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Fractional differential equations under Poissonian white noise input: Transient response and probability density function

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ABSTRACT

Fractional differential equations driven by stochastic excitation are widely used in several engineering and physical applications, as they provide an effective framework for modelling systems characterized by hereditary behavior, memory effects, and nonlocal interactions under random inputs.

Although the response probability density function (PDF) of fractional differential equations excited by Gaussian white noise has been extensively investigated, this is not the case for fractional differential equations driven by Poissonian white noise. Indeed, while the analysis of integer-order differential equations subjected to Poissonian white noise has been widely addressed, the few available studies concerning fractional differential equations under Poissonian white noise excitation are mainly restricted to the stationary component of the response PDF.

To overcome these limitations, this paper proposes a continuous-time method to obtain the PDF of the response of two-term fractional differential equations subjected to Poissonian white noise excitation. In the proposed method, the response process is first represented by means of the method of iterated kernels. This representation is then employed to derive the characteristic function of the response process, from which the PDF is evaluated through inverse Fourier transform. To assess the accuracy of the proposed method, numerical simulations have been performed on two different structural systems. The results show an excellent agreement between the PDF obtained through the proposed method and that evaluated by Monte Carlo simulation.

1. Introduction

Structural systems excited by stochastic input are widely investigated in engineering applications, ranging from aerospace [1] and civil engineering [2,3] to mechanical [4,5] and biomedical fields. For instance, ambient vibration excitation is commonly modeled as a Gaussian white noise process [6], whereas vehicular traffic on bridges is typically represented as a Poissonian white noise process [7,8].

The probabilistic characterization of the response of structural systems subjected to a stochastic excitation typically relies on the determination of the probability density function (PDF) of the state variables or, equivalently, of the Fourier transform of PDF, i.e. the characteristic function (CF). Alternatively, the probabilistic characterization of the response process can also be carried out through the computation of the statistical moments or cumulants [9]. Several approaches can be employed to perform the probabilistic characterization: from the Monte Carlo simulation to more refined techniques based on path integral methods [10–15] or on complex fractional moments [16–21].

In several dynamical systems subjected to random excitation, the response is often formulated in a Markovian framework by introducing an appropriate finite-dimensional state vector. For instance, in standard second-order mechanical systems, the state is usually defined by the displacement and velocity, so that the future evolution of the response is fully determined by the current state, without requiring knowledge of the complete past trajectory. By contrast, the response of fractional-order dynamical systems driven by stochastic excitation is not Markovian. Therefore, the evaluation of the structural response at a given time instant requires the entire previous time evolution, which considerably increases the complexity of the problem and the computational effort required for its solution. This is due to the fact that fractional derivatives and integrals are nonlocal operators which introduce memory effects into the system dynamics. Although fractional operators introduce the aforementioned difficulties, they have emerged as an effective mathematical tool for describing physical processes in which memory, hereditary effects, and nonlocal interactions play a significant role. In fact, they enable an accurate representation of experimental observations that cannot be satisfactorily captured within the classical Newtonian framework, while often retaining a relatively limited number of model parameters. Owing to these features, fractional-order differential equations have been extensively employed in the modelling of a wide range of real-world problems, including non-Newtonian fluid dynamics [22,23], heat transport [24–28], structural

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vibration control [29], biomechanics [30,31], nonlocal continuum mechanics [32], and viscoelasticity [33–37].

Although the PDF of the response of fractional differential equations excited by Gaussian white noise has been extensively investigated [15,37,38], this is not the case for fractional differential equations driven by Poissonian white noise. In fact, while the analysis of integer-order differential equations subjected to Poissonian white noise has been widely addressed in the literature [7,13,39–43], only a limited number of studies have considered fractional differential equations under Poissonian white noise excitation. Moreover, these contributions are mainly restricted to the stationary component of the response PDF [44,45].

Recently, a new method for finding the PDF of the response of a single-term fractional differential equation subjected to Poissonian white noise excitation has been proposed [46]. The main advantage of this method is that it provides the PDF directly in terms of the displacement response, thereby avoiding the need for a state-variable description and, consequently, the construction of a joint probability density function. This leads to a reduction in the analytical and computational complexity of the problem. However, despite these advantages, the method still presents some limitations. In particular, it is developed within a discrete-time setting and becomes approximate when the amplitudes of the Poissonian impulses are not Gaussian-distributed. Moreover, the method is restricted to the response of a single-term fractional differential equation and is therefore not directly suitable for more complex dynamical systems.

The objective of this paper is to propose a continuous-time probabilistic method to obtain the PDF of the transient response of two-term fractional differential equations driven by Poissonian white noise. The proposed method extends the existing framework beyond single-term fractional models, thereby enabling the treatment of a broader class of fractional viscoelastic systems. By deriving the CF of the response process directly from a continuous-time representation, the response PDF can be evaluated through the inverse Fourier transform of the CF without introducing a state-space description or any time discretization of the stochastic excitation. Furthermore, the proposed method is not restricted to Gaussian-distributed amplitudes, but is applicable to any type of PDF of the Poissonian impulse amplitudes, thereby providing a more general basis for the transient probabilistic characterization of non-Markovian fractional-order systems subjected to Poissonian stochastic excitation.

The accuracy of the proposed method is assessed through numerical simulations performed on two different structural systems, by comparing the response PDF obtained from the proposed method with that estimated by Monte Carlo simulation.

2. Mathematical formulation

2.1. Preliminary remarks

In this section, some preliminary concepts and definitions on fractional Compound motion (fCm) are reported for clarity's sake as well as for introducing appropriate notation.

