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A second order Hamiltonian neural model

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ABSTRACT

In this paper we establish the existence of one bounded periodic weak solution for a nonlinear parametric differential problem via variational methods.

1. Introduction

The development of mathematical models of neurons for potential applications in artificial intelligence marks a significant advancement in our understanding and replication of brain function. As the fundamental units of the nervous system, neurons operate in complex, highly interconnected ways, making their study essential for the progress of AI technologies. Accurate mathematical models enable the simulation of neuronal behavior, offering new insights into processes like information processing, learning, and memory (see [1,2]). A building block of this framework is given by single neuron models, whose behavior is usually described by nonlinear ordinary differential equations (ODEs). A deep understanding of how a single-cell takes part to the whole process of propagating the information is then of fundamental interest. Several models have appeared in the literature (see for sake of simplicity [3,4] and references therein) and the most successful example is the Hodgkin–Huxley model for cell excitability [5]. These results concern differential models of the first order, whereas Wheeler and Schieve [6] improved the Hopfield equation in [7] adding an inertial term. Therefore, with the inclusion of inertia, the new master neuron equation becomes a second order differential equation.

In this paper, we study the existence of a T-periodic solution for the following class of nonlinear differential problem

$$\begin{cases} -(p(t)u'(t))' + q(t)u(t) = \lambda f(t, u(t)) & \text{in }]0, T[, \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \tag{P}$$

where we assume the following assumption

(H) T, λ are positive constants, $p, q \in L^\infty([0,T])$ are such that $p(0) = p(T)$, $\operatorname{ess\,inf}_{[0,T]} p > 0$, $\operatorname{ess\,inf}_{[0,T]} q > 0$ and $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is L^1-Carathéodory.

Our approach is based on critical point theory and in particular we apply a local minimum theorem established by Bonanno [8].

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2. Variational framework

Let $L^2([0,T])$ be the classical Lebesgue space equipped with the usual norm $\|\cdot\|_2$ and denote by C_T^∞ the space of indefinitely differentiable T-periodic functions from $\mathbb{R}$ to $\mathbb{R}$. Given $u \in L^1([0,T])$, we say that $\varphi \in L^1([0,T])$ is the *weak derivative* of u if

$$\int_0^T u(t)v'(t)\mathrm{d}t = -\int_0^T \varphi(t)v(t)\mathrm{d}t \qquad \text{for all } v \in C_T^\infty,$$

and we denote $u' = \varphi$. Following the notation in the book of Mawhin-Willem [9], we denote with $W_T^{1,2}$ the periodic Sobolev space defined by

$$W_T^{1,2} = \{u \in L^2([0,T]) : u' \in L^2([0,T])\},$$

endowed with the usual norm $\|u\|_{1,2} = \left(\|u\|_2^2 + \|u'\|_2^2\right)^{\frac{1}{2}}$. It is well known that $W_T^{1,2}$ is a reflexive Banach space and $C_T^\infty \subset W_T^{1,2}$. We assume that assumption (H) holds and we indicate with H_T^1 the Hilbert space $W_T^{1,2}$ with the inner product

$$\langle u, v\rangle = \int_0^T p(t)u'(t)v'(t)\,\mathrm{d}t + \int_0^T q(t)u(t)v(t)\,\mathrm{d}t,$$

for all $u, v \in H_T^1$, which induces the following equivalent norm

$$\|u\| = \left(\int_0^T p(t)|u'(t)|^2\,\mathrm{d}t + \int_0^T q(t)|u(t)|^2\,\mathrm{d}t\right)^{\frac{1}{2}}.$$

Also, it is known that H_T^1 is compactly embedded in $C([0,T])$ and

$$\|u\|_\infty \le \bar{\alpha}\|u\| \quad \text{for some } \bar{\alpha} > 0. \tag{1}$$

