J. Math. Anal. Appl. 547 (2025) 129369

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Regular Articles

Generalized sampling operators with derivative samples

Rosario Corso

Dipartimento di Matematica e Informatica, Università degli Studi di Palermo, Via Archirafi 34, Palermo, 90123, Italy

A R T I C L E I N F O

Article history:
Received 3 November 2024
Available online 12 February 2025
Submitted by S. Tikhonov

Keywords:
Generalized sampling operators
Hermite sampling
Rate of convergence
Simultaneous approximation
Voronovskaya-type formula

A B S T R A C T

The generalized sampling operator is able to approximate bounded continuous functions. It is modeled on the sampling expansion for band-limited functions given by the Whittaker-Kotel'nikov-Shannon theorem. During the decades, some variations of this classical theorem have been proposed. One of them (dating back to Jagerman and Fogel and, in a more general form, to Linden and Abramson) takes into consideration also the derivative samples for the reconstruction of band-limited functions, with a consequent benefit of a larger sampling rate compared to the Whittaker-Kotel'nikov-Shannon theorem. Motivated by this new reconstruction, we modify the generalized sampling operator including the samplings of derivatives up to a generic order to approximate non necessarily band-limited functions. One of the main features of this new operator (which we call an Hermite-type sampling operator) is the faster order of approximation. Besides the convergence and its rate, we study well-posedness, regularity, simultaneous approximation and Voronovskaya-type formula.

1. Introduction

The Whittaker-Kotel'nikov-Shannon (WKS) theorem [44,32,42] is a celebrated result about the reconstruction of band-limited functions from their samples at a uniform grid. More precisely, it affirms that if $f : \mathbb{R} \to \mathbb{R}$ is square integrable (in symbols, $f \in L^2(\mathbb{R})$) band-limited and continuous function with Fourier transform[1] vanishing outside $]-\pi W, \pi W[$, then

$$f(x) = \sum_{k\in\mathbb{Z}} f(k\tau)\,\mathrm{sinc}\left(\frac{x}{\tau} - k\right), \qquad \tau = \frac{1}{W}, \tag{1}$$

for every $x \in \mathbb{R}$. Here, sinc is the function defined by

E-mail address: rosario.corso02@unipa.it.

[1] In literature, the Fourier transform can be found in more versions with different factors in the expression. In this paper the Fourier transform is defined for an integrable function f (in symbols, $f \in L^1(\mathbb{R})$) as $\widehat{f}(v) = \int_{-\infty}^{+\infty} f(x)e^{-ivx}dx$, $v \in \mathbb{R}$.

https://doi.org/10.1016/j.jmaa.2025.129369

$$\operatorname{sinc}(u) = \begin{cases} \frac{\sin(\pi u)}{\pi u} & u \neq 0 \\ 1 & u = 0 \end{cases}$$

and τ is the so-called sampling rate. In the same paper [42] concerning this theorem, Shannon stated that the reconstruction is possible also considering the samples of f and its derivative. The result, which was proved later by Jagerman and Fogel [28], says that if $f \in L^2(\mathbb{R})$ is a band-limited and continuous[2] function with Fourier transform vanishing outside the interval $]-\pi W, \pi W[$, then

$$f(x) = \sum_{k\in\mathbb{Z}} \left(f(k\tau) + f'(k\tau)(x - k\tau)\right) \operatorname{sinc}^2\left(\frac{x}{\tau} - k\right), \qquad \tau = \frac{2}{W}, \tag{2}$$

for every $x \in \mathbb{R}$. We remark that in (2) the sampling rate is doubled in comparison with (1). The new rate has a clear advantage: being larger than the case of (1), it allows to approximate the function using a less number of sampling points over an interval of the same length.

Subsequently, Linden and Abramson [33] considered the more general framework of the derivatives up to an order $n > 1$. The result (with a rigorous proof given by Rawn [39]) can be stated as follows: let $f \in L^2(\mathbb{R})$ be a band-limited and continuous function with Fourier transform vanishing outside the interval $]-\pi W, \pi W[$, then

$$f(x) = \sum_{k\in\mathbb{Z}} \left(\sum_{j=0}^{n} \frac{1}{j!} f_j(k\tau)(x - k\tau)^j\right) \operatorname{sinc}^{n+1}\left(\frac{x}{\tau} - k\right), \qquad \tau = \frac{n+1}{W}, \tag{3}$$

for every $x \in \mathbb{R}$ and for an even greater sampling rate τ. Here,

$$f_j(x) = \sum_{i=0}^{j} \binom{j}{i} \left(\frac{\pi W}{n+1}\right)^{j-i} \Gamma_{n+1}^{j-i}(0) f^{(i)}(x), \qquad \Gamma_{n+1}^{m}(x) = \frac{d^m}{dx^m}\left(\frac{1}{\operatorname{sinc}(\frac{x}{\pi})}\right)^{n+1}.$$

Similarly to the Hermite interpolation [43], equations (2) and (3) are called Hermite sampling expansions in literature, see e.g., [11,26]. Other relevant works on the reconstruction of band-limited functions with derivative sampling are [37,7,24,10,38].

Coming back to WKS theorem, in order to extend the convergence of (1) to more general functions, the *generalized sampling operator* was introduced in [40] and it defined for a bounded function f by

$$(G_w f)(x) = \sum_{k\in\mathbb{Z}} f\left(\frac{k}{w}\right) \chi(wx - k), \qquad x \in \mathbb{R}, \tag{4}$$

where χ is a *kernel*, i.e. a continuous function $\chi : \mathbb{R} \to \mathbb{R}$ such that $\sum_{k\in\mathbb{Z}} \chi(u-k) = 1$ for every $u \in \mathbb{R}$ and $\sum_{k\in\mathbb{Z}} |\chi(u-k)|$ is convergent uniformly for $u \in \mathbb{R}$.
Despite the similarity between the expression of $G_w f$ and the series in (1), a result of the type $f(x) = (G_w f)(x)$ for some $w > 0$ is not available when f is just a bounded continuous function. Actually, as proved in [19] the convergence is asymptotic, i.e. if f is a bounded function then

$$(G_w f)(x) \to f(x) \text{ as } w \to +\infty$$

at any point x of continuity of f. In addition, the convergence is uniform, i.e.

[2] As well-known, a band-limited and continuous function in $L^2(\mathbb{R})$ is also smooth.

$$\lim_{w\to+\infty} \|G_w f - f\|_\infty = 0,$$

if f is bounded and uniformly continuous [40,19]. Generalized sampling operators have been further studied in relation to Voronovskaya-type results [13], approximation of curves [21], simultaneous approximation [3] and have been modified in the multivariate case in [16,17] and for L^p functions in [12].

Inspired by the derivative samplings in the expressions of (2) and (3), in this paper we introduce a new class of operators and study approximation results. Namely, for a kernel χ (satisfying some assumptions below), $n \in \mathbb{N}$ and an n times differentiable function $f : \mathbb{R} \to \mathbb{R}$ we define the following operators

$$(G_{n,w}f)(x) = \sum_{k\in\mathbb{Z}} \left(\sum_{j=0}^{n} \frac{1}{j!} f^{(j)}\left(\frac{k}{w}\right)\left(x - \frac{k}{w}\right)^j \right) \chi(wx - k), \qquad x \in \mathbb{R}, \tag{5}$$

if the series is absolutely convergent for any x. We call $G_{n,w}$ an *Hermite-type sampling operator of order* n in coherence to the name of expansions (2) and (3) above.[3] In the case $n = 0$ we recover the generalized sampling operator $G_{0,w} = G_w$, while $G_{1,w}$ is related to the series in (2), taking $w = \frac{1}{\tau}$. Note that, in the general case, the expression in the brackets of $G_{n,w}$ has a different form compared to (3) since the functions f_j in (3) are linear combinations of the derivatives. We decide to consider the expression of (4) because it is more natural.
Considering the operator $G_{n,w}$, we assume in particular that χ is *kernel of order* n, i.e. a continuous function satisfying the following conditions

(i) for every $u \in \mathbb{R}$,

$$\sum_{k\in\mathbb{Z}} \chi(u - k) = 1; \tag{6}$$

(ii) the series

$$\sum_{k\in\mathbb{Z}} |u - k|^n |\chi(u - k)| \tag{7}$$

is convergent uniformly[4] for $u \in \mathbb{R}$.

The new operator approximates bounded and n times differentiable functions f with continuous and bounded derivatives up to the order n, in the sense that $\lim\limits_{w\to+\infty} \|G_{n,w}f - f\|_\infty = 0$ (see Theorem 6). Moreover, the fact that the sampling rate in (2) is larger than in (1) gives a strong indication that the rate of convergence of (5) may be higher for larger values of n. This is exactly the case; indeed in Theorem 8 we prove that $\|G_{n,w}f - f\|_\infty = o(w^{-n})$ for $w \to +\infty$, when f and its first n derivatives are uniformly continuous. Therefore, our study was not motivated only by the inspiration of adding derivative samplings in the expression of the operator, but also by the improved order of approximation which provides a faster convergence and smaller errors.

We remark that the Hermite-type sampling operators are different to the m-th order Kantorovich type sampling series [2] which is constructed starting from the generalized sampling operator, the differential and the anti-differential operators. In [41] approximations by series similar to (5) were studied but with a linear combination of derivatives with complex scalars instead of the terms $f\left(\frac{k}{w}\right)\left(x - \frac{k}{w}\right)^j$ (see also the related

[3] The name “Hermite-type sampling operator of order n” is also simpler than “generalization of the n-th order of the generalized sampling operator”, terminology that could be adopted from [29].