The fCm, labeled as $C_\beta(t)$, is defined as the solution of the fractional differential equation [46]

$$\begin{cases} \left({}_0D_t^\beta C_\beta\right)(t) = W_P(t)\mathrm{U}(t) \\ C_\beta(t) = 0 \ \ \forall t \le 0 \quad \text{w.p.1} \end{cases}, \tag{1}$$

in which $\mathrm{U}(t)$ is the unit step function, $W_P(t)$ is a zero-mean Poissonian white noise process, while $\left({}_0D_t^\beta C_\beta\right)(t)$ represents the β-order Riemann-Liouville (RL) fractional derivative of $C_\beta(t)$, calculated with respect to time t. The latter is defined as

$$\left({}_0D_t^\beta C_\beta\right)(t) = \frac{1}{\Gamma(n-\beta)}\frac{d^n}{dt^n}\left[\int_0^t (t-\tau)^{n-\beta-1}C_\beta(\tau)d\tau\right], \tag{2}$$

where n represents the smallest integer greater than β and $\Gamma(\cdot)$ is the Euler's Gamma function.

Further, the Poissonian white noise process $W_P(t)$ is a stochastic process constituted by a train of impulses, with zero-mean random amplitudes Y_k occurring at random time instants T_k, expressed in the form [47]

$$W_P(t) = \sum_{k=1}^{N(t)} Y_k \delta(t - T_k). \tag{3}$$

In Eq. (3), $\delta(\cdot)$ is the Dirac's delta function and $N(t)$ is the Poisson counting process, that gives the number of spikes in the interval $[0, t)$. The Poisson counting process $N(t)$ has independent increments and, for $t = 0$, it is equal to zero. Moreover, the stochastic average of $N(t)$, is computed as

$$E[N(t)] = \lambda t, \tag{4}$$

where λ represents the number of spikes per unit time. The probability of occurrence of b events in $[0, t)$ is given as

$$\Pr\{N(t) = b\} = \frac{(\lambda t)^b}{b!}\exp(-\lambda t), \tag{5}$$

being b a positive integer.

Since $C_\beta(t) = 0 \ \forall t \le 0$ w.p.1, the solution of Eq. (1) for $t \ge 0$ can be expressed as the β-order RL fractional integral of $W_P(t)$, calculated with respect to time and expressed as

$$C_\beta(t) = \left({}_0I_t^\beta W_p\right)(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}W_P(\tau)d\tau. \tag{6}$$

By substituting Eq. (3) into Eq. (6) and evaluating the RL fractional integral, the fCm assumes the form

$$C_\beta(t) = \sum_{k=1}^{N(t)} \frac{Y_k(t-T_k)^{\beta-1}U(t-T_k)}{\Gamma(\beta)}. \tag{7}$$

The fCm $C_\beta(t)$ is a zero-mean process that, for $\beta = 1$, coincides with the classical Compound motion. Moreover, if $\lambda \to \infty$ and the impulse amplitudes (Gaussian) are scaled so that $\lambda E\left[Y_k^2\right] \to q$ (where q is a positive and finite constant), the Poissonian white noise converges in distribution to Gaussian white noise [46].

2.2. Transient response and PDF of two-term fractional differential equations under Poissonian white noise: proposed method

In this section, the transient response and the PDF of a two-term fractional differential equation under Poissonian white noise is described in detail.

Let us consider the fractional differential equation

$$\begin{cases} \rho_1\left({}_0D_t^{\beta_1}X\right)(t) + \rho_2\left({}_0D_t^{\beta_2}X\right)(t) = W_P(t)\mathrm{U}(t) \\ X(t) = 0 \ \ \forall t \le 0 \quad \text{w.p.1} \end{cases}, \tag{8}$$

where ρ_1 and ρ_2 are two coefficients whose physical dimensions depend on β_1 and β_2, respectively as $[\rho_1] = \left[M\,T^{\beta_1-2}\right]$ and $[\rho_2] = \left[M\,T^{\beta_2-2}\right]$.

Dividing both sides of Eq. (8) for ρ_1, it leads to

$$\begin{cases} \left({}_0D_t^{\beta_1}X\right)(t) + \overline{\rho}\left({}_0D_t^{\beta_2}X\right)(t) = W_P(t)\mathrm{U}(t)\big/\rho_1 \\ X(t) = 0 \ \ \forall t \leq 0 \quad \text{w.p.1} \end{cases}, \tag{9}$$

being $\overline{\rho} = \rho_2/\rho_1$ a constant having physical dimensions $[\overline{\rho}] = \left[T^{\beta_2-\beta_1}\right]$.

The method developed in the following assumes $\beta_1 > \beta_2 > 0$, $X(t) = 0 \ \forall t \leq 0$, a linear system, and additive compound Poisson excitation with finite rate λ. The impulse amplitudes Y_k are assumed to be independent and identically distributed, independent of the Poisson counting process, and zero-mean, consistently with the zero-mean Poissonian white noise. The characteristic function and the inverse Fourier transform are also assumed to satisfy convergence conditions.

The response process $X(t)$, for $t \geq 0$, can be evaluated as

$$X(t) = \frac{1}{\rho_1}\left({}_0I_t^{\beta_1}W_p\right)(t) - \overline{\rho}\left({}_0I_t^{\gamma}X\right)(t), \tag{10}$$

in which $\gamma = \beta_1 - \beta_2$. By applying the method of iterated kernels [48], Eq. (10) can be expressed in the form

$$X(t) = \frac{1}{\rho_1}\sum_{r=0}^{\infty}(-\overline{\rho})^r\left({}_0I_t^{\nu_r}W_P\right)(t) = \frac{1}{\rho_1}\sum_{r=0}^{\infty}(-\overline{\rho})^r C_{\nu_r}(t), \tag{11}$$

where $\nu_r = \beta_1 + r\gamma$. From a close observation of Eq. (11), it can be noted that the process $X(t)$ is a linear combination of fCms. The convergence of the expansion in Eq. (11) is guaranteed by classical results on Neumann series for Volterra integral equations with weakly singular kernels. In particular, for any finite time interval, the iterated fractional-integral terms remain bounded and the corresponding series is absolutely convergent.

