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On the critical curve for systems of hyperbolic inequalities in an exterior domain of the half-space

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ABSTRACT

We establish blow-up results for a system of semilinear hyperbolic inequalities in an exterior domain of the half-space. The considered system is investigated under an inhomogeneous Dirichlet-type boundary condition depending on both time and space variables. In certain cases, an optimal criterium of Fujita-type is derived. Our results yield naturally sharp nonexistence criteria for the corresponding stationary wave system and equation.

Contents

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1. Introduction

In this paper, we consider a system of wave inequalities in an exterior domain of the half-space, under inhomogeneous Dirichlet-type boundary conditions. Let $N \geq 2$, we study the following problem

$$\begin{cases} \partial_{tt}u - \Delta u & \geq & |v|^p & \text{in} & (0,\infty)\times\Omega, \\ \partial_{tt}v - \Delta v & \geq & |u|^q & \text{in} & (0,\infty)\times\Omega, \\ (u(t,x),v(t,x)) & \succeq & (0,0) & \text{on} & (0,\infty)\times\Gamma_0, \\ (u(t,x),v(t,x)) & \succeq & (a(t)f(x),b(t)g(x)) & \text{on} & (0,\infty)\times\Gamma_1, \end{cases} \tag{1.1}$$

where $\Omega = \{x \in \overline{\mathbb{R}^N_+} : |x| \geq 1\}$, $\mathbb{R}^N_+ = \{x = (x_1, x_2, \cdots, x_N) \in \mathbb{R}^N : x_N > 0\}$, $\Gamma_0 = \{x \in \Omega : x_N = 0\}$, $\Gamma_1 = \{x \in \Omega : x_N > 0, |x| = 1\}$, $p, q > 1$, $f, g \in L^1(\Gamma_1)$, and $a(t)$, $b(t)$ are nonnegative locally integrable functions to be specified later. Here, by ν_i $(i = 0, 1)$ we will denote the outward unit normal vector on Γ_i, relative to Ω, and by $\succeq$ we mean the partial order in $\mathbb{R}^2$ given as

$$(w_1, w_2) \succeq (z_1, z_2) \Longleftrightarrow w_i \geq z_i,\ i = 1, 2.$$

Furthermore, for $w, z \in \mathbb{R}^2$ we write $w \succ z$ to indicate that $w \succeq z$ and $w \neq z$. Theoretically we are interested in establishing whether global weak solutions to problem (1.1) do not exist. Some motivations for studying problems of type (1.1) are mentioned below.

In the case of the whole space, the large-time behavior of solutions to the wave equation

$$\partial_{tt}u - \Delta u = |u|^p \quad \text{in } (0,\infty)\times\mathbb{R}^N \tag{1.2}$$

has been investigated in several works, see e.g. [4–6,12,15,21,22,24,25,28] and the references therein. For example, in [4] the authors discuss the existence of unique global solution under suitable weighted Strichartz estimates and without spherical symmetry, and [25] adds information about the solution to the so-called Strauss conjecture for (1.2), with dimension $N \geq 4$. The similar result as in [25] can be established by different method (see Zhou [28] for more details and information). In [15], the authors study an initial boundary value problem of semilinear wave equation in exterior domain in two space dimensions with critical power. Hence, they show that the solution will blow-up in a finite time. Thanks to these works, we know that for every $N \geq 2$, (1.2) admits a Fujita-type critical exponent (Strauss exponent)

$$p_S(N) = \frac{N + 1 + \sqrt{N^2 + 10N - 7}}{2(N-1)}.$$

More precisely, we note that

(i) if $1 < p \leq p_S(N)$, then for any compactly supported initial values with positive average, the solution to (1.2) blows-up in a finite time;
(ii) if $p > p_S(N)$, then the solution to (1.2) exists globally in time for suitable compactly supported initial values.

In [2], the authors investigate the system of wave equations

$$\begin{cases} \partial_{tt}u - \Delta u & = & |v|^p & \text{in} & (0,\infty)\times\mathbb{R}^N, \\ \partial_{tt}v - \Delta v & = & |u|^q & \text{in} & (0,\infty)\times\mathbb{R}^N, \end{cases} \tag{1.3}$$

where $p, q > 1$. Namely, it was shown that, if

$$\frac{N-1}{2} < \max\left\{\frac{p+2+q^{-1}}{pq-1}, \frac{q+2+p^{-1}}{pq-1}\right\},$$

then (under certain conditions on the initial values) (1.3) has no global solution. Moreover, for (p,q) belonging to a subset of the $p\&q$ plane

$$p,q>1, \ \frac{N-1}{2} > \max\left\{\frac{p+2+q^{-1}}{pq-1}, \frac{q+2+p^{-1}}{pq-1}\right\},$$

(1.3) has a global solution, provided the initial values are sufficiently small. For other works related to (1.3), see e.g. [1,3,7,14] and the references therein. For example, in [7] the authors study the impact of a framework of test functions on obtaining sharp estimates of solutions to both nonlinear wave equations and systems of nonlinear wave equations. A class of variational inequalities of Kirchhoff-type is studied in [29], where the authors establish the existence of infinite radial solutions in $\mathbb{R}^N$, by the non-smooth critical point theory based on Szulkin functionals. Before continuing the discussion of our setting, we also mention the work [26], where the authors consider a wide class of evolutionary variational-hemivariational inequalities of hyperbolic types, with the functional framework given in an evolution triple of spaces. By exploiting the Rothe approximation method, the authors establish results on existence, uniqueness, and regularity of solution to inequalities involving both a convex potential and a locally Lipschitz superpotential. Now, the study of wave inequalities in the whole space was first considered in [13] in the following form:

$$\partial_{tt}u - \Delta u \geq |u|^p \quad \text{in } (0,\infty)\times\mathbb{R}^N. \tag{1.4}$$

In [13], another critical exponent (namely, Kato exponent) was obtained in the following form

$$p_K(N) = \frac{N+1}{N-1}.$$

In [20], the authors generalize the result in [13] and point out the sharpness of $p_K(N)$. In fact, it was shown that for $N \geq 2$, we distinguish the following two cases:

(i) if $1 < p \leq p_K(N)$ and

$$\int_{\mathbb{R}^N} \partial_t u(0,x)\,dx > 0, \tag{1.5}$$

then (1.4) admits no global weak solution;

(ii) if $p > p_K(N)$, then there are positive global solutions to (1.4) satisfying (1.5).

For other contributions related to hyperbolic inequalities in the whole space, see e.g. [9,17,19] and the references therein. In the recent work [9], the authors investigate the effect of suitable gradient terms on the large-time behavior of solutions to certain classes of hyperbolic inequalities. Some nonexistence results for hyperbolic inequalities on Riemannian manifolds can be found in [10,18]. In [18], the authors obtain necessary conditions for the existence of solutions, extending previous nonexistence results for the wave operator with power nonlinearity on $\mathbb{R}^N$. In [10], test functions are used to study higher order evolution inequalities, with respect to the time variable, hence the authors need also to estimate the second derivatives of the test functions.

In [16], among other problems, the author considers the hyperbolic inequality

$$\partial_{tt}u - \Delta u \geq |u|^p \quad \text{in } (0,\infty)\times K, \tag{1.6}$$

under the Dirichlet-type boundary condition

$$u(t,x) \geq 0, \quad \text{on } (0,\infty) \times \partial K, \tag{1.7}$$

where K is the cone defined by

$$K = \{(r,\omega) : r > 0,\, \omega \in \Omega\}$$

and Ω is a domain of S^{N-1} ($N \geq 3$). It was shown that, if the condition

$$1 < p \leq 1 + \frac{2}{s^* + 1}$$

holds, where

$$s^* = \frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \lambda_1}$$

and λ_1 is the first eigenvalue of the Laplace Beltrami operator on Ω, then problem (1.6) under the boundary condition (1.7) has no nontrivial global weak solution. Notice that in the special case $K = \mathbb{R}^N_+$, one has $\lambda_1 = N - 1$ and $1 + \frac{2}{s^*+1} = 1 + \frac{2}{N}$.