[4] Actually, it is sufficient that the series (7) is convergent uniformly for $u \in [0, 1[$ since it is periodic in u with period 1.

works [30,31]). Reconstruction in shift-invariant spaces with derivative samplings were considered in [24]; the authors gave also approximation results but assuming as additional hypotheses the Strang–Fix conditions of a certain order on the generator of the space. In [38] the authors have studied the expansions in shift-invariant spline spaces by operators involving derivatives and reconstruction functions, which are obtained from the generator of the space and a block Laurent operator. They also studied the approximation under the condition that the operators fix the polynomials up to a certain order. Finally, the values of derivatives have been considered also in the expression of other types of operators, see for instance [8,9,14,29].

The structure of the paper is as follows. In Section 2 we give some properties and examples of kernels. Next, in Section 3 we prove preliminary results on the Hermite-type sampling operators about sufficient hypotheses on f in order that (5) is well-defined, about continuity and regularity. Moreover, we show the convergence results for $w \to +\infty$ to the function and to its first derivatives (Theorems 6 and 7). In Section 4 we investigate the rates of convergence (Theorem 8) and, in particular, estimates for $G_{n,w}f - f$ in terms of powers of w. After that, we make a discussion comparing the rates of convergence of the operators $G_{n,w}$ and G_w. A result from [20] establishes high rates of convergence $\mathcal{O}(w^{-r})$, $r > 1$, for the generalized sampling operator G_w imposing opportune assumptions (the so-called Strang-Fix conditions) on the kernels. In contrast, our operators have the advantage of reaching higher rates without specific hypothesis on the kernels besides the finite moments. Finally, in Section 5 we prove Theorem 9 about a Voronovskaya-type formula (an asymptotic and pointwise equality). Throughout the paper graphical examples and plots can be found to support the results obtained from the theory.

2. Preliminaries

First of all, we fix the notation used in the paper. We denote by $C_b^n(\mathbb{R})$ (respectively, $C^n(\mathbb{R})$) the space of the bounded and n times differentiable functions $f : \mathbb{R} \to \mathbb{R}$ with bounded and continuous (respectively, bounded and uniformly continuous) derivatives up to the order n. We remark that a differentiable function with bounded and continuous derivative is uniformly continuous. Therefore, $f \in C^n(\mathbb{R})$ if and only if $f \in C_b^n(\mathbb{R})$ and $f^{(n)}$ is uniformly continuous. For $n = 0$ we simply write $C_b(\mathbb{R})$ and $C(\mathbb{R})$, respectively. Moreover, for a bounded function $f : \mathbb{R} \to \mathbb{R}$ we write, as usually, $\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|$.

For any kernel χ and $\alpha > 0$ we denote by

$$m_\alpha(u, \chi) := \sum_{k\in\mathbb{Z}} (u - k)^\alpha \chi(u - k), \qquad u \in \mathbb{R},$$

the *discrete moment* of χ of order α and by

$$M_\alpha(\chi) := \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |u - k|^\alpha |\chi(u - k)|, \tag{8}$$

the *absolute discrete moment* of order α. Actually, the supremum can be calculated in the interval $[0, 1[$ because of the periodicity of the series in (8).

In general, there is no guarantee that $M_\alpha(\chi)$ is finite. Anyway, if χ is a kernel of order $\alpha = n$ or higher, in the sense specified in the introduction, then it is clearly true. This is stated in the following lemma alongwith other useful properties of kernels.

Lemma 1. *Let χ be a kernel of order α. The following statements hold.*

(i) $M_\alpha(\chi) < +\infty$.
(ii) *χ is a kernel of order β, for any $0 \le \beta \le \alpha$. In particular, $M_\beta(\chi) < +\infty$ for any $0 \le \beta \le \alpha$.*
(iii) *If $\chi(u) = \mathcal{O}(|u|^{-\beta-\gamma})$ as $|u| \to +\infty$ for some $\gamma > 1$, then χ is of order β.*

(iv) $\lim_{R\to+\infty} \sum_{k\in\mathbb{Z}:|u-k|\geq R} |u-k|^\alpha|\chi(u-k)| = 0$ *uniformly in* $u \in \mathbb{R}$.

Proof. Statement (i) is simply due to the fact that (7) defines a continuous periodic function. Statement (ii) follows from the inequality $|u-k|^\beta|\chi(u-k)| \leq |u-k|^\alpha|\chi(u-k)|$ for $|u-k| > 1$ and $0 \leq \beta < \alpha$ and from (i). The third statement is a clear consequence of the convergence condition of the generalized harmonic series.

Finally, let $\epsilon > 0$. Since (7) is convergent uniformly for $u \in \mathbb{R}$, there exists $\tilde{R} > 0$ such that

$$\sum_{|k|\geq\tilde{R}} |u-k|^\alpha|\chi(u-k)| < \epsilon, \qquad \forall u \in \mathbb{R}. \tag{9}$$

Now let $R := \tilde{R} + 1$ and any $u \in \mathbb{R}$. Denoting by $\lfloor u \rfloor$ the floor of u, we define $u_0 = u - \lfloor u \rfloor$. Thus, writing $k' = k - \lfloor u \rfloor$, we have

$$\sum_{k\in\mathbb{Z}:|u-k|\geq R} |u-k|^\alpha|\chi(u-k)| = \sum_{k'\in\mathbb{Z}:|u_0-k'|\geq R} |u_0-k'|^\alpha|\chi(u_0-k')|. \tag{10}$$

Taking into account that $0 \leq u_0 < 1$, the inequality $|u_0 - k'| \geq R$ implies $|k'| \geq R - 1 = \tilde{R}$. Hence, by (10) and (9)

$$\sum_{k\in\mathbb{Z}:|u-k|\geq R} |u-k|^\alpha|\chi(u-k)| \leq \sum_{|k'|\geq\tilde{R}} |u_0-k'|^\alpha|\chi(u_0-k')| < \epsilon.$$

This proves statement (iv).

If $M_\alpha(\chi) < +\infty$, then clearly $m_\alpha(u,\chi)$ is a well-defined real function on u. A useful criterion to establish the validity of condition (6) of approximate identity is given by of [20, Lemma 2]. Indeed, if $\chi \in L^1(\mathbb{R})$, then (6) is equivalent to

$$\widehat{\chi}(2\pi k) = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0. \end{cases} \tag{11}$$

We now present classical examples of kernel with their properties (see for instance [12,18,40]). The *B-spline of order* $m \in \mathbb{N}^+$ is the function defined for $m = 1$ by

$$B_1(u) = \begin{cases} 1 & u \in \left[-\frac{1}{2}, \frac{1}{2}\right] \\ 0 & \text{otherwise,} \end{cases}$$

and for $m \geq 2$ by

$$B_m(u) = \frac{1}{(m-1)!} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \max\left(\frac{m}{2} + u - j, 0\right)^{m-1},$$

or equivalently by $B_m = B_{m-1} * B_1$, where $*$ is the convolution product. The Fourier transform of B_m is

$$\widehat{B_m}(v) = \operatorname{sinc}^m\left(\frac{v}{2\pi}\right).$$

Consequently, (11) holds and then (6) is satisfied. Also (7) is true since B_m has compact support (that is $[-\frac{m}{2}, \frac{m}{2}]$). In conclusion, B_m is a kernel of any order.

For $m \in \mathbb{N}$, $m \geq 2$ we define

$$F_m(u) = c_m \operatorname{sinc}^m\left(\frac{u}{m}\right), \qquad \text{where } c_m = \left(\int_{-\infty}^{+\infty} \operatorname{sinc}^m\left(\frac{u}{m}\right) du\right)^{-1}.$$

In the particular case $m = 2$ we have the *Fejér kernel* $F_2(u) = \frac{1}{2}\operatorname{sinc}^2\left(\frac{u}{2}\right)$ and, in general, if m is even, then F_m is a *Jackson*-type kernel. The Fourier transform of F_m is

$$\widehat{F_m}(v) = c_m B_m\left(\frac{mv}{2\pi}\right),$$

which says that F_m is bandlimited and that (6) holds, since (11) is satisfied. Moreover, for the fact that $F_m(u) = \mathcal{O}(|u|^{-m})$ as $|u| \to +\infty$, the kernel F_m is of order α for any $0 \leq \alpha < m-1$, by Lemma 1.

The *Bochner–Riesz type kernels* are

$$b_\eta(u) = \frac{2^\eta}{\sqrt{2\pi}}\Gamma(\eta+1)|u|^{-\eta-\frac{1}{2}} J_{\eta+\frac{1}{2}}(|u|)$$

where $\eta > 1$ and J_λ is the Bessel function of order λ and Γ is the Euler gamma function. The functions b_η are in fact kernels of order $\eta - 1$. Indeed,

$$\widehat{b_\eta}(v) = \begin{cases} (1-v^2)^\eta & |v| \leq 1 \\ 0 & |v| > 1, \end{cases}$$

so (11) is true and hence (6) is satisfied. Moreover, $b_\eta(u) = \mathcal{O}(|u|^{-\eta-1})$ as $|u| \to +\infty$ since $J_\lambda(u) = \mathcal{O}(|u|^{-\frac{1}{2}})$ as $|u| \to +\infty$ (see [15]). Thus, Lemma 1 implies that (7) holds with $\beta = \eta - 1$. In addition, b_η is bandlimited.

