Communications in Analysis and Mechanics

https://www.aimspress.com/journal/cam

Communications in Analysis and Mechanics, 16(1): 1–23.
DOI: 10.3934/cam.2024001
Received: 22 September 2023
Revised: 15 December 2023
Accepted: 02 January 2024
Published: 08 January 2024

Research article

Anisotropic $(\vec{p}, \vec{q})$-Laplacian problems with superlinear nonlinearities

Eleonora Amoroso[1], **Angela Sciammetta**[2] **and Patrick Winkert**[3,*]

[1] Department of Engineering, University of Messina, 98166 Messina, Italy
[2] Department of Mathematics and Computer Science, University of Palermo, 90123 Palermo, Italy
[3] Technische Universität Berlin, Institut für Mathematik, Straße des 17. Juni 136, 10623 Berlin, Germany

* **Correspondence:** Email: winkert@math.tu-berlin.de.

Abstract: In this paper we consider a class of anisotropic $(\vec{p}, \vec{q})$-Laplacian problems with nonlinear right-hand sides that are superlinear at $\pm\infty$. We prove the existence of two nontrivial weak solutions to this kind of problem by applying an abstract critical point theorem under very general assumptions on the data without supposing the Ambrosetti-Rabinowitz condition.

Keywords: anisotropic problems; critical point theory; existence; multiplicity results; location of the solutions; parametric problem; $(\vec{p}, \vec{q})$-Laplacian; superlinear nonlinearity
Mathematics Subject Classification: 35A01, 35D30, 35J62, 35J66

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$ be a bounded domain with boundary $\partial\Omega$ of class C^1. In this paper we consider the following anisotropic differential equation involving the $(\vec{p}, \vec{q})$-Laplacian given by

$$\begin{aligned} -\Delta_{\vec{p}} u - \Delta_{\vec{q}} u &= \lambda f(x,u) \quad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{$D_\lambda^{\vec{p},\vec{q}}$}$$

where λ is a positive parameter, $\vec{p} = (p_1, p_2, \dots, p_N)$, $\vec{q} = (q_1, q_2, \dots, q_N)$, $\vec{p}, \vec{q} \in \mathbb{R}^N$ are real vectors such that

$$\max\{q_1, q_2, \dots, q_N\} < \min\{p_1, p_2, \dots, p_N\} < N,$$

and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a nonlinearity with subcritical growth that is superlinear at $\pm\infty$, see ($\mathrm{H_f}$) for the precise assumptions. Moreover, for any $\vec{s} = (s_1, s_2, \dots, s_N) \in \mathbb{R}^N$, we denote by

$$\Delta_{\vec{s}} u = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{s_i-2} \frac{\partial u}{\partial x_i}\right)$$

the anisotropic $\vec{s}$-Laplace differential operator. If $s_i = 2$ for all $i = 1, \ldots, N$, we get

$$\sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2} = \Delta u,$$

that is the usual Laplacian and if $\vec{s}$ is constant (i.e. $s_i = s$ for each $i = 1, \ldots, N$) we obtain

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{s-2} \frac{\partial u}{\partial x_i} \right) = \tilde{\Delta}_s u,$$

which is called the pseudo-s-Laplace operator, see Belloni-Kawohl [1] or Brasco-Franzina [2]. More information about anisotropic operators and in particular about the theory of anisotropic Sobolev spaces can be found in Kufner-Rákosník [3], Nikol'skiĭ [4] and Rákosník [5, 6].

Anisotropic differential problems have a large background in several applications, for example, the study of an epidemic disease in heterogeneous habitat can be expressed by an anisotropic nonlinear system. In general, anisotropic operators are used for modeling in which partial differential derivatives vary with the direction. Also, the anisotropic Laplacian, given by

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\frac{\partial}{\partial \xi_i} \left(\frac{1}{2} F^2 \right) (\nabla u) \right)$$

where $F(\xi) = (\sum_{i=1}^{N} |\xi_i|^2)^{\frac{1}{2}}$ for $\xi \in \mathbb{R}^N$, plays a key role in physical models of crystal growth in the context of the so-called Wulff shape of F (also known as equilibrium crystal shape), see the work of Wulff [7]. For more information on applications in different disciplines we refer to Antontsev-Díaz-Shmarev [8], Bendahmane-Chrif-El Manouni [9], Bendahmane-Langlais-Saad [10], Vétois [11] and the references therein.

Although there are some works for $\vec{p}$-Laplacian problems, only a few exist for the anisotropic $(\vec{p}, \vec{q})$-Laplacian. Recently, Razani-Figueiredo [12] studied the anisotropic Dirichlet problem

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{\partial u}{\partial x_i} \right) - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{q_i-2} \frac{\partial u}{\partial x_i} \right) = \lambda u^{\gamma-1} \tag{1.1}$$

in a bounded and regular domain Ω of $\mathbb{R}^N$ with $\gamma > 1$ and $\lambda > 0$. Based on a sub-supersolution approach the authors show the existence of at least one positive solution of (1.1). In Razani [13] nonstandard competing anisotropic $(\vec{p}, \vec{q})$-Laplacian problems with convolution of the form

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} - \mu \left| \frac{\partial u}{\partial x_i} \right|^{q_i-2} \right) \frac{\partial u}{\partial x_i} = f(x, \phi * u, \nabla(\phi * u)) \tag{1.2}$$

have been considered, where $\phi \in L^1(\mathbb{R}^N)$. If $\mu > 0$, then the existence of a generalized solution of (1.2) is shown by using a Galerkin base for the function space and if $\mu \leq 0$, then any generalized solution turns out to be a weak solution. We also mention the recent work of Tavares [14] who considered existence and multiplicity of nonnegative solutions for the problem given by

$$\begin{aligned} -\Delta_{\vec{p}} u - \Delta_{\vec{q}} u &= k(x) u^{\alpha-1} + f(x, u) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where $\alpha > 1$, $k \in L^{\infty}(\Omega)$ with $k(x) > 0$ for a.a. $x \in \Omega$ and a Carathéodory nonlinearity $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ with subcritical growth. Existence and regularity results for anisotropic problems driven by the $\vec{p}$-Laplacian have been obtained by several authors. Without guarantee of completeness, we mention just some of them and refer to the papers of Bonanno-D'Aguì-Sciammetta [15], Ciani-Figueiredo-Suárez [16], Ciraolo-Figalli-Roncoroni [17], Ciraolo-Sciammetta [18], DiBenedetto-Gianazza-Vespri [19], dos Santos-Figueiredo-Tavares [20], Fragalà-Gazzola-Kawohl [21], Perera-Agarwal-O'Regan [22], Ragusa-Razani-Safari [23], see also the references therein. Related results for the (p,q)-Laplacian, double phase equations, anisotropic problems or the discrete p-Laplacian can be found in the works of Bai-Papageorgiou-Zeng [24], Bohner-Caristi-Ghobadi-Heidarkhani [25], El Manouni-Marino-Winkert [26], Leggat-Miri [27], He-Ousbika-El Allali-Zuo [28], Ju-Molica Bisci-Zhang [29], Liu-Motreanu-Zeng [30], Papageorgiou [31], Son-Sim [32], Vetro-Vetro [33], Zeng-Bai-Gasiński-Winkert [34], Zeng-Papageorgiou [35] and Zeng-Rădulescu-Winkert [36].

Motivated by the above mentioned works for $\vec{p}$-Laplacian problems, we consider in our paper so-called $(\vec{p},\vec{q})$-Laplacian problems with general right-hand sides. Our main goal is to apply a two critical point result due to Bonanno-D'Aguì [37, Theorem 2.1] in order to get the existence of two positive solutions for problem $(D_\lambda^{\vec{p},\vec{q}})$ with different energy sign. Note that the results in [37] are given in a very general setting which can be applied to a large number of problems. Our paper can be seen as an extension of the work of Bonanno-D'Aguì-Sciammetta [38] for $(\vec{p},\vec{q})$-Laplacian problems. But not only the differential operator is more general than in [38], also the condition required on f are weaker. Indeed, instead of assuming the Ambrosetti-Rabinowitz condition, we only assume that the nonlinear term on the right-hand side of $(D_\lambda^{\vec{p},\vec{q}})$ is $(p^+ - 1)$-superlinear at $\pm\infty$ (with p^+ defined in (2.2) replacing h by p) and fulfills in addition a suitable behavior at $\pm\infty$, see $(\mathrm{H_f})$. These hypotheses are weaker than the Ambrosetti-Rabinowitz condition. Under these conditions, together with the subcritical growth, we prove the existence of two weak solutions for problem $(D_\lambda^{\vec{p},\vec{q}})$ that have opposite energy sign related to the energy functional of $(D_\lambda^{\vec{p},\vec{q}})$.

The paper is organized as follows. In Section 2 we present the main properties of anisotropic Sobolev spaces and consider the main features of the anisotropic $(\vec{p},\vec{q})$-Laplacian, see Propositions 2.3 and 2.4. Moreover, we recall the main abstract critical point theorem which will play a key role in our treatment, see Theorem 2.7. In Section 3 we first state the precise assumptions on the data of problem $(D_\lambda^{\vec{p},\vec{q}})$ and prove that the corresponding energy functional fulfills the C-condition. Then we give our main result about the existence of two nontrivial weak solutions of $(D_\lambda^{\vec{p},\vec{q}})$ (see Theorem 3.3) and some consequences in which the solutions are nonnegative, see Theorems 3.4 and 3.5. Finally, we consider the autonomous problem $(AD_\lambda^{\vec{p},\vec{q}})$, providing an existence result (see Corollary 3.6) and an example.