Since

$$C_{\nu_r}(t) = \sum_{k=1}^{N(t)}\frac{Y_k(t-T_k)^{\nu_r-1}U(t-T_k)}{\Gamma(\nu_r)}, \tag{12}$$

Eq. (11) can be rewritten as

$$X(t) = \sum_{k=1}^{N(t)} Y_k h(t-T_k), \tag{13}$$

being

$$h(t-T_k) = \frac{1}{\rho_1}\sum_{r=0}^{\infty}\frac{(-\overline{\rho})^r}{\Gamma(\nu_r)}(t-T_k)^{\nu_r-1}U(t-T_k). \tag{14}$$

It is to be stressed that $h(t)$ represents the impulse response function of the structural system described in Eq. (8), which can be expressed in closed form as $\rho_1^{-1}t^{\beta_1-1}E_{\gamma,\beta_1}(-\overline{\rho}t^{\gamma})U(t)$ being $E_{\cdot,\cdot}(\cdot)$ the two-parameter Mittag-Leffler function [37]. The characteristic function of the response process $X(t)$ is defined as

$$\Phi_X(\theta,t) = E[\exp(i\theta X(t))]. \tag{15}$$

By substituting Eq. (13) into Eq. (15) and by conditioning on both $N(t)$ and the arrival times $T_1, \ldots, T_{N(t)}$, yields

$$\Phi_{X|N(t),T_1,\ldots,T_{N(t)}}(\theta,t) = E\left[\exp\left(i\theta\sum_{k=1}^{N(t)}Y_k h(t-T_k)\right)|N(t),T_1,\ldots,T_{N(t)}\right]. \tag{16}$$

Since Y_k are independent identically distributed random variables [49], Eq. (16) can be expressed as

$$\Phi_{X|N(t),T_1,\ldots,T_{N(t)}}(\theta,t) = \prod_{k=1}^{N(t)}E[\exp(i\theta Y_k h(t-T_k))|N(t),T_1,\ldots,T_{N(t)}]. \tag{17}$$

Moreover, by considering that Y_k are independent on both $N(t)$ and the arrival times $T_1, \ldots, T_{N(t)}$, Eq. (17) assumes the form

$$\Phi_{X|N(t),T_1,\ldots,T_{N(t)}}(\theta,t) = \prod_{k=1}^{N(t)}E[\exp(i\theta Y_k h(t-T_k))], \tag{18}$$

which can be rewritten as

$$\Phi_{X|N(t),T_1,\ldots,T_{N(t)}}(\theta,t) = \prod_{k=1}^{N(t)}\Phi_{Y_k}(\theta h(t-T_k)), \tag{19}$$

being $\Phi_{Y_k}(\cdot)$ the characteristic function of Y_k. Since the impulse amplitudes are identically distributed, $\Phi_{Y_k}(\cdot)$ does not depend on k. Therefore, it can be written, for simplicity, as $\Phi_Y(\cdot)$.

By conditioning only on $N(t)$ and exploiting the fact that, given $N(t)$, the arrival times are distributed as the order statistics of independent identically distributed uniform random variables over $(0,t)$, Eq. (19) becomes (see Appendix A)

$$\Phi_{X|N(t)}(\theta,t) = E\left[\prod_{k=1}^{N(t)}\Phi_Y(\theta h(t-T_k))|N(t)\right] = \left(\frac{1}{t}\int_0^t\Phi_Y(\theta h(t-\tau))d\tau\right)^{N(t)}. \tag{20}$$

By removing the conditioning with respect to $N(t)$, since $N(t)$ has Poissonian distribution, it follows that (see Appendix A)

$$\Phi_X(\theta,t) = \exp\left(\lambda\int_0^t(\Phi_Y(\theta h(t-\tau)) - 1)d\tau\right). \tag{21}$$

Once the characteristic function has been evaluated, the PDF of the response process can be computed as the inverse Fourier transform of Eq. (21) in the form

$$p_X(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\theta x}\Phi_X(\theta,t)d\theta. \tag{22}$$

It is worth observing that Eq. (21) represents the complete characteristic function of the response process, including the contribution associated with the event $N(t) = 0$. Accordingly, the inverse Fourier transform in Eq. (22) has to be understood in the generalized sense, i.e. composed by both a continuous component and a Dirac's delta function.

3. Numerical applications

The purpose of this section is to assess the accuracy and applicability of the proposed method in evaluating the transient PDF of the response of two-term fractional differential equations subjected to Poissonian white noise excitation. To this aim, two representative structural systems are considered. The first one is a fractional Kelvin-Voigt model, in which the response is governed by a classical viscous contribution and a fractional viscoelastic term. The second one is a fractional oscillator, where the fractional viscoelastic contribution is coupled with an inertial term. These two examples allow the performance of the proposed method to be examined for systems characterized by different dynamical features, while preserving the same probabilistic structure of the external excitation. Although in both examples one of the two derivatives present in the equation of motion assumes an integer value, the considered systems are still two-term fractional differential equations, since the classical viscous or inertial contribution is coupled with a fractional hereditary term. These benchmarks were selected because they correspond to standard mechanical models with a clear physical

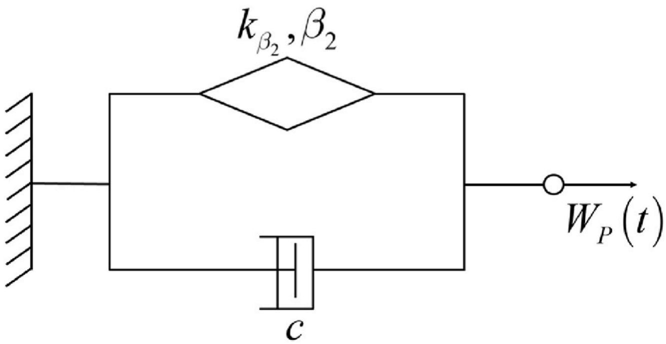

Fig. 1. Fractional Kelvin-Voigt model excited by a Poissonian white noise process.

interpretation. Nevertheless, the proposed formulation is not limited to these cases, as the derivation remains valid for a non-integer order of both β_1 and β_2.