3. Basic properties and convergence theorems

In this section we start to give results about the Hermite-type sampling operator $G_{n,w}$. In particular, we provide sufficient conditions on χ and f under which $G_{n,w}f$ is well-defined, we study the continuity and regularity of the operators and we also prove convergence results.

Proposition 2. *Let χ be a kernel order $n \in \mathbb{N}$, $w > 0$, and $f : \mathbb{R} \to \mathbb{R}$ be an n times differentiable function. Then $G_{n,w}f$ is well-defined if one of the following statements holds:*

(i) for any $j = 0, \dots, n$, $f^{(j)}$ is bounded;
(ii) for any $j = 0, \dots, n$, $f^{(j)}$ grows no more than a polynomial of order $n - j$, i.e.

$$|f^{(j)}(x)| \leq p_{n-j}(x), \qquad \forall x \in \mathbb{R}, \tag{12}$$

where p_{n-j} is a polynomial of order $n - j$;
(iii) the limits $\lim_{x\to\pm\infty} \frac{f(x)}{x^n}$ are finite and not null.

Proof. We firstly note that, by Lemma 1, $M_j(\chi) < \infty$ for every $j = 0, \dots, n$.

(i) If $f^{(j)}$ is bounded for any $j = 0, \dots, n$, then a straightforward calculation

$$\sum_{k\in\mathbb{Z}} \left(\sum_{j=0}^{n} \frac{1}{j!}\left|f^{(j)}\left(\frac{k}{w}\right)\right|\left|x - \frac{k}{w}\right|^j\right)|\chi(wx-k)| \leq \sum_{j=0}^{n} \frac{\|f^{(j)}\|_\infty M_j(\chi)}{j!w^j} \tag{13}$$

shows that the series in (5) is absolutely convergent for every $x \in \mathbb{R}$.

(ii) For any $j = 0, 1, \ldots, n$, let p_{n-j} be as in (12). We write $p_{n-j}(x) = a_{j,n-j}x^{n-j} + a_{j,n-j-1}x^{n-j-1} + \cdots + a_{j,0}$. Making use of Jensen's inequality, we have for $k \in \mathbb{Z}$ and $j < n$

$$\left|f^{(j)}\left(\frac{k}{w}\right)\right| \leq |a_{j,n-j}|\left|\frac{k}{w}\right|^{n-j} + \cdots + |a_{j,0}|$$
$$\leq 2^{n-j-1}|a_{j,n-j}|\left(|x|^{n-j} + \left|x - \frac{k}{w}\right|^{n-j}\right) + \cdots + |a_{j,0}|$$

and, for $j = n$, $\left|f^{(n)}\left(\frac{k}{w}\right)\right| \leq a_{0,0}$.

Therefore, for some positive numbers $b_{j,n-j}, b_{j,n-j-1}, \ldots, b_{j,0}$, we have

$$\left|f^{(j)}\left(\frac{k}{w}\right)\right|\left|x - \frac{k}{w}\right|^j \leq \left(b_{j,n-j}|x|^{n-j} + b_{j,n-j-1}|x|^{n-j-1} + \cdots + b_{j,1}|x|\right)\frac{|wx-k|^j}{w^j}$$
$$+ b_{j,n-j}\frac{|wx-k|^n}{w^n} + b_{j,n-j-1}\frac{|wx-k|^{n-1}}{w^{n-1}} + \cdots + b_{j,0}\frac{|wx-k|^j}{w^j}.$$

In conclusion, the series in (5) is absolutely convergent for any $x \in \mathbb{R}$, since

$$\sum_{k\in\mathbb{Z}}\left(\sum_{j=0}^{n}\frac{1}{j!}\left|f^{(j)}\left(\frac{k}{w}\right)\right|\left|x - \frac{k}{w}\right|^j\right)|\chi(wx-k)| \leq \sum_{j=0}^{n} q_{n-j}(|x|)\frac{M_j(\chi)}{w^j}$$

for some polynomials q_{n-j}.

(iii) Since $\lim\limits_{x\to\pm\infty}\frac{f(x)}{x^n}$ are finite there exist $\alpha_0, R > 0$ such that $|f(x)| \leq \alpha_0|x|^n$ for $|x| > R$. Moreover, exploiting the continuity of f and denoting by $\beta_0 = \max\limits_{[-R,R]}|f(x)|$, we can write $|f(x)| \leq \alpha_0|x|^n + \beta_0$ for every $x \in \mathbb{R}$.

The hypothesis that $\lim\limits_{x\to\pm\infty}\frac{f(x)}{x^n} \neq 0$ implies that $\lim\limits_{x\to\pm\infty}|f(x)| = +\infty$. Thus, by the L'Hôpital's rule, also $\lim\limits_{x\to\pm\infty}\frac{f'(x)}{x^{n-1}}$ is finite and not null. With similar considerations as above, there exist $\alpha_1, \beta_1 > 0$ such that $|f'(x)| \leq \alpha_1|x|^{n-1} + \beta_1$ for every $x \in \mathbb{R}$.

Repeating the argument, also bounds for the other derivatives can be obtained, i.e. for $j = 0, 1, \ldots, n-1$ there exist $\alpha_j, \beta_j > 0$ such that $|f^{(j)}(x)| \leq \alpha_j|x|^{n-j} + \beta_j$ and there exists $\beta_n > 0$ such that $|f^{(n)}(x)| \leq \beta_n$ for every $x \in \mathbb{R}$.

Therefore, the conclusion is obtained arguing as in the step (ii).

As stated above, $G_{n,w}f$ exists if f and its derivatives grow less than polynomials of certain orders. Actually, by Taylor formula and (6), the operator fixes the polynomials of degrees less or equal to its order.

Proposition 3. *Let f be a polynomial of order less or equal to $n \in \mathbb{N}$. Then $G_{n,w}f = f$ for every $w > 0$.*

We now move on the continuity question of $G_{n,w}f$. The property essentially holds because the kernel is continuous and the boundedness of f and of its derivatives is sufficient as requirement.

Proposition 4. *Let χ be a kernel of order n, $w > 0$ and $f \in C_b^n(\mathbb{R})$. Then $G_{n,w}f$ is continuous. If, in addition, χ is uniformly continuous, then the same holds for $G_{n,w}f$.*

Proof. The kernel χ is of order n and then also of lower order j by Lemma 1. In particular, the series in (7) is, with j instead of n, convergent uniformly for $u \in \mathbb{R}$. This means that for $\epsilon > 0$ it is possible to find $K > 0$ such that

$$\sum_{|k|>K} |u-k|^j |\chi(u-k)| < \epsilon, \qquad \forall u \in \mathbb{R}, \forall j = 0,1,\dots,n. \tag{14}$$

Now let

$$s_K(x) = \sum_{|k|\le K} \left(\sum_{j=0}^{n} \frac{1}{j!} f^{(j)}\left(\frac{k}{w}\right)\left(x - \frac{k}{w}\right)^j \right) \chi(wx-k) \text{ for } x \in \mathbb{R}.$$

By (14),

$$|(G_{n,w}f)(x) - s_K(x)| \le \sum_{j=0}^{n} \frac{\|f^{(j)}\|_\infty}{j!w^j} \sum_{|k|>K} |wx-k|^j \, |\chi(wx-k)| < \epsilon \sum_{j=0}^{n} \frac{\|f^{(j)}\|_\infty}{j!w^j}$$

for every $x \in \mathbb{R}$. Thus, the sequence of functions $(s_K)_{K\in\mathbb{N}^+}$ is uniformly convergent to $G_{n,w}f$. Furthermore, s_K is a continuous function because sums of continuous functions (we recall that, as defined in the introduction, a kernel is continuous). Therefore, $G_{n,w}f$ is a continuous function, too. The second part of the statement follows by similar arguments.

Further regularity requires additional assumptions on the kernel. More precisely, we can state the following result.

Proposition 5. *Let $\chi \in C_b^m(\mathbb{R})$ be a kernel of order n such that, for every $\ell = 0,1,\dots,m$, the series*

$$\sum_{k\in\mathbb{Z}} |u-k|^n |\chi^{(\ell)}(u-k)| \text{ is convergent uniformly for } u \in \mathbb{R}.$$

Moreover, let $f \in C_b^n(\mathbb{R})$. Then $G_{n,w}f \in C_b^m(\mathbb{R})$ for every $w > 0$. If, in addition, $\chi \in C^m(\mathbb{R})$, then $G_{n,w}f \in C^m(\mathbb{R})$ for every $w > 0$.

Proof. The proof is similar to that of Proposition 4 so we give only a sketch. The assumptions and Lemma 1 imply that for $r = 0,1,\dots,n$ and $\ell = 0,1,\dots,m$

$$\sum_{k\in\mathbb{Z}} |u-k|^r |\chi^{(\ell)}(u-k)| \text{ is convergent uniformly for } u \in \mathbb{R}. \tag{15}$$

Let $\epsilon > 0$. The function s_K belongs to $C_b^m(\mathbb{R})$ and its derivative of order m is a linear combination of functions of type $(wx-k)^r\chi^{(\ell)}(wx-k)$ for opportune values of $r = 0,1,\dots,n$ and $\ell = 0,1,\dots,m$. Hence, by (15) and by the boundedness of f and its derivatives, the sequence (s'_K) is uniformly convergent. Since (s_K) converges to $G_{n,w}f$, then the latter must be differentiable. Moreover, the hypothesis ensures that $(G_{n,w}f)'$ is bounded, so $G_{n,w}f$ belongs to $C_b(\mathbb{R})$. Iterating the arguments for the greater orders it is possible to prove that $G_{n,w}f \in C_b^m(\mathbb{R})$.
In the case in which $\chi \in C^m(\mathbb{R})$, then s_K is in $C^m(\mathbb{R})$ and with the same steps also $G_{n,w}f$ belongs to $C^m(\mathbb{R})$.

