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EXISTENCE AND APPROXIMATION OF A SOLUTION FOR A TWO POINT NONLINEAR DIRICHLET PROBLEM

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Abstract. The existence of at least one positive solution to a second-order nonlinear two-point boundary value problem, is established. Combining difference methods with Brouwer fixed point and Ascolì-Arzelà theorems, we get a solution as the limit of an appropriate sequence of piecewise linear interpolations. Furthermore, a priori bounds on the infinite norm of a solution and its derivatives are pointed out. Some examples are also discussed to illustrate our results.

1. **Introduction.** Consider the following two point nonlinear Dirichlet boundary value problem

$$\begin{cases} -u'' = f(x,u,u'), & x \in [a,b], \\ u(a) = u(b) = 0, \end{cases} \tag{P_f}$$

where $f : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

The aim of this paper is to establish the existence of at least one classical positive solution for problem (P_f) combining some tools of functional analysis on a finite dimensional normed space, as Brouwer fixed point and Ascolì-Arzelà theorems, with the discrete difference methods and a priori estimates given in Theorem 2.2. Roughly speaking, our goal is obtained in two main steps.

First, putting together Brouwer's fixed point theorem with some ideas arising from [2], we get the existence of at least one solution for the associated standard difference Dirichlet problem

$$\begin{cases} -\Delta^2 u_n(k-1) = h_n^2 f_k^n\left(u_n(k), \dfrac{\Delta u_n(k-1)}{h_n}\right), & k \in [1,n], \\ u(0) = u(n+1) = 0, \end{cases} \tag{$D^n_{h_n^2 f}$}$$

where $h_n = \frac{b-a}{n+1}$ is the step size, $x_k^n = a + kh_n$ are the grid points, and $f_k^n(t,s) = f(x_k^n, t, s)$ for all $k = 0, \dots, n+1$ and for every $t, s \in \mathbb{R}$, while $u_n(k) := u(x_k^n)$,

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$\Delta u_n(k-1) := u_n(k) - u_n(k-1)$, $\Delta^2 u_n(k-1) := \Delta(\Delta u_n(k-1))$ are the forward and the second order differences, respectively, for $k = 1, \dots, n$.

Next, the existence of at least one solution to problem (P_f) is obtained, by applying Ascolì-Arzelà theorem, as the limit of a sequence of piecewise linear interpolations of the solution u_n of the discrete problem $(D^n_{h_n^2 f})$ (Theorem 3.1). Finally, the sign information on a solution has been obtained with truncation techniques in the case in which the problem does not admit the trivial one (Theorem 3.3). In particular, as consequence of our main results, we have the following

Theorem 1.1. *Let $f : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous and bounded function such that $f(x,0,0) > 0$ for every $x \in [a,b]$. Then, problem (P_f) admits at least one positive classical solution $u \in C^2([a,b])$ with*

$$\|u\|_\infty \leq \frac{(b-a)^2}{8} M, \qquad \|u'\|_\infty \leq \frac{(b-a)}{2} M,$$

where $M := \sup_{(x,t,s)\in[a,b]\times\mathbb{R}\times\mathbb{R}} |f(x,t,s)|$ and fulfills condition (U) of Theorem 3.1.

It is understood that Theorem 1.1 (Theorems 3.1 and 3.3), as existence results are well known, while the approximation of the solution by using the solutions of the corresponding sequence of difference equations seems new, see for instance, [15, Theorem 9.2]. More precisely, we can see that our conclusion, concerning mere existence, is a special case of the result given in the vector case in [14, Theorem 4.2, p. 424] as an application of Schauder fixed point theorem, and it can essentially be traced to [11], where the author proves it by using Euler approximations satisfying the boundary conditions obtained through the intermediate value theorem. The bounds of the solutions are exactly the same, although obtained with different approaches. Moreover, in the one dimensional case, Theorem 1.1 gives a more precise version of [17, Theorem 2.3] obtained by using set-valued methods.