In both applications, the response PDF obtained from the proposed method is compared with that estimated by Monte Carlo simulation. The latter is used as a reference solution and is computed by considering 2.5×10^4 independent samples having time duration equal to 5 s discretized with a time sampling step $\Delta t = 0.05$ s. Each sample is directly generated from the response representation reported in Eq. (11) by considering the first 10^4 terms of the series. To verify that the method is not restricted to a specific probability law for the impulse amplitudes, two different distributions are considered for Y_k: a zero-mean Gaussian distribution and a zero-mean uniform distribution.

3.1. Fractional Kelvin-Voigt model

As a first application, the fractional Kelvin-Voigt model shown in Fig. 1 is considered.

The governing equation is

$$\begin{cases} c\dot{X}(t) + k_{\beta_2}\left({}_0D_t^{\beta_2}X\right)(t) = W_P(t)\mathrm{U}(t) \\ X(t) = 0 \quad \forall t \le 0 \quad \text{w.p.1} \end{cases}, \tag{23}$$

where c is the damping coefficient of the dashpot, $\dot{X}(t)$ denotes the first-order derivative of the response process with respect to time, and k_{β_2} is the springpot coefficient. This model is particularly suitable as a first benchmark since it represents a basic fractional viscoelastic system in which a local dissipative mechanism is coupled with a hereditary contribution. Thus, it provides a direct test of the capability of the proposed method to account for memory effects induced by the fractional derivative.

The parameters adopted in the numerical analysis are $c = 100.00$ kg s^{-1}, $\beta_2 = 0.21$, and $k_{\beta_2} = 200.00$ kg s$^{-1.79}$.

The excitation is modeled as a Poissonian white noise process $W_P(t)$ with average number of impulses per unit time $\lambda = 1.44$. The impulse amplitudes are first assumed to be zero-mean Gaussian random variables. In this case, the characteristic function of Y_k is

$$\Phi_{Y_k}(\theta) = \exp\left(-\frac{\theta^2}{2}\sigma_{Y_k}^2\right), \tag{24}$$

where $\sigma_{Y_k} = 4$. Subsequently, the same analysis is repeated by assuming that the impulse amplitudes are uniformly distributed over the interval $[-a, a]$. The corresponding characteristic function is

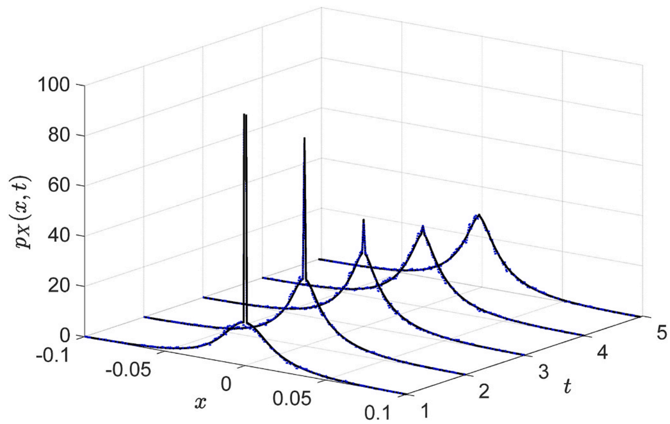

Fig. 2. Comparison between the response PDF obtained with the proposed method (continuous line) and that obtained from the Monte Carlo simulation (dotted line): Gaussian distribution of Y_k.

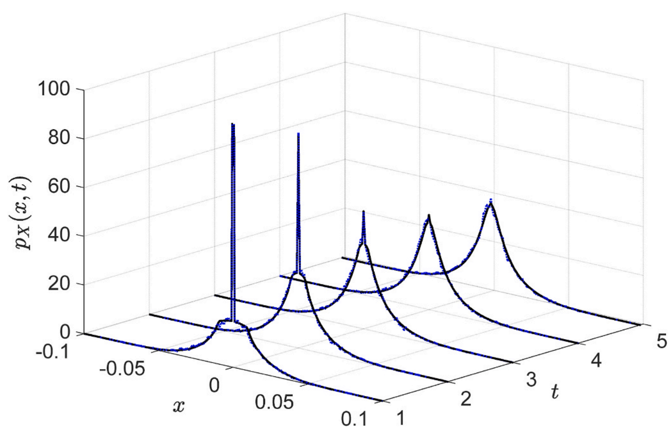

Fig. 3. Comparison between the response PDF obtained with the proposed method (continuous line) and that obtained from the Monte Carlo simulation (dotted line): Uniform distribution of Y_k.

$$\Phi_{Y_k}(\theta) = \frac{\sin(a\,\theta)}{a\,\theta}, \tag{25}$$

with $a = 5$.

For both distributions of the impulse amplitudes, the inverse Fourier transform was evaluated through the inverse fast Fourier transform. The domain θ, limited in the interval $[-3141.55, 3141.55]$, was discretized with a sampling step $\Delta\theta = 0.0785388$, which corresponds to a spatial discretization with $\Delta x = 0.001$ over the interval $[-40, 40]$. Therefore, no direct numerical quadrature of highly oscillatory Fourier integrals was required.

The response PDFs obtained by means of the proposed method are compared with those estimated through Monte Carlo simulation in Figs. 2 and 3, for the Gaussian and the uniform distributions of Y_k, respectively.

The comparison shows an excellent agreement between the two solutions for all the considered time instants. This agreement is observed for both distributions of the impulse amplitudes, thus confirming that the proposed method is not affected by the Gaussian or non-Gaussian nature of the impulse amplitudes.