In particular, setting $p_- = \operatorname{ess\,inf}_{t\in[0,T]} p(t)$ and $q_- = \operatorname{ess\,inf}_{t\in[0,T]} q(t)$, by [10, Proposition 2.1] we have a possible numerical estimation of the embedding constant

$$\alpha = \sqrt{\frac{p_- + T^2 q_-}{T p_- q_-}}. \tag{2}$$

Although α may not be the best constant for inequality (1), it is useful for our study. Put $F(t,x) = \int_0^x f(t,s)\,\mathrm{d}s$ for every $(t,x) \in [0,T]\times\mathbb{R}$ and consider the functionals $\Phi, \Psi : H_T^1 \to \mathbb{R}$ defined as

$$\Phi(u) = \frac{1}{2}\|u\|^2, \qquad \Psi(u) = \int_0^T F(t,u(t))\,\mathrm{d}t,$$

for all $u \in H_T^1$. Therefore, the so-called energy functional related to problem (P) is given by $I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$ for all $u \in H_T^1$. Standard computations show that Φ and Ψ are C^1-functionals and one has

$$I_\lambda'(u)(v) = \int_0^T \left(p(t)u'(t)v'(t) + q(t)u(t)v(t)\right)\mathrm{d}t - \int_0^T f(t,u(t))v(t)\mathrm{d}t,$$

for all $u, v \in H_T^1$. In this paper we are interested in the existence of weak solutions for problem (P).

Definition 1. A function $u \in H_T^1$ is called *weak solution* of (P) if

$$\int_0^T p(t)u'(t)v'(t)\mathrm{d}t + \int_0^T q(t)u(t)v(t)\mathrm{d}t = \lambda\int_0^T f(t,u(t))v(t)\mathrm{d}t,$$

for all $v \in C_T^\infty$.

Note that if $u \in H_T^1$, then u has a weak derivative u' and as a consequence of the Fundamental Lemma in [9], it holds that $u(0) = u(T)$. Moreover, by the definition of weak solution it follows that pu' has a weak derivative, which ensures that $u'(0) = u'(T)$. Therefore, if $u \in H_T^1$ is a weak solution for problem (P), then it satisfies the boundary conditions. In addition, from Definition 1 it follows that u is a weak solution of (P) if and only if u is a critical point of I_λ, namely $I_\lambda'(u)(v) = 0$ for all $v \in H_T^1$.

Hence, in order to investigate the existence of solutions for problem (P), we can study the existence of the critical points of the energy functional. To this aim, our main tool is a local minimum theorem established by Bonanno in [8]. However, we use the version of the result given in [11, Theorem 2.3].

3. Main results

In this section we present our main result on the existence of at least a non-zero bounded weak solution for problem (P) and, in particular, we determine an interval of the parameter λ for which the problem admits such a solution.

Theorem 2. *Suppose that (H) holds and assume that there exist two positive constants c, d, with $d < c$, such that*

$$\frac{\int_0^T \max_{|x|\le c} F(t,x)\,\mathrm{d}t}{c^2} < \frac{1}{T\alpha^2\|q\|_\infty}\frac{\int_0^T F(t,d)\,\mathrm{d}t}{d^2}. \tag{h$_1$}$$

Then, for each $\lambda \in \Lambda_{c,d} := \left]\frac{T\|q\|_\infty}{2}\frac{d^2}{\int_0^T F(t,d)\,\mathrm{d}t}, \frac{1}{2\alpha^2}\frac{c^2}{\int_0^T \max_{|x|\le c} F(t,x)\,\mathrm{d}t}\right[$, problem (P) admits at least a non-trivial weak solution $u_\lambda \in H^1_T$ such that $\|u_\lambda\|_\infty < c$ and $\|u_\lambda\| < \frac{c}{\alpha}$, with α given in (2).