2. Preliminaries and basic properties

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with boundary $\partial\Omega$ of class C^1. For any real vector $\vec{h} = (h_0, h_1, h_2, \dots, h_N)$ with $h_i \geq 1$ for every $i = 0, 1, \dots, N$, we indicate with $W^{1,\vec{h}}(\Omega)$ the anisotropic Sobolev space defined by

$$W^{1,\vec{h}}(\Omega) = \left\{ u \in L^{h_0}(\Omega) \ : \ \frac{\partial u}{\partial x_i} \in L^{h_i}(\Omega) \text{ for } i = 1, \dots, N \right\},$$

equipped with the norm

$$\|u\|_{W^{1,\vec{h}}(\Omega)} = \|u\|_{L^{h_0}(\Omega)} + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{h_i}(\Omega)}. \tag{2.1}$$

Moreover, set

$$h^- = \min\{h_1, h_2, \ldots, h_N\} \quad \text{and} \quad h^+ = \max\{h_1, h_2, \ldots, h_N\}, \tag{2.2}$$

and suppose that $h^- < N$ and $h_0 < (h^-)^* = \frac{Nh^-}{N-h^-}$. Denote by $W_0^{1,\vec{h}}(\Omega)$ the closure of $C_0^\infty(\Omega)$ endowed with the following norm

$$\|u\|_{W_0^{1,\vec{h}}(\Omega)} := \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{h_i}(\Omega)}, \tag{2.3}$$

which is equivalent to the usual one given in (2.1). Indeed, taking into account that $W^{1,h^-}(\Omega)$ is compactly embedded in $L^{h_0}(\Omega)$ and using Hölder's inequality (see (2.4) in Proposition 2.1), we have that

$$\|u\|_{L^{h_0}(\Omega)} \leq c\|u\|_{W^{1,h^-}(\Omega)} \leq \tilde{k} \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{h_i}(\Omega)}.$$

It is well known that $W_0^{1,\vec{h}}(\Omega)$, endowed with the norm defined in (2.3), is a separable Banach space and it is also reflexive if $h_i > 1$ for all $i = 1, \ldots, N$, see Rákosník [5, Theorem 1].

Given $\vec{p} = (p_0, p_1, p_2, \ldots, p_N)$ and $\vec{q} = (q_0, q_1, q_2, \ldots, q_N)$, with $p_i, q_i \geq 2$ for every $i = 1, \ldots, N$, we suppose that

(H) $q^+ < p^- < N$, $p_0 < (p^-)^*$ and $q_0 < (q^-)^*$, where $(\cdot)^* = \frac{N(\cdot)}{N-(\cdot)}$ denotes the critical Sobolev exponent.

In the following proposition, we give a relation between the spaces $W_0^{1,\vec{p}}(\Omega)$ and $W_0^{1,\vec{q}}(\Omega)$ and their norms. In particular, we underline that p_0 and q_0 are necessary only for the definition of the anisotropic Sobolev spaces, but since we endow the spaces with the equivalent norm given in (2.3), from now on we will only deal with the components $(p_1, \ldots, p_N)$ and $(q_1, \ldots, q_N)$.

Proposition 2.1. *If $q^+ < p^-$, then $W_0^{1,\vec{p}}(\Omega) \subseteq W_0^{1,\vec{q}}(\Omega)$ and*

$$\|u\|_{W_0^{1,\vec{q}}(\Omega)} \leq \max_{1\leq i\leq N} \left\{ |\Omega|^{\frac{p_i - q_i}{p_i q_i}} \right\} \|u\|_{W_0^{1,\vec{p}}(\Omega)},$$

where $|\Omega|$ is the Lebesgue measure of Ω.

Proof. Fix $u \in W_0^{1,\vec{p}}(\Omega)$ and $i \in \{1, \ldots, N\}$. In particular, $\frac{\partial u}{\partial x_i} \in L^{p_i}(\Omega)$ and $\left|\frac{\partial u}{\partial x_i}\right|^{p^-} \in L^{\frac{p_i}{p^-}}(\Omega)$. If $p_i > p^-$, by Hölder's inequality, we get

$$\left(\int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p^-} \mathrm{d}x \right)^{\frac{1}{p^-}} \leq |\Omega|^{\frac{p_i - p^-}{p_i p^-}} \left(\int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p_i} \mathrm{d}x \right)^{\frac{1}{p_i}}, \tag{2.4}$$

while if $p_i = p^-$, then (2.4) is an equality. Moreover, $\left|\frac{\partial u}{\partial x_i}\right|^{q^+} \in L^{\frac{p^-}{q^+}}(\Omega)$, then again from Hölder's inequality, we obtain

$$\left(\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{q^+} \mathrm{d}x\right)^{\frac{1}{q^+}} \leq |\Omega|^{\frac{p^- - q^+}{p^- q^+}} \left(\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p^-} \mathrm{d}x\right)^{\frac{1}{p^-}}. \tag{2.5}$$

Thus, combining (2.4) and (2.5) we derive

$$\left(\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{q^+} \mathrm{d}x\right)^{\frac{1}{q^+}} \leq |\Omega|^{\frac{p_i - q^+}{p_i q^+}} \left(\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i} \mathrm{d}x\right)^{\frac{1}{p_i}}, \tag{2.6}$$

for all $i = 1, \ldots, N$. Furthermore, if $q^+ > q_i$ and using Hölder's inequality we have

$$\left(\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{q_i} \mathrm{d}x\right)^{\frac{1}{q_i}} \leq |\Omega|^{\frac{q^+ - q_i}{q^+ q_i}} \left(\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{q^+} \mathrm{d}x\right)^{\frac{1}{q^+}}, \tag{2.7}$$

and if $q^+ = q_i$ the previous inequality becomes an equality. Then, from (2.6) and (2.7) it follows that

$$\left(\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{q_i} \mathrm{d}x\right)^{\frac{1}{q_i}} \leq |\Omega|^{\frac{p_i - q_i}{p_i q_i}} \left(\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i} \mathrm{d}x\right)^{\frac{1}{p_i}} \quad \text{for all } i = 1, \ldots, N.$$

Hence,

$$\|u\|_{W_0^{1,\vec{q}}(\Omega)} \leq \max_{1 \leq i \leq N} \left\{|\Omega|^{\frac{p_i - q_i}{p_i q_i}}\right\} \|u\|_{W_0^{1,\vec{p}}(\Omega)},$$

and the proof is complete.

In the sequel, we estimate the embedding constant of $W_0^{1,\vec{p}}(\Omega)$ into $L^r(\Omega)$ for each $r \in [1, (p^-)^*]$ with $p^- < N$, where $(p^-)^*$ is the critical Sobolev exponent to p^-, that is

$$(p^-)^* = \frac{Np^-}{N - p^-}. \tag{2.8}$$

Proposition 2.2. *If* $1 \leq p^- < N$, *then for any* $r \in [1, (p^-)^*]$, $W_0^{1,\vec{p}}(\Omega) \hookrightarrow L^r(\Omega)$ *is continuous and one has*

$$\|u\|_{L^r(\Omega)} \leq T_r \|u\|_{W_0^{1,\vec{p}}(\Omega)} \tag{2.9}$$

for all $u \in W_0^{1,\vec{p}}(\Omega)$, *where*

$$\begin{aligned} T_r &= c_r 2^{\frac{(N-1)(p^- - 1)}{p^-}} \max_{1 \leq i \leq N} \left\{|\Omega|^{\frac{p_i - p^-}{p_i p^-}}\right\}, \\ c_r &= T\, |\Omega|^{\frac{(p^-)^* - r}{(p^-)^* r}}, \\ T &\leq \frac{N^{-\frac{1}{p^-}}}{\sqrt{\pi}} \left(\frac{p^- - 1}{N - p^-}\right)^{1 - \frac{1}{p^-}} \left(\frac{\Gamma(N)\Gamma\left(1 + \frac{N}{2}\right)}{\Gamma\left(\frac{N}{p^-}\right)\Gamma\left(1 + N - \frac{N}{p^-}\right)}\right)^{\frac{1}{N}}, \end{aligned} \tag{2.10}$$

see Talenti [39, formula (2)] and Γ *is the Euler function. Moreover, for any* $r \in [1, (p^-)^*[$ *the embedding* $W_0^{1,\vec{p}}(\Omega) \hookrightarrow L^r(\Omega)$ *is compact.*

Proof. From the Sobolev embedding theorem there exists a positive constant $T \in \mathbb{R}$ such that for all $u \in W_0^{1,p^-}(\Omega)$ the following holds

$$\|u\|_{L^{(p^-)^*}(\Omega)} \leq T\|u\|_{W_0^{1,p^-}(\Omega)}, \tag{2.11}$$

where $(p^-)^*$ and T are defined in (2.8) and (2.10), respectively.