For the reader's convenience, we mention the historical paper [16] for a complete and exhaustive survey on the above mentioned classical existence results, as well as for an overview on classic methods developed to study boundary value problems for nonlinear ordinary differential equations.

Concerning second order nonlinear differential problems with nonlinearities depending on the derivative there is a huge literature, involving more general quasilinear operators, as $p-$Laplacian, $\phi-$laplacian, strongly nonlinear differential operators, and/or functional boundary value conditions covering, amongst others, many interesting settings as Sturm-Liouville and multipoint boundary data. On these topics, deeper results are established in [3, 6, 13, 18, 20]. It is worth noting that compared to the approach developed in the above mentioned papers, to obtain a priori bounds on the derivative we do not make any assumptions about the existence of lower and upper solutions for problem (P_f) as well as there are no requests of some Bernstein–Nagumo–Hartman type condition. We get the a priori estimates on the derivative adapting some arguments contained in [2], where the non-linearity does not involve u' and the discrete problem $(D^n_{h_n^2 f})$ is solved by variational methods.

More precisely, exploiting the a priori estimates occurring for the unique solution of the linear problem $D^n_{h_n^2 w}$, $w \in \mathbb{R}^n$, and its first differences, cfr. Theorem 2.2 below, we give a slightly more refined version of [12, Lemma 2.4], see also [15, 21, 22, 23], (Proposition 2.3). The latter result not only ensures that the discrete problem does not generate "irrelevant" solutions which tend to $+\infty$ as $n \to \infty$, but also allows us to obtain a solution u of the continuous problem (P_f), by using

Ascoli-Arzelà theorem, as a uniform limit of an appropriate sequence of piecewise linear interpolations. In addition, in our setting, it gives also a priori estimates on the L^∞ norms of a solution and its derivatives.

For further insights on second order differential problems with nonlinearities depending of the derivative of the solution, we also refer the reader to [1, 9, 10, 19] and the references therein.

2. **Preliminaries.** Consider the n-dimensional space

$$X = \{u \in \mathbb{R}^{n+2} : u(0) = u(n+1) = 0\},$$

endowed with the norm

$$\|u\|_X := \left(\sum_{k=1}^{n+1} |\Delta u(k-1)|^2\right)^{\frac{1}{2}} \quad \forall\, u \in X, \tag{1}$$

which is equivalent to the norm

$$\|u\|_2 = \left(\sum_{k=1}^{n} |u(k)|^2\right)^{\frac{1}{2}} \quad \forall\, u \in X.$$

In the sequel, we will also use the following (equivalent) norm

$$\|u\|_\infty = \max_{k\in[1,n]} |u(k)| \quad \forall\, u \in X.$$

To obtain the existence of at least one solution for problem ($D^n_{h_n^2 f}$), we will use the following consequence of Brouwer's fixed point theorem, see also [5] for an infinite dimensional version involving set-valued analysis.

Theorem 2.1. *Let X, Y be two finite dimensional normed spaces, let K be a bounded, closed and convex subset of X and let $L, G : K \to Y$ be two continuous functions. Assume that*

(j) L is one-to-one with $G(K) \subseteq L(K)$.

Then, there exists a point of coincidence $x_0 \in K$ for L and G, i.e. $L(x_0) = G(x_0)$.

Proof. We apply Brouwer's fixed point theorem, see [9], to the map $F : K \to K$ defined by

$$F(x) = L^{-1}(G(x)) \quad \text{for all } x \in K,$$

which owing to (j) is well-posed and continuous. □

Now, we arrange to our goals some results contained in [2], see also [15, Theorem 6.8], which in addition to a priori estimations of the solution u_n for a linear difference problem of type ($D^n_{h_n^2 f}$) give also a useful information on the sup-norm of its differences Δu_n.