Under the assumptions about the kernel of Proposition 5, and from estimate (13) we find that $G_{n,w} : C_b^n(\mathbb{R}) \to C_b^m(\mathbb{R})$ is a well-defined and bounded operator. The next result is one of the main theorems of the paper and concerns the convergence for $w \to +\infty$.

Theorem 6. *Let χ be a kernel of order n and $f \in C_b^n(\mathbb{R})$. Then*

$$\lim_{w\to+\infty} \|G_{n,w}f - f\|_\infty = 0.$$

Proof. Let $w > 0$ and $x \in \mathbb{R}$. By (6) we have

$$(G_{n,w}f)(x) - f(x) = \sum_{k\in\mathbb{Z}} \left(\sum_{j=0}^{n} \frac{1}{j!} f^{(j)}\left(\frac{k}{w}\right)\left(x - \frac{k}{w}\right)^j - f(x) \right) \chi(wx - k). \tag{16}$$

For any $k \in \mathbb{Z}$, by Taylor formula, there exists $\xi_{x,\frac{k}{w}}$ between x and $\frac{k}{w}$, such that

$$f(x) - \sum_{j=0}^{n-1} \frac{1}{j!} f^{(j)}\left(\frac{k}{w}\right)\left(x - \frac{k}{w}\right)^j = \frac{1}{n!} f^{(n)}\left(\xi_{x,\frac{k}{w}}\right)\left(x - \frac{k}{w}\right)^n. \tag{17}$$

Inserting (17) in (16) we obtain

$$(G_{n,w}f)(x) - f(x) = \frac{1}{n!} \sum_{k\in\mathbb{Z}} \left(f^{(n)}\left(\frac{k}{w}\right) - f^{(n)}\left(\xi_{x,\frac{k}{w}}\right)\right)\left(x - \frac{k}{w}\right)^n \chi(wx - k).$$

Thus

$$|(G_{n,w}f)(x) - f(x)| \le \frac{1}{n!w^n} \sum_{k\in\mathbb{Z}} \left| f^{(n)}\left(\frac{k}{w}\right) - f^{(n)}\left(\xi_{x,\frac{k}{w}}\right)\right| |wx - k|^n \, |\chi(wx - k)| \tag{18}$$

and by the boundedness of $f^{(n)}$ we have

$$\|G_{n,w}f - f\|_\infty \le \frac{2\|f^{(n)}\|_\infty M_n(\chi)}{n!w^n} \tag{19}$$

which shows that $\lim_{w\to+\infty} \|G_{n,w}f - f\|_\infty = 0.$

Example 1. We can view the convergence result of Theorem 6 with the help of Fig. 1. The function $f(x) = \frac{1+x}{1+x^2}$ is considered and its plot is showed on the top of Fig. 1. We applied the operator $G_{2,w}$ on f for increasing values of w, namely $w = 3, 4, 5$, and with kernel F_3. Some details of the plots of these functions are in the bottom of Fig. 1 and the approximation errors are $\|G_{2,3}f - f\|_\infty = 0.0072$, $\|G_{2,4}f - f\|_\infty = 0.0015$ and $\|G_{2,5}f - f\|_\infty = 0.0003$.

Since the Hermite-type sampling operators $G_{n,w}$, $n \ge 1$, are defined for functions with a certain degree of smoothness, it is natural to ask if the derivatives of the $G_{n,w}f$ converge to the derivatives of f. For the generalized sampling operator a result about this topic, i.e. about the simultaneous approximation, has been given in [3]. The following theorem gives an answer to this question.

Theorem 7. *Let $\chi \in C_b^n(\mathbb{R})$ be a kernel of order n such that, for every $\ell = 0, 1, \dots, n$, the series*

$$\sum_{k\in\mathbb{Z}} |u - k|^n |\chi^{(\ell)}(u - k)| \text{ is convergent uniformly for } u \in \mathbb{R}.$$

Moreover, let $f \in C^n(\mathbb{R})$. Then, for every $\ell = 0, 1, \dots, n$,

$$\lim_{w\to+\infty} \|(G_{n,w}f)^{(\ell)} - f^{(\ell)}\|_\infty = 0. \tag{20}$$

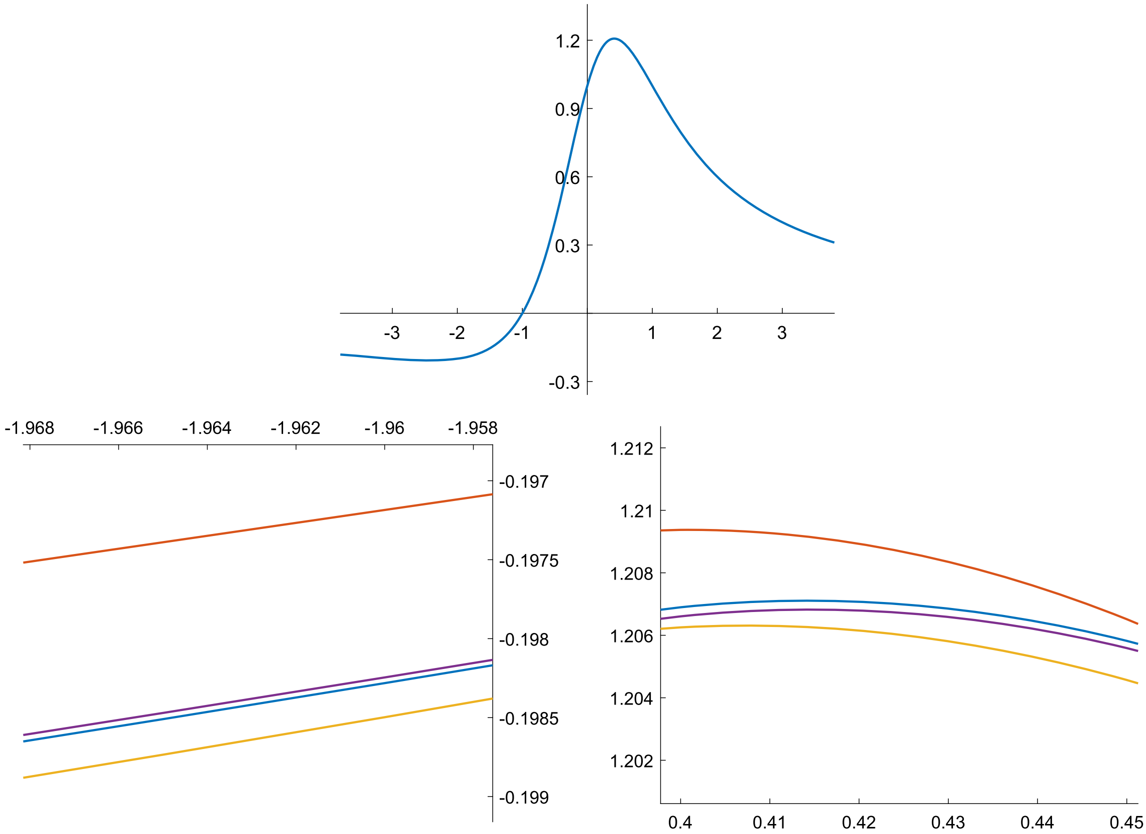

Fig. 1. Details of the plots of f given by $f(x) = \frac{1+x}{1+x^2}$ (in blue), $G_{2,3}f$ (in red), $G_{2,4}f$ (in orange) and $G_{2,5}f$ (in purple), with kernel F_3. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

Proof. In this proof, we indicate for convenience the kernel in the operator symbol, so we write $G^{\chi}_{n,w}$ instead of $G_{n,w}$. By Proposition 5, $G^{\chi}_{n,w}f$ is n times differentiable. Let us fix $\ell = 1, \dots, n$ (the case $\ell = 0$ is covered by Theorem 6). Arguing as in Lemma 1, the hypotheses ensure that $M_r(\chi^{(\ell)}) < \infty$ for every $r = 0, \dots, n$. Moreover, the following relation holds

$$(G^{\chi}_{n,w}f)^{(\ell)} = \sum_{i=0}^{\ell} \binom{\ell}{i} w^i G^{\chi^{(i)}}_{n-\ell+i,w} f^{(\ell-i)}, \tag{21}$$

where $G^{\chi^{(i)}}_{n-\ell+i,w}$ has the same expression of the Hermite-type sampling operator but with $\chi^{(i)}$ instead of χ. We note that, by (6), for every $i = 0, \dots, n$,

$$\sum_{k\in\mathbb{Z}} \chi^{(i)}(u-k) = 0, \qquad \forall u \in \mathbb{R}. \tag{22}$$

Taking $i = 0$ in (21), we have exactly the Hermite-type sampling operator $G^{\chi}_{n-\ell,w}$, with kernel χ, and order $n - \ell$. Hence, by Theorem 6,

$$\lim_{w\to+\infty} \|G^{\chi}_{n-\ell,w} f^{(\ell)} - f^{(\ell)}\|_\infty = 0. \tag{23}$$