Fix $r \in [1,(p^-)^*]$. Since $\frac{1}{r} = \frac{1}{(p^-)^*} + \frac{(p^-)^*-r}{(p^-)^* r}$, by Hölder's inequality and (2.11), we have

$$\|u\|_{L^r(\Omega)} \leq \|u\|_{L^{(p^-)^*}(\Omega)}|\Omega|^{\frac{(p^-)^*-r}{(p^-)^* r}} \leq T|\Omega|^{\frac{(p^-)^*-r}{(p^-)^* r}}\|u\|_{W_0^{1,p^-}(\Omega)},$$

that is

$$\|u\|_{L^r(\Omega)} \leq c_r\|u\|_{W_0^{1,p^-}(\Omega)}. \tag{2.12}$$

Arguing as in the proof of Proposition 2.1 in Bonanno-D'Aguì-Sciammetta [38], we derive

$$\|u\|_{W_0^{1,p^-}(\Omega)} \leq 2^{\frac{(N-1)(p^- -1)}{p^-}} \max_{1\leq i\leq N}\left\{|\Omega^{\frac{p_i-p^-}{p_i p^-}}\right\}\|u\|_{W_0^{1,\vec{p}}(\Omega)},$$

and, taking (2.12) into account, we get that (2.9) holds for every $r \in [1,(p^-)^*]$.

Finally, combining the continuous embedding $W_0^{1,\vec{p}}(\Omega) \hookrightarrow W_0^{1,p^-}(\Omega)$ with the compact embedding $W_0^{1,p^-}(\Omega) \hookrightarrow L^r(\Omega)$, it follows that $W_0^{1,\vec{p}}(\Omega) \hookrightarrow L^r(\Omega)$ is compact for any $r \in [1,(p^-)^*[$.

Now, we define

$$F(x,t) = \int_0^t f(x,\xi)\,\mathrm{d}\xi \quad \text{for all } (x,t) \in \Omega\times\mathbb{R},$$

and we introduce the functionals $\Phi, \Psi\colon W_0^{1,\vec{p}}(\Omega) \to \mathbb{R}$ given by

$$\begin{aligned} \Phi(u) &= \sum_{i=1}^N\left(\frac{1}{p_i}\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\mathrm{d}x + \frac{1}{q_i}\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^{q_i}\mathrm{d}x\right),\\ \Psi(u) &= \int_\Omega F(x,u)\,\mathrm{d}x, \end{aligned} \tag{2.13}$$

for every $u \in W_0^{1,\vec{p}}(\Omega)$. Clearly, Φ and Ψ are Gâteaux differentiable and one has

$$\begin{aligned} \Phi'(u)(v) &= \sum_{i=1}^N\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,\mathrm{d}x + \sum_{i=1}^N\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^{q_i-2}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,\mathrm{d}x,\\ \Psi'(u)(v) &= \int_\Omega f(x,u)v\,\mathrm{d}x, \end{aligned} \tag{2.14}$$

for every $u, v \in W_0^{1,\vec{p}}(\Omega)$. Also, we consider the energy functional $I_\lambda\colon W_0^{1,\vec{p}}(\Omega) \to \mathbb{R}$ associated to our problem $(D_\lambda^{\vec{p},\vec{q}})$, that is given by $I_\lambda = \Phi - \lambda\Psi$ for all $\lambda > 0$.

We recall that $u\colon \Omega \to \mathbb{R}$ is a weak solution of problem $(D_\lambda^{\vec{p},\vec{q}})$ if the following condition holds for all $v \in W_0^{1,\vec{p}}(\Omega)$

$$\sum_{i=1}^{N} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,\mathrm{d}x + \sum_{i=1}^{N} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{q_i-2} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,\mathrm{d}x = \lambda \int_\Omega f(x,u)v\,\mathrm{d}x.$$

Then, from (2.14) it follows that $u \in W_0^{1,\vec{p}}(\Omega)$ is a weak solution of problem $(D_\lambda^{\vec{p},\vec{q}})$ if and only if u is a critical point for I_λ. Consequently, our study is based on critical point theory and in particular on a critical point theorem due to Bonanno-D'Aguì [37] that we state later in Theorem 2.7.

In the following, we deal with some properties of the Gâteaux derivative of the functional Φ that are needed in our investigation.

Proposition 2.3. *The functional* $\Phi'\colon W_0^{1,\vec{p}}(\Omega) \to \left(W_0^{1,\vec{p}}(\Omega)\right)^*$ *defined in* (2.14) *is monotone and coercive.*

Proof. First we prove that Φ' is monotone, i.e.

$$\langle \Phi'(u) - \Phi'(v), u - v\rangle \geq 0 \quad \text{for all } u, v \in W_0^{1,\vec{p}}(\Omega). \tag{2.15}$$

To this end, we use the following inequality

$$\left(\left|\frac{\partial u}{\partial x_i}\right|^{r-2} \frac{\partial u}{\partial x_i} - \left|\frac{\partial v}{\partial x_i}\right|^{r-2} \frac{\partial v}{\partial x_i}\right)\frac{\partial(u-v)}{\partial x_i} \geq C\left|\frac{\partial(u-v)}{\partial x_i}\right|^r$$

for each $r \geq 2$ and for some constant $C > 0$, see Simon [40] or Lindqvist [41]. Indeed, using the previous inequality we have

$$\begin{aligned}
&\langle \Phi'(u) - \Phi'(v), u - v\rangle \\
&= \sum_{i=1}^{N} \int_\Omega \left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i} - \left|\frac{\partial v}{\partial x_i}\right|^{p_i-2} \frac{\partial v}{\partial x_i}\right)\frac{\partial(u-v)}{\partial x_i}\,\mathrm{d}x \\
&\quad + \sum_{i=1}^{N} \int_\Omega \left(\left|\frac{\partial u}{\partial x_i}\right|^{q_i-2} \frac{\partial u}{\partial x_i} - \left|\frac{\partial v}{\partial x_i}\right|^{q_i-2} \frac{\partial v}{\partial x_i}\right)\frac{\partial(u-v)}{\partial x_i}\,\mathrm{d}x \\
&\geq \sum_{i=1}^{N} \int_\Omega \left(C_0\left|\frac{\partial(u-v)}{\partial x_i}\right|^{p_i} + C_1\left|\frac{\partial(u-v)}{\partial x_i}\right|^{q_i}\right)\mathrm{d}x \\
&\geq 0,
\end{aligned}$$

and (2.15) is achieved.

Now, we prove that Φ' is coercive. We observe that

$$\langle \Phi'(u), u\rangle = \sum_{i=1}^{N} \int_\Omega \left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i} + \left|\frac{\partial u}{\partial x_i}\right|^{q_i}\right)\mathrm{d}x \geq \sum_{i=1}^{N} \left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(\Omega)}^{p_i}. \tag{2.16}$$

Moreover, let $j \in \{1, \ldots, N\}$ be such that

$$\left(\int_\Omega \left|\frac{\partial u}{\partial x_j}\right|^{p_j}\mathrm{d}x\right)^{\frac{1}{p_j}} := \max_{1\leq i\leq N}\left(\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i}\mathrm{d}x\right)^{\frac{1}{p_i}}.$$

Then,

$$\begin{aligned}\|u\|_{W_0^{1,\vec{p}}(\Omega)} &= \sum_{i=1}^{N}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(\Omega)} \le N\left(\int_\Omega\left|\frac{\partial u}{\partial x_j}\right|^{p_j}\,\mathrm{d}x\right)^{\frac{1}{p_j}}\\ &\le N\left(\int_\Omega\left|\frac{\partial u}{\partial x_1}\right|^{p_1}\,\mathrm{d}x+\ldots+\int_\Omega\left|\frac{\partial u}{\partial x_N}\right|^{p_i}\,\mathrm{d}x\right)^{\frac{1}{p_j}}\\ &= N\left(\sum_{i=1}^{N}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(\Omega)}^{p_i}\right)^{\frac{1}{p_j}}.\end{aligned}$$

Thus, we get

$$\sum_{i=1}^{N}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(\Omega)}^{p_i} \ge \frac{1}{N^{p_j}}\|u\|_{W_0^{1,\vec{p}}(\Omega)}^{p_j}. \tag{2.17}$$

From (2.16) and (2.17), we derive

$$\langle\Phi'(u),u\rangle \ge \frac{1}{N^{p_j}}\|u\|_{W_0^{1,\vec{p}}(\Omega)}^{p_j},$$

namely

$$\frac{\langle\Phi'(u),u\rangle}{\|u\|_{W_0^{1,\vec{p}}(\Omega)}} \ge \frac{1}{N^{p_j}}\|u\|_{W_0^{1,\vec{p}}(\Omega)}^{p_j-1},$$

and this implies that Φ' is coercive.

Proposition 2.4. *The map* $\Phi'\colon W_0^{1,\vec{p}}(\Omega)\to\left(W_0^{1,\vec{p}}(\Omega)\right)^*$ *has the* (S_+)*-property, that is*

$$\textit{if}\quad u_n\rightharpoonup u\quad \textit{in } W_0^{1,\vec{p}}(\Omega)\quad \textit{and}\quad \limsup_{n\to\infty}\langle\Phi'(u_n),u_n-u\rangle\le 0,$$

then $u_n\to u$ *in* $W_0^{1,\vec{p}}(\Omega)$.