Theorem 2.2. *Let $n \geq 1$, and $w \in \mathbb{R}^n$. Then, the problem*

$$\begin{cases} -\Delta^2 u_n(k-1) = h_n^2 w(k)\,, & k \in [1,n]\,, \\ u(0) = u(n+1) = 0\,, \end{cases} \tag{$D^n_{h_n^2 w}$}$$

admits a unique solution $u_n \in X$, with

$$u_n(k) = h_n^2 \sum_{j=1}^{n} G(k,j) w(j) \qquad \forall k \in [1,n]\,, \tag{2}$$

where $G(k,j)$ is the discrete Green's function $G : [1,n] \times [1,n] \to \mathbb{R}$, defined as follows

$$G(k,j) = \begin{cases} \dfrac{j(n+1-k)}{n+1}, & \text{if } j \leq k, \\[2ex] \dfrac{k(n+1-j)}{n+1}, & \text{if } j \geq k. \end{cases} \tag{3}$$

Moreover, one has

$$\|u_n\|_\infty \leq \frac{(b-a)^2}{8}\|w\|_\infty, \tag{4}$$

and

$$\left\|\frac{\Delta u_n}{h_n}\right\|_\infty \leq \frac{(b-a)}{2}\|w\|_\infty, \tag{5}$$

Proof. Fix $n \geq 1$, put

$$B_n = \begin{cases} \dfrac{(n+1)^2}{8}, & \text{if } n \text{ is odd}, \\ \dfrac{n(n+2)}{8}, & \text{if } n \text{ is even}, \end{cases} \tag{6}$$

and let $u_n \in X$ be the unique solution of the linear problem ($D^n_{h_n^2 w}$). Bearing in mind [2, Theorem 2.1, Remark 4.1], see also [15, Theorem 6.8], it remains to prove (5). To this end, it easy to show that, for every $k \in [2,n]$, one has

$$\Big|\sum_{j=1}^{n}[G(k,j) - G(k-1,j)]\Big| \leq \frac{1}{n+1}\left(\frac{k(k-1)}{2} + \frac{(n+2-k)(n+1-k)}{2}\right),$$

while, if $k = 1$ or $k = n+1$, if we agree that $G(0,j) = G(n+1,j) = 0$,

$$\Big|\sum_{j=1}^{n}[G(k,j) - G(k-1,j)]\Big| \leq \frac{1}{n+1}\frac{n(n+1)}{2}.$$

From this, for every $k \in [1, n+1]$, we get

$$\begin{aligned} |\Delta u_n(k-1)| &= h_n^2 \left|\sum_{j=1}^{n}[G(k,j) - G(k-1,j)]\right| |w(k)| \\ &\leq \frac{(b-a)^2}{(n+1)^3}\left(\frac{n(n+1)}{2}\right)\|w\|_\infty \\ &= \frac{(b-a)}{(n+1)}\frac{n}{n+1}\frac{(b-a)}{2}\|w\|_\infty, \end{aligned}$$

which clearly ensures (5) and our conclusion is achieved. ◻

Now, we recall a slightly more refined version of [12, Lemma 2.4], which represents the core of the so-called constructive method for determining a solution to problem (P_f) as the limit of piecewise linear interpolations generated starting from the solutions of the discrete problem ($D^n_{h_n^2 f}$). On this argument, for a detailed proof, we refer to [22, 23], see also [2, Proposition 4.1].

Hereafter, when we fix n, unless explicitly stated, it means that n is as large as needed.

For $n \geq 1$ and $k = 1, ..., n+1$, we put

$$v_n(k-1) := \frac{\Delta u_n(k-1)}{h_n}, \tag{7}$$

and we define two functions $\alpha_n, \beta_n : [a,b] \to \mathbb{R}$ as follows:

$$\alpha_n(x) := u_n(k-1) + v_n(k-1)(x - x_{k-1}^n), \qquad \forall\, x_{k-1}^n \leq x \leq x_k^n, \tag{8}$$

$$\beta_n(x) := v_n(k-1) + \frac{\Delta v_n(k-1)}{h_n}(x - x_{k-1}^n), \qquad \forall\, x_{k-1}^n \leq x \leq x_k^n, \tag{9}$$

for all $x \in [a,b]$.