Now, a natural question is to understand the wave equation or inequality on other unbounded domains of $\mathbb{R}^N$. The study of blow-up for wave equation on exterior domains was initialized in [27]. Namely, the author considers the inhomogeneous problem

$$\partial_{tt} u - \Delta u = |x|^{\alpha} |u|^p \quad \text{in } (0,\infty) \times \mathcal{D}^c, \tag{1.8}$$

under the Neumann boundary condition

$$\frac{\partial u}{\partial \nu}(t,x) = f(x) \quad \text{on } (0,\infty) \times \partial \mathcal{D}, \tag{1.9}$$

where $\mathcal{D}$ is a smooth bounded set of $\mathbb{R}^N$, $N \geq 3$, $\mathcal{D}^c$ is the complement of $\mathcal{D}$, $\alpha > -2$ and $f(x) \geq 0$. In this case, it was shown that the critical exponent is equal to $\frac{N+\alpha}{N-2}$. More precisely, it was shown that the following are the cases:

(i) if $1 < p < \frac{N+\alpha}{N-2}$ and $f \not\equiv 0$, then (1.8)-(1.9) admits no global solution;
(ii) if $p > \frac{N+\alpha}{N-2}$, (1.8)-(1.9) has global solutions for some $f > 0$.

In [8], among other results, it was shown that $p = \frac{N+\alpha}{N-2}$ belongs to the blow-up case. In [11], the authors consider the system of wave inequalities

$$\begin{cases} \partial_{tt} u - \Delta u & \geq & |x|^a |v|^p & \text{in} & (0,\infty) \times \mathcal{D}^c, \\ \partial_{tt} v - \Delta v & \geq & |x|^b |u|^q & \text{in} & (0,\infty) \times \mathcal{D}^c, \end{cases} \tag{1.10}$$

where $p, q > 1$, $(a,b) \succ (-2,-2)$ and $N \geq 2$, under three types of boundary conditions:
the Dirichlet-type condition:

$$(u(t,x), v(t,x)) \succeq (f(x), g(x)) \quad \text{on } (0,\infty) \times \partial \mathcal{D}; \tag{1.11}$$

the Neumann-type condition:

$$\left(\frac{\partial u}{\partial \nu}(t,x), \frac{\partial v}{\partial \nu}(t,x)\right) \succeq (f(x), g(x)) \quad \text{on } (0,\infty)\times \partial\mathcal{D}; \tag{1.12}$$

the mixed-type boundary condition:

$$\left(u(t,x), \frac{\partial v}{\partial \nu}(t,x)\right) \succeq (f(x), g(x)) \quad \text{on } (0,\infty)\times \partial\mathcal{D}, \tag{1.13}$$

where $f,g \in L^1(\partial\mathcal{D})$ and $\left(\int_{\partial\mathcal{D}} f\,d\sigma, \int_{\partial\mathcal{D}} g\,d\sigma\right) \succ (0,0)$. It was shown that all the above problems share the same critical behavior. Namely, we note that, if $N=2$; or $N\geq 3$ and

$$N < \max\left\{ \operatorname{sgn}\left(\int\limits_{\partial\mathcal{D}} f\,d\sigma\right) \times \frac{2p(q+1)+pb+a}{pq-1}, \operatorname{sgn}\left(\int\limits_{\partial\mathcal{D}} g\,d\sigma\right) \times \frac{2q(p+1)+qa+b}{pq-1} \right\}, \tag{1.14}$$

then we get the following conclusions:

(i) problem (1.10)-(1.11) admits no global weak solution if $f,g\geq 0$;
(ii) problem (1.10)-(1.12) admits no global weak solution;
(iii) problem (1.10)-(1.13) admits no global weak solution if $p>2$ and $f\geq 0$.

Moreover, if $\mathcal{D}$ is a ball, the sign condition for f and g can be erased in (i) and (iii). Notice that the sharpness of (1.14) was justified in [11] (see [11, Remarks 1.5-1.6)].

As far as we know, the study of the large-time behavior of evolution inequalities in an exterior domain of the half-space was not addressed in the literature. Motivated by this fact and the above mentioned works, problem (1.1) is investigated in this paper.

Before stating our obtained results, let us mention in which sense the solutions to (1.1) are considered. Just before, let

$$D=(0,\infty)\times\Omega,\ \Gamma^0=(0,\infty)\times\Gamma_0,\ \Gamma^1=(0,\infty)\times\Gamma_1.$$

We introduce the functional space

$$\Phi=\left\{\varphi\in C_c^2(D):\varphi\geq 0,\ \varphi|_{\Gamma^i}=0,\ \frac{\partial\varphi}{\partial\nu_i}|_{\Gamma^i}\leq 0,\ i=0,1\right\},$$

where $C_c^2(D)$ is the space of C^2 functions compactly supported in D. Notice that $\Gamma^i\subset D$ for all $i=0,1$.

Definition 1.1. We say that $(u,v)\in L^q_{loc}(D)\times L^p_{loc}(D)$ is a global weak solution to (1.1), if

$$\int\limits_D |v|^p\varphi\,dx\,dt - \int\limits_{\Gamma^1} a(t)\frac{\partial\varphi}{\partial\nu_1}f(x)\,d\sigma_1\,dt \leq \int\limits_D u\left(\partial_{tt}\varphi-\Delta\varphi\right)\,dx\,dt \tag{1.15}$$

and

$$\int\limits_D |u|^q\varphi\,dx\,dt - \int\limits_{\Gamma^1} b(t)\frac{\partial\varphi}{\partial\nu_1}g(x)\,d\sigma_1\,dt \leq \int\limits_D v\left(\partial_{tt}\varphi-\Delta\varphi\right)\,dx\,dt \tag{1.16}$$

for every $\varphi\in\Phi$.

For $h \in L^1(\Gamma_1)$, we introduce the integral

$$\mathcal{I}_h = \int_{\Gamma_1} x_N h(x)\, d\sigma_1.$$

Then, our main result for problem (1.1) is the following existence result.

Theorem 1.2. *Assume that $a(t) \sim t^\alpha$ and $b(t) \sim t^\beta$ near infinity, where $\alpha, \beta \in \mathbb{R}$. Let $f, g \in L^1(\Gamma_1)$ be such that $(\mathcal{I}_f, \mathcal{I}_g) \succ (0,0)$. If the following condition is satisfied*

$$N+1 < \max\left\{\operatorname{sgn}(\mathcal{I}_f)\left(\alpha + \frac{2p(q+1)}{pq-1}\right), \operatorname{sgn}(\mathcal{I}_g)\left(\beta + \frac{2q(p+1)}{pq-1}\right)\right\}, \tag{1.17}$$

then (1.1) admits no global weak solution.

Remark 1.3. Notice that the condition (1.17) is equivalent to the following assumptions

$$\mathcal{I}_f > 0 \text{ and } N+1 < \alpha + \frac{2p(q+1)}{pq-1},$$

or

$$\mathcal{I}_g > 0 \text{ and } N+1 < \beta + \frac{2q(p+1)}{pq-1}.$$

Remark 1.4. Observe that for suitable values $K_1, K_2 > 0$, we get that

$$(u,v)(t,x) = \left(K_1(t+1)^{\frac{-2(p+1)}{pq-1}}, K_2(t+1)^{\frac{-2(q+1)}{pq-1}}\right)$$

is a global solution to (1.1) with $f = g \equiv 0$. This shows the necessity of the assumption $(\mathcal{I}_f, \mathcal{I}_g) \succ (0,0)$ in Theorem 1.2.

In the special case $a = b \equiv 1$ (so $\alpha = \beta = 0$), we deduce from Theorem 1.2 the following nonexistence result.

Corollary 1.5. *Let $a = b \equiv 1$ and $f, g \in L^1(\Gamma_1)$ be such that $(\mathcal{I}_f, \mathcal{I}_g) \succ (0,0)$. If the following condition is satisfied*

$$N+1 < \frac{2}{pq-1}\max\left\{\operatorname{sgn}(\mathcal{I}_f)p(q+1), \operatorname{sgn}(\mathcal{I}_g)q(p+1)\right\}, \tag{1.18}$$

then (1.1) admits no global weak solution.

