

Extracting work from random collisions: A model of a quantum heat engine

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We study the statistical distribution of the ergotropy and of the efficiency of a single-qubit battery and of a single-qubit Otto engine, respectively fuelled by random collisions. The single qubit, our working fluid, is assumed to exchange energy with two reservoirs, a non-equilibrium "hot" reservoir and a zero temperature cold reservoir. The interactions between the qubit and the reservoirs is described in terms of a collision model of open system dynamics. The qubit interacts with the non-equilibrium reservoir (a large ensemble of qudits all prepared in the same pure state) via random unitary collisions and with the cold reservoir (a large ensemble of qubits in their ground state) via a partial swap. Due to the random nature of the interaction with the hot reservoir, fluctuations in ergotropy, heat, and work are present, shrinking with the size of the qudits in the hot reservoir. While the mean, "macroscopic" efficiency of the Otto engine is the same as in the case in which the hot reservoir is a thermal one, the distribution of efficiencies does not support finite moments, so that the mean of efficiencies does not coincide with the macroscopic efficiency.

I. INTRODUCTION

The development of quantum technologies [1] is pushing the realm of thermodynamics to nanoscale heat engines [2–7]. Here the working fluid can be a particle with discrete energy levels, or in the extreme case even a single qubit. Such setups challenge the validity of thermodynamic concepts and raise fundamental questions related to the discreteness of energy levels, the relevance of quantum coherences, the (possibly strong) coupling to non-equilibrium reservoirs [8–20], and the same definition of heat and work, to name but a few.

Collision models [21–31] (see [32] for a succinct entry point to this framework and [33] for an extensive review) are an important tool for quantum thermodynamics, as they can be used to conveniently model the interaction with reservoirs as unitary transformations, even in the regime of strong system-reservoir coupling. It is then possible to address fundamental questions like the relaxation to equilibrium, the link between information and thermodynamics, the efficiency of thermodynamic cycles in multi-level engines [34], and non-Markovian effects.

In this work, we consider a particular kind of non-equilibrium, "hot" reservoir, whose interaction with the working fluid is modeled by random collisions [35–39]. Such possibility is quite appealing since by definition random collisions are a "cheap" resource, meaning that no control of type and duration of system-environment interaction is needed. Here the question is whether random collisions can be exploited as a useful resource to perform quantum thermodynamic tasks.

More specifically, we consider a working medium consisting of a single qubit, alternating collisions with a hot, non-equilibrium reservoir and a cold, thermal reservoir. The collisions with the hot reservoir, consisting of qudits, are modeled as random unitaries, while the collisions with

the qubits of the cold reservoir are, as usual in the literature, modeled by partial swap operations. In this protocol, the qubit acts as a quantum battery [40] (see [41, 42] for reviews), which is charged (discharged) via collisions with the hot (cold) reservoir. We fully characterize the process by looking at the statistical distribution of the ergotropies after each collision.

The non-equilibrium reservoir enhances coherences in the qubit system and we analyze their impact in a quantum Otto cycle. We show that the macroscopic mean efficiency of this quantum engine after many cycles is the same as for a standard quantum Otto cycle (i.e., with a high-temperature reservoir rather than the non-equilibrium one). On the other hand, fluctuations in work and heat depend on the dimensionality of the qudits in the non-equilibrium reservoir. Moreover, the distribution of single-cycle efficiencies is size-invariant. We derive the efficiency distribution and show that it does not support finite moments, so that the mean of efficiencies does not coincide with the macroscopic efficiency.

II. ERGOTROPY FLOW

A. Charge and discharge of the single-qubit battery

Let us first analyse the process of charging and discharging of a single qubit battery interacting with two reservoirs. The qubit is charged via an exchange of energy - and coherence - with a "hot", non-equilibrium, reservoir consisting of a large number of qudits (each belonging to a Hilbert space of dimension μ), all identically prepared in a pure state described by the density operator $\hat{\chi}$. The qubit interacts with such reservoir via a sequence of pairwise random collisions with the individual

qudit of the environment. Such collisions are modelled as a random unitary $\hat{\mathcal{R}}(L)$ ($L = 2\mu$ is the dimension of the joint qubit-qudit Hilbert space), drawn from the invariant Haar measure on the unitary group $U(L)$ and conveniently parametrized in terms of the Hurwitz representation [43]. It is important to note that such collisions are not ‘weak’, i.e. they can strongly change both the energy and the coherences of the single qubit battery.