Proof. Let $\{u_n\}_{n\in\mathbb{N}}\subset W_0^{1,\vec{p}}(\Omega)$ be such that $u_n\rightharpoonup u$ in $W_0^{1,\vec{p}}(\Omega)$ and

$$\limsup_{n\to+\infty}\langle\Phi'(u_n),u_n-u\rangle\le 0. \tag{2.18}$$

First, we observe that

$$\lim_{n\to+\infty}\langle\Phi'(u),u_n-u\rangle=0, \tag{2.19}$$

since $\Phi'(u)$ is a linear operator in $W_0^{1,\vec{p}}(\Omega)$ and $u_n\rightharpoonup u$ in $W_0^{1,\vec{p}}(\Omega)$. Hence, from (2.18) and (2.19) it follows that

$$\limsup_{n\to+\infty}\langle\Phi'(u_n)-\Phi'(u),u_n-u\rangle\le 0. \tag{2.20}$$

Now, for all $i = 1, \dots, N$ and for all $w, v \in W_0^{1,\vec{p}}(\Omega)$ we set

$$\begin{aligned}
\mathcal{A}_i^{p_i}(w)(v) &= \int_\Omega \left|\frac{\partial w}{\partial x_i}\right|^{p_i-2} \frac{\partial w}{\partial x_i}\frac{\partial v}{\partial x_i}\,\mathrm{d}x,\\
\mathcal{A}_i^{q_i}(w)(v) &= \int_\Omega \left|\frac{\partial w}{\partial x_i}\right|^{q_i-2} \frac{\partial w}{\partial x_i}\frac{\partial v}{\partial x_i}\,\mathrm{d}x,\\
\mathcal{B}_i^{p_i}(w)(v) &= \frac{1}{\left\|\frac{\partial w}{\partial x_i}\right\|_{L^{p_i}(\Omega)}^{p_i-2}}\mathcal{A}_i^{p_i}(w)(v),\\
\mathcal{B}_i^{q_i}(w)(v) &= \frac{1}{\left\|\frac{\partial w}{\partial x_i}\right\|_{L^{q_i}(\Omega)}^{q_i-2}}\mathcal{A}_i^{q_i}(w)(v),
\end{aligned}$$

and

$$\mathcal{B}_i(w)(v) = \mathcal{B}_i^{p_i}(w)(v) + \mathcal{B}_i^{q_i}(w)(v).$$

Then, we can write Φ' as follows

$$\Phi'(w)(v) = \sum_{i=1}^N \left(\mathcal{A}_i^{p_i}(w)(v) + \mathcal{A}_i^{q_i}(w)(v)\right) \quad \text{for every } w, v \in W_0^{1,\vec{p}}(\Omega).$$

By (2.20), we get

$$\limsup_{n\to+\infty} \langle \mathcal{B}_i(u_n) - \mathcal{B}_i(u), u_n - u\rangle \le 0 \quad \text{for all } i = 1, \dots, N. \tag{2.21}$$

Moreover, we have

$$\begin{aligned}
&\langle \mathcal{B}_i(u_n) - \mathcal{B}_i(u), u_n - u\rangle\\
&= \left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{p_i}(\Omega)}^2 + \left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{q_i}(\Omega)}^2 + \left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(\Omega)}^2 + \left\|\frac{\partial u}{\partial x_i}\right\|_{L^{q_i}(\Omega)}^2\\
&\quad - \langle \mathcal{B}_i(u_n), u\rangle - \langle \mathcal{B}_i(u), u_n\rangle\\
&= \left(\left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{p_i}(\Omega)} - \left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(\Omega)}\right)^2 + \left(\left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{q_i}(\Omega)} - \left\|\frac{\partial u}{\partial x_i}\right\|_{L^{q_i}(\Omega)}\right)^2\\
&\quad + 2\left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{p_i}(\Omega)}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(\Omega)} + 2\left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{q_i}(\Omega)}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{q_i}(\Omega)}\\
&\quad - \langle \mathcal{B}_i(u_n), u\rangle - \langle \mathcal{B}_i(u), u_n\rangle\\
&= \left(\left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{p_i}(\Omega)} - \left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(\Omega)}\right)^2 + \left(\left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{q_i}(\Omega)} - \left\|\frac{\partial u}{\partial x_i}\right\|_{L^{q_i}(\Omega)}\right)^2\\
&\quad + \left(\left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{p_i}(\Omega)}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(\Omega)} - \langle \mathcal{B}_i^{p_i}(u_n), u\rangle\right)\\
&\quad + \left(\left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{p_i}(\Omega)}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(\Omega)} - \langle \mathcal{B}_i^{p_i}(u), u_n\rangle\right)
\end{aligned}$$

$$+\left(\left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{q_i}(\Omega)}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{q_i}(\Omega)}-\langle\mathcal{B}_i^{q_i}(u_n),u\rangle\right)$$
$$+\left(\left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{q_i}(\Omega)}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{q_i}(\Omega)}-\langle\mathcal{B}_i^{q_i}(u),u_n\rangle\right).$$

Also, applying Hölder's inequality one has

$$\left|\langle\mathcal{B}_i^{p_i}(u_n),u\rangle\right|\le\left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{p_i}(\Omega)}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(\Omega)},$$
$$\left|\langle\mathcal{B}_i^{p_i}(u),u_n\rangle\right|\le\left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{p_i}(\Omega)}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(\Omega)},$$
$$\left|\langle\mathcal{B}_i^{q_i}(u_n),u\rangle\right|\le\left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{q_i}(\Omega)}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{q_i}(\Omega)},$$
$$\left|\langle\mathcal{B}_i^{q_i}(u),u_n\rangle\right|\le\left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{q_i}(\Omega)}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{q_i}(\Omega)}.$$

Hence, we derive that

$$\langle\mathcal{B}_i(u_n)-\mathcal{B}_i(u),u_n-u\rangle$$
$$\ge\left(\left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{p_i}(\Omega)}-\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(\Omega)}\right)^2+\left(\left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{q_i}(\Omega)}-\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{q_i}(\Omega)}\right)^2$$
$$\ge\left(\left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{p_i}(\Omega)}-\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(\Omega)}\right)^2,$$

which, taking (2.21) into account, implies that

$$\lim_{n\to+\infty}\left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{p_i}(\Omega)}=\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(\Omega)}\quad\text{for all } i=1,\dots,N.$$

Then, from Papageorgiou-Winkert [42, Proposition 4.1.11], since $L^{p_i}(\Omega)$ is uniformly convex, one has

$$\lim_{n\to+\infty}\left\|\frac{\partial u_n}{\partial x_i}-\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(\Omega)}=0\quad\text{for all } i=1,\dots,N.$$

Thus, it follows that

$$\lim_{n\to+\infty}\|u_n-u\|_{W_0^{1,\vec{p}}(\Omega)}=0,$$

and our claim is proved.

Finally, we point out the following result in order to get information on the sign of the solutions of $(D_\lambda^{\vec{p},\vec{q}})$. For this purpose, let

$$f^+(x,t)=\begin{cases}f(x,0) & \text{if } t<0,\\ f(x,t) & \text{if } t\ge 0,\end{cases}$$

for all $(x,t)\in\Omega\times\mathbb{R}$ and consider the following problem

$$\begin{aligned}-\Delta_{\vec{p}}u-\Delta_{\vec{q}}u&=\lambda f^+(x,u) && \text{in } \Omega,\\ u&=0 && \text{on } \partial\Omega.\end{aligned}\tag{$D_{\lambda,f^+}^{\vec{p},\vec{q}}$}$$

Lemma 2.5. *Assume that $f(x,0) \geq 0$ for a.a. $x \in \Omega$. Then, any weak solution of problem $(D^{\vec{p},\vec{q}}_{\lambda,f^+})$ is nonnegative and it is also a weak solution of problem $(D^{\vec{p},\vec{q}}_{\lambda})$.*

Proof. Let u_0 be a weak solution of problem $(D^{\vec{p},\vec{q}}_{\lambda,f^+})$, namely

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega\left|\frac{\partial u_0}{\partial x_i}\right|^{p_i-2}\frac{\partial u_0}{\partial x_i}\frac{\partial v}{\partial x_i}\,\mathrm{d}x+\sum_{i=1}^{N}\int_\Omega\left|\frac{\partial u_0}{\partial x_i}\right|^{q_i-2}\frac{\partial u_0}{\partial x_i}\frac{\partial v}{\partial x_i}\,\mathrm{d}x\\&=\lambda\int_\Omega f^+(x,u_0)v\,\mathrm{d}x,\end{aligned}\tag{2.22}$$

for all $v \in W^{1,\vec{p}}_0(\Omega)$.

In order to prove that u_0 is nonnegative, put $A = \{x \in \Omega \,:\, u_0(x) < 0\}$ and $u_0^- = \min\{u_0, 0\}$. Clearly, $u_0^- \in W^{1,\vec{p}}_0(\Omega)$ (see, for example, Papageorgiou-Winkert [42, Corollary 4.5.19]). Choosing $v = u_0^-$ in (2.22), one has

$$\begin{aligned}0&\leq\sum_{i=1}^{N}\int_A\left|\frac{\partial u_0}{\partial x_i}\right|^{p_i}\mathrm{d}x\\&\leq\sum_{i=1}^{N}\int_A\left|\frac{\partial u_0}{\partial x_i}\right|^{p_i-2}\frac{\partial u_0}{\partial x_i}\frac{\partial u_0^-}{\partial x_i}\,\mathrm{d}x+\sum_{i=1}^{N}\int_A\left|\frac{\partial u_0}{\partial x_i}\right|^{q_i-2}\frac{\partial u_0}{\partial x_i}\frac{\partial u_0^-}{\partial x_i}\,\mathrm{d}x\\&=\sum_{i=1}^{N}\int_\Omega\left|\frac{\partial u_0}{\partial x_i}\right|^{p_i-2}\frac{\partial u_0}{\partial x_i}\frac{\partial u_0^-}{\partial x_i}\,\mathrm{d}x+\sum_{i=1}^{N}\int_\Omega\left|\frac{\partial u_0}{\partial x_i}\right|^{q_i-2}\frac{\partial u_0}{\partial x_i}\frac{\partial u_0^-}{\partial x_i}\,\mathrm{d}x\\&=\lambda\int_\Omega f^+(x,u_0)u_0^-\,\mathrm{d}x\\&=\lambda\int_A f(x,0)u_0^-\,\mathrm{d}x\leq 0\,,\end{aligned}$$

that is,

$$\sum_{i=1}^{N}\int_A\left|\frac{\partial u_0}{\partial x_i}\right|^{p_i}\mathrm{d}x=0.$$