Remark 1.6. At this time, we do not know whether the condition (1.17) is sharp or not. However, in the special case $a = b \equiv 1$, our condition (1.18) is sharp. Namely, assume that

$$N+1 > \frac{2}{pq-1}\max\left\{p(q+1), q(p+1)\right\}. \tag{1.19}$$

Furthermore, let

$$(u_*, v_*)(x) = \epsilon x_N(|x|^{\delta_1}, |x|^{\delta_2}),$$

where $\delta_2 = d\delta_1$, with

$$\frac{p+1+p(q+1)}{pq-1} < \delta_1 < \max\{N, (N-1)p-1\}, \tag{1.20}$$

$$\frac{1}{p} + \frac{p+1}{\delta_1 p} < d < \max\left\{\frac{N}{\delta_1}, q - \frac{q+1}{\delta_1}\right\} \tag{1.21}$$

and

$$0 < \epsilon \leq \min\left\{(\delta_1(N-\delta_1))^{\frac{1}{p-1}}, (\delta_2(N-\delta_2))^{\frac{1}{q-1}}\right\}. \tag{1.22}$$

Then, we can check that (u_*, v_*) is a stationary solution to (1.1) for suitable $f, g \geq 0$. Notice that under the condition (1.19), the set of values δ_1 satisfying (1.20) is non-empty. Moreover, under the condition (1.20), the set of values d satisfying (1.21) is non-empty. Notice also that from (1.20) and (1.21), we have $0 < \delta_i < N$, $i = 1, 2$. Thus, the set of values ϵ satisfying (1.22) is non-empty.

If $p = q$ in Corollary 1.5, we have the following nonexistence result.

Theorem 1.7. *Let $a = b \equiv 1$, $p = q$ and $f, g \in L^1(\Gamma_1)$ be such that $(\mathcal{I}_f, \mathcal{I}_g) \succ (0,0)$. If the following condition is satisfied*

$$N + 1 = \frac{2p}{p-1}, \tag{1.23}$$

then (1.1) admits no global weak solution.

Clearly, Corollary 1.5 and Theorem 1.7 yield nonexistence results for the corresponding stationary problem

$$\begin{cases} -\Delta u & \geq & |v|^p & \text{in} & \Omega, \\ -\Delta v & \geq & |u|^q & \text{in} & \Omega, \\ (u(x), v(x)) & \succeq & (0,0) & \text{on} & \Gamma_0, \\ (u(x), v(x)) & \succeq & (f(x), g(x)) & \text{on} & \Gamma_1. \end{cases} \tag{1.24}$$

We state this result in the form of the following corollary.

Corollary 1.8. *Let $f, g \in L^1(\Gamma_1)$ be such that $(\mathcal{I}_f, \mathcal{I}_g) \succ (0,0)$. If one of the following conditions is satisfied:*

(i) (1.18) *holds;*
(ii) $p = q$ *and* (1.23) *holds,*

then (1.24) admits no weak solution.

Remark 1.9. Consider the case of a single inequality

$$\begin{cases} \partial_{tt} u - \Delta u & \geq & |u|^p & \text{in} & (0,\infty) \times \Omega, \\ u(t,x) & \geq & 0 & \text{on} & (0,\infty) \times \Gamma_0, \\ u(t,x) & \geq & f(x) & \text{on} & (0,\infty) \times \Gamma_1. \end{cases} \tag{1.25}$$

By Corollary 1.5 and Theorem 1.7, we deduce that, if $\mathcal{I}_f > 0$ and

$$1 < p \leq \frac{N+1}{N-1},$$

then (1.25) admits no global weak solution. Moreover, by Remark 1.6, we deduce that $\frac{N+1}{N-1}$ is the critical exponent (in the sense of Fujita) for problem (1.25). The same result holds for the corresponding stationary problem

$$\begin{cases} -\Delta u & \geq & |u|^p & \text{in} & \Omega, \\ u(x) & \geq & 0 & \text{on} & \Gamma_0, \\ u(x) & \geq & f(x) & \text{on} & \Gamma_1. \end{cases}$$

It is interesting to observe that $\frac{N+1}{N-1}$ is exactly the Kato critical exponent for (1.4).

Finally, we mention a related work of nonlocal nature, see the paper of Straughan [23] where the author discusses a computational procedure to get the neutral curves for instability associated with thermal convective phenomena. This study shows how an investigation of critical cases and nonexistence criteria of solutions can be successfully applied to control certain physical systems in hydrodynamics. Indeed Straughan's work depicts an useful strategy to employ in the minimization process over all wave numbers.

The rest of the paper is organized as follows. In Section 2, we establish some estimates that will play a crucial role in the proof of our main results. Section 3 is devoted to the proof of Theorems 1.2 and 1.7. Finally, some open questions are raised in Section 4.

2. Preliminaries

Throughout this paper, the symbol C denotes always a generic positive constant, which is independent of the scaling parameter T and the solutions u, v. Its value could be changed from one line to another. First we derive two useful a priori estimates of integral type, then we introduce some appropriate test functions to obtain other auxiliary estimates.

2.1. A priori estimates

For $m > 1$ and $\varphi \in \Phi$, let

$$I_m(\varphi) = \int_D \varphi^{\frac{-1}{m-1}} |\partial_{tt}\varphi|^{\frac{m}{m-1}} \, dx \, dt \tag{2.1}$$

and

$$J_m(\varphi) = \int_D \varphi^{\frac{-1}{m-1}} |\Delta\varphi|^{\frac{m}{m-1}} \, dx \, dt. \tag{2.2}$$

The following a priori estimates for problem (1.1) will play a crucial role in the proof of Theorems 1.2 and 1.7.

Lemma 2.1. *Let $(u, v) \in L^q_{loc}(D) \times L^p_{loc}(D)$ be a global weak solution to (1.1). Assume that there exists $\varphi \in \Phi$ such that*

$$\int_{\Gamma^1} a(t) \frac{\partial \varphi}{\partial \nu_1} f(x) \, d\sigma_1 \, dt \leq 0. \tag{2.3}$$

Then, we have

$$-\int_{\Gamma^1} b(t)\frac{\partial\varphi}{\partial\nu_1}g(x)\,d\sigma_1\,dt \leq C\left(I_q(\varphi)^{\frac{q-1}{q}}+J_q(\varphi)^{\frac{q-1}{q}}\right)^{\frac{q}{pq-1}}\left(I_p(\varphi)^{\frac{p-1}{p}}+J_p(\varphi)^{\frac{p-1}{p}}\right)^{\frac{pq}{pq-1}}, \tag{2.4}$$

provided that $I_m(\varphi)<\infty$ *and* $J_m(\varphi)<\infty$, $m\in\{p,q\}$.

Proof. Let $(u,v)\in L^q_{loc}(D)\times L^p_{loc}(D)$ be a global weak solution to (1.1). Let $\varphi\in\Phi$ be such that (2.3) holds. Using (1.15) and (2.3), we obtain

$$\int_D |v|^p\varphi\,dx\,dt \leq \int_D |u||\partial_{tt}\varphi|\,dx\,dt + \int_D |u||\Delta\varphi|\,dx\,dt. \tag{2.5}$$

On the other hand, by means of Hölder's inequality, we get

$$\int_D |u||\partial_{tt}\varphi|\,dx\,dt \leq \left(\int_D |u|^q\varphi\,dx\,dt\right)^{\frac{1}{q}} I_q(\varphi)^{\frac{q-1}{q}} \tag{2.6}$$

and

$$\int_D |u||\Delta\varphi|\,dx\,dt \leq \left(\int_D |u|^q\varphi\,dx\,dt\right)^{\frac{1}{q}} J_q(\varphi)^{\frac{q-1}{q}}. \tag{2.7}$$

In view of the inequalities (2.5), (2.6) and (2.7), we obtain that

$$\int_D |v|^p\varphi\,dx\,dt \leq \left(\int_D |u|^q\varphi\,dx\,dt\right)^{\frac{1}{q}}\left(I_q(\varphi)^{\frac{q-1}{q}}+J_q(\varphi)^{\frac{q-1}{q}}\right). \tag{2.8}$$

Similarly, by (1.16) and using Hölder's inequality, we deduce that

$$\int_D |u|^q\varphi\,dx\,dt - \int_{\Gamma^1} b(t)\frac{\partial\varphi}{\partial\nu_1}g(x)\,d\sigma_1\,dt \leq \left(\int_D |v|^p\varphi\,dx\,dt\right)^{\frac{1}{p}}\left(I_p(\varphi)^{\frac{p-1}{p}}+J_p(\varphi)^{\frac{p-1}{p}}\right). \tag{2.9}$$

Combining (2.8) with (2.9), we get the following inequality

$$\begin{aligned}&\int_D |u|^q\varphi\,dx\,dt - \int_{\Gamma^1} b(t)\frac{\partial\varphi}{\partial\nu_1}g(x)\,d\sigma_1\,dt\\ &\leq \left(\int_D |u|^q\varphi\,dx\,dt\right)^{\frac{1}{pq}}\left(I_q(\varphi)^{\frac{q-1}{q}}+J_q(\varphi)^{\frac{q-1}{q}}\right)^{\frac{1}{p}}\left(I_p(\varphi)^{\frac{p-1}{p}}+J_p(\varphi)^{\frac{p-1}{p}}\right).\end{aligned} \tag{2.10}$$