The battery then dumps its energy into a cold reservoir, consisting of a large number of qubits, all identically prepared in a thermal state $\hat{\vartheta}$. Again the system-environment interaction takes place via a sequence of pairwise collisions between the battery qubit and the environment qubits. We assume each collision to be described by a (unitary) partial swap operation:

$$\hat{\mathcal{P}}(\alpha) = \cos\alpha\hat{I} + i\sin\alpha\hat{\mathcal{S}}, \quad \left(0 \leq \alpha \leq \frac{\pi}{2}\right), \quad (1)$$

where $\hat{\mathcal{S}}$ is the swap operator: $\hat{\mathcal{S}}(|\phi\rangle \otimes |\psi\rangle) \equiv |\psi\rangle \otimes |\phi\rangle$. If the system state before a collision with the cold reservoir is $\hat{\rho}$, then after the collision it is

$$\hat{\rho}' = \cos^2\alpha\hat{\rho} + \sin^2\alpha\hat{\vartheta} + i\sin\alpha\cos\alpha[\hat{\vartheta}, \hat{\rho}], \quad (2)$$

where we have traced over the environment degrees of freedom. Note that, in the case of complete swap, i.e. $\alpha = \frac{\pi}{2}$, the qubit state $\hat{\rho}'$ after the collision is a Gibbs state, which is, as we will explain shortly, passive. In that case, battery discharging is complete.

Since the ‘‘hot’’ reservoir is a non-thermal environment consisting of pure states interacting with the battery via random unitaries, as we mentioned already the system-environment collisions will modify the battery coherences. In this scenario a convenient quantity to analyse the flux of energy from the hot to the cold reservoir is the battery ergotropy [44], i.e. the maximum amount of work that can be extracted from the battery via a suitable unitary evolution \hat{U} (in our case a single qubit non-dissipative evolution).

For a system described by a density operator $\hat{\rho}$ and Hamiltonian \hat{H} , the ergotropy is defined as

$$\mathcal{E}(\hat{\rho}, \hat{H}) = \text{Tr}(\hat{\rho}\hat{H}) - \min[\text{Tr}(\hat{U}\hat{\rho}\hat{U}^\dagger\hat{H})], \quad (3)$$

where the minimum is taken over all possible unitary transformations \hat{U} . Given the state $\hat{\rho} = \sum_n r_n |r_n\rangle\langle r_n|$ and the Hamiltonian $\hat{H} = \sum_n \epsilon_n |\epsilon_n\rangle\langle \epsilon_n|$, with $r_0 \geq r_1 \geq \dots$, and $\epsilon_0 \leq \epsilon_1 \leq \dots$, there is a unique state

$$\hat{\pi} = \hat{U}\hat{\rho}\hat{U}^\dagger = \sum_n r_n |\epsilon_n\rangle\langle \epsilon_n|$$

which minimizes $\text{Tr}(\hat{U}\hat{\rho}\hat{U}^\dagger\hat{H})$. The state $\hat{\pi}$ is called passive, since it cannot deliver any work via the above unitary dynamics.

Given the Bloch sphere representation of the qubit state, $\hat{\rho} = \frac{1}{2}(\hat{I} + \mathbf{r} \cdot \hat{\boldsymbol{\sigma}})$, with $\mathbf{r} = (x, y, z)$ Bloch vector and $\hat{\boldsymbol{\sigma}} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ the vector of Pauli matrices, and the

Hamiltonian $\hat{H} = \frac{1}{2}\Delta\hat{\sigma}_z$, we have $\mathcal{E} = \Delta(r + z)$. The qubit acts as a quantum battery, which can be charged or discharged via unitary interactions (collisions) with qubits (or qudits) of the environment.

