977), no. 1, 8–96.
[18] A. R. Kemer, *Varieties of finite rank*, Proc. 15th All the Union Algebraic Conf., Krasnoyarsk, vol. 2 (1979) pp. 73 (in Russian).
[19] P. Koshlukov, D. La Mattina, *Graded algebras with polynomial growth of their codimensions*, J. Algebra **434** (2015), 115–137.
[20] D. La Mattina, T. S. Nascimento, A. C. Vieira, *Minimal star-varieties of polynomial growth and bounded colength*, J. Pure Appl. Algebra **222** (2018), 1765–1785.
[21] S. P. Mishchenko, A. Regev, M. V. Zaicev, *A characterization of P.I. algebras with bounded multiplicities of the cocharacters*, J. Algebra **219** (1999), no. 1, 356–368.
[22] S. Mishchenko, A. Valenti, *A star-variety with almost polynomial growth*, J. Algebra **223** (2000), no. 1, 66–84.
[23] F. C. Otera, *Finitely generated PI-superalgebras with bounded multiplicities of the cocharacters*, Comm. Algebra **33** (2005), no. 6, 1693–1707.
[24] M. L. Racine, *Primitive superalgebras with superinvolution*, J. Algebra **206** (1998), no. 2, 588–614.
[25] M. L. Racine, E. I. Zelmanov, *Simple Jordan superalgebras with semisimple even part*, J. Algebra **270** (2003), no. 2, 374–444.
[26] A. Regev, *Existence of identities in $A \otimes B$*, Israel J. Math. **11** (1972), 131–152.
[27] A. C. Vieira, *Supervarieties of small graded colength*, J. Pure Appl. Algebra **217** (2013), 322–333.
[28] A. C. Vieira, *Finitely generated algebras with involution and multiplicities bounded by a constant*, J. Algebra **422** (2015), 487–503.

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