Hence, it holds that

$$\int_A\left|\frac{\partial u_0}{\partial x_i}\right|^{p_i}\mathrm{d}x=0\quad\text{for all } i=1,\ldots,N,$$

which yields

$$\int_\Omega\left|\frac{\partial u_0{}^-}{\partial x_i}\right|^{p_i}\mathrm{d}x=\int_A\left|\frac{\partial u_0}{\partial x_i}\right|^{p_i}\mathrm{d}x+\int_{\Omega\setminus A}\left|\frac{\partial u_0{}^-}{\partial x_i}\right|^{p_i}\mathrm{d}x=0$$

for all $i = 1, \ldots, N$. Therefore, we obtain

$$\|u_0^-\|_{W_0^{1,\vec{p}}(\Omega)}=\sum_{i=1}^{N}\left(\int_\Omega\left|\frac{\partial u_0{}^-}{\partial x_i}\right|^{p_i}\mathrm{d}x\right)^{\frac{1}{p_i}}=0,$$

so $u_0^- = 0$ in Ω, which means $u_0 \geq 0$ in Ω.

Now, we prove that u_0 is a weak solution for problem $(D_\lambda^{\vec{p},\vec{q}})$. Indeed, from (2.22) one has

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega \left|\frac{\partial u_0}{\partial x_i}\right|^{p_i-2}\frac{\partial u_0}{\partial x_i}\frac{\partial v}{\partial x_i}\,\mathrm{d}x + \sum_{i=1}^{N}\int_\Omega \left|\frac{\partial u_0}{\partial x_i}\right|^{q_i-2}\frac{\partial u_0}{\partial x_i}\frac{\partial v}{\partial x_i}\,\mathrm{d}x\\ &= \lambda\int_\Omega f^+(x,u_0)v\,\mathrm{d}x = \lambda\int_\Omega f(x,u_0)v\,\mathrm{d}x,\end{aligned}$$

for all $v \in W_0^{1,\vec{p}}(\Omega)$, and the conclusion is achieved.

The proofs of our main results are based on the following two critical point theorem due to Bonanno-D'Aguì [37, see Theorem 2.1 and Remark 2.2]. First we recall the definition of the Cerami condition.

Definition 2.6. *Let X be a Banach space and X^* be its topological dual space. Given $I_\lambda \in C^1(X)$, we say that I_λ satisfies the Cerami-condition (C-condition for short), if every sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that*

(C_1) $|I_\lambda(u_n)| \leq c_1$ *for some $c_1 > 0$ and for all $n \in \mathbb{N}$,*

(C_2) $(1 + \|u_n\|_X)\, I'_\lambda(u_n) \to 0$ *in X^* as $n \to \infty$,*

admits a strongly convergent subsequence in X.

Theorem 2.7. *Let X be a real Banach space and let $\Phi, \Psi\colon X \to \mathbb{R}$ be two functionals of class C^1 such that* $\inf_X \Phi(u) = \Phi(0) = \Psi(0) = 0$. *Assume that there are $r \in \mathbb{R}$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$, such that*

$$\frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}, \tag{2.23}$$

and, for each

$$\lambda \in \tilde{\Lambda} = \left]\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}\right[,$$

the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the C-condition and it is unbounded from below. Moreover, Φ is supposed to be coercive.

Then, for each $\lambda \in \tilde{\Lambda}$, the functional I_λ admits at least two nontrivial critical points $u_{\lambda,1}$, $u_{\lambda,2} \in X$ such that $I(u_{\lambda,1}) < 0 < I(u_{\lambda,2})$.

3. Main results

In this section we formulate and prove our main results. We suppose the following assumptions on the nonlinearity.

(H_f) $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that the conditions below are satisfied:

(f_1) there exist $\alpha < (p^-)^*$ and constants $a_1, a_2 > 0$ such that

$$|f(x,t)| \leq a_1 + a_2|t|^{\alpha-1}$$

for a.a. $x \in \Omega$ and for all $t \in \mathbb{R}$;

(f_2) if $F(x,s) = \int_0^s f(x,\xi)\,\mathrm{d}\xi$, then

$$\lim_{t\to\pm\infty} \frac{F(x,t)}{|t|^{p^+}} = +\infty$$

uniformly for a. a. $x \in \Omega$;

(f_3) there exist $\beta, \gamma \in \mathbb{R}$, with

$$\min\{\beta,\gamma\} \in \left((\alpha - p^-)\frac{N}{p^-}, \alpha\right)$$

such that

$$0 < m \leq \liminf_{t\to+\infty} \frac{f(x,t)t - p^+F(x,t)}{|t|^\beta}$$

uniformly for a.a. $x \in \Omega$, and

$$0 < m \leq \liminf_{t\to-\infty} \frac{f(x,t)t - p^+F(x,t)}{|t|^\gamma}$$

uniformly for a.a. $x \in \Omega$.

Remark 3.1. *We observe that from hypotheses* (f_1) *and* (f_2) *it follows that*

$$p^+ < \alpha < (p^-)^*.$$

Furthermore, we emphasize that such β and γ in (f_3) *exists, since*

$$(\alpha - p^-)\frac{N}{p^-} = \alpha\frac{N}{p^-} - (p^-)^*\frac{N-p^-}{p^-} < \alpha\frac{N}{p^-} - \alpha\frac{N-p^-}{p^-} = \alpha.$$

Finally, we underline that in this context it is possible to choose two different exponents β and γ for going to $+\infty$ and $-\infty$, respectively.

In the following we give a preliminary result on the energy functional associated to our problem.

Proposition 3.2. *Let hypotheses* (H) *and* (H_f) *be satisfied. Then the functional $I_\lambda \colon W_0^{1,\vec{p}}(\Omega) \to \mathbb{R}$ satisfies the* C*-condition for each $\lambda > 0$.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\vec{p}}(\Omega)$ be a sequence such that (C_1) and (C_2) hold, see Definition 2.6. First, we prove that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $L^\beta(\Omega)$.
From (C_2), we get

$$\left|\langle \Phi'(u_n), v\rangle - \lambda \int_\Omega f(x,u_n)\,v\,\mathrm{d}x\right| \leq \frac{\varepsilon_n\|v\|}{1+\|u_n\|}, \tag{3.1}$$

for all $v \in W_0^{1,\vec{p}}(\Omega)$ and with $\varepsilon_n \to 0^+$. Fix $n \in \mathbb{N}$ and choose $v = u_n \in W_0^{1,\vec{p}}(\Omega)$. Substituting in (3.1), we derive

$$-\sum_{i=1}^{N} \int_\Omega \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i} \mathrm{d}x - \sum_{i=1}^{N} \int_\Omega \left|\frac{\partial u_n}{\partial x_i}\right|^{q_i} \mathrm{d}x + \lambda \int_\Omega f(x, u_n)u_n \,\mathrm{d}x \leq \varepsilon_n, \tag{3.2}$$

for all $n \in \mathbb{N}$. From (C_1) we have

$$\sum_{i=1}^{N} \int_\Omega \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i} \mathrm{d}x + \frac{p^+}{q^+} \sum_{i=1}^{N} \int_\Omega \left|\frac{\partial u_n}{\partial x_i}\right|^{q_i} \mathrm{d}x - \lambda \int_\Omega p^+ F(x, u_n) \,\mathrm{d}x \leq p^+ c_1. \tag{3.3}$$

Adding (3.2) and (3.3) and taking into account that $q^+ < p^+$, we obtain

$$\lambda \int_\Omega \left(f(x, u_n)\, u_n - p^+ F(x, u_n)\right) \mathrm{d}x \leq c_2 \tag{3.4}$$

for some $c_2 > 0$ and for all $n \in \mathbb{N}$.

Without loss of generality, we may assume $\beta \leq \gamma$. Then, assumptions (f_1) and (f_3) imply that there exist $c_3 \in (0, m)$ and $c_4 > 0$ such that

$$c_3|s|^\beta - c_4 \leq f(x, s)s - p^+ F(x, s)$$

for a.a. $x \in \Omega$ and for all $s \in \mathbb{R}$. Exploiting this in (3.4), we derive

$$\|u_n\|_{L^\beta(\Omega)}^\beta \leq c_5 \quad \text{for some } c_5 > 0 \text{ and for all } n \in \mathbb{N},$$

namely $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $L^\beta(\Omega)$.