We consider a sequence of charging/discharging cycles. The reservoirs are assumed to be so large that the system never collides twice with the same environment qudit (qubit). We can have a pictorial view of the model by considering a single qubit colliding in sequence with the individual qudits (qubits) of two long chains, corresponding to the hot and cold reservoirs, respectively. If $\hat{\rho}_n$ denotes the system’s density operator after n cycles (collisions with each reservoir), we have the map

$$\hat{\rho}_{n+1} = \text{Tr}_{\text{HC}} \left\{ \hat{\mathcal{P}}\hat{\mathcal{R}} \left(\hat{\rho}_n \otimes \hat{\chi} \otimes \hat{\theta} \right) \hat{\mathcal{R}}^\dagger \hat{\mathcal{P}}^\dagger \right\}, \quad (4)$$

where the trace is over both [hot (H) and cold (C)] reservoirs.

B. Statistical distribution of the battery ergotropy

We numerically investigate the mean and the statistical distribution of ergotropy, as a function of the number of collisions with the reservoirs. Hereafter we set the state of the cold reservoir qubits as $\hat{\theta} = |\downarrow\rangle\langle \downarrow|$. This ideal case of a zero-temperature reservoir leads, after a collision with complete swap, to the passive state with the lowest energy (i.e., ground state energy) for the system.

We first consider the case in which the hot reservoir consists of qubits ($\mu = 2$). Initially the system is prepared in a pure state (which one is irrelevant when averaging over random collisions). In Fig. 1 we show the numerically generated histograms of statistical distributions of the system ergotropies, after each of the first three collisions with the hot and the cold reservoir. Hereafter, we consider 10^4 trajectories, i.e., each one with random unitaries drawn from the invariant Haar measure on the unitary group $U(L = 2\mu) = 4$.

We then show in Fig. 2 the mean ergotropy as a function of the number of collisions, for different values of the swap parameter α . We can see that a periodic steady state is approached, with a period of two collisions, one with the hot and one with the cold reservoir. The value of α affects the time needed to practically achieve the periodic steady state, as well as the working of the quantum battery. Indeed, the effectiveness of the discharging process increases with the swap parameter, and a passive state is obtained for the limiting case of complete swap, $\alpha = \frac{\pi}{2}$.

In what follows, we change the dimension μ of the qudits in the hot reservoir. The statistical distributions of the ergotropies is shown in Fig. 3 after six cycles, so that for the used value of the swap parameter ($\alpha = \frac{\pi}{10}$) the periodic steady-state is in practice achieved. We can see that the distribution shrinks with increasing the dimension μ . The mean ergotropy, shown in Fig. 4 as a

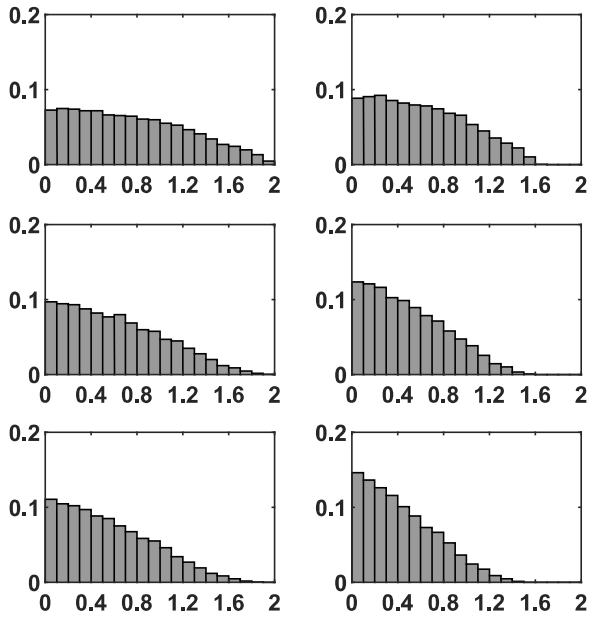


FIG. 1. Histograms showing the statistical distribution of the ergotropies of the system, after collisions with the hot reservoir (left panels) or the cold reservoir (right panels). From top to bottom: distributions after the first, the second, and the third collision with the (hot or cold) reservoir. The hot reservoir consists of qubits ($\mu = 2$); the swap parameter $\alpha = \frac{\pi}{10}$.

function of the number of collisions, is smaller at larger μ . Indeed, in the limit $\mu \rightarrow \infty$ the non-equilibrium reservoir becomes an infinite-temperature reservoir, leading the system qubit after each collision to the completely mixed thermal state $\rho = \frac{I}{2}$, for which the ergotropy vanishes.