Proposition 2.3. *Assume that there exists $n_0 \geq 1$ such that problem $(D^n_{h_n^2 f})$ admits at least one solution $u_n \in X$, for all $n \geq n_0$. In addition, we suppose that there exist two positive constants R and Q such that*

(A_1) $\|u_n\|_\infty \leq R$ *and* $\|v_n\|_\infty \leq Q$*, for all* $n \geq n_0$.

Then, arguing by sub-sequences if necessary, there exist two functions $\alpha, \beta \in C([a,b])$ fulfilling the following conditions:

(U_1) $\lim_{n\to\infty} \|\alpha_n - \alpha\|_\infty = 0$,
(U_2) $\lim_{n\to\infty} \|\beta_n - \beta\|_\infty = 0$,

where, for all $x \in [a,b]$,

$$\alpha(x) = \int_a^x \beta(s)\, ds \quad \text{and} \quad \beta(x) = \beta(a) - \int_a^x f(s, \alpha(s), \alpha'(s))\, ds. \tag{10}$$

In particular, $\alpha \in C^2([a,b])$ and it turns out that α is a solution of problem (P_f) with

$$\|\alpha\|_\infty \leq R, \qquad \|\alpha'\|_\infty \leq Q, \qquad \|\alpha''\|_\infty \leq M, \tag{11}$$

where $M = \max_{(x,t,s)\in[a,b]\times[-R,R]\times[-Q,Q]} |f(x,t,s)|$.

Proof. Since we are arguing in a standard way, for the reader's convenience we limit ourselves to recalling the fundamental steps of the proof. First, we point out that the sequences $\{\alpha_n\}_n$ and $\{\beta_n\}_n$ are equi-bounded and equi-continuous in $[a,b]$, that is, for $n \in \mathbb{N}$, we can show that

(a_1) $\|\alpha_n\|_\infty \leq R$, $\|\beta_n\|_\infty \leq Q$;
(a_2) $|\alpha_n(s_1) - \alpha_n(s_2)| \leq Q|s_1 - s_2|$ for all $s_1, s_2 \in [a,b]$;
(a_3) $|\beta_n(s_1) - \beta_n(s_2)| \leq M|s_1 - s_2|$ for all $s_1, s_2 \in [a,b]$.

Next, (U_1) and (U_2) are easily proved by applying Ascoli-Arzelà's Theorem, while (10) can be easily proved by using (U_1), (U_2) and bearing in mind that

$$\begin{aligned} \alpha_n(x_k^n) &= u_n(k) \\ &= u_n(0) + h_n \sum_{i=1}^k v_n(i-1) \\ &= \alpha_n(a) + h_n \sum_{i=1}^k \beta_n(x_{i-1}^n), \end{aligned}$$

and

$$\begin{aligned}\beta_n(x_k^n) &= v_n(k)\\ &= v_n(0) + h_n \sum_{i=1}^{k} \frac{\Delta v_n(i-1)}{h_n}\\ &= \beta_n(a) - h_n \sum_{i=1}^{k} f_i^n(u_n(i), v_n(i-1))\\ &= \beta_n(a) - h_n \sum_{i=1}^{k} f_i^n(\alpha_n(x_i^n)), \beta_n(x_i^n)).\end{aligned}$$

Moreover, (10), clearly ensures that α is a solution of problem (P_f). Finally, the estimates (11) are a direct consequence of conditions (A_1) above. □

3. **Main results.** Our main results are the following.