On the other hand, by means of ε-Young inequality with $0<\varepsilon<1$, we have

$$\begin{aligned}&\left(\int_D |u|^q\varphi\,dx\,dt\right)^{\frac{1}{pq}}\left(I_q(\varphi)^{\frac{q-1}{q}}+J_q(\varphi)^{\frac{q-1}{q}}\right)^{\frac{1}{p}}\left(I_p(\varphi)^{\frac{p-1}{p}}+J_p(\varphi)^{\frac{p-1}{p}}\right)\\&\le \varepsilon\int_D |u|^q\varphi\,dx\,dt+C\left(I_q(\varphi)^{\frac{q-1}{q}}+J_q(\varphi)^{\frac{q-1}{q}}\right)^{\frac{q}{pq-1}}\left(I_p(\varphi)^{\frac{p-1}{p}}+J_p(\varphi)^{\frac{p-1}{p}}\right)^{\frac{pq}{pq-1}}.\end{aligned}\tag{2.11}$$

Thus, it follows from (2.10) and (2.11) that

$$\begin{aligned}&(1-\varepsilon)\int_D |u|^q\varphi\,dx\,dt-\int_{\Gamma^1} b(t)\frac{\partial\varphi}{\partial\nu_1}g(x)\,d\sigma_1\,dt\\&\le C\left(I_q(\varphi)^{\frac{q-1}{q}}+J_q(\varphi)^{\frac{q-1}{q}}\right)^{\frac{q}{pq-1}}\left(I_p(\varphi)^{\frac{p-1}{p}}+J_p(\varphi)^{\frac{p-1}{p}}\right)^{\frac{pq}{pq-1}}.\end{aligned}$$

Since $0<\varepsilon<1$, we conclude that (2.4) holds true. □

Proceeding as in the proof of Lemma 2.1, we obtain the following a priori estimate.

Lemma 2.2. *Let $(u,v)\in L^q_{loc}(D)\times L^p_{loc}(D)$ be a global weak solution to (1.1). Assume that there exists $\varphi\in\Phi$ such that*

$$\int_{\Gamma^1} b(t)\frac{\partial\varphi}{\partial\nu_1}g(x)\,d\sigma_1\,dt\le 0.$$

Then, we get

$$-\int_{\Gamma^1} a(t)\frac{\partial\varphi}{\partial\nu_1}f(x)\,d\sigma_1\,dt\le C\left(I_p(\varphi)^{\frac{p-1}{p}}+J_p(\varphi)^{\frac{p-1}{p}}\right)^{\frac{p}{pq-1}}\left(I_q(\varphi)^{\frac{q-1}{q}}+J_q(\varphi)^{\frac{q-1}{q}}\right)^{\frac{pq}{pq-1}},$$

provided that $I_m(\varphi)<\infty$ and $J_m(\varphi)<\infty$, $m\in\{p,q\}$.

2.2. Test functions and some estimates

We introduce the function

$$H(x)=x_N\left(1-|x|^{-N}\right),\quad x=(x_1,x_2,\cdots,x_N)\in\Omega.\tag{2.12}$$

Now, it can be easily seen that $H\ge 0$ and it satisfies the following

$$\begin{cases}-\Delta H &= 0 \text{ in } \Omega,\\ H &= 0 \text{ on } \Gamma_0\cup\Gamma_1.\end{cases}\tag{2.13}$$

We need also two cut-off functions. So, let $\xi,\eta\in C^\infty(\mathbb{R})$ be such that

$$0\le\xi\le 1,\quad \xi(s)=1 \text{ if } |s|\le 1,\quad \xi(s)=0 \text{ if } |s|\ge 2.\tag{2.14}$$

and

$$\eta\ge 0,\quad \mathrm{supp}(\eta)\subset\subset(0,1).\tag{2.15}$$

For $T > 0$ and sufficiently large ℓ, we introduce the functions

$$\rho(t) = \eta\left(\frac{t}{T^\theta}\right)^\ell, \quad t > 0, \tag{2.16}$$

$$\mu(x) = H(x)\xi\left(\frac{|x|^2}{T^2}\right)^\ell, \quad x \in \Omega,$$

and

$$\varphi(t,x) = \rho(t)\mu(x), \quad (t,x) \in D. \tag{2.17}$$

Here, $\theta > 0$ is a constant to be chosen later.

Lemma 2.3. *For sufficiently large T and ℓ, the function φ defined by (2.17), belongs to Φ.*

Proof. It is clear that $\varphi \geq 0$ and for sufficiently large ℓ, we have $\varphi \in C_c^2(D)$. Moreover, by (2.13), we have $\varphi|_{\Gamma^i} = 0$ for all $i = 0, 1$. Hence, we need just to show that

$$\left.\frac{\partial \varphi}{\partial \nu_i}\right|_{\Gamma^i} \leq 0, \quad i = 0, 1. \tag{2.18}$$

On the other hand, we have

$$\begin{aligned}
\nabla\mu(x) &= \nabla\left(H(x)\xi\left(\frac{|x|^2}{T^2}\right)^\ell\right)\\
&= \xi\left(\frac{|x|^2}{T^2}\right)^\ell \nabla H(x) + H(x)\nabla\left[\xi\left(\frac{|x|^2}{T^2}\right)^\ell\right]\\
&= \xi\left(\frac{|x|^2}{T^2}\right)^\ell \left(\left(1-|x|^{-N}\right)e_N + Nx_N|x|^{-N-2}x\right) + H(x)\nabla\left[\xi\left(\frac{|x|^2}{T^2}\right)^\ell\right],
\end{aligned} \tag{2.19}$$

where $e_N = (0, \cdots, 0, 1) \in \mathbb{R}^N$. Then, by (2.19), for $x \in \Gamma_0$, we get

$$\nabla\mu(x) = \xi\left(\frac{|x|^2}{T^2}\right)^\ell \left(1-|x|^{-N}\right)e_N,$$

which yields

$$\frac{\partial \mu}{\partial \nu_0}(x) = -\xi\left(\frac{|x|^2}{T^2}\right)^\ell \left(1-|x|^{-N}\right) \leq 0.$$

Thus, by (2.15) and (2.17), we obtain

$$\frac{\partial \varphi}{\partial \nu_0}(t,x) = \rho(t)\frac{\partial \mu}{\partial \nu_0}(x) \leq 0, \quad (t,x) \in \Gamma^0.$$

Again, by (2.19), for $x \in \Gamma_1$ we get

$$\nabla\mu(x) = Nx_N\xi\left(\frac{1}{T^2}\right)^\ell x.$$

On the other hand, by (2.14), for sufficiently large T, we deduce that

$$\xi\left(\frac{1}{T^2}\right)=1.$$

Then, we note that for sufficiently large T, the following is the case

$$\nabla\mu(x)=Nx_N x,$$

which yields

$$\frac{\partial\varphi}{\partial\nu_1}(t,x)=-Nx_N\rho(t)\leq 0,\quad (t,x)\in\Gamma_1. \tag{2.20}$$

Thus (2.18) is proved. □

Lemma 2.4. *Let $a(t)\sim t^\alpha$ near infinity, $\alpha\in\mathbb{R}$ and $\mathcal{I}_f\geq 0$. Then, for sufficiently large T, the following inequality holds*

$$-\int_{\Gamma^1} a(t)\frac{\partial\varphi}{\partial\nu_1}f(x)\,d\sigma_1\,dt\geq CT^{\theta(\alpha+1)}\mathcal{I}_f. \tag{2.21}$$

Proof. In view of (2.20), we obtain

$$\begin{aligned}-\int_{\Gamma^1} a(t)\frac{\partial\varphi}{\partial\nu_1}f(x)\,d\sigma_1\,dt&=N\int_{\Gamma^1} a(t)\rho(t)x_N f(x)\,d\sigma_1\,dt\\&=N\left(\int_0^\infty a(t)\eta\left(\frac{t}{T^\theta}\right)^\ell dt\right)\mathcal{I}_f.\end{aligned} \tag{2.22}$$

On the other hand, by (2.15), for sufficiently large T, we have (notice that $a(t)\geq 0$)

$$\begin{aligned}\int_0^\infty a(t)\eta\left(\frac{t}{T^\theta}\right)^\ell dt&=\int_0^{T^\theta} a(t)\eta\left(\frac{t}{T^\theta}\right)^\ell dt\\&\geq\int_{\frac{T^\theta}{2}}^{T^\theta} a(t)\eta\left(\frac{t}{T^\theta}\right)^\ell dt\\&\geq C\int_{\frac{T^\theta}{2}}^{T^\theta} t^\alpha\eta\left(\frac{t}{T^\theta}\right)^\ell dt\\&=CT^{\theta(\alpha+1)}\int_{\frac{1}{2}}^{1} s^\alpha\eta(s)^\ell\,ds,\end{aligned}$$

and hence we conclude that

$$\int_0^\infty a(t)\eta\left(\frac{t}{T^\theta}\right)^\ell dt \geq CT^{\theta(\alpha+1)}. \tag{2.23}$$

Combining (2.22) with (2.23), we obtain the inequality (2.21). □

Following the proof of Lemma 2.4, we deduce the following estimate.