Proof. In order to apply [11, Theorem 2.3] to the energy functional $I_\lambda = \Phi - \lambda\Psi$ defined in Section 2, we notice that the functionals Φ and Ψ satisfy the required regularity assumptions. So, we first prove that there exist $r \in \mathbb{R}$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$, such that

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}. \tag{3}$$

Set $r = \frac{c^2}{2\alpha^2}$ and note that for any $u \in H^1_T$ such that $\Phi(u) < r$, by (1) it follows that $\|u\|_\infty \le \alpha\|u\| \le \alpha\sqrt{2r} = c$. Hence, we get

$$\frac{\sup_{\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r} \le \frac{\sup_{|x|\le c}\int_0^T F(t,x)\,\mathrm{d}t}{r} \le 2\alpha^2\frac{\int_0^T \max_{|x|\le c} F(t,x)\,\mathrm{d}t}{c^2}. \tag{4}$$

Now, let $\tilde{u}(t) = d$ for all $t \in [0,T]$. Clearly, $\tilde{u} \in H^1_T$ and $\Phi(\tilde{u}) \le \frac{d^2T\|q\|_\infty}{2}$, so that

$$\frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} \ge \frac{2}{T\|q\|_\infty}\frac{\int_0^T F(t,d)\,\mathrm{d}t}{d^2}. \tag{5}$$

Combining (4), (5) and (h$_1$), we get that

$$\frac{\sup_{\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r} \le 2\alpha^2\frac{\int_0^T \max_{|x|\le c} F(t,x)\,\mathrm{d}t}{c^2} < \frac{2}{T\|q\|_\infty}\frac{\int_0^T F(t,d)\,\mathrm{d}t}{d^2} \le \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}.$$

It remains to prove that $0 < \Phi(\tilde{u}) < r$. Clearly, we know that $0 < \Phi(\tilde{u}) \le \frac{d^2T\|q\|_\infty}{2}$, so we only need to verify that $d^2 < \frac{1}{T\alpha^2\|q\|_\infty}c^2$. Arguing by contradiction, suppose that $d^2 \ge \frac{1}{T\alpha^2\|q\|_\infty}c^2$ and exploiting this together with the assumption $d < c$, we obtain

$$\frac{\int_0^T \max_{|x|\le c} F(t,x)\,\mathrm{d}t}{c^2} \ge \frac{\int_0^T F(t,d)\,\mathrm{d}t}{c^2} \ge \frac{1}{T\alpha^2\|q\|_\infty}\frac{\int_0^T F(t,d)\,\mathrm{d}t}{d^2},$$

that is in contradiction with hypothesis (h$_1$). Then, assumption (3) is verified. In order to prove that the energy functional I_λ satisfies the $(\mathrm{PS})^{[r]}$-condition, one can follow the proof of [12, Theorem 3.1].

Finally, note that $\Lambda_{c,d}$ is nonempty because of (h$_1$). So, Theorem 2.3 in [11] ensures the existence of at least a bounded weak solution u_λ for each $\lambda \in \Lambda_{c,d}$. □

Finally, we provide an existence result for the autonomous problem, i.e. when the nonlinear term does not depend explicitly on t, which is the case of many applications. Therefore, we deal with the following problem

$$\begin{cases} -(p(t)u'(t))' + q(t)u(t) = \lambda g(u(t)) & t \in]0,T[, \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \tag{AP}$$

where $g : [0,T] \to \mathbb{R}$ is continuous and nonnegative. In this case, we set

$$G(x) = \int_0^x g(s)\,\mathrm{d}s \quad \forall\, x \in \mathbb{R} \quad \text{and} \quad \Psi(u) = \int_0^T G(u(t))\,\mathrm{d}t \quad \forall\, u \in H^1_T,$$

and, arguing as in the proof of Theorem 2, the following result holds.

Theorem 3. *Suppose that (H) holds and assume that there exist two positive constants c, d, with $d < c$, such that*

$$\frac{G(c)}{c^2} < \frac{1}{T\alpha^2\|q\|_\infty}\frac{G(d)}{d^2}. \tag{h$_2$}$$

Then, for each $\lambda \in \left]\frac{\|q\|_\infty}{2}\frac{d^2}{G(d)}, \frac{1}{2T\alpha^2}\frac{c^2}{G(c)}\right[$ problem (AP) admits at least a non-trivial weak solution $u_\lambda \in H^1_T$ such that $\|u_\lambda\|_\infty < c$ and $\|u\| < \frac{c}{\alpha}$, with α given in (2).

Data availability

No data was used for the research described in the article.

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