Theorem 3.1. *Assume that there exist two positive constants r_1 and r_2 such that*

$$\max_{(x,t,s)\in[a,b]\times[-r_1,r_1]\times[-r_2,r_2]} |f(x,t,s)| \leq \min\left\{\frac{8r_1}{(b-a)^2}, \frac{2r_2}{(b-a)}\right\}. \tag{12}$$

Then, problem (P_f) admits at least one classical solution $u \in C^2([a,b])$ with

$$\|u\|_\infty \leq r_1, \qquad \|u'\|_\infty \leq r_2, \qquad \|u''\|_\infty \leq \min\left\{\frac{8r_1}{(b-a)^2}, \frac{2r_2}{(b-a)}\right\}.$$

Moreover, the following conditions hold:

(U) *$u(x) = \lim_{n\to+\infty} \alpha_n(x)$ and $u'(x) = \lim_{n\to+\infty} \beta_n(x)$ uniformly in $[a,b]$, α_n and β_n being the functions described in (8) and (9), respectively.*

Proof. **Claim.** *Let $n \geq 1$. Then, problem ($D^n_{h_n^2 f}$) has a solution $u_n \in X$ fulfilling*

$$\|u_n\|_\infty \leq r_1\,, \tag{13}$$

$$\left\|\frac{\Delta u_n}{h_n}\right\|_\infty \leq r_2. \tag{14}$$

Fix $n \geq 1$ and let r_1 and r_2 be as in (12). In the finite dimensional normed space X, we consider the bounded closed and convex subset

$$K_n := \left\{u_n \in X : \|u_n\|_\infty \leq r_1, \left\|\frac{\Delta u_n}{h_n}\right\|_\infty \leq r_2.\right\} \tag{15}$$

We look for a solution $u_n \in X$ of problem ($D^n_{h_n^2 f}$) as a point of coincidence $u_0 \in K_n$ between the continuous vector fields $L, G : K_n \to \mathbb{R}^n$ defined by putting

$$L(u)_k := -\Delta^2 u(k-1) \quad \text{and} \quad G(u)_k := h_n^2 f_k^n\left(u_n(k), \frac{\Delta u_n(k-1)}{h_n}\right), \tag{16}$$

for all $k = 1, ..., n$.

Clearly, Theorem 2.2 ensures that L is one-to-one. To show that $G(K_n) \subseteq L(K_n)$, take $z \in G(K_n)$ and prove that there exist a unique $u \in K_n$ such that $L(u) = z$.

Taking into account that $z \in G(K_n)$ one has that there exists $y \in K_n$ such that $z = G(y)$, so to achieve our goal we need to prove that the linear system $L(u) = G(y)$ admits a unique solution $u \in K_n$. Applying Theorem 2.2, with

$w(k) = f_k^n\left(y_n(k), \dfrac{\Delta y_n(k-1)}{h_n}\right)$, for all $k = 1, ..., n$, one has that the previous system admits an unique solution u.

Moreover, since $y \in K_n$, by (4) and (5), it is easy to see that

$$\|u\|_\infty \le h_n^2 \|w\|_\infty \le \frac{(b-a)^2}{8} \max_{(x,t,s)\in[a,b]\times[-r_1,r_1]\times[-r_2,r_2]} |f(x,t,s)| \le r_1,$$
$$\left\|\frac{\Delta u}{h_n}\right\|_\infty \le \frac{(b-a)}{2} \max_{(t,s,\xi)\in[a,b]\times[-r_1,r_1]\times[-r_2,r_2]} |f(x,t,s)| \le r_2,$$

that is, $u \in K_n$ and the claim is proved.

Therefore, combining the previous claim with (12), our conclusions follow by applying Proposition 2.3 with $R = r_1$, $Q = r_2$. □

Remark 3.2. Theorem 3.1 in addition to [5, Theorem 3.1] shows that u fulfills condition (U). In particular, in [5], the main result is shown by applying an abstract coincidence point theorem and the estimates on the L^∞ norms are obtained involving the embedding of the space $W_0^{2,2}(0,1)$ in $C^1(0,1)$.

To look for a positive solution, we need to combine Theorem (3.1) with some truncation techniques as in [7] and [8].