Now, we prove that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,\vec{p}}(\Omega)$.
From hypotheses (f_1) and (f_3) it follows that $\beta < \alpha < (p^-)^*$. Hence, there exists $t \in (0, 1)$ such that

$$\frac{1}{\alpha} = \frac{1-t}{\beta} + \frac{t}{(p^-)^*}. \tag{3.5}$$

By the interpolation inequality (see Papageorgiou-Winkert [42, Proposition 2.3.17]), we get

$$\|u_n\|_{L^\alpha(\Omega)} \leq \|u_n\|_{L^\beta(\Omega)}^{1-t} \|u_n\|_{L^{(p^-)^*}(\Omega)}^{t} \quad \text{for all } n \in \mathbb{N}.$$

Since $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $L^\beta(\Omega)$, using also Proposition 2.2, one has

$$\|u_n\|_{L^\alpha(\Omega)}^\alpha \leq c_6 \|u_n\|_{W_0^{1,\vec{p}}(\Omega)}^{t\alpha} \quad \text{for all } n \in \mathbb{N}, \tag{3.6}$$

with some $c_6 > 0$. Choosing $v = u_n \in W_0^{1,\vec{p}}(\Omega)$ in (3.1), we have

$$\sum_{i=1}^{N} \int_\Omega \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i} \mathrm{d}x + \sum_{i=1}^{N} \int_\Omega \left|\frac{\partial u_n}{\partial x_i}\right|^{q_i} \mathrm{d}x - \lambda \int_\Omega f(x, u_n)u_n \,\mathrm{d}x \leq \varepsilon_n$$

for all $n \in \mathbb{N}$. By (2.17), (f_1) and (3.6) we obtain that there exists $j \in \{1, \ldots, N\}$ such that

$$\begin{aligned}\frac{1}{N^{p_j}}\|u_n\|^{p_j}_{W_0^{1,\vec{p}}(\Omega)} &\le \sum_{i=1}^{N}\left\|\frac{\partial u_n}{\partial x_i}\right\|^{p_i}_{L^{p_i}(\Omega)}\\ &\le \lambda\int_\Omega f(x,u_n)u_n\,\mathrm{d}x + \varepsilon_n\\ &\le \lambda c_7\left[1+\|u_n\|^{t\alpha}_{W_0^{1,\vec{p}}(\Omega)}\right]+\varepsilon_n,\end{aligned} \tag{3.7}$$

for some $c_7 > 0$ and for all $n \in \mathbb{N}$.

From (3.5) and (f_3) follows that $t\alpha < p_j$, indeed

$$\begin{aligned}t\alpha = \frac{(p^-)^*(\alpha-\beta)}{(p^-)^*-\beta} &= \frac{Np^-(\alpha-\beta)}{Np^- - N\beta + \beta p^-}\\ &< \frac{Np^-(\alpha-\beta)}{Np^- - N\beta + (\alpha-p^-)\frac{N}{p^-}p^-} = p^-\\ &\le p_j.\end{aligned} \tag{3.8}$$

Then, (3.7) and (3.8) imply that $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\vec{p}}(\Omega)$ is bounded.

Finally, we prove that $\{u_n\}_{n\in\mathbb{N}}$ admits a strongly convergent subsequence in $W_0^{1,\vec{p}}(\Omega)$. Because of the boundedness of $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\vec{p}}(\Omega)$, there exists a subsequence, not relabeled, such that

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,\vec{p}}(\Omega) \quad \text{and} \quad u_n \to u \quad \text{in } L^\alpha(\Omega). \tag{3.9}$$

We choose $v = u_n - u \in W_0^{1,\vec{p}}(\Omega)$ in (3.1). Passing to the limit as $n \to \infty$ and using (3.9), we derive

$$\lim_{n\to\infty}\langle \Phi'(u_n), u_n - u\rangle = 0.$$

Since Φ' has the (S_+)-property (see Proposition 2.4), it follows that $u_n \to u$ in $W_0^{1,\vec{p}}(\Omega)$ and this completes the proof.

Now, we present our main result. To this aim, put

$$R := \sup_{x\in\Omega}\operatorname{dist}(x, \partial\Omega). \tag{3.10}$$

Standard computations show that there exists $x_0 \in \Omega$ such that $B(x_0, R) \subseteq \Omega$ and we denote by

$$\omega_R := |B(x_0,R)| = \frac{\pi^{\frac{N}{2}}}{\Gamma(1+\frac{N}{2})}R^N,$$

the measure of the N-dimensional ball of radius R. Finally, we set

$$\mathcal{K} = \max_{1\le i\le N}\left\{\frac{1}{R^{p_i}}, \frac{1}{R^{q_i}}\right\}\frac{2^N-1}{q^-\left(2^{N-p^+}\right)}\omega_R, \tag{3.11}$$

$$\delta = \max\left\{r^{\frac{1}{p^-}}, r^{\frac{1}{p^+}}\right\}\sum_{i=1}^{N}p_i^{\frac{1}{p_i}}. \tag{3.12}$$

Theorem 3.3. *Let hypotheses* (H) *and* ($\mathrm{H_f}$) *be satisfied. Assume that there exist two constants $r, d > 0$ such that*

(h_1) $\mathcal{K}\sum_{i=1}^{N}(d^{p_i}+d^{q_i}) < r$,

(h_2) $F(x,s) \geq 0 \quad$ *for a.a.* $x \in \Omega$ *and for all* $s \in [0,d]$,

(h_3) $$\frac{a_1 T_1 \delta + \frac{a_2}{\alpha} T_\alpha^\alpha \delta^\alpha}{r} < \frac{1}{\mathcal{K}} \frac{\int_{B\left(x_0,\frac{R}{2}\right)} F(x,d)\,\mathrm{d}x}{\sum_{i=1}^{N}(d^{p_i}+d^{q_i})},$$

where a_1, a_2, α are given in (f_1)*, T_1 and T_α are defined by formula* (2.10)*, R is given in* (3.10)*, and $\mathcal{K}$ as well as δ are defined by* (3.11) *and* (3.12)*, respectively.*

Then, for each

$$\lambda \in \Lambda := \left]\mathcal{K}\frac{\sum_{i=1}^{N}(d^{p_i}+d^{q_i})}{\int_{B\left(x_0,\frac{R}{2}\right)} F(x,d)\,\mathrm{d}x}, \frac{r}{a_1 T_1 \delta + \frac{a_2}{\alpha} T_\alpha^\alpha \delta^\alpha}\right[,$$

problem $(D_\lambda^{\vec{p},\vec{q}})$ *has at least two nontrivial weak solutions $u_\lambda, v_\lambda \in W_0^{\vec{p}}(\Omega)$ such that $I_\lambda(u_\lambda) < 0 < I_\lambda(v_\lambda)$.*

Proof. Our aim is to apply Theorem 2.7 for $X = W_0^{1,\vec{p}}(\Omega)$ and Φ, Ψ defined as in (2.13). The functionals Φ and Ψ satisfy all the required regularity properties, since Φ is coercive by construction (see Proposition 2.3), the energy functional I_λ satisfies the C-condition due to Proposition 3.2 and it is unbounded from below by (f_2) and finally

$$\inf_{u \in W_0^{1,\vec{p}}(\Omega)} \Psi(u) = \Psi(0) = \Phi(0).$$

Moreover, the interval Λ is nonempty because of assumption (h_3). Thus, it remains to verify hypothesis (2.23). First, we observe that

$$\Phi^{-1}\left(]-\infty, r[\right) \subseteq \left\{u \in W_0^{1,\vec{p}}(\Omega) \,:\, \|u\|_{W_0^{1,\vec{p}}(\Omega)} \leq \delta\right\}. \tag{3.13}$$

Then, from (f_1), (2.9) and (3.13) we estimate that

$$\begin{aligned}
&\frac{\sup\limits_{u \in \Phi^{-1}(]-\infty,r])} \Psi(u)}{r}\\
&\leq \frac{\sup\limits_{u \in \Phi^{-1}(]-\infty,r])} \left(a_1\|u\|_{L^1(\Omega)} + \frac{a_2}{\alpha}\|u\|_{L^\alpha(\Omega)}^\alpha\right)}{r}\\
&\leq \frac{\sup\limits_{u \in \Phi^{-1}(]-\infty,r])} \left[a_1 T_1\|u\|_{W_0^{1,\vec{p}}(\Omega)} + \frac{a_2}{\alpha} T_\alpha^\alpha \|u\|_{W_0^{1,\vec{p}}(\Omega)}^\alpha\right]}{r}\\
&\leq \frac{a_1 T_1 \delta + \frac{a_2}{\alpha} T_\alpha^\alpha \delta^\alpha}{r}.
\end{aligned} \tag{3.14}$$

On the other hand, we introduce the following function

$$\tilde{u}(x)=\begin{cases}0 & \text{if } x\in\Omega\setminus B(x_0,R),\\ \dfrac{2d}{R}(R-|x-x_0|) & \text{if } B(x_0,R)\setminus B\left(x_0,\frac{R}{2}\right),\\ d & \text{if } x\in B\left(x_0,\frac{R}{2}\right),\end{cases}$$

where R is given in (3.10). Clearly, $\tilde{u}\in W_0^{1,\vec{p}}(\Omega)$. From assumption ($h_2$), we get

$$\begin{aligned}\Psi(\tilde{u}) &= \int_{B(x_0,R)\setminus B\left(x_0,\frac{R}{2}\right)} F\left(x,\frac{2d}{R}(R-|x-x_0|)\right)\mathrm{d}x\\ &\quad+\int_{B\left(x_0,\frac{R}{2}\right)} F(x,d)\,\mathrm{d}x\\ &\geq \int_{B\left(x_0,\frac{R}{2}\right)} F(x,d)\,\mathrm{d}x.\end{aligned}\tag{3.15}$$