Lemma 2.5. *Let $b(t) \sim t^\beta$ near infinity, $\beta \in \mathbb{R}$ and $\mathcal{I}_g \geq 0$. Then, for sufficiently large T, the following inequality holds*

$$-\int_{\Gamma^1} b(t)\frac{\partial\varphi}{\partial\nu_1}g(x)\,d\sigma_1\,dt \geq CT^{\theta(\beta+1)}\mathcal{I}_g.$$

We give next result with complete proof.

Lemma 2.6. *Let $m > 1$. For sufficiently large T and ℓ, the following inequality holds*

$$I_m(\varphi) \leq CT^{N+1-\theta\left(\frac{m+1}{m-1}\right)}. \tag{2.24}$$

Proof. By (2.1) and (2.17), we have

$$I_m(\varphi) = \left(\int_\Omega \mu(x)\,dx\right)\int_0^\infty \rho(t)^{\frac{-1}{m-1}}|\rho''(t)|^{\frac{m}{m-1}}\,dt. \tag{2.25}$$

On the other hand, we have

$$\begin{aligned}\int_\Omega \mu(x)\,dx &= \int_\Omega H(x)\xi\left(\frac{|x|^2}{T^2}\right)^\ell dx\\ &= \int_\Omega x_N\left(1-|x|^{-N}\right)\xi\left(\frac{|x|^2}{T^2}\right)^\ell dx.\end{aligned}$$

Using (2.14), for sufficiently large T, we obtain the following chain of inequalities

$$\begin{aligned}\int_\Omega \mu(x)\,dx &= \int_{1<|x|<\sqrt{2}T, x_N>0} x_N\left(1-|x|^{-N}\right)\xi\left(\frac{|x|^2}{T^2}\right)^\ell dx\\ &\leq \int_{1<|x|<\sqrt{2}T, x_N>0} x_N\,dx\\ &\leq \int_{1<|x|<\sqrt{2}T, x_N>0} |x|\,dx\\ &\leq \int_{1<|x|<\sqrt{2}T} |x|\,dx\\ &\leq CT^{N+1}.\end{aligned} \tag{2.26}$$

Moreover, by (2.15), for sufficiently large ℓ, we have

$$\begin{aligned}\int_0^\infty \rho(t)^{\frac{-1}{m-1}}|\rho''(t)|^{\frac{m}{m-1}}\,dt &= \int_0^{T^\theta} \eta\left(\frac{t}{T^\theta}\right)^{\frac{-\ell}{m-1}}\left|\frac{d^2}{dt^2}\eta\left(\frac{t}{T^\theta}\right)^\ell\right|^{\frac{m}{m-1}}\\ &\le CT^{\frac{-2\theta m}{m-1}}\int_0^{T^\theta}\eta\left(\frac{t}{T^\theta}\right)^{\ell-\frac{2m}{m-1}}\,dt\\ &= CT^{\frac{-2\theta m}{m-1}+\theta}\int_0^1 \eta(s)^{\ell-\frac{km}{m-1}}\,ds,\end{aligned}$$

that is,

$$\int_0^\infty \rho(t)^{\frac{-1}{m-1}}|\rho''(t)|^{\frac{m}{m-1}}\,dt \le CT^{-\theta\left(\frac{m+1}{m-1}\right)}. \tag{2.27}$$

Hence, (2.24) follows from (2.25), (2.26) and (2.27). □

Now, we provide an estimate for $J_m(\varphi)$.

Lemma 2.7. *Let $m > 1$. For sufficiently large T and ℓ, the following inequality holds*

$$J_m(\varphi) \le CT^{N+1-\frac{2m}{m-1}+\theta}. \tag{2.28}$$

Proof. By (2.2) and (2.17), we have

$$J_m(\varphi) = \left(\int_0^\infty \rho(t)\,dt\right)\int_\Omega \mu(x)^{\frac{-1}{m-1}}|\Delta\mu(x)|^{\frac{m}{m-1}}\,dx. \tag{2.29}$$

On the other hand, by (2.15), we have

$$\begin{aligned}\int_0^\infty \rho(t)\,dt &= \int_0^\infty \eta\left(\frac{t}{T^\theta}\right)^\ell\,dt\\ &= \int_0^{T^\theta}\eta\left(\frac{t}{T^\theta}\right)^\ell\,dt\\ &= T^\theta\int_0^1 \eta(s)^\ell\,ds,\end{aligned}$$

that gives us

$$\int_0^\infty \rho(t)\,dt = CT^\theta. \tag{2.30}$$

Moreover, using (2.13), for $x \in \Omega$ we obtain

$$
\begin{aligned}
\Delta\mu(x) &= \Delta\left(H(x)\xi\left(\frac{|x|^2}{T^2}\right)^{\ell}\right)\\
&= \xi\left(\frac{|x|^2}{T^2}\right)^{\ell}\Delta H(x) + H(x)\Delta\left[\xi\left(\frac{|x|^2}{T^2}\right)^{\ell}\right] + 2\nabla\left[\xi\left(\frac{|x|^2}{T^2}\right)^{\ell}\right]\cdot\nabla H(x)\\
&= H(x)\Delta\left[\xi\left(\frac{|x|^2}{T^2}\right)^{\ell}\right] + 2\nabla\left[\xi\left(\frac{|x|^2}{T^2}\right)^{\ell}\right]\cdot\nabla H(x),
\end{aligned}
\tag{2.31}
$$

where "$\cdot$" denotes the inner product in $\mathbb{R}^N$. On the other hand, by (2.14), for $x \in \Omega$ with $T < |x| < \sqrt{2}T$, we have

$$
\left|H(x)\Delta\left[\xi\left(\frac{|x|^2}{T^2}\right)^{\ell}\right]\right| \le CT^{-2}H(x)\xi\left(\frac{|x|^2}{T^2}\right)^{\ell-2} \le CT^{-2}\xi\left(\frac{|x|^2}{T^2}\right)^{\ell-2} x_N
\tag{2.32}
$$

and

$$
\begin{aligned}
\nabla\left[\xi\left(\frac{|x|^2}{T^2}\right)^{\ell}\right]\cdot\nabla H(x) &= 2\ell T^{-2}\xi\left(\frac{|x|^2}{T^2}\right)^{\ell-1}\xi'\left(\frac{|x|^2}{T^2}\right) x\cdot\left(\left(1-|x|^{-N}\right)e_N + Nx_N|x|^{-N-2}x\right)\\
&= 2\ell T^{-2}\xi\left(\frac{|x|^2}{T^2}\right)^{\ell-1}\xi'\left(\frac{|x|^2}{T^2}\right)\left(\left(1-|x|^{-N}\right)x_N + Nx_N|x|^{-N}\right)\\
&= 2\ell T^{-2}\xi\left(\frac{|x|^2}{T^2}\right)^{\ell-1}\xi'\left(\frac{|x|^2}{T^2}\right) x_N\left(1+(N-1)|x|^{-N}\right),
\end{aligned}
$$

which yield

$$
\left|\nabla\left[\xi\left(\frac{|x|^2}{T^2}\right)^{\ell}\right]\cdot\nabla H(x)\right| \le CT^{-2}\xi\left(\frac{|x|^2}{T^2}\right)^{\ell-2} x_N.
\tag{2.33}
$$