For $i = 1, \dots, \ell$, we apply Taylor formula and (22) to obtain

$$
\begin{aligned}
w^i (G^{\chi^{(i)}}_{n-\ell+i,w} f^{(\ell-i)})(x) &= w^i \sum_{k\in\mathbb{Z}} \left(\sum_{j=0}^{n-\ell+i} \frac{1}{j!} f^{(\ell-i+j)}\left(\frac{k}{w}\right)\left(x-\frac{k}{w}\right)^j \right) \chi^{(i)}(wx-k) \\
&= w^i \sum_{k\in\mathbb{Z}} \left(f^{(\ell-i)}(x) + \frac{1}{(n-\ell+i)!}\left(f^{(n)}\left(\frac{k}{w}\right) - f^{(n)}\left(\xi_{i,x,\frac{k}{w}}\right)\right)\left(x-\frac{k}{w}\right)^{n-\ell+i} \right)\chi^{(i)}(wx-k) \qquad (24)\\
&= \frac{1}{(n-\ell+i)!w^{n-\ell}} \sum_{k\in\mathbb{Z}} \left(f^{(n)}\left(\frac{k}{w}\right) - f^{(n)}\left(\xi_{i,x,\frac{k}{w}}\right)\right)(wx-k)^{n-\ell+i}\chi^{(i)}(wx-k)
\end{aligned}
$$

where $\xi_{i,x,\frac{k}{w}}$ is between x and $\frac{k}{w}$. Clearly,

$$
\left| w^i (G^{\chi^{(i)}}_{n-\ell+i,w} f^{(\ell-i)})(x) \right| \le \frac{2\|f^{(n)}\|_\infty}{(n-\ell+i)!w^{n-\ell}} M_{n-\ell+i}(\chi^{(i)}), \qquad i = 1, \dots, \ell. \qquad (25)
$$

Therefore, putting together (21), (23) and (25), we prove (20) for $\ell < n$.
Now, we consider the final case $\ell = n$. Firstly, thanks to the hypothesis on χ and similarly to the proof of Lemma 1(iv), we can establish that

$$
\lim_{R\to+\infty} \sum_{k\in\mathbb{Z}:|u-k|\ge R} |u-k|^i |\chi^{(i)}(u-k)| = 0 \text{ uniformly in } u \in \mathbb{R}. \qquad (26)
$$

Since $f^{(n)}$ is uniformly continuous, for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f^{(n)}(y) - f^{(n)}(z)| < \epsilon$ when $|y-z| < \delta$. Now, we write the last row of (24) as $I_1 + I_2$ where

$$
\begin{aligned}
I_1 &= \frac{1}{i!} \sum_{k\in\mathbb{Z}:|x-\frac{k}{w}|<\delta} \left(f^{(n)}\left(\frac{k}{w}\right) - f^{(n)}\left(\xi_{i,x,\frac{k}{w}}\right)\right)(wx-k)^i \chi^{(i)}(wx-k), \\
I_2 &= \frac{1}{i!} \sum_{k\in\mathbb{Z}:|x-\frac{k}{w}|\ge\delta} \left(f^{(n)}\left(\frac{k}{w}\right) - f^{(n)}\left(\xi_{i,x,\frac{k}{w}}\right)\right)(wx-k)^i \chi^{(i)}(wx-k).
\end{aligned}
$$

Thus, by the uniform continuity and by (26) (with $u = wx$ and $R = w\delta$), we can write for w sufficiently large

$$
|I_1| \le \epsilon \frac{M_i(\chi^{(i)})}{i!}, \qquad |I_2| \le \epsilon \frac{2\|f^{(n)}\|_\infty}{i!}.
$$

This proved that for every $i = 1, \dots, \ell$, (24) vanishes as $w \to +\infty$ and uniformly in $x \in \mathbb{R}$. Together with (21) and (23), statement (20) is proved also for $\ell = n$.

4. Rate of convergence

In approximation theory much interest is placed not only on the convergence but also the rate of convergence or, in different words, order of approximation (for some literature about various operators see [3,9,14,22,23,25,27,29,35,36,45]). Indeed, higher order means faster convergence and reduced approximation errors.

In the following theorem we prove quantitative estimates for the Hermite-type sampling operators, which state in particular that the rate of convergence increases for higher value of n. We recall that, for $\delta > 0$, the *modulus of continuity* of a uniformly continuous function $g : \mathbb{R} \to \mathbb{R}$ is

$$
\omega(g, \delta) = \sup_{|x-y|<\delta} |g(x) - g(y)|.
$$

The modulus of continuity has the following properties

(i) $\omega(g,\delta') \le \omega(g,\delta)$ if $\delta' < \delta$;
(ii) $\lim_{\delta\to 0} \omega(g,\delta) = 0$;
(iii) $\omega(g,\lambda\delta) \le (1+\lambda)\omega(g,\delta)$ if $\lambda > 0$.

Theorem 8. *Let χ be a kernel of order n.*

(i) If $f \in C^n(\mathbb{R})$, then $\|G_{n,w}f - f\|_\infty = o(w^{-n})$ for $w \to +\infty$.
(ii) If $f \in C^n(\mathbb{R})$, $M_{n+1}(\chi) < \infty$ and $w > 0$, then

$$\|G_{n,w}f - f\|_\infty \le \frac{\omega(f^{(n)}, \frac{1}{w})(M_n(\chi) + M_{n+1}(\chi))}{n! w^n}.$$

(iii) If $f \in C_b^{n+1}(\mathbb{R})$, $M_{n+1}(\chi) < \infty$ and $w > 0$, then

$$\|G_{n,w}f - f\|_\infty \le \frac{\|f^{(n+1)}\|_\infty M_{n+1}(\chi)}{(n+1)! w^{n+1}}. \tag{27}$$

In particular, $\|G_{n,w}f - f\|_\infty = \mathcal{O}(w^{-(n+1)})$ for $w \to +\infty$.

Proof. Statement (i) follows from (18) by showing that

$$\lim_{w\to+\infty} \sum_{k\in\mathbb{Z}} \left| f^{(n)}\left(\frac{k}{w}\right) - f^{(n)}\left(\xi_{x,\frac{k}{w}}\right)\right| |wx-k|^n \, |\chi(wx-k)| = 0.$$

The latter is proved, as in the final part of the proof of Theorem 7, by splitting the series and by applying the uniform continuity of $f^{(n)}$ and Lemma 1(iv).
For statement (ii), we start again from (18) and make the following steps applying the definition and properties of the modulus of continuity,

$$\begin{aligned}
|(G_{n,w}f)(x) - f(x)| &\le \frac{1}{n!w^n} \sum_{k\in\mathbb{Z}} \left| f^{(n)}\left(\frac{k}{w}\right) - f^{(n)}\left(\xi_{x,\frac{k}{w}}\right)\right| |wx-k|^n \, |\chi(wx-k)| \\
&\le \frac{1}{n!w^n} \sum_{k\in\mathbb{Z}} \omega\left(f^{(n)}, \left|\frac{k}{w} - \xi_{x,\frac{k}{w}}\right|\right) |wx-k|^n \, |\chi(wx-k)| \\
&\le \frac{1}{n!w^n} \sum_{k\in\mathbb{Z}} \omega\left(f^{(n)}, \left|\frac{k}{w} - x\right|\right) |wx-k|^n \, |\chi(wx-k)| \\
&= \frac{1}{n!w^n} \sum_{k\in\mathbb{Z}} \omega\left(f^{(n)}, \left|\frac{wx-k}{w}\right|\right) |wx-k|^n \, |\chi(wx-k)| \\
&\le \frac{1}{n!w^n} \sum_{k\in\mathbb{Z}} (1 + |wx-k|)\, \omega\left(f^{(n)}, \frac{1}{w}\right) |wx-k|^n \, |\chi(wx-k)| \\
&= \frac{\omega\left(f^{(n)}, \frac{1}{w}\right)(M_n(\chi) + M_{n+1}(\chi))}{n!w^n}.
\end{aligned}$$

Finally, to prove statement (iii), we suppose that $f \in C_b^{n+1}(\mathbb{R})$, thus working with the Taylor formula as in Theorem 6 we can write (16) as

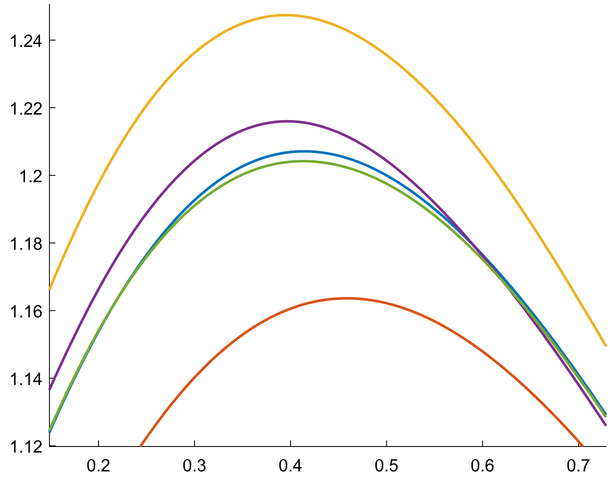

Fig. 2. Comparison between plots of f (in blue), the approximation $G_{0,3}f = G_3 f$ by the generalized sampling operator (in red), the approximations $G_{1,3}f$ (in orange), $G_{2,3}f$ (in purple) and $G_{3,3}f$ (in green). The function f is given by $f(x) = \frac{1+x}{1+x^2}$ (the same function of Fig. 1) and for all the Hermite-type sampling operators the kernel is the B-spline B_5.