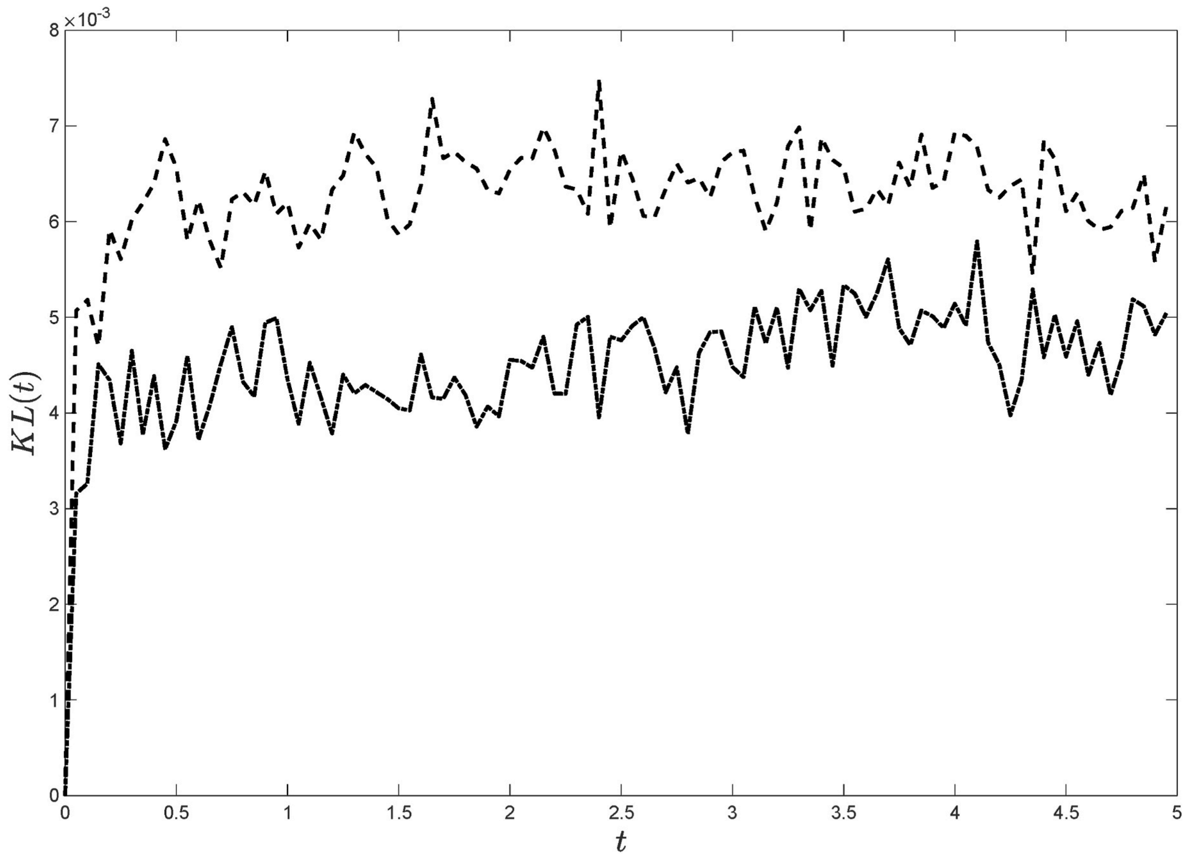

Fig. 4. Kullback-Leibler divergence between the response PDF obtained with the proposed method and that obtained from the Monte Carlo simulation: Gaussian distribution of Y_k (dashed line) and uniform distribution of Y_k (dash-dotted line).

Moreover, the sharp peaks present near $x = 0$, especially at early time instants, are a direct consequence of the compound Poisson nature of the excitation. Indeed, the probability that no impulse occurs is not zero. Therefore, a non-negligible fraction of the response probability is concentrated at the origin, and the corresponding Dirac contribution appears in the numerical representation as a very sharp peak around $x = 0$.

To provide a quantitative measure of the discrepancy between the response PDF obtained from the proposed method and the response PDF estimated by Monte Carlo simulation, respectively labeled as $p_X^{(PM)}(x,t)$ and $p_X^{(MC)}(x,t)$, the Kullback-Leibler (KL) divergence is evaluated as

$$KL(t) = \int_{-\infty}^{\infty} p_X^{(MC)}(x,t) \ln\left(\frac{p_X^{(MC)}(x,t)}{p_X^{(PM)}(x,t)}\right) dx. \tag{26}$$

The results obtained in terms of KL divergence are reported in Fig. 4. It is to be stressed that the KL divergence has been calculated directly from the results obtained, including the Dirac contribution, without any type of smoothing.

From the results reported in Fig. 4, it can be observed that the KL divergence assumes very small values throughout the whole time interval, confirming that the discrepancy between the proposed method and the Monte Carlo simulation is negligible. Therefore, the results of the first numerical application demonstrate that the proposed method provides an accurate description of the response PDF of a fractional Kelvin-Voigt system under Poissonian white noise excitation.

The maximum value and the mean value of the KL divergence are reported in Table 1.

From the results reported in Table 1 it can be noted that both the maximum and the mean value of the KL divergence are very low.

Table 1
Maximum value and mean value of the KL divergence.

	Maximum value	Mean value
Gaussian distribution of Y_k	7.5×10^{-3}	6.3×10^{-3}
Uniform distribution of Y_k	5.8×10^{-3}	4.5×10^{-3}

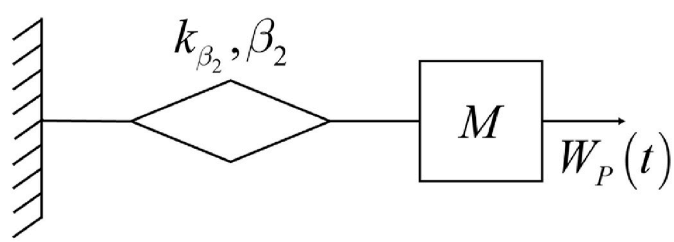

Fig. 5. Fractional oscillator excited by a Poissonian white noise.

3.2. Fractional oscillator

The second application concerns the fractional oscillator shown in Fig. 5.