Furthermore, it holds

$$\begin{aligned}\Phi(\tilde{u}) &= \sum_{i=1}^{N}\frac{1}{p_i}\int_{B(x_0,R)\setminus B\left(x_0,\frac{R}{2}\right)}\left(\frac{2d}{R}\right)^{p_i}\mathrm{d}x\\ &\quad+\sum_{i=1}^{N}\frac{1}{q_i}\int_{B(x_0,R)\setminus B\left(x_0,\frac{R}{2}\right)}\left(\frac{2d}{R}\right)^{q_i}\mathrm{d}x\\ &\leq \frac{2^{p^+}}{q^-}\sum_{i=1}^{N}(d^{p_i}+d^{q_i})\max_{1\leq i\leq N}\left\{\frac{1}{R^{p_i}},\frac{1}{R^{q_i}}\right\}\left[\frac{2^N-1}{2^N}\omega_R\right]\\ &= \mathcal{K}\sum_{i=1}^{N}(d^{p_i}+d^{q_i}).\end{aligned}\tag{3.16}$$

Hypothesis (h_1) implies that $0<\Phi(\tilde{u})<r$ and combining (3.14), (3.15) as well as (3.16) we have

$$\begin{aligned}\frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{r} &\leq \frac{a_1T_1\delta+\frac{a_2}{\alpha}T_\alpha^\alpha\delta^\alpha}{r}\\ &< \frac{1}{\mathcal{K}}\frac{\int_{B\left(x_0,\frac{R}{2}\right)}F(x,d)\,\mathrm{d}x}{\sum_{i=1}^{N}(d^{p_i}+d^{q_i})}\\ &\leq \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}.\end{aligned}$$

This proves (2.23). Hence, since $\Lambda\subseteq\tilde{\Lambda}$, Theorem 2.7 ensures that problem $(D_\lambda^{\vec{p},\vec{q}})$ admits at least two nontrivial weak solutions $u_\lambda, v_\lambda\in W_0^{1,\vec{p}}(\Omega)$ with opposite energy sign.

The following result is a consequence of Lemma 2.5 and of Theorem 3.3.

Theorem 3.4. *Let hypotheses* (H) *and* ($\mathrm{H_f}$) *be satisfied. Assume* ($\mathrm{h_1}$)–($\mathrm{h_3}$) *and that* $f(x,0) \geq 0$ *for a.a.* $x \in \Omega$*. Then, for every*

$$\lambda \in \Lambda := \left]\mathcal{K}\frac{\sum_{i=1}^{N}(d^{p_i}+d^{q_i})}{\int_{B\left(x_0,\frac{R}{2}\right)} F(x,d)\,\mathrm{d}x}, \frac{r}{a_1T_1\delta + \dfrac{a_2}{\alpha}T_\alpha^\alpha\delta^\alpha}\right[,$$

problem ($D_\lambda^{\vec{p},\vec{q}}$) *has at least two nonnegative weak solutions* $u_\lambda, v_\lambda \in W_0^{\vec{p}}(\Omega)$ *such that* $I_\lambda(u_\lambda) < 0 < I_\lambda(v_\lambda)$*.*

Now, we consider the particular case in which the nonlinearity is nonnegative.

Theorem 3.5. *Let hypotheses* (H) *and* ($\mathrm{H_f}$) *be satisfied. Assume that* f *is nonnegative and*

($\mathrm{h_4}$) $\displaystyle\limsup_{t\to 0^+}\frac{\inf_{x\in\Omega}F(x,t)}{t^{q^-}} = +\infty$

Then, for each $\lambda \in]0,\lambda^*[$*, with*

$$\lambda^* = \sup_{r>0}\frac{r}{a_1T_1\delta + \frac{a_2}{\alpha}T_\alpha^\alpha\delta^\alpha}, \tag{3.17}$$

problem ($D_\lambda^{\vec{p},\vec{q}}$) *admits at least two nonnegative weak solutions* u_λ*,* $v_\lambda \in W_0^{1,\vec{p}}(\Omega)$ *such that* $I_\lambda(u_\lambda) < 0 < I_\lambda(v_\lambda)$*.*

Proof. Fix $\lambda \in]0,\lambda^*[$, then there exists $r > 0$ such that

$$\lambda < \frac{r}{a_1T_1\delta + \frac{a_2}{\alpha}T_\alpha^\alpha\delta^\alpha}.$$

From ($\mathrm{h_4}$) follows that

$$\limsup_{t\to 0^+}\frac{1}{\mathcal{K}}\frac{\int_{B\left(x_0,\frac{R}{2}\right)}F(x,t)\,\mathrm{d}x}{\sum_{i=1}^{N}(t^{p_i}+t^{q_i})} \geq \frac{\omega_{\frac{R}{2}}}{\mathcal{K}}\limsup_{t\to 0^+}\frac{\inf_{x\in\Omega}F(x,t)}{\sum_{i=1}^{N}(t^{p_i}+t^{q_i})} = +\infty,$$

since

$$\frac{1}{\sum_{i=1}^{N}(t^{p_i}+t^{q_i})} \geq \frac{1}{2Nt^{q^-}}.$$

Then, in correspondence of $\frac{1}{\lambda}$, there exists $t > 0$ small enough such that

$$\frac{1}{\mathcal{K}}\frac{\int_{B\left(x_0,\frac{R}{2}\right)}F(x,t)\,\mathrm{d}x}{\sum_{i=1}^{N}(t^{p_i}+t^{q_i})} > \frac{1}{\lambda} > \frac{a_1T_1\delta + \dfrac{a_2}{\alpha}T_\alpha^\alpha\delta^\alpha}{r},$$

namely assumption ($\mathrm{h_3}$) is satisfied. Since ($\mathrm{h_2}$) follows from the sign assumption on the nonlinearity, we can apply Theorem 3.3 and Lemma 2.5 to complete the proof.

Finally, we deal with the autonomous case and we present an existence result which is a consequence of Theorem 3.5. Consider the autonomous problem

$$\begin{aligned} -\Delta_{\vec{p}}u - \Delta_{\vec{q}}u &= \lambda g(u) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{$AD_\lambda^{\vec{p},\vec{q}}$}$$

where $g\colon \mathbb{R} \to \mathbb{R}$ is a nonnegative continuous function. From Lemma 2.5 it follows that we can consider the nonlinearity only in $[0, +\infty)$. We assume the following:

($\mathrm{H_g}$) ($\mathrm{g_1}$) there exist $\alpha < (p^-)^*$ and constants $a_1, a_2 > 0$ such that

$$g(t) \le a_1 + a_2 |t|^{\alpha-1}$$

for all $t \ge 0$;

($\mathrm{g_2}$) if $G(s) = \int_0^s g(\xi)\,\mathrm{d}\xi$, then

$$\lim_{t\to+\infty} \frac{G(t)}{t^{p^+}} = +\infty;$$

($\mathrm{g_3}$) there exists $\beta \in \mathbb{R}$, with

$$\beta \in \left((\alpha - p^-)\frac{N}{p^-}, \alpha \right),$$

such that

$$0 < m \le \liminf_{t\to+\infty} \frac{g(t)t - p^+ G(t)}{t^\beta}.$$

The following result holds.

Corollary 3.6. *Let hypotheses* (H) *and* ($\mathrm{H_g}$) *be satisfied. Assume that*

$$\limsup_{t\to 0^+} \frac{G(t)}{t^{q^-}} = +\infty. \tag{$\mathrm{h_4}$'}$$

Then, for each $\lambda \in]0, \lambda^*[$, *with* λ^* *defined in* (3.17)*, the problem* ($AD_\lambda^{\vec{p},\vec{q}}$) *admits at least two nonnegative weak solutions* u_λ, $v_\lambda \in W_0^{1,\vec{p}}(\Omega)$ *such that* $I_\lambda(u_\lambda) < 0 < I_\lambda(v_\lambda)$.

In conclusion, we provide an example.

Example 3.7. *Consider two constants* c, κ *such that*

$$c \ge 1, \quad p^+ < \kappa < (p^-)^* \quad \textit{and} \quad \frac{\kappa}{p^-} - \frac{\kappa}{N} < 1.$$

Let $g\colon [0, +\infty) \to \mathbb{R}$ *be a function defined by*

$$g(t) = (t + c)^{\kappa-1} \left(\kappa \log(t + c) + 1 \right) \quad \textit{for all } t \ge 0.$$

Then, g satisfies assumptions ($\mathrm{H_g}$) *with* $\beta = \kappa$ *and* $\alpha = \kappa + \sigma$*, with* $\sigma > 0$ *small enough such that*

$$\alpha < (p^-)^* \quad \textit{and} \quad \frac{\alpha}{p^-} - \frac{\kappa}{N} < 1.$$

Moreover, the function g satisfies assumption ($\mathrm{h_4}$')*, hence we can apply Corollary 3.6 to get the existence of two nonnegative weak solutions of problem* ($AD_\lambda^{\vec{p},\vec{q}}$) *with opposite energy sign.*

Acknowledgments

The first two authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). The authors have been partially supported by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). The paper is partially supported by PRIN 2017 – Progetti di Ricerca di rilevante Interesse Nazionale, "Nonlinear Differential Problems via Variational, Topological and Set-valued Methods" (2017AYM8XW) and by FFR-2023-Sciammetta.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflict of interest.