III. A QUANTUM OTTO ENGINE FUELLED BY RANDOM COLLISIONS

In the previous section we have introduced a model where the hot reservoir is a non-equilibrium reservoir, which can enhance coherences in the working medium, which in our case is a qubit system. It is therefore appealing to investigate whether these coherences could be used to improve the performance of a heat engine. For that purpose, we consider a model of a quantum Otto cycle, consisting of four strokes.

Stroke A: the system, initially in the state $\hat{\rho}$, interacts with the hot, non-equilibrium reservoir, while its Hamiltonian remains unchanged, $\hat{H}_1 = \frac{1}{2}\Delta_1\hat{\sigma}_z$ ($\Delta_1 > 0$). The collision is modeled as a random unitary transformation, after which the system density matrix becomes $\hat{\rho}'$. The heat absorbed by the system is then given by

$$Q_{\text{in}} = \text{Tr}(\hat{H}_1\hat{\rho}') - \text{Tr}(\hat{H}_1\hat{\rho}) = \frac{1}{2}\Delta_1(z' - z), \quad (5)$$

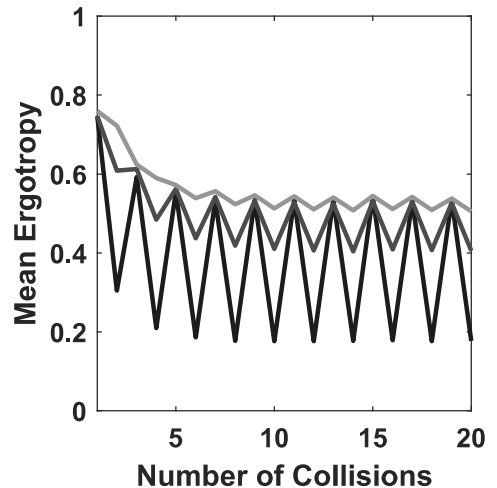


FIG. 2. Ensemble-averaged ergotropy as a function of the number of collisions (alternating collisions with the hot $-\mu = 2$ - and the cold reservoir). Swap parameter $\alpha = \frac{\pi}{20}$ (green line), $\frac{\pi}{10}$ (red line), and $\frac{\pi}{5}$ (blue line).

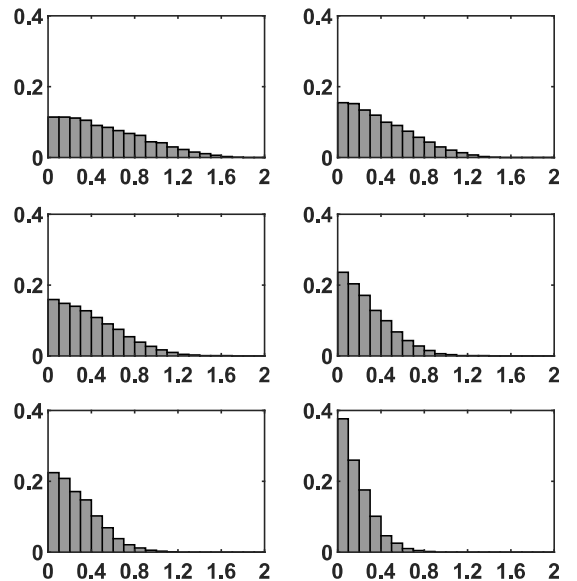


FIG. 3. Histograms showing the statistical distribution of the ergotropies of the system, after 6 cycles, a collision with the hot reservoir (left panels) and a further collision with the cold reservoir (right panels). From top to bottom: qudits of dimension $\mu = 2, 4$, and 8 . Swap parameter $\alpha = \frac{\pi}{10}$.

where z and z' are the z -components of the Bloch vectors of $\hat{\rho}$ and of $\hat{\rho}'$, respectively.