The governing equation is

$$\begin{cases} \overline{M}\ddot{X}(t) + k_{\beta_2}\left({}_0D_t^{\beta_2}X\right)(t) = W_P(t)\mathrm{U}(t) \\ X(t) = 0 \ \ \forall t \le 0 \quad \text{w.p.1} \end{cases}, \tag{27}$$

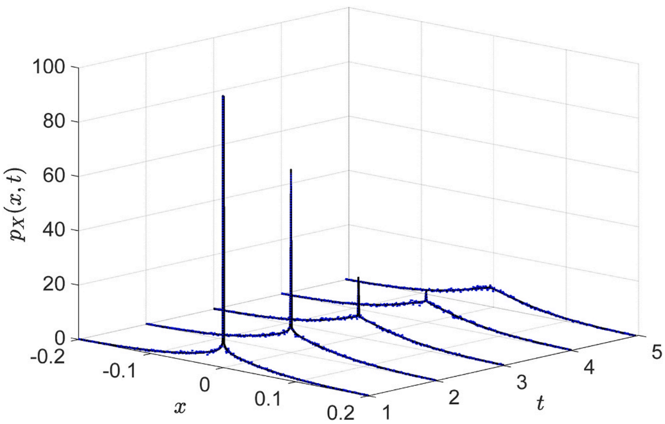

Fig. 6. Comparison between the response PDF obtained with the proposed method (continuous line) and that obtained from the Monte Carlo simulation (dotted line): Gaussian distribution of Y_k.

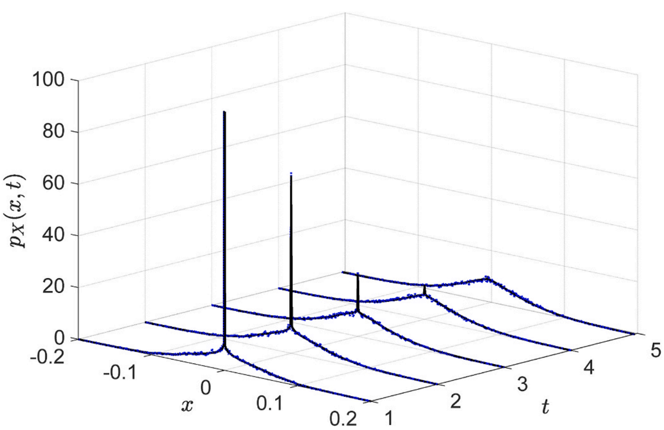

Fig. 7. Comparison between the response PDF obtained with the proposed method (continuous line) and that obtained from the Monte Carlo simulation (dotted line): Uniform distribution of Y_k.

where $\overline{M}$ is the mass of the system, $\ddot{X}(t)$ represents the second derivative of the response process calculated with respect to time, and k_{β_2} is the springpot coefficient. Differently from the previous case, the presence of the inertial term introduces an oscillatory dynamical behavior. Therefore, the response PDF is affected by the interaction between the impulsive excitation, the inertial contribution, and the fractional hereditary term.

The parameters adopted in the numerical analysis are $\overline{M} = 10.00$ kg, $\beta_2 = 0.12$ and $k_{\beta_2} = 200.00$ kg s$^{-1.88}$.

The excitation process is the same as that considered in section 3.1, namely a Poissonian white noise process with $\lambda = 1.44$. Moreover, to allow a consistent comparison between the two numerical applications, the adopted distributions of the impulse amplitudes are: a zero-mean Gaussian distribution with $\sigma_{Y_k} = 4$ and a zero-mean uniform distribution over $[-5, 5]$.

The same discrete Fourier inversion procedure adopted in the previous example was used. In particular, the inverse Fourier transform was evaluated through the inverse fast Fourier transform. The domain θ, limited in the interval $[-3141.55, 3141.55]$, was discretized with a sampling step $\Delta\theta = 0.0785388$, This choice corresponds to a spatial discretization with $\Delta x = 0.001$ over the interval $[-40, 40]$. Therefore, also in this case, no direct numerical quadrature of highly oscillatory Fourier integrals was required.

Figs. 6 and 7 show the response PDFs obtained from the proposed method and from Monte Carlo simulation for the Gaussian and uniform distributions of Y_k, respectively.

Also in this case, the two solutions are in excellent agreement. Therefore, the proposed method is able to capture the time evolution of the response PDF and its dependence on the probability distribution of the impulse amplitudes.

The quantitative comparison is performed again by means of the Kullback-Leibler divergence defined in Eq. (26), and the corresponding results are reported in Fig. 8. Also in this case, the KL divergence has been calculated directly from the results obtained, including the Dirac contribution, without any type of smoothing.

Although the values of $KL(t)$ are slightly larger than those obtained for the fractional Kelvin-Voigt model, they remain very low over the entire time interval, confirming that the proposed method retains a high accuracy also when applied to a fractional oscillatory system. The maximum value and the mean value of the KL divergence are reported in Table 2.

From the results reported in Table 2 it can be noted that both the maximum and the mean value of the KL divergence remain very low.

4. Concluding remarks

In this paper, a continuous-time method to obtain the probability density function of the response of two-term fractional differential equations subjected to Poissonian white noise excitation has been proposed. The proposed method is based on the representation of the response through the method of iterated kernels, followed by the derivation of the characteristic function and the evaluation of the probability density function by inverse Fourier transformation.

The reliability of the proposed method has been assessed through numerical applications. The excellent agreement between the results obtained by using the proposed method and those obtained from Monte Carlo simulations has been observed for two different fractional structural systems and for both Gaussian and non-Gaussian impulse amplitudes. These results confirm that the proposed method can be effectively employed for the probabilistic characterization of non-Markovian fractional-order systems subjected to Poissonian stochastic excitation, without resorting to a state-space representation, time-discretization of the excitation, or Gaussian assumptions on the impulse amplitudes.