Hence, by (2.14), (2.31), (2.32) and (2.33), for sufficiently large T and ℓ, we obtain

$$
\begin{aligned}
&\int_{\Omega} \mu(x)^{\frac{-1}{m-1}}|\Delta\mu(x)|^{\frac{m}{m-1}}\,dx\\
&\le CT^{\frac{-2m}{m-1}} \int_{x\in\Omega, T<|x|<\sqrt{2}T} x_N\left(1-|x|^{-N}\right)^{\frac{-1}{m-1}}\xi\left(\frac{|x|^2}{T^2}\right)^{\ell-\frac{2m}{m-1}}\,dx\\
&\le CT^{\frac{-2m}{m-1}} \int_{x\in\Omega, T<|x|<\sqrt{2}T} x_N\xi\left(\frac{|x|^2}{T^2}\right)^{\ell-\frac{2m}{m-1}}\,dx\\
&= CT^{N+1-\frac{2m}{m-1}} \int_{1<|y|<\sqrt{2}, y_N>0} y_N\xi(|y|^2)^{\ell-\frac{2m}{m-1}}\,dy,
\end{aligned}
$$

that is,

$$
\int_{\Omega} \mu(x)^{\frac{-1}{m-1}}|\Delta\mu(x)|^{\frac{m}{m-1}}\,dx \le CT^{N+1-\frac{2m}{m-1}}.
\tag{2.34}
$$

Hence, (2.28) follows from (2.29), (2.30) and (2.34). $\square$

The following lemma in some sense is a byproduct of Lemmas 2.6 and 2.7.

Lemma 2.8. *Let $m > 1$ and $\theta = 1$. For sufficiently large T and ℓ, the following inequality holds*

$$I_m(\varphi)^{\frac{m-1}{m}} + J_m(\varphi)^{\frac{m-1}{m}} \leq CT^{\left(N+2-\frac{2m}{m-1}\right)\left(\frac{m-1}{m}\right)}. \tag{2.35}$$

Proof. By (2.24) and (2.28), for sufficiently large T and ℓ, there holds

$$I_m(\varphi)^{\frac{m-1}{m}} + J_m(\varphi)^{\frac{m-1}{m}} \leq C\left(T^{\lambda_1} + T^{\lambda_2}\right),$$

where

$$\lambda_1 = \left(N + 1 + \theta\left(\frac{-m-1}{m-1}\right)\right)\left(\frac{m-1}{m}\right)$$

and

$$\lambda_2 = \left(N + 1 - \frac{2m}{m-1} + \theta\right)\left(\frac{m-1}{m}\right).$$

Observe that

$$\lambda_2 - \lambda_1 = 2(\theta - 1).$$

So, taking $\theta = 1$, we obtain

$$\lambda_1 = \lambda_2 = \left(N + 2 - \frac{2m}{m-1}\right)\left(\frac{m-1}{m}\right),$$

which yields (2.35). $\square$

Appealing to Lemma 2.8 we can obtain the following result.

Lemma 2.9. *Let $\theta = 1$. For sufficiently large T and ℓ, the following inequality holds*

$$\left(I_q(\varphi)^{\frac{q-1}{q}} + J_q(\varphi)^{\frac{q-1}{q}}\right)^{\frac{q}{pq-1}} \left(I_p(\varphi)^{\frac{p-1}{p}} + J_p(\varphi)^{\frac{p-1}{p}}\right)^{\frac{pq}{pq-1}} \leq CT^{\frac{1}{pq-1}((N+2)(pq-1)-2q(p+1))}. \tag{2.36}$$

Proof. Using Lemma 2.8 with $m = q$, we obtain

$$\left(I_q(\varphi)^{\frac{q-1}{q}} + J_q(\varphi)^{\frac{q-1}{q}}\right)^{\frac{q}{pq-1}} \leq CT^{\left(N+2-\frac{2q}{q-1}\right)\left(\frac{q-1}{pq-1}\right)}. \tag{2.37}$$

Similarly, using Lemma 2.8 with $m = p$, we obtain

$$\left(I_p(\varphi)^{\frac{p-1}{p}} + J_p(\varphi)^{\frac{p-1}{p}}\right)^{\frac{pq}{pq-1}} \leq CT^{\left(N+2-\frac{2p}{p-1}\right)\left(\frac{(p-1)q}{pq-1}\right)}. \tag{2.38}$$

Hence, (2.36) follows from (2.37) and (2.38). $\square$

Similarly, using again Lemma 2.8, we get the following estimate.

Lemma 2.10. *Let $\theta = 1$. For sufficiently large T and ℓ, the following inequality holds*

$$\left(I_p(\varphi)^{\frac{p-1}{p}} + J_p(\varphi)^{\frac{p-1}{p}}\right)^{\frac{p}{pq-1}} \left(I_q(\varphi)^{\frac{q-1}{q}} + J_q(\varphi)^{\frac{q-1}{q}}\right)^{\frac{pq}{pq-1}} \leq CT^{\frac{1}{pq-1}((N+2)(pq-1)-2p(q+1))}.$$

For the study of the critical case, we need to introduce another cut-off function. So, let $\Lambda : \mathbb{R} \to [0,1]$ be a smooth function satisfying the conditions:

$$\Lambda(s) = 1 \text{ if } s \leq 0, \quad \Lambda(s) = 0 \text{ if } s \geq 1. \tag{2.39}$$

For $T > 0$ and sufficiently large ℓ, we consider the function

$$\mu_*(x) = H(x)\Lambda\left(\frac{\ln\left(\frac{|x|}{\sqrt{T}}\right)}{\ln(\sqrt{T})}\right)^{\ell}, \quad x \in \Omega,$$

where H is the function defined by (2.12). Hence, we introduce the new test function

$$\varphi_*(t,x) = \rho(t)\mu_*(x), \quad (t,x) \in D, \tag{2.40}$$

where ρ is the function defined by (2.16).

Lemma 2.11. *For sufficiently large T and ℓ, the function φ_* defined by (2.40) belongs to Φ.*

Proof. We need just to show that

$$\left.\frac{\partial \varphi_*}{\partial \nu_i}\right|_{\Gamma^i} \leq 0, \quad i = 0, 1. \tag{2.41}$$

We have the following calculations

$$\begin{aligned}
\nabla\mu_*(x) &= \nabla\left(H(x)\Lambda\left(\frac{\ln\left(\frac{|x|}{\sqrt{T}}\right)}{\ln(\sqrt{T})}\right)^{\ell}\right) \\
&= \Lambda\left(\frac{\ln\left(\frac{|x|}{\sqrt{T}}\right)}{\ln(\sqrt{T})}\right)^{\ell} \nabla H(x) + H(x)\nabla\left(\Lambda\left(\frac{\ln\left(\frac{|x|}{\sqrt{T}}\right)}{\ln(\sqrt{T})}\right)^{\ell}\right) \\
&= \Lambda\left(\frac{\ln\left(\frac{|x|}{\sqrt{T}}\right)}{\ln(\sqrt{T})}\right)^{\ell} \left(\left(1-|x|^{-N}\right)e_N + Nx_N|x|^{-N-2}x\right) \\
&\quad + H(x)\nabla\left(\Lambda\left(\frac{\ln\left(\frac{|x|}{\sqrt{T}}\right)}{\ln(\sqrt{T})}\right)^{\ell}\right).
\end{aligned} \tag{2.42}$$

Then, by (2.42), for $x \in \Gamma_0$ we get

$$\nabla\mu_*(x) = \Lambda\left(\frac{\ln\left(\frac{|x|}{\sqrt{T}}\right)}{\ln(\sqrt{T})}\right)^{\ell} \left(1-|x|^{-N}\right)e_N,$$

which yields

$$\frac{\partial \mu_*}{\partial \nu_0}(x) = -\Lambda\left(\frac{\ln\left(\frac{|x|}{\sqrt{T}}\right)}{\ln(\sqrt{T})}\right)^{\ell}\left(1-|x|^{-N}\right) \leq 0.$$

Thus, by (2.15) and (2.40), we deduce that

$$\frac{\partial \varphi_*}{\partial \nu_0}(t,x) = \rho(t)\frac{\partial \mu_*}{\partial \nu_0}(x) \leq 0, \quad (t,x) \in \Gamma^0.$$

Again, by (2.42), for $x \in \Gamma_1$ we get

$$\nabla \mu_*(x) = N x_N \Lambda\left(\frac{\ln\left(\frac{1}{\sqrt{T}}\right)}{\ln(\sqrt{T})}\right)^{\ell} x.$$

On the other hand, by (2.39), for sufficiently large T, we conclude that

$$\Lambda\left(\frac{\ln\left(\frac{1}{\sqrt{T}}\right)}{\ln(\sqrt{T})}\right) = 1.$$

Then, for sufficiently large T, we deduce that

$$\nabla \mu_*(x) = N x_N x,$$

which yields

$$\frac{\partial \varphi_*}{\partial \nu_1}(t,x) = -N x_N \rho(t) \leq 0, \quad (t,x) \in \Gamma_1. \tag{2.43}$$

Thus the conditions in (2.41) are proved. □

Using (2.43) and following the proof of Lemma 2.4, we obtain the following estimates.