$$(G_{n,w}f)(x) - f(x) = \frac{1}{(n+1)!} \sum_{k\in\mathbb{Z}} f^{(n+1)}\left(\theta_{x,\frac{k}{w}}\right)\left(x - \frac{k}{w}\right)^{n+1} \chi(wx-k), \tag{28}$$

where, for every $k \in \mathbb{Z}$, $\theta_{x,\frac{k}{w}}$ is a number between x and $\frac{k}{w}$. In conclusion, the statement follows by the inequalities

$$\begin{aligned} |(G_{n,w}f)(x) - f(x)| &\leq \frac{1}{(n+1)!w^{n+1}} \sum_{k\in\mathbb{Z}} \left|f^{(n+1)}\left(\theta_{x,\frac{k}{w}}\right)\right| |wx-k|^{n+1} |\chi(wx-k)| \\ &\leq \frac{\|f^{(n+1)}\|_\infty M_{n+1}(\chi)}{(n+1)!w^{n+1}}. \end{aligned}$$

Example 2. Fig. 2 shows the plots of the function $f(x) = \frac{1+x}{1+x^2}$ and of the approximations $G_{n,w}f$, for values of n from 0 to 3 and for $w = 3$ fixed. The kernel used is the B-spline of order 5 in every case. We recall that $G_{0,w}$ corresponds to the generalized sampling operator G_w. As can be observed in the figure, increasing the order produces the effect of a better approximation in accordance to Theorem 8. This can be checked also from the numerical errors:

$$\|G_{0,3}f - f\|_\infty = 0.054, \quad \|G_{1,3}f - f\|_\infty = 0.044,$$
$$\|G_{2,3}f - f\|_\infty = 0.013, \quad \|G_{3,3}f - f\|_\infty = 0.003.$$

Considering once again the same function f and the same kernel B_5, in Fig. 3 we can see the errors $E_n(w) = \|G_{n,w}f - f\|_\infty$ in terms of w (for $n = 1, 2$) and the corresponding bounds given by Theorem 8(iii).

A faster rate implies a smaller value of w to reach a given error. This is shown in Table 1 comparing the generalized sampling operator G_w and the Hermite-type sampling operator $G_{2,w}$ (applied on the same function f of this example and with the kernel B_5). Table 1 puts in comparison also the total number of samples in an interval of length one that we need to calculate $G_w f$ and $G_{2,w}f$. For the total number of samples we mean just the number of samples $f(\frac{k}{w})$ for G_w and the number of samples $f(\frac{k}{w}), f'(\frac{k}{w}), f''(\frac{k}{w})$ for $G_{2,w}$.

In the first column of Table 1 two L^∞ errors are fixed. In the second (resp., third) column the required value of w for G_w (resp., for $G_{2,w}$) to have the corresponding error is indicated. In the fourth (resp., fifth) column the total number of samples required by G_w (resp., by $G_{2,w}$) on a interval of unitary length is shown. As

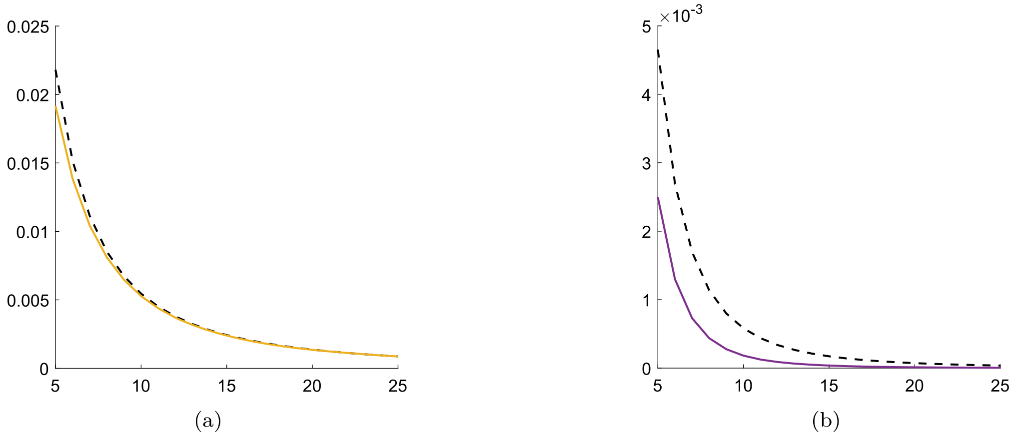

Fig. 3. In (a): the solid line is the plot of the error $E_1(w) = \|G_{1,w}f - f\|_\infty$ depending on w; the dashed line is the plot of the bound $T_1(w) = \frac{\|f''\|_\infty M_2(\chi)}{2w^2}$ of Theorem 8(iii).
In (b): the solid line is the plot of the error $E_2(w) = \|G_{2,w}f - f\|_\infty$ depending on w; the dashed line is the plot of the bound $T_2(w) = \frac{\|f'''\|_\infty M_3(\chi)}{6w^3}$ of Theorem 8(iii).

Table 1
Values of w and total numbers of samples in an interval of length one required by the operators G_w and $G_{2,w}$ to obtain a given approximation error. The values of w are rounded to the nearest integer.

L^∞ error	w for G_w	w for $G_{2,w}$	n. of samples for G_w	n. of samples for $G_{2,w}$
0.001	24	7	24	21
0.0001	73	13	73	39

one can read in Table 1, the values of w for the Hermite-type sampling operator are much smaller than those for the generalized sampling operator. To reach the error equals to 0.001 we need more or less the same number of samples, while for the error equals to 0.0001 the operator $G_{2,w}$ requires a significant lower number of samples.

Remark 1. The Hermite-type sampling operator $G_{n,w}$ is a generalization of G_w, therefore a discussion of their rates of convergence is meaningful. By [20, Theorem 2], if for some $r \in \mathbb{N}$, $r > 1$,

$$m_j(u, \chi) = 0, \qquad \forall u \in \mathbb{R}, j = 1, \dots, r-1, \tag{29}$$

or equivalently (in the case where $\widehat{\chi}$ is $r-1$ times differentiable) if the *Strang–Fix type conditions*

$$\widehat{\chi}^{(j)}(2\pi k) = \begin{cases} 1 & k = j = 0, \\ 0 & k \in \mathbb{Z}\setminus\{0\}, j = 0 \\ 0 & k \in \mathbb{Z}, j = 1, \dots, r-1 \end{cases}$$

hold (see [20, Lemma 3]), and $M_r(\chi) < \infty$, then for every $f \in C^r(\mathbb{R})$ we have

$$\|G_w f - f\|_\infty \leq \frac{\|f^{(r)}\|_\infty M_r(\chi)}{r! w^r}. \tag{30}$$

For a first comparison, if (29) is satisfied for $r = n$ and $f \in C^n(\mathbb{R})$, then $\|G_w f - f\|_\infty = \mathcal{O}(w^{-n})$ for $w \to +\infty$, while by Theorem 8(i) we have a stronger rate, $\|G_w f - f\|_\infty = o(w^{-n})$ for $w \to +\infty$.

In the case where (29) is satisfied for $r = n+1$ and $f \in C^{n+1}(\mathbb{R})$, then the bound in (30) is the same of (27) for the Hermite-type sampling operator of order n. Anyway, (30) holds under the conditions (29) which are quite strong if we require a great value of r. Indeed, the common kernels χ we presented in Section 2, $\chi = B_m$ and $\chi = F_m$ (with $m > 2$) and $\chi = b_\eta$ (with $\eta > 2$), satisfy (29) only for the low value $r = 2$, since

$$\widehat{\chi}(0) = 1, \quad \widehat{\chi}(2\pi k) = 0 \text{ for every } k \in \mathbb{Z}\backslash\{0\},\ \widehat{\chi}'(2\pi k) = 0 \text{ for every } k \in \mathbb{Z} \text{ and } \quad \widehat{\chi}''(0) \neq 0.$$

Thus, for a function $f \in C^{n+1}(\mathbb{R})$, $n \geq 2$, and for a kernel among those above, we have

$$\|G_w f - f\|_\infty = \mathcal{O}(w^{-2}) \text{ as } w \to +\infty,$$

while the rate for the Hermite-type sampling operator with the same kernel (with m, η sufficiently large, in the cases of F_m or β_η) is higher, i.e.

$$\|G_{n,w} f - f\|_\infty = \mathcal{O}(w^{-(n+1)}) \text{ as } w \to +\infty.$$

The operators $G_{n,w}$ have then an advantage about the order of approximation in comparison to the generalized sampling operator G_w. Of course, it is possible to construct new kernels with more null initial moments and then the operator G_w provides a greater order of approximation. Anyway, it will still be limited to a specific order. For instance, if we want a rate at least of $\mathcal{O}(w^{-4})$, then we can construct a kernel χ with $m_1(\cdot, \chi) = m_2(\cdot, \chi) = m_3(\cdot, \chi) = 0$ and have indeed $\|G_w f - f\|_\infty = \mathcal{O}(w^{-4})$ as $w \to +\infty$. Anyway, if in a next phase we desire a faster rate, say $\mathcal{O}(w^{-5})$, then χ may not be good (exactly if $m_4(\cdot, \chi)$ turns out to be not null). Hence, in this case we have to construct another kernel for the rate $\mathcal{O}(w^{-5})$.

This problem is not present if we use the Hermite-type sampling operator $G_{n,w}$, because the rate given by Theorem 8 does not require particular conditions on the kernel besides that $M_{n+1}(\chi) < \infty$ (which is satisfied, for what said in Section 2, by any B-spline, no matter the value of n, by the kernels F_m for $m > n + 2$ and b_η for $\eta \geq n + 2$).