Stroke B: the system is decoupled from the hot reservoir and its Hamiltonian is adiabatically changed up to $\hat{H}_2 = \frac{1}{2}\Delta_2\hat{\sigma}_z$ (with $\Delta_1 > \Delta_2 > 0$), whereas the system density matrix $\hat{\rho}'$ remains unchanged. The work per-

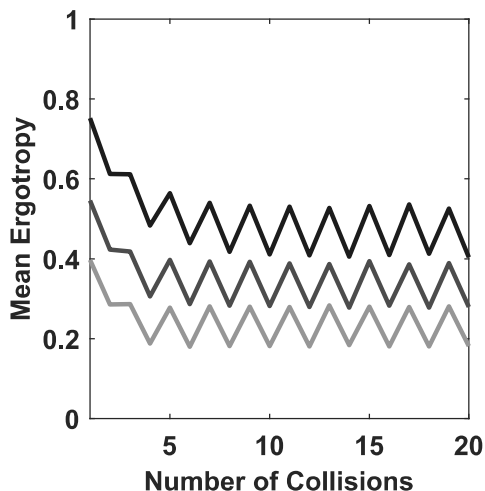


FIG. 4. Ensemble-averaged ergotropy as a function of the number of collisions (alternating collisions with the hot and the cold reservoir). Swap parameter $\alpha = \frac{\pi}{10}$, qudits of dimension $\mu = 2$ (blue line), 4 (red line), and 8 (green line).

formed by the system is given by

$$W_{\text{out}} = \text{Tr}(\hat{H}_1 \hat{\rho}') - \text{Tr}(\hat{H}_2 \hat{\rho}') = \frac{1}{2} z' (\Delta_1 - \Delta_2). \quad (6)$$

Stroke C: the system Hamiltonian remains unchanged, and the interaction of the system with the cold, thermal reservoir is modeled by a partial swap collision, after which the system density matrix becomes $\hat{\rho}''$. The heat absorbed by the system (on average negative) is

$$Q_{\text{out}} = \text{Tr}(\hat{H}_2 \hat{\rho}'') - \text{Tr}(\hat{H}_2 \hat{\rho}') = \frac{1}{2} \Delta_2 (z'' - z'). \quad (7)$$

Stroke D: the system is decoupled from the cold reservoir and its Hamiltonian adiabatically returns to $\hat{H}_1 = \frac{1}{2} \Delta_1 \hat{\sigma}_z$, whereas the system density matrix $\hat{\rho}''$ remain unchanged. The work performed by the system (on average negative) is

$$W_{\text{in}} = \text{Tr}(\hat{H}_2 \hat{\rho}'') - \text{Tr}(\hat{H}_1 \hat{\rho}'') = \frac{1}{2} z'' (\Delta_2 - \Delta_1). \quad (8)$$

We point out that each single realization of the Otto cycle is not strictly speaking a true cycle since in general the final state $\hat{\rho}''$ is different from the initial state $\hat{\rho}$. However, by a suitable ensemble average it is possible to define a “macroscopic” efficiency.

If we use the standard formula for the efficiency η of a heat engine we obtain

$$\eta = \frac{W}{Q_{\text{in}}} = \frac{z' - z''}{z' - z} \left(1 - \frac{\Delta_2}{\Delta_1} \right), \quad (9)$$

where $W = W_{\text{in}} + W_{\text{out}}$. After ensemble averaging, we obtain a periodic steady-state (with the period of the

Otto cycle), and therefore $\langle \hat{\rho}'' \rangle = \langle \hat{\rho} \rangle$ ($\langle \dots \rangle$ denotes ensemble averaging). Consequently we have $\langle z'' \rangle = \langle z \rangle$, which leads to the standard Otto cycle efficiency:

$$\eta_m = \frac{\langle W \rangle}{\langle Q_{\text{in}} \rangle} = 1 - \frac{\Delta_2}{\Delta_1}. \quad (10)$$

This macroscopic efficiency can be obtained after averaging work and input heat over a large number of cycles and/or over an ensemble of random collisions. While efficiency assumes a clear thermodynamic meaning only after one of these two averages, it is nevertheless interesting, when considering the engine constancy, to investigate efficiency fluctuations. In what follows, we shall perform such study.