References

1. M. Belloni, B. Kawohl, The pseudo-p-Laplace eigenvalue problem and viscosity solutions as $p \to \infty$, *ESAIM Control Optim. Calc. Var.*, **10** (2004), 28–52. https://doi.org/10.1051/cocv:2003035
2. L. Brasco, G. Franzina, An anisotropic eigenvalue problem of Stekloff type and weighted Wulff inequalities, *NoDEA Nonlinear Differential Equations Appl*, **20** (2013), 1795–1830. https://doi.org/10.1007/s00030-013-0231-4
3. A. Kufner, J. Rákosník, Boundary value problems for nonlinear partial differential equations in anisotropic Sobolev spaces, *Časopis Pěst. Mat.*, **106** (1981), 170–185. http://dx.doi.org/10.21136/CPM.1981.118087
4. S. M. Nikol'skiĭ, An imbedding theorem for functions with partial derivatives considered in different metrics, *Izv. Akad. Nauk SSSR Ser. Mat.*, **22** (1958), 321–336.
5. J. Rákosník, Some remarks to anisotropic Sobolev spaces I, *Beiträge Anal.*, **13** (1979), 55–68.
6. J. Rákosník, Some remarks to anisotropic Sobolev spaces II, *Beiträge Anal.*, **15** (1980), 127–140.
7. G. Wulff, Zur Frage der Geschwindigkeit des Wachsthums und der Auflösung der Krystallflächen, *Zeitschrift für Kristallographie – Crystalline Materials*, **34** (1901), 449–530.
8. S. N. Antontsev, J. I. Díaz, S. Shmarev, *Energy methods for free boundary problems*, Birkhäuser Boston, Inc., Boston, MA, 2002. https://doi.org/10.1007/978-1-4612-0091-8
9. M. Bendahmane, M. Chrif, S. El Manouni, An approximation result in generalized anisotropic Sobolev spaces and applications, *Z. Anal. Anwend.*, **30** (2011), 341–353. https://doi.org/10.4171/ZAA/1438
10. M. Bendahmane, M. Langlais, M. Saad, On some anisotropic reaction-diffusion systems with L^1-data modeling the propagation of an epidemic disease, *Nonlinear Anal.*, **54** (2003), 617–636. https://doi.org/10.1016/S0362-546X(03)00090-7

11. J. Vétois, The blow-up of critical anisotropic equations with critical directions, *NoDEA Nonlinear Differential Equations Appl*, **18** (2011), 173–197. https://doi.org/10.1007/s00030-010-0090-1

12. A. Razani, G. M. Figueiredo, A positive solution for an anisotropic (p, q)-Laplacian, *Discrete Contin. Dyn. Syst. Ser. S.*, **16** (2023), 1629–1643. https://doi.org/10.3934/dcdss.2022147

13. A. Razani, Nonstandard competing anisotropic (p, q)-Laplacians with convolution, *Bound. Value Probl.*, **2022**, (2022), 1–10. https://doi.org/10.1186/s13661-022-01669-z

14. L. Tavares, Solutions for a class of problems driven by an anisotropic (p, q)-Laplacian type operator, *Commun. Anal. Mech.*, **15** (2023), 533–550. https://doi.org/10.3934/cam.2023026

15. G. Bonanno, G. D'Aguì, A. Sciammetta, Multiple solutions for a class of anisotropic $\vec{p}$-Laplacian problems, *Bound. Value Probl.*, accepted 2023.

16. S. Ciani, G. M. Figueiredo, A. Suárez, Existence of positive eigenfunctions to an anisotropic elliptic operator via the sub-supersolution method, *Arch. Math.*, **116** (2021), 85–95. https://doi.org/10.1007/s00013-020-01518-4

17. G. Ciraolo, A. Figalli, A. Roncoroni, Symmetry results for critical anisotropic p-Laplacian equations in convex cones, *Geom. Funct. Anal.*, **30** (2020), 770–803. https://doi.org/10.1007/s00039-020-00535-3

18. G. Ciraolo, A. Sciammetta, Gradient estimates for the perfect conductivity problem in anisotropic media, *J. Math. Pures Appl.*, **127** (2019), 268–298. https://doi.org/10.1016/j.matpur.2018.09.006

19. E. DiBenedetto, U. Gianazza, V. Vespri, Remarks on local boundedness and local Hölder continuity of local weak solutions to anisotropic p-Laplacian type equations, *J. Elliptic Parabol. Equ.*, **2** (2016), 157–169. https://doi.org/10.1007/BF03377399

20. G. C. G. dos Santos, G. M. Figueiredo, L. S. Tavares, Existence results for some anisotropic singular problems via sub-supersolutions, *Milan J. Math.*, **87** (2019), 249–272. https://doi.org/10.1007/s00032-019-00300-8

21. I. Fragalà, F. Gazzola, B. Kawohl, Existence and nonexistence results for anisotropic quasi-linear elliptic equations, *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, **21** (2004), 715–734. https://doi.org/10.1016/j.anihpc.2003.12.001

22. K. Perera, R. P. Agarwal, D. O'Regan, Nontrivial solutions of p-superlinear anisotropic p-Laplacian systems via Morse theory, *Topol. Methods Nonlinear Anal.*, **35** (2010), 367–378.

23. M. A. Ragusa, A. Razani, F. Safari, Existence of radial solutions for a $p(x)$-Laplacian Dirichlet problem, *Adv. Difference Equ.*, **2021** (2021), 1–14. https://doi.org/10.1186/s13662-021-03369-x

24. Y. Bai, N. S. Papageorgiou, S. Zeng, A singular eigenvalue problem for the Dirichlet (p, q)-Laplacian, *Math. Z.*, **300** (2021), 325–345. https://doi.org/10.1007/s00209-021-02803-w

25. M. Bohner, G. Caristi, A. Ghobadi, S. Heidarkhani, Three solutions for discrete anisotropic Kirchhoff-type problems, *Demonstr. Math.*, **56** (2023), 20220209. https://doi.org/10.1515/dema-2022-0209

26. S. El Manouni, G. Marino, P. Winkert, Existence results for double phase problems depending on Robin and Steklov eigenvalues for the p-Laplacian, *Adv. Nonlinear Anal.*, **11** (2022), 304–320. https://doi.org/10.1515/anona-2020-0193

27. A. R. Leggat, S. E. Miri, An existence result for a singular-regular anisotropic system, *Rend. Circ. Mat. Palermo*, **72** (2023), 977–996. https://doi.org/10.1007/s12215-022-00718-x

28. H. He, M. Ousbika, Z. El Allali, J. Zuo, Non-trivial solutions for a partial discrete Dirichlet nonlinear problem with p-Laplacian, *Commun. Anal. Mech.*, **15** (2023), 598–610. http://dx.doi.org/10.3934/cam.2023030

29. C. Ju, G. Molica Bisci, B. Zhang, On sequences of homoclinic solutions for fractional discrete p-Laplacian equations, *Commun. Anal. Mech.*, **15** (2023), 586–597. https://doi.org/10.3934/cam.2023029

30. Z. Liu, D. Motreanu, S. Zeng, Positive solutions for nonlinear singular elliptic equations of p-Laplacian type with dependence on the gradient, *Calc. Var. Partial Differential Equations*, **58** (2019), 28. https://doi.org/10.1007/s00526-018-1472-1

31. N. S. Papageorgiou, Double phase problems: a survey of some recent results, *Opuscula Math*, **42** (2022), 257–278. https://doi.org/10.7494/OpMath.2022.42.2.257

32. B. Son, I. Sim, Analysis of positive solutions to one-dimensional generalized double phase problems, *Adv. Nonlinear Anal.*, **11** (2022), 1365–1382. https://doi.org/10.1515/anona-2022-0240

33. C. Vetro, F. Vetro, Three solutions to mixed boundary value problem driven by $p(z)$-Laplace operator, *Math. Nachr.*, **294** (2021),1175–1185. https://doi.org/10.1002/mana.201900123

34. S. Zeng, Y. Bai, L. Gasiński, P. Winkert, Existence results for double phase implicit obstacle problems involving multivalued operators, *Calc. Var. Partial Differential Equations*, **59** (2020), no. 5, Paper No. 176, 18 pp. https://doi.org/10.1007/s00526-020-01841-2

35. S. Zeng, N. S. Papageorgiou, Positive solutions for (p, q)-equations with convection and a sign-changing reaction, *Adv. Nonlinear Anal.*, **11** (2022), 40–57. https://doi.org/10.1515/anona-2020-0176

36. S. Zeng, V. D. Rădulescu, P. Winkert, Double phase implicit obstacle problems with convection and multivalued mixed boundary value conditions, *SIAM J. Math. Anal.*, **54** (2022), no. 2, 1898–1926. https://doi.org/10.1137/21M1441195

37. G. Bonanno, G. D'Aguì, Two non-zero solutions for elliptic Dirichlet problems, *Z. Anal. Anwend.*, **35** (2016), 449–464. https://doi.org/10.4171/zaa/1573

38. G. Bonanno, G. D'Aguì, A. Sciammetta, Existence of two positive solutions for anisotropic nonlinear elliptic equations, *Adv. Differential Equations*, **26** (2021), 229–258. https://doi.org/10.57262/ade026-0506-229

39. G. Talenti, Best constant in Sobolev inequality, *Ann. Mat. Pura Appl.*, **110** (1976), 353–372. https://doi.org/10.1007/BF02418013

40. J. Simon, Régularité de la solution d'une équation non linéaire dans $\mathbb{R}^N$, *Journées d'Analyse Non Linéaire (Proc. Conf. Besançon, 1977)*, Springer, Berlin, **665** (1978), 205–227. https://doi.org/10.1007/BFb0061807

41. P. Lindqvist, *Notes on the stationary p-Laplace equation*, Springer, Cham, 2019. https://doi.org/10.1007/978-3-030-14501-9

42. N. S. Papageorgiou, P. Winkert, *Applied Nonlinear Functional Analysis*, De Gruyter, Berlin, 2018. https://doi.org/10.1515/9783110532982-201