Furthermore, we can make another remark on the aspect above. In [20, Section 5.2.2] a way to construct kernels as linear combinations of B-splines and with null discrete moments up to a prefixed value is provided. In particular, these examples can be found:

(i) $\chi_2(u) = 3B_2(u - 2) - 2B_2(u - 3)$, giving

$$\|G_w f - f\|_\infty \leq 15 \frac{\|f''\|_\infty}{w^2}, \qquad \forall f \in C^2(\mathbb{R});$$

(ii) $\chi_3(u) = \frac{1}{8}(47B_3(u - 2) - 62B_3(u - 3) + 23B_3(u - 4))$, giving

$$\|G_w f - f\|_\infty \leq 54 \frac{\|f'''\|_\infty}{w^3}, \qquad \forall f \in C^3(\mathbb{R});$$

(iii) $\chi_4(u) = \frac{1}{6}(115B_4(u - 3) - 256B_4(u - 4) + 203B_4(u - 5) - 56B_4(u - 6))$, giving

$$\|G_w f - f\|_\infty \leq 968 \frac{\|f^{(4)}\|_\infty}{w^4}, \qquad \forall f \in C^4(\mathbb{R}).$$

These estimates derive from (30) by rounding the factors $\frac{M_r(\chi)}{r!}$ to the nearest larger integer number; in fact, we have

$$\frac{M_2(\chi_2)}{2!} = 15 \qquad \frac{M_3(\chi_3)}{3!} = 53.29... \qquad \frac{M_4(\chi_4)}{4!} = 967.37...$$

Even though the rate of convergence of the generalized sampling operator is improved by constructing these new kernels, we see that the factors are relatively high. The estimates in (i)-(iii) turn out to be less optimal than the ones for the Hermite sampling operator with, for instance, the simple kernel $\chi = B_3$. Indeed, since $M_2(B_3) = M_3(B_3) = M_4(B_3) = \frac{1}{4}$, estimates (27) become

$$\|G_{1,w}f - f\|_\infty \leq \frac{1}{8}\frac{\|f''\|_\infty}{w^2}, \qquad \forall f \in C^2(\mathbb{R});$$
$$\|G_{2,w}f - f\|_\infty \leq \frac{1}{24}\frac{\|f'''\|_\infty}{w^3}, \qquad \forall f \in C^3(\mathbb{R});$$
$$\|G_{3,w}f - f\|_\infty \leq \frac{1}{96}\frac{\|f^{(4)}\|_\infty}{w^4}, \qquad \forall f \in C^4(\mathbb{R}).$$

5. Voronovskaya-type formula

Voronovskaya-type formulas are asymptotic relations for the pointwise approximation by operators of functions in terms of derivatives (see [1,4–6,13,27,34,45] for the Voronovskaya formulas of some operators in approximation theory). For the Hermite type sampling operator $G_{n,w}$ we prove the following result.

Theorem 9. *Let χ be a kernel of order $n+1$ with constant discrete moment m_{n+1} and not null,*

$$m_{n+1}(u,\chi) = A \neq 0, \qquad \forall u \in \mathbb{R}. \tag{31}$$

Moreover, let $w > 0$ and $f \in C_b^{n+1}(\mathbb{R})$. Then, for any $x \in \mathbb{R}$,

$$\lim_{w\to+\infty} w^{n+1}((G_{n,w}f)(x) - f(x)) = \frac{f^{(n+1)}(x)}{(n+1)!}A.$$

Proof. First of all, (31) implies that $M_{n+1}(\chi) = |A| < \infty$. Let $x \in \mathbb{R}$. From (28) we get

$$w^{n+1}((G_{n,w}f)(x) - f(x)) = \frac{1}{(n+1)!}\sum_{k\in\mathbb{Z}} f^{(n+1)}\left(\theta_{x,\frac{k}{w}}\right)(wx-k)^{n+1}\chi(wx-k),$$

where, for every $k \in \mathbb{Z}$, $\theta_{x,\frac{k}{w}}$ is a number between x and $\frac{k}{w}$. Therefore, by (31),

$$\begin{aligned} &w^{n+1}((G_{n,w}f)(x) - f(x)) - \frac{f^{(n+1)}(x)}{(n+1)!}A = \\ &= \frac{1}{(n+1)!}\sum_{k\in\mathbb{Z}}\left(f^{(n+1)}\left(\theta_{x,\frac{k}{w}}\right) - f^{(n+1)}(x)\right)(wx-k)^{n+1}\chi(wx-k). \end{aligned} \tag{32}$$

Now let $\epsilon > 0$. By the definition of continuity of $f^{(n+1)}$, there exists $\delta > 0$ such that $|f^{(n+1)}(x) - f^{(n+1)}(y)| < \epsilon$ for every $|x-y| < \delta$. Hence, we can write the series in (32) as $I_1 + I_2$ where

$$I_1 = \frac{1}{(n+1)!}\sum_{k\in\mathbb{Z}:|x-\frac{k}{w}|<\delta}\left(f^{(n+1)}\left(\theta_{x,\frac{k}{w}}\right) - f^{(n+1)}(x)\right)(wx-k)^{n+1}\chi(wx-k),$$
$$I_2 = \frac{1}{(n+1)!}\sum_{k\in\mathbb{Z}:|x-\frac{k}{w}|\geq\delta}\left(f^{(n+1)}\left(\theta_{x,\frac{k}{w}}\right) - f^{(n+1)}(x)\right)(wx-k)^{n+1}\chi(wx-k).$$

The first object can be estimated as follows

$$|I_1| \leq \frac{\epsilon}{(n+1)!}\sum_{k\in\mathbb{Z}:|x-\frac{k}{w}|<\delta}|wx-k|^{n+1}|\chi(wx-k)| \leq \frac{M_{n+1}}{(n+1)!}\epsilon. \tag{33}$$

Moreover, by Lemma 1(iv) with $u = wx$ and $R = w\delta$ we have

$$|I_2| \leq \frac{2\|f^{(n+1)}\|_\infty}{(n+1)!} \sum_{k\in\mathbb{Z}:|x-\frac{k}{w}|\geq\delta} |wx-k|^{n+1} |\chi(wx-k)| \leq \frac{2\|f^{(n+1)}\|_\infty}{(n+1)!}\epsilon. \tag{34}$$

Summarizing, by (32)-(34), we find that

$$\left| w^{n+1}((G_{n,w}f)(x) - f(x)) - \frac{f^{(n+1)}(x)}{(n+1)!}A \right| \leq \frac{M_{n+1} + 2\|f^{(n+1)}\|_\infty}{(n+1)!}\epsilon$$

which proves the statement.

An equivalent condition of constant moment can be given employing the Poisson's summation formula (see the proof of [20, Lemma 3]): if the function $g(u) = u^{n+1}\chi(u)$ belongs to $L^1(\mathbb{R})$, then (31) holds if and only if

$$\widehat{\chi}^{(n+1)}(2\pi k) = \begin{cases} (-i)^{n+1}A & k = 0, \\ 0 & k \in \mathbb{Z}\backslash\{0\}. \end{cases}$$

A direct consequence of Theorem 9 is a saturation result. Actually, we prove it relaxing the condition of constant moment. We recall that, if $f \in C^n(\mathbb{R})$, then $\|G_{n,w}f - f\|_\infty = o(w^{-n})$ for $w \to +\infty$ (Theorem 8).

Corollary 10. *Let χ be a kernel of order $n+1$ such that $\inf_{u\in\mathbb{R}} |m_{n+1}(u,\chi)| > 0$. Moreover, let $w > 0$ and $f \in C_b^{n+1}(\mathbb{R})$. If $\|G_{n,w}f - f\|_\infty = o(w^{-(n+1)})$ as $w \to +\infty$, then f is constant.*

Proof. By (28), for any $x \in \mathbb{R}$ we have the following equality analogous to (32)

$$w^{n+1}((G_{n,w}f)(x) - f(x)) - \frac{f^{(n+1)}(x)}{(n+1)!} \sum_{k\in\mathbb{Z}} (wx-k)^{n+1} \chi(wx-k) =$$
$$= \frac{1}{(n+1)!} \sum_{k\in\mathbb{Z}} \left(f^{(n+1)}\left(\theta_{x,\frac{k}{w}}\right) - f^{(n+1)}(x) \right) (wx-k)^{n+1} \chi(wx-k),$$

which can be written also as

$$\begin{aligned} \frac{f^{(n+1)}(x)}{(n+1)!} \sum_{k\in\mathbb{Z}} (wx-k)^{n+1} \chi(wx-k) = \\ &= w^{n+1}((G_{n,w}f)(x) - f(x)) \\ &\quad - \frac{1}{(n+1)!} \sum_{k\in\mathbb{Z}} \left(f^{(n+1)}\left(\theta_{x,\frac{k}{w}}\right) - f^{(n+1)}(x) \right) (wx-k)^{n+1} \chi(wx-k). \end{aligned}$$

Since $\inf_{u\in\mathbb{R}} |m_{n+1}(u,\chi)| > 0$, there exists $c > 0$ such that

$$\begin{aligned} c|f^{(n+1)}(x)| \leq &|w^{n+1}((G_{n,w}f)(x) - f(x))| + \\ &+ \frac{1}{(n+1)!} \left| \sum_{k\in\mathbb{Z}} \left(f^{(n+1)}\left(\theta_{x,\frac{k}{w}}\right) - f^{(n+1)}(x) \right) (wx-k)^{n+1} \chi(wx-k) \right|. \end{aligned} \tag{35}$$

The hypothesis $\|G_{n,w}f - f\|_\infty = o(w^{-(n+1)})$ as $w \to +\infty$ means that

$$\lim_{w\to+\infty} w^{n+1}((G_{n,w}f)(x) - f(x)) = 0, \qquad \forall x \in \mathbb{R}.$$

Furthermore, as in the proof of Theorem 9, the term on the second line in (35) goes to 0 when $w \to +\infty$. Hence, from (35) we have $f^{(n+1)}(x) = 0$ for every $x \in \mathbb{R}$, so f is a polynomial of degree less or equal to n. In particular, since f is bounded, f is necessarily constant.