CRediT authorship contribution statement

Salvatore Russotto: Conceptualization, Data curation, Formal analysis, Investigation, Methodology, Software, Validation, Visualization, Writing – original draft, Writing – review & editing. **Mario Di Paola:** Supervision, Writing – original draft. **Antonina Pirrotta:** Methodology, Supervision, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

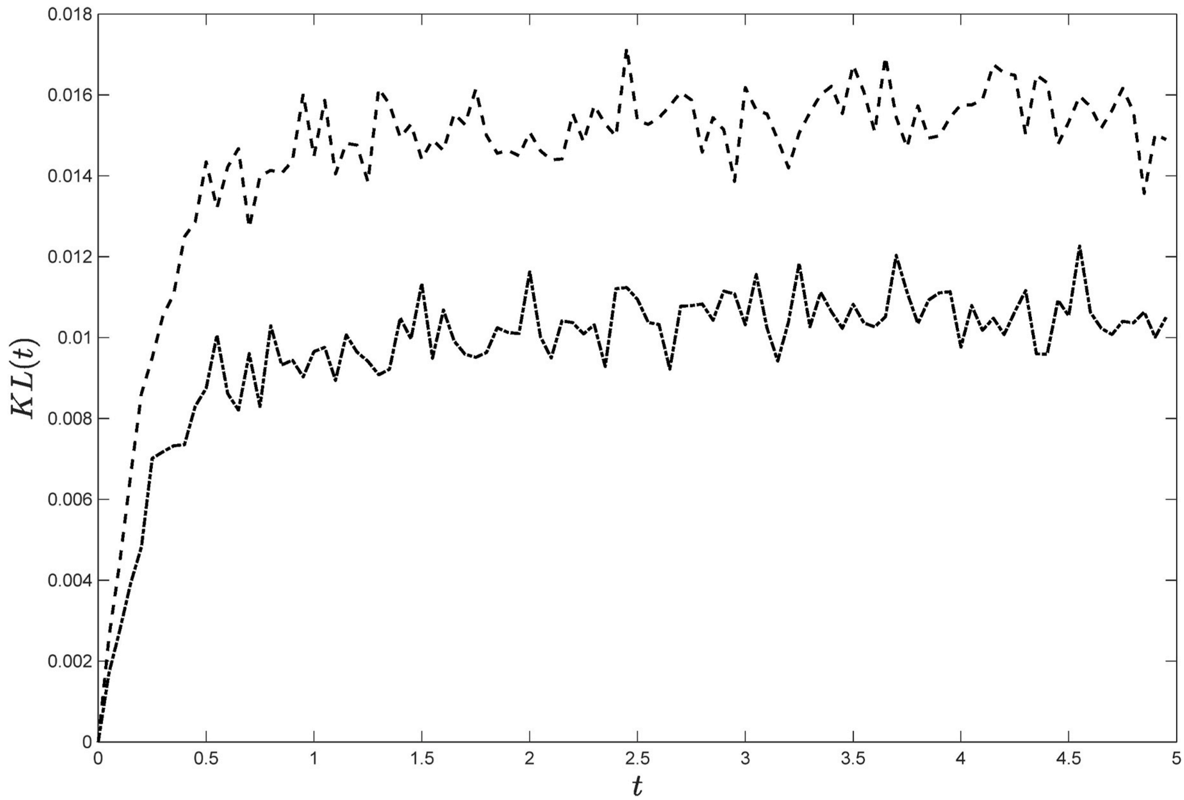

Fig. 8. – Kullback-Leibler divergence between the response PDF obtained with the proposed method and that obtained from the Monte Carlo simulation: Gaussian distribution of Y_k (dashed line) and uniform distribution of Y_k (dash-dotted line).

Table 2
– Maximum value and mean value of the KL divergence.

	Maximum value	Mean value
Gaussian distribution of Y_k	1.72×10^{-2}	1.45×10^{-2}
Uniform distribution of Y_k	1.23×10^{-2}	0.97×10^{-2}

Appendix A. Derivation of Eq. (20) and Eq. (21) from Eq. (19)

The characteristic function of the response process, conditioned on the number of impulses $N(t)$ and on the corresponding arrival times $T_1,\dots,T_{N(t)}$, is written, as reported in Eq. (19), in the form

$$\Phi_{X|N(t),T_1,\dots,T_{N(t)}}(\theta,t)=\prod_{k=1}^{N(t)}\Phi_{Y_k}(\theta h(t-T_k)), \tag{A.1}$$

where $\Phi_{Y_k}(\cdot)$ denotes the characteristic function of the impulse amplitude Y_k. Since the impulse amplitudes are identically distributed, $\Phi_{Y_k}(\cdot)$ does not depend on k. Therefore, it can be written, for simplicity, as $\Phi_Y(\cdot)$.

By conditioning only on the event $N(t)=\widetilde{n}$, Eq. (A.1) gives

$$\Phi_{X|N(t)=\widetilde{n}}(\theta,t)=E\left[\prod_{k=1}^{N(t)}\Phi_Y(\theta h(t-T_k))|N(t)=\widetilde{n}\right]. \tag{A.2}$$

It is well known that, conditionally on $N(t)=\widetilde{n}$, the arrival times $T_1,\dots,T_{\widetilde{n}}$ of a Poissonian process in the interval $(0,t)$, ordered as

$$0<T_1<\dots<T_{\widetilde{n}}<t, \tag{A.3}$$

are distributed as the order statistics of $\widetilde{n}$ independent random variables uniformly distributed over $(0,t)$. Therefore, their joint probability density function is

$$p_{T_1,\dots,T_{\widetilde{n}}|N(t)=\widetilde{n}}(\tau_1,\dots,\tau_{\widetilde{n}})=\frac{\widetilde{n}!}{t^{\widetilde{n}}} \tag{A.4}$$

for

$$0 < \tau_1 < \ldots < \tau_{\tilde{n}} < t, \tag{A.5}$$

and it is equal to zero elsewhere. Consequently, Eq. (A.2) can be written as

$$\Phi_{X|N(t)=\tilde{n}}(\theta, t) = \frac{\tilde{n}!}{t^{\tilde{n}}} \int_{0<\tau_1<\ldots<\tau_{\tilde{n}}<t} \prod_{k=1}^{\tilde{n}} \Phi_Y(\theta h(t-\tau_k))\, d\tau_1 \ldots d\tau_{\tilde{n}}. \tag{A.6}$$