Theorem 3.3. *Let $f : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x,0,0) > 0$ for every $x \in [a,b]$. Assume that there exist two positive constants r_1 and r_2 such that*

$$\max_{(x,t,s,)\in[a,b]\times[0,r_1]\times[-r_2,r_2]} |f(x,t,s)| \le \min\left\{\frac{8r_1}{(b-a)^2}, \frac{2r_2}{(b-a)}\right\}. \tag{17}$$

Then, problem (P_f) admits at least one positive classical solution $u \in C^2([a,b])$ fulfilling condition (U) and with

$$\|u\|_\infty \le r_1, \qquad \|u'\|_\infty \le r_2, \qquad \|u''\|_\infty \le \min\left\{\frac{8r_1}{(b-a)^2}, \frac{2r_2}{(b-a)}\right\}.$$

Proof. Let $\hat{f} : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the function defined by

$$\hat{f}(x,t,s) = \begin{cases} f(x,t,s) & \text{if } t \ge 0, \\ f(x,0,s) & \text{if } t < 0. \end{cases} \tag{18}$$

Clearly, applying Theorem 3.1, problem $(P_{\hat{f}})$, admits at least one non-trivial classical solution $\tilde{u} \in C^2([a,b])$ which satisfies our conclusions provided that $\tilde{u}$ is positive in (a,b). Since, $\tilde{u}$ is continuous in $[a,b]$, Weierstrass Theorem ensures that there exists $x_1 \in [a,b]$ such that $\tilde{u}(x_1) = \min_{x\in[a,b]} \tilde{u}(x)$. If $x_1 = a$ or $x_1 = b$, we are done. If $x_1 \in (a,b)$ we have $\tilde{u}(x_1) \le 0$ and, since Fermat's Theorem implies that $\tilde{u}'(x_1) = 0$, one has

$$0 \ge -\tilde{u}''(x_1) = \hat{f}(x_1,0,0) = f(x_1,0,0) > 0.$$

We have obtained a contradiction and this completes the proof. □

In the applications of the previous theorem, the following is very useful, where roughly speaking we work only with one parameter.

Corollary 3.4. *Let $f : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x,0,0) > 0$ for every $x \in [a,b]$. Assume that there exists $r > 0$ such that*

$$\left(\frac{b-a}{2}\right) \max\left\{\frac{(b-a)}{4}, 1\right\} \leq \frac{r}{\max\limits_{(x,t,s,) \in [a,b] \times [0,r] \times [-r,r]} |f(x,t,s)|} \tag{19}$$

Then, problem (P_f) admits at least one positive classical solution $u \in C^2([a,b])$ fulfilling condition (U) with

$$\|u\|_\infty \leq r, \qquad \|u'\|_\infty \leq r, \qquad \|u''\|_\infty \leq \frac{2}{b-a} \min\left\{\frac{4}{b-a}, 1\right\} r.$$

Proof. Clearly, since $f(x,0,0) > 0$, (19) is well posed. Moreover, (19) implies that (17) is satisfied with $r_1 = r_2 = r$. So, applying Theorem 3.3 the proof is completed. $\square$

Remark 3.5. In the spirit of the above results, we explicitly observe that the conclusions of Theorem (3.3) continue to hold provided that at least one of the following conditions is verified:

$$\frac{(b-a)^2}{8} < \sup_{r>0} \frac{r}{\max\limits_{(x,t,s,) \in [a,b] \times [0,r] \times \left[-\frac{4}{b-a} r, \frac{4}{b-a} r\right]} |f(x,t,s)|}; \tag{20}$$

$$\frac{(b-a)}{2} < \sup_{r>0} \frac{r}{\max\limits_{(x,t,s,) \in [a,b] \times \left[0, \frac{b-a}{4} r\right] \times [-r,r]} |f(x,t,s)|}. \tag{21}$$

More precisely, we have

$$\|u\|_\infty \leq r, \qquad \|u'\|_\infty \leq \frac{4}{b-a} r, \qquad \|u''\|_\infty \leq \frac{8}{(b-a)^2} r,$$

and

$$\|u\|_\infty \leq \frac{b-a}{4} r, \qquad \|u'\|_\infty \leq r, \qquad \|u''\|_\infty \leq \frac{2}{b-a} r,$$

according to either (20) or (21) is satisfied, respectively. Indeed, in the first case, it is enough to apply Theorem 3.3 with $r = r_1$ and $r_2 = \frac{4}{b-a} r$ and, in the second one, $r_1 = \frac{b-a}{4} r$ and $r_2 = r$.