Funding sources

The author was partially supported by the Università degli Studi di Palermo (FFR2024).

Declaration of competing interest

The author has no competing interests to declare that are relevant to the content of this article.

Acknowledgments

The author is a member of the "Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni" (GNAMPA-INdAM), of the "Research ITalian network on Approximation" (RITA) network and of the UMI group - "Teoria dell'Approssimazione e Applicazioni".

References

[1] F.G. Abdullayev, V.V. Savchuk, Fejér-type positive operator based on Takenaka–Malmquist system on unit circle, J. Math. Anal. Appl. 529 (2) (2024) 127298.
[2] T. Acar, D. Costarelli, G. Vinti, Linear prediction and simultaneous approximation by m-th order Kantorovich type sampling series, Banach J. Math. Anal. 14 (2020) 1481–1508.
[3] T. Acar, B.R. Draganov, A characterization of the rate of the simultaneous approximation by generalized sampling operators and their Kantorovich modification, J. Math. Anal. Appl. 530 (2024) 127740.
[4] T. Acar, M. Cappelletti Montano, P. Garrancho, V. Leonessa, Voronovskaya type results for Bernstein-Chlodovsky operators preserving e^{2x}, J. Math. Anal. Appl. 491 (11) (2020) 124307.
[5] T. Acar, S. Kursun, Pointwise convergence of generalized Kantorovich exponential sampling series, Dolomites Res. Notes Approx. 16 (1) (2023) 1–10.
[6] A.-M. Acu, A.-I. Măduţa, I. Rasa, Voronovskaya type results and operators fixing two functions, Math. Model. Anal. 26 (3) (2021) 395–410.
[7] B. Adcock, M. Gataric, A.C. Hansen, Density theorems for nonuniform sampling of bandlimited functions using derivatives or bunched measurements, J. Fourier Anal. Appl. 23 (6) (2017) 1311–1347.
[8] M. Ali Özarslan, O. Duman, Approximation theorems by Meyer-König and Zeller type operators, Chaos Solitons Fractals 41 (1) (2009) 451–456.
[9] A. Altin, O. Doğru, F. Taşdelen, The generalization of Meyer-König and Zeller operators by generating functions, J. Math. Anal. Appl. 312 (2005) 181–194.
[10] M.H. Annaby, R.M. Asharabi, Truncation, amplitude, and jitter errors on $\mathbb{R}$ for sampling series derivatives, J. Approx. Theory 163 (3) (2011) 336–362.
[11] R.M. Asharabi, J. Prestin, A modification of Hermite sampling with a Gaussian multiplier, Numer. Funct. Anal. Optim. 36 (4) (2015) 419–437.
[12] C. Bardaro, P.L. Butzer, R.L. Stens, G. Vinti, Kantorovich-type generalized sampling series in the setting of Orlicz spaces, Sampl. Theory Signal Image Process., Int. J. 6 (1) (2007) 29–52.
[13] C. Bardaro, I. Mantellini, A Voronovskaya-type theorem for a general class of discrete operators, Rocky Mt. J. Math. 39 (5) (2009) 1411–1442.
[14] B. Baxhaku, P.N. Agrawal, R. Shukla, Bivariate positive linear operators constructed by means of q-Lagrange polynomials, J. Math. Anal. Appl. 491 (2) (2020) 124337.
[15] F. Bowman, Introduction to Bessel Function, Dover Publications Inc., New York, 1958.
[16] P.L. Butzer, A. Fischer, R.L. Stens, Generalized sampling approximation of multivariate signals; general theory, in: Proc. Fouth Meeting on Real Analysis and Measure Theory, Capri, 1990, in: Atti Sem. Mat. Fis. Univ. Modena, vol. 39, 1990.
[17] P.L. Butzer, A. Fischer, R.L. Stens, Generalized sampling approximation of multivariate signals; theory and some applications, Note Mat. 10 (suppl. 1) (1990) 173–191.
[18] P.L. Butzer, R.J. Nessel, Fourier Analysis and Approximation I, Academic Press, New York-London, 1971.
[19] P.L. Butzer, W. Splettstößer, R.L. Stens, The sampling theorem and linear prediction in signal analysis, Jahresber. Dtsch. Math.-Ver. 90 (1988) 1–70.

[20] P.L. Butzer, R.L. Stens, Linear prediction by samples from the past, in: Advanced Topics in Shannon Sampling and Interpolation Theory, Springer, 1993, pp. 157–183.
[21] R. Corso, G. Gucciardi, Curves defined by a class of discrete operators: approximation result and applications, Math. Methods Appl. Sci. 48 (2) (2025) 2388–2403.
[22] D. Costarelli, Convergence and high order of approximation by Steklov sampling operators, Banach J. Math. Anal. 18 (4) (2024) 70.
[23] D. Costarelli, G. Vinti, Order of approximation for sampling Kantorovich operators, J. Integral Equ. Appl. 26 (3) (2014) 345–368.
[24] A.G. García, G. Pérez-Villalón, Approximation from shift-invariant spaces by generalized sampling formulas, Appl. Comput. Harmon. Anal. 24 (1) (2008) 58–69.
[25] M. Goyal, V. Gupta, P.N. Agrawal, Quantitative convergence results for a family of hybrid operators, Appl. Math. Comput. 271 (2015) 893–904.
[26] K. Gröchenig, J.L. Romero, J. Stöckler, Sharp results on sampling with derivatives in shift-invariant spaces and multi-window Gabor frames, Constr. Approx. 51 (2020) 1–25.
[27] V. Gupta, A.-M. Acu, D.F. Sofonea, Approximation of Baskakov type Pólya–Durrmeyer operators, Appl. Math. Comput. 294 (2017) 318–331.
[28] D.L. Jagerman, L.J. Fogel, Some general aspects of the sampling theorem, IRE Trans. Inf. Theory 2 (4) (1956) 139–146.
[29] G. Kirov, I. Popova, A generalization of linear positive operators, Math. Balk. 7 (1993) 149–162.
[30] Y. Kolomoitsev, A. Krivoshein, M. Skopina, Differential and falsified sampling expansions, J. Fourier Anal. Appl. 24 (5) (2018) 1276–1305.
[31] Y. Kolomoitsev, M. Skopina, Uniform approximation by multivariate quasi-projection operators, Anal. Math. Phys. 12 (2022) 68.
[32] V. Kotel'nikov, On the carrying capacity of the "ether" and wire in telecommunications, in: First All-Union Conference on Questions of Communications, Izd. Red. Upr. Svyazi RKKA, Moscow, Russian, 1933.
[33] D.A. Linden, N.M. Abramson, A generalization of the sampling theorem, Inf. Control 3 (1) (1960) 26–31, see also Errata: Inf. Control 4 (1) (1961) 95–96.
[34] S.A. Mohiuddine, B. Alamri, Generalization of equi-statistical convergence via weighted lacunary sequence with associated Korovkin and Voronovskaya type approximation theorems, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. 113 (3) (2019) 1955–19731.
[35] S.A. Mohiuddine, K.K. Singh, A. Alotaibi, On the order of approximation by modified summation-integral-type operators based on two parameters, Demonstr. Math. 56 (1) (2023) 20220182.
[36] M. Nasiruzzaman, N. Rao, S. Wazir, R. Kumar, Approximation on parametric extension of Baskakov–Durrmeyer operators on weighted spaces, J. Inequal. Appl. (2019) 103.
[37] A. Papoulis, Generalized sampling expansion, IEEE Trans. Circuits Syst. 24 (11) (1977) 652–654.
[38] K. Priyanka, A.A. Selvan, Derivative sampling expansions in shift-invariant spaces with error estimates covering discontinuous signals, IEEE Trans. Inf. Theory 70 (8) (2024) 5453–5470.
[39] M.D. Rawn, A stable nonuniform sampling expansion involving derivatives, IEEE Trans. Inf. Theory 35 (6) (1989) 1223–1227.
[40] S. Ries, R.L. Stens, Approximation by generalized sampling series, in: Constructive Theory of Functions '84, Sofia, 1984, pp. 746–756.
[41] M. Skopina, Band-limited scaling and wavelet expansions, Appl. Comput. Harmon. Anal. 36 (1) (2014) 143–157.
[42] C.E. Shannon, Communication in the presence of noise, Proc. IRE 37 (1949) 10–21.
[43] J. Stoer, R. Bulirsch, Introduction to Numerical Analysis, Springer-Verlag, New York, 2002.
[44] E.T. Whittaker, On the functions which are represented by the expansions of the interpolation-theory, Proc. R. Soc. Edinb. 35 (1915) 181–194.
[45] R. Yadav, V. Narayan Mishra, R. Meher, Approximation on Durrmeyer modification of generalized Szász–Mirakjan operators, Math. Methods Appl. Sci. 47 (10) (2024) 8226–8248.