We first consider the statistical distributions of work W and input heat Q_{in} , shown in Fig. 5. Hereafter, histograms are constructed on 10^5 cycles. It can be seen that all histograms are nicely fitted by a Gaussian distribution, of width decreasing with increasing the dimension μ of the qudits in the non-equilibrium reservoirs. More precisely, we have seen that the standard deviations of both W and Q_{in} decrease, for a given swap parameter α , as $1/\sqrt{\mu}$. We have also checked the validity of the Gaussian fit for other values of α , with the standard deviation of the work distribution increasing with α .

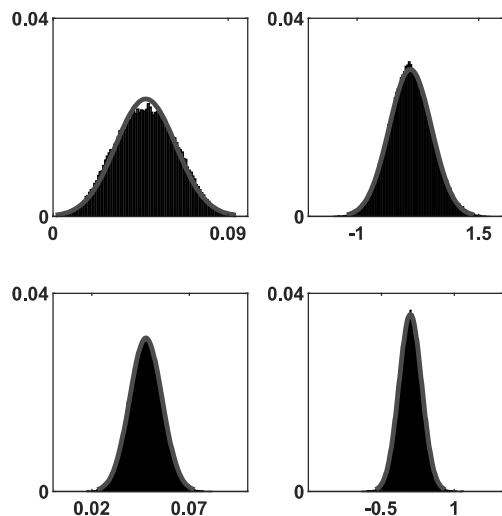


FIG. 5. Histogram showing the statistical distribution of work W (left panels) and input heat Q_{in} (right panels) for the Otto cycle described in the text. The hot reservoir consists of qudits of dimension $\mu = 2$ (top panels) and 8 (bottom panels). Swap parameter $\alpha = \frac{\pi}{10}$, qubit gaps during the cycle: $\Delta_1 = 2$, $\Delta_2 = 1$. Red lines show Gaussian fits.

Finally, we consider the efficiency distributions, shown (for the cases of Fig. 5) in Fig. 6. Assuming, as confirmed by numerical simulations, Gaussian distributions for both W and Q_{in} , we obtain a ratio distribution $p(\eta)$ for $\eta = W/Q_{\text{in}}$ known from the literature (see appendix for details). Such distribution predicts $p(\eta \rightarrow \pm\infty) \propto \eta^{-2}$,

a decay consistent with the power-law fits shown in the right plots of Fig. 6. Since the standard deviation of both W and Q_{in} drops with μ as $1/\sqrt{\mu}$, we can conclude that the statistical distribution of the efficiency is independent of the size of qudits in the non-equilibrium reservoir. Finally, we point out that a decay $p(\eta \rightarrow \pm\infty) \propto \eta^{-2}$ implies that moments of any order of $p(\eta)$ are not finite. In particular, the mean efficiency $\langle \eta \rangle$ is not finite and therefore does not coincide with the macroscopic efficiency $\eta_m = \langle W \rangle / \langle Q_{\text{in}} \rangle$ [45].

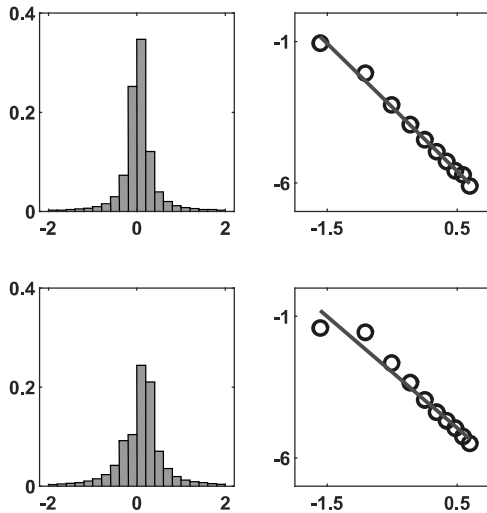