Lemma 2.12. *Assume that $\mathcal{I}_f \geq 0$. Then, for sufficiently large T, the following inequality holds*

$$-\int_{\Gamma^1} \frac{\partial \varphi_*}{\partial \nu_1} f(x)\, d\sigma_1\, dt \geq C T^{\theta} \mathcal{I}_f.$$

Lemma 2.13. *Assume that $\mathcal{I}_g \geq 0$. Then, for sufficiently large T, the following inequality holds*

$$-\int_{\Gamma^1} \frac{\partial \varphi_*}{\partial \nu_1} g(x)\, d\sigma_1\, dt \geq C T^{\theta} \mathcal{I}_g.$$

For the next result, we provide complete proof.

Lemma 2.14. *Let $m > 1$. For sufficiently large T and ℓ, the following inequality holds*

$$I_m(\varphi_*) \leq C T^{N+1-\theta\left(\frac{m+1}{m-1}\right)}. \tag{2.44}$$

Proof. By (2.1) and (2.40), we have

$$I_m(\varphi_*) = \left(\int_\Omega \mu_*(x)\,dx\right)\int_0^\infty \rho(t)^{\frac{-1}{m-1}}|\rho''(t)|^{\frac{m}{m-1}}\,dt. \tag{2.45}$$

On the other hand, we have

$$\begin{aligned}\int_\Omega \mu_*(x)\,dx &= \int_\Omega H(x)\Lambda\left(\frac{\ln\left(\frac{|x|}{\sqrt{T}}\right)}{\ln(\sqrt{T})}\right)^\ell dx\\ &= \int_\Omega x_N\left(1-|x|^{-N}\right)\Lambda\left(\frac{\ln\left(\frac{|x|}{\sqrt{T}}\right)}{\ln(\sqrt{T})}\right)^\ell dx.\end{aligned}$$

Using (2.39) and the fact that $0 \le \Lambda \le 1$, for sufficiently large T, we obtain the following chain of inequalities

$$\begin{aligned}\int_\Omega \mu_*(x)\,dx &= \int_{1<|x|<T,x_N>0} x_N\left(1-|x|^{-N}\right)\Lambda\left(\frac{\ln\left(\frac{|x|}{\sqrt{T}}\right)}{\ln(\sqrt{T})}\right)^\ell dx\\ &\le \int_{1<|x|<T,x_N>0} x_N\Lambda\left(\frac{\ln\left(\frac{|x|}{\sqrt{T}}\right)}{\ln(\sqrt{T})}\right)^\ell dx\\ &\le \int_{1<|x|<T,x_N>0} x_N\,dx\\ &\le CT^{N+1}.\end{aligned} \tag{2.46}$$

Hence, inequality (2.44) follows from (2.45), (2.46) and (2.27). □

Lemma 2.15. *Let $m = \frac{N+1}{N-1}$. For sufficiently large T and ℓ, the following holds*

$$J_m(\varphi_*) \le CT^\theta(\ln T)^{\frac{-2}{N-1}}. \tag{2.47}$$

Proof. By (2.2) and (2.40), we have

$$J_m(\varphi) = \left(\int_0^\infty \rho(t)\,dt\right)\int_\Omega \mu_*(x)^{\frac{-1}{m-1}}|\Delta\mu_*(x)|^{\frac{m}{m-1}}\,dx. \tag{2.48}$$

Moreover, using (2.13), for $x \in \Omega$, we obtain

$$\begin{aligned}\Delta\mu_*(x) &= \Delta\left(H(x)\Lambda\left(\frac{\ln\left(\frac{|x|}{\sqrt{T}}\right)}{\ln(\sqrt{T})}\right)^\ell\right)\\ &= \Lambda\left(\frac{\ln\left(\frac{|x|}{\sqrt{T}}\right)}{\ln(\sqrt{T})}\right)^\ell \Delta H(x) + H(x)\Delta\left[\Lambda\left(\frac{\ln\left(\frac{|x|}{\sqrt{T}}\right)}{\ln(\sqrt{T})}\right)^\ell\right]\end{aligned}$$

$$+2\nabla\left[\Lambda\left(\frac{\ln\left(\frac{|x|}{\sqrt{T}}\right)}{\ln(\sqrt{T})}\right)^{\ell}\right]\cdot\nabla H(x)$$

$$= H(x)\Delta\left[\Lambda\left(\frac{\ln\left(\frac{|x|}{\sqrt{T}}\right)}{\ln(\sqrt{T})}\right)^{\ell}\right] + 2\nabla\left[\Lambda\left(\frac{\ln\left(\frac{|x|}{\sqrt{T}}\right)}{\ln(\sqrt{T})}\right)^{\ell}\right]\cdot\nabla H(x). \tag{2.49}$$

On the other hand, by (2.39), for $x \in \Omega$ with $\sqrt{T} < |x| < T$, we have

$$\left|H(x)\Delta\left[\Lambda\left(\frac{\ln\left(\frac{|x|}{\sqrt{T}}\right)}{\ln(\sqrt{T})}\right)^{\ell}\right]\right| \le C(\ln T)^{-1}|x|^{-2}\Lambda\left(\frac{\ln\left(\frac{|x|}{\sqrt{T}}\right)}{\ln(\sqrt{T})}\right)^{\ell-2} x_N \tag{2.50}$$

and

$$\nabla\left[\Lambda\left(\frac{\ln\left(\frac{|x|}{\sqrt{T}}\right)}{\ln(\sqrt{T})}\right)^{\ell}\right]\cdot\nabla H(x)$$

$$= \frac{\ell}{|x|^2\ln\sqrt{T}}\Lambda\left(\frac{\ln\left(\frac{|x|}{\sqrt{T}}\right)}{\ln(\sqrt{T})}\right)^{\ell-1}\Lambda'\left(\frac{\ln\left(\frac{|x|}{\sqrt{T}}\right)}{\ln(\sqrt{T})}\right) x\cdot\left(\left(1-|x|^{-N}\right)e_N + N x_N|x|^{-N-2}x\right)$$

$$= \frac{\ell}{\ln\sqrt{T}}\Lambda\left(\frac{\ln\left(\frac{|x|}{\sqrt{T}}\right)}{\ln(\sqrt{T})}\right)^{\ell-1}\Lambda'\left(\frac{\ln\left(\frac{|x|}{\sqrt{T}}\right)}{\ln(\sqrt{T})}\right)|x|^{-2}\left(\left(1-|x|^{-N}\right)x_N + N x_N|x|^{-N}\right)$$

$$= \frac{\ell}{\ln\sqrt{T}}\Lambda\left(\frac{\ln\left(\frac{|x|}{\sqrt{T}}\right)}{\ln(\sqrt{T})}\right)^{\ell-1}\Lambda'\left(\frac{\ln\left(\frac{|x|}{\sqrt{T}}\right)}{\ln(\sqrt{T})}\right)|x|^{-2}x_N\left(1+(N-1)|x|^{-N}\right).$$

It follows that

$$\left|\nabla\left[\Lambda\left(\frac{\ln\left(\frac{|x|}{\sqrt{T}}\right)}{\ln(\sqrt{T})}\right)^{\ell}\right]\cdot\nabla H(x)\right| \le C(\ln T)^{-1}|x|^{-2}\Lambda\left(\frac{\ln\left(\frac{|x|}{\sqrt{T}}\right)}{\ln(\sqrt{T})}\right)^{\ell-2} x_N. \tag{2.51}$$