Finally, it is worth noticing that (20) or (21) are always satisfied if the size of the interval $[a,b]$ is small enough.

Proof of Theorem 1.1. It is enough to apply Theorem (3.3) with $r_1 = \frac{(b-a)^2}{8} M$ and $r_2 = \frac{(b-a)}{2} M$ that is $M = \frac{8}{(b-a)^2} r_1 = \frac{2}{(b-a)} r_2$. $\square$

Remark 3.6. In the previous results, in order to study the sign of the solution found, the principle of the discrete strong maximum is not used, as for example in [4, Theorem 3.1 and Proposition 2.2], but the dependence of the non-linearity from the convective term u' is fully exploited.

To illustrate the results obtained we present some examples. For simplicity, we drop the dependence on x. Clearly, for the autonomous equation $-u'' = f(u,u')$, because translates of solutions are again solutions, without loss of generality, problem (P_f) may be set in an interval $[-a,a]$, for some $a > 0$.

Example 3.1. Let $a > 0$. The function $u(x) = \frac{M}{2}(a^2 - x^2)$ is the unique solution of the problem

$$-u''(x) = M, \quad u(-a) = u(a) = 0, \quad \text{for all}\, x \in [-a, a],$$

where $M > 0$, so that

$$\|u\|_\infty = \frac{M}{2}a^2, \qquad \|u'\|_\infty = Ma.$$

So, the estimates furnished by Theorem 1.1 seem to be sharp.

Example 3.2. If $a \leq \frac{1}{2e}$, then problem

$$\begin{cases} -u'' = e^{u+u'}, & \text{in } (-a, a), \\ u(-a) = u(a) = 0, \end{cases}$$

admits at least one classical and positive solution u fulfilling condition (U) and with

$$\|u\|_\infty \leq \frac{1}{2}, \qquad \|u'\|_\infty \leq \frac{1}{2}, \qquad \|u''\|_\infty \leq \frac{1}{2a}.$$

To this end, it is enough to apply Corollary 3.4, with $r = \frac{1}{2}$.

Example 3.3. If $0 < a \leq \frac{\sqrt{17}-1}{4}$, then problem

$$\begin{cases} -u'' = \frac{u(u')^2+1}{1+u^2}, & \text{in } (-a, a), \\ u(-a) = u(a) = 0, \end{cases}$$

admits at least one classical and positive solution u fulfilling condition (U) with

$$\|u\|_\infty \leq a, \qquad \|u'\|_\infty \leq 2, \qquad \|u''\|_\infty \leq \frac{2}{a}.$$

To apply Theorem 3.3 with $r_1 = a$ and $r_2 = 2$ we need to have

$$1 \leq \frac{4a+1}{1+a^2} \leq \frac{2}{a},$$

which is satisfied for $0 < a \leq \frac{\sqrt{17}-1}{4}$.

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REFERENCES

[1] R. P. Agarwal, *Difference Equations and Inequalities*, Marcel Dekker, New York, Basel, 2000.
[2] E. Amoroso, P. Candito and J. Mawhin, Existence of a priori bounded solutions for discrete two-point boundary value problems, *J. Math. Anal. Appl.*, **519** (2023), Paper No. 126807, 18 pp.
[3] P. Amster and J. Haddad, A Hartman-Nagumo type condition for a class of contractible domains, *Topol. Methods Nonlinear Anal.*, **41** (2013), 287-304.
[4] G. Bonanno, P. Candito and G. D'Aguì, Variational methods on finite dimensional Banach spaces and discrete problems, *Adv. Nonlinear Stud.*, **14** (2014), 915-939.
[5] G. Bonanno, P. Candito and D. Motreanu, A coincidence point theorem for sequentially continuous mappings, *J. Math. Anal. Appl.*, **435**, (2016), 606-615.