For compactness, let us define

$$F(\tau_1, \ldots, \tau_{\tilde{n}}) = \prod_{k=1}^{\tilde{n}} \Phi_Y(\theta h(t-\tau_k)). \tag{A.7}$$

The function $F(\tau_1, \ldots, \tau_{\tilde{n}})$ is symmetric with respect to its arguments. Indeed, for any permutation α of the set $\{1, \ldots, \tilde{n}\}$,

$$F(\tau_{\alpha(1)}, \ldots, \tau_{\alpha(\tilde{n})}) = \prod_{k=1}^{\tilde{n}} \Phi_Y(\theta h(t-\tau_{\alpha(k)})) = \prod_{k=1}^{\tilde{n}} \Phi_Y(\theta h(t-\tau_k)) = F(\tau_1, \ldots, \tau_{\tilde{n}}). \tag{A.8}$$

This symmetry implies that the integral of the function $F(\tau_1, \ldots, \tau_{\tilde{n}})$ over the full hypercube $(0,t)^{\tilde{n}}$ is equal to $\tilde{n}!$ times its integral over the ordered simplex $0 < \tau_1 < \ldots < \tau_{\tilde{n}} < t$. In fact, the hypercube $(0,t)^{\tilde{n}}$ can be decomposed into $\tilde{n}!$ disjoint regions, each corresponding to one of the possible orderings of the variables $\tau_1, \ldots, \tau_{\tilde{n}}$. Since the function $F(\tau_1, \ldots, \tau_{\tilde{n}})$ is invariant under any permutation of its arguments, the integral over each of these regions is the same. Therefore,

$$\int_{(0,t)^{\tilde{n}}} F(\tau_1, \ldots, \tau_{\tilde{n}}) d\tau_1 \ldots d\tau_{\tilde{n}} = \tilde{n}! \int_{0<\tau_1<\ldots<\tau_{\tilde{n}}<t} F(\tau_1, \ldots, \tau_{\tilde{n}}) d\tau_1 \ldots d\tau_{\tilde{n}}. \tag{A.9}$$

Using Eq. (A.9), Eq. (A.6) becomes

$$\Phi_{X|N(t)=\tilde{n}}(\theta, t) = \frac{1}{t^{\tilde{n}}} \int_{(0,t)^{\tilde{n}}} \prod_{k=1}^{\tilde{n}} \Phi_Y(\theta h(t-\tau_k))\, d\tau_1 \ldots d\tau_{\tilde{n}}. \tag{A.10}$$

Since the integration domain is now the hypercube $(0,t)^{\tilde{n}}$, the variables $\tau_1, \ldots, \tau_{\tilde{n}}$ are not constrained by any ordering condition. Moreover, the integrand is a product of functions, each depending only on one integration variable. Hence, the multiple integral can be factorized as

$$\Phi_{X|N(t)=\tilde{n}}(\theta, t) = \frac{1}{t^{\tilde{n}}} \prod_{k=1}^{\tilde{n}} \int_0^t \Phi_Y(\theta h(t-\tau_k)) d\tau_k. \tag{A.11}$$

Since all the integrals in Eq. (A.11) are identical, the latter can be rewritten in the form

$$\Phi_{X|N(t)=\tilde{n}}(\theta, t) = \left(\frac{1}{t} \int_0^t \Phi_Y(\theta h(t-\tau)) d\tau \right)^{\tilde{n}} \tag{A.12}$$

that, by replacing the fixed value $\tilde{n}$ with $N(t)$, yields

$$\Phi_{X|N(t)}(\theta, t) = \left(\frac{1}{t} \int_0^t \Phi_Y(\theta h(t-\tau)) d\tau \right)^{N(t)}, \tag{A.13}$$

which corresponds to Eq. (20).

For compactness, let us define

$$A(\theta, t) = \frac{1}{t} \int_0^t \Phi_Y(\theta h(t-\tau)) d\tau. \tag{A.14}$$

Thus, Eq. (A.13) becomes

$$\Phi_{X|N(t)=\tilde{n}}(\theta, t) = (A(\theta, t))^{N(t)}. \tag{A.15}$$

The unconditional characteristic function $\Phi_X(\theta, t)$ is obtained by removing the conditioning with respect to the Poisson counting process. Therefore, by applying the law of total expectation, one obtains

$$\Phi_X(\theta, t) = \sum_{\tilde{n}=0}^{\infty} (A(\theta, t))^{\tilde{n}} \Pr\{N(t) = \tilde{n}\}. \tag{A.16}$$

Since $N(t)$ has Poissonian distribution, then Eq. (A.16) can be rewritten as

$$\Phi_X(\theta, t) = \exp(-\lambda t) \sum_{\tilde{n}=0}^{\infty} (A(\theta, t))^{\tilde{n}} \frac{(\lambda t)^{\tilde{n}}}{\tilde{n}!}. \tag{A.17}$$

Moreover, by considering that the series presents at the right hand side of Eq. (A.17) is the exponential series, Eq. (A.17) assumes the form

$$\Phi_X(\theta,t) = \exp(-\lambda t)\exp(\lambda t A(\theta,t)) = \exp(\lambda t(A(\theta,t)-1)). \tag{A.18}$$

By substituting Eq. (A.14) into Eq. (A.18), yields

$$\Phi_X(\theta,t) = \exp\left(\lambda\left(\int_0^t (\Phi_Y(\theta h(t-\tau)))d\tau - t\right)\right), \tag{A.19}$$

that, as reported in Eq. (21), can be rewritten in the form

$$\Phi_X(\theta,t) = \exp\left(\lambda\int_0^t (\Phi_Y(\theta h(t-\tau))-1)d\tau\right). \tag{A.20}$$

Data availability

Data will be made available on request.

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