Hence, involving (2.49), (2.50) and (2.51) we deduce that

$$\int_{\Omega}\mu_*(x)^{\frac{-1}{m-1}}|\Delta\mu_*(x)|^{\frac{m}{m-1}}\,dx$$

$$\le C(\ln T)^{\frac{-m}{m-1}}\int_{x\in\Omega,\sqrt{T}<|x|<T} x_N|x|^{\frac{-2m}{m-1}}\left(1-|x|^{-N}\right)^{\frac{-1}{m-1}}\Lambda\left(\frac{\ln\left(\frac{|x|}{\sqrt{T}}\right)}{\ln(\sqrt{T})}\right)^{\ell-\frac{2m}{m-1}}\,dx$$

$$\le C(\ln T)^{\frac{-m}{m-1}}\int_{x\in\Omega,\sqrt{T}<|x|<T}|x|^{1-\frac{2m}{m-1}}\,dx$$

$$\le C(\ln T)^{\frac{-m}{m-1}}\int_{r=\sqrt{T}}^{T} r^{N-\frac{2m}{m-1}}\,dr$$

$$= C(\ln T)^{\frac{-m}{m-1}} \int_{r=\sqrt{T}}^{T} r^{-1}\, dr$$

$$\leq C(\ln T)^{\frac{-1}{m-1}},$$

which gives us the inequality

$$\int_{\Omega} \mu_*(x)^{\frac{-1}{m-1}} |\Delta\mu_*(x)|^{\frac{m}{m-1}}\, dx \leq C(\ln T)^{\frac{-2}{N-1}}. \tag{2.52}$$

Hence, we can conclude that the estimate (2.47) follows from (2.48), (2.52) and (2.30). □

3. Proof of the main results

In this section, we prove Theorems 1.2 and 1.7. We recall that both these results concern the nonexistence of global weak solutions to (1.1).

Proof of Theorem 1.2. We argue by contradiction, supposing that $(u, v) \in L^q_{loc}(D) \times L^p_{loc}(D)$ is a global weak solution to (1.1). We first consider the case

$$\mathcal{I}_f > 0 \text{ and } N + 1 < \alpha + \frac{2p(q+1)}{pq-1}. \tag{3.1}$$

For sufficiently large T and ℓ, let φ be the test function defined by (2.17). Since $\mathcal{I}_g \geq 0$, by Lemma 2.5, we get

$$\int_{\Gamma^1} b(t) \frac{\partial\varphi}{\partial\nu_1} g(x)\, d\sigma_1\, dt \leq 0.$$

Hence, by Lemmas 2.2, 2.3 and 2.4, we obtain

$$T^{\theta(\alpha+1)}\mathcal{I}_f \leq C\left(I_p(\varphi)^{\frac{p-1}{p}} + J_p(\varphi)^{\frac{p-1}{p}}\right)^{\frac{p}{pq-1}} \left(I_q(\varphi)^{\frac{q-1}{q}} + J_q(\varphi)^{\frac{q-1}{q}}\right)^{\frac{pq}{pq-1}}.$$

Taking $\theta = 1$ in the above inequality, we deduce from Lemma 2.10 that

$$\mathcal{I}_f \leq CT^{\lambda}, \tag{3.2}$$

where

$$\lambda = \frac{1}{pq-1}\left((N+2)(pq-1) - 2p(q+1)\right) - \alpha - 1.$$

Observe that by (3.1), we have $\mathcal{I}_f > 0$ and $\lambda < 0$. Hence, passing to the limit as $T \to \infty$ in (3.2), we obtain a contradiction with the assumption that $\mathcal{I}_f > 0$.

Now, we focus on the case

$$\mathcal{I}_g > 0 \text{ and } N + 1 < \beta + \frac{2q(p+1)}{pq-1}. \tag{3.3}$$

As in the previous case, for sufficiently large T and ℓ, we use the same test function φ defined by (2.17). Since $\mathcal{I}_f \geq 0$, by Lemma 2.4, we get

$$\int_{\Gamma^1} a(t) \frac{\partial \varphi}{\partial \nu_1} f(x) \, d\sigma_1 \, dt \leq 0.$$

Hence, by Lemmas 2.1, 2.3 and 2.5, we obtain

$$T^{\theta(\beta+1)} \mathcal{I}_g \leq C \left(I_q(\varphi)^{\frac{q-1}{q}} + J_q(\varphi)^{\frac{q-1}{q}} \right)^{\frac{q}{pq-1}} \left(I_p(\varphi)^{\frac{p-1}{p}} + J_p(\varphi)^{\frac{p-1}{p}} \right)^{\frac{pq}{pq-1}}.$$

Taking $\theta = 1$ in the above inequality, we deduce from Lemma 2.9 that

$$\mathcal{I}_g \leq C T^{\kappa}, \tag{3.4}$$

where

$$\kappa = \frac{1}{pq-1} \left((N+2)(pq-1) - 2q(p+1) \right) - \beta - 1.$$

On the other hand, in view of (3.3), we have $\mathcal{I}_g$ and $\kappa < 0$. Hence, passing to the limit as $T \to \infty$ in (3.4), we obtain a contradiction with the assumption that $\mathcal{I}_g > 0$. This completes the proof of Theorem 1.2. □

Now, we present the complete proof of Theorem 1.7.

Proof of Theorem 1.7. We use also the contradiction argument. Namely, suppose that $(u, v) \in L^p_{loc}(D) \times L^p_{loc}(D)$ is a global weak solution to (1.1). Without restriction of the generality, we may assume that $\mathcal{I}_f > 0$. At this time, for sufficiently large T and ℓ, we use the test function φ_* defined by (2.40). Since $\mathcal{I}_g \geq 0$, by Lemma 2.13, we get

$$\int_{\Gamma^1} \frac{\partial \varphi_*}{\partial \nu_1} g(x) \, d\sigma_1 \, dt \leq 0.$$

Hence, by Lemma 2.2 (with $a = b \equiv 1$ and $p = q$), Lemma 2.11 and Lemma 2.12, we obtain

$$T^{\theta} \mathcal{I}_f \leq C \left(I_p(\varphi_*)^{\frac{p-1}{p}} + J_p(\varphi_*)^{\frac{p-1}{p}} \right)^{\frac{p}{p-1}},$$

which yields

$$T^{\theta} \mathcal{I}_f \leq C \left(I_p(\varphi_*) + J_p(\varphi_*) \right). \tag{3.5}$$

On the other hand, by Lemma 2.14 (with $m = p$) and Lemma 2.15 (with $m = p$; notice that by (1.23), we have $p = \frac{N+1}{N-1}$), we obtain

$$I_p(\varphi_*) + J_p(\varphi_*) \leq C \left(T^{N+1-\theta\left(\frac{p+1}{p-1}\right)} + T^{\theta} (\ln T)^{\frac{-2}{N-1}} \right). \tag{3.6}$$

Then, in view of (3.5) and (3.6), we get

$$\mathcal{I}_f \leq C \left(T^{N+1-\frac{2\theta p}{p-1}} + (\ln T)^{\frac{-2}{N-1}} \right). \tag{3.7}$$

Thus, taking $\theta > \frac{(N+1)(p-1)}{2p} = 1$ (i.e., $N + 1 - \frac{2\theta p}{p-1} < 0$) and passing to the limit as $T \to \infty$ in (3.7), we obtain a contradiction with the assumption that $\mathcal{I}_f > 0$. This completes the proof of Theorem 1.7. □

4. Further remarks

In Theorem 1.2, the critical case

$$N \geq 2,\ N+1 = \max\left\{\operatorname{sgn}(\mathcal{I}_f)\left(\alpha + \frac{2p(q+1)}{pq-1}\right), \operatorname{sgn}(\mathcal{I}_g)\left(\beta + \frac{2q(p+1)}{pq-1}\right)\right\} \tag{4.1}$$

for system (1.1) is not completely investigated here. Namely, by Corollary 1.5 and Theorem 1.7, we know only that, if $p = q$ and $\alpha = \beta = 0$, then (4.1) belongs to blow-up case. It should be interesting to decide whether in general, the critical curve (4.1) in $p\&q$ plan belongs to the blow-up situation.

In Theorem 1.2, the sharpness of the condition (1.17) was established only in the special case $a = b \equiv 1$ (see Remark 1.6). It should be interesting to study the existence of global solutions to system (1.1) in the general case when

$$N+1 > \max\left\{\operatorname{sgn}(\mathcal{I}_f)\left(\alpha + \frac{2p(q+1)}{pq-1}\right), \operatorname{sgn}(\mathcal{I}_g)\left(\beta + \frac{2q(p+1)}{pq-1}\right)\right\}.$$

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