[6] A. Cabada, D. O'Regan and R. L. Pouso, Second order problems with functional conditions including Sturm-Liouville and multipoint conditions, *Math. Nachr.*, **281** (2008), 1254-1263.

[7] P. Candito and R. Livrea, An existence result for a Neumann problem, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.*, **22** (2015), 481-488.

[8] P. Candito and R. Livrea, Differential and difference equations with applications, *Existence Results for Periodic Boundary Value Problems with a Convection Term*, Springer Proc. Math. Stat., **333**, Springer, Cham, 2020, 593-602,

[9] G. Dinca and J. Mawhin, *Brouwer Degree. The Core of Nonlinear Analysis*, Birkhäuser, Basel, 2021.

[10] G. Feltrin and F. Zanolin, Bound sets for a class of $\phi-$Laplacian operators, *J. Differential Equations*, **297** (2021), 508-535.

[11] M. Fukuhara, Quelques recherches sur les équations différentielles du second ordre, *Japanese J. Math.*, **5** (1928), 351-367.

[12] R. Gaines, Difference equations associated with boundary value problems for second order nonlinear ordinary differential equations, *SIAM J. Numer. Anal.*, **11** (1974), 411-434.

[13] M. R. Grossinho, F. Minhós and A. I. Santos, A note on a class of problems for a higher-order fully nonlinear equation under one-sided Nagumo-type condition, *Nonlinear Anal.*, **70** (2009), 4027-4038.

[14] P. Hartman, *Ordinary Differential Equations*, The Society for Industrial and Applied Mathematics, 2^{nd}, Philadelphia, 2002.

[15] W. G. Kelley and A. C. Peterson, *Difference Equations: An Introduction with Applications*, Academic Press, Inc., Boston, MA, 1991.

[16] J. Mahwin, Boundary value problems for nonlinear ordinary differential equations: From successive approximations to topology, in *Development of Mathematics 1900-1950*, Birkhäuser, Basel, 1994, 443-477.

[17] S. A. Marano, Existence theorems for a multivalued boundary value problem, *Bull. Austral. Math. Soc.*, **45** (1992), 249-260.

[18] C. Marcelli and F. Papalini, Boundary value problems for strongly nonlinear equations under a Wintner-Nagumo growth condition, *Bound. Value Probl.*, (2017), Paper No. 183, 15 pp.

[19] J. Mawhin, The Bernstein-Nagumo problem and two-point boundary value problems for ordinary differential equations, *Qualitative Theory of Differential Equations*, **I, II** Szeged, (1979), 709-740, *Colloq. Math. Soc. János Bolyai*, **30**, North-Holland, Amsterdam-New York, 1981.

[20] J. Mawhin and H. B. Thompson, Nagumo conditions and second-order quasilinear equations with compatible nonlinear functional boundary conditions, *Rocky Mountain J. Math.*, **41** (2011), 573-596.

[21] I. Rachůnková and C. C. Tisdell, Existence of non-spurious solutions to discrete Dirichlet problems with lower and upper solutions, *Nonlinear Anal.*, **67** (2007), 1236-1245.

[22] C. C. Tisdell, A discrete approach to continuous second-order boundary value problems via monotone iterative techniques, *Int. J. Difference Equ.*, **12** (2017), 145-160.

[23] C. C. Tisdell, Y. Liu and Z. Liu, Existence of solutions to discrete and continuous second-order boundary value problems via Lyapunov functions and a priori bounds, *Electron. J. Qual. Theory Differ. Equ.*, (2019), Paper No. 42, 11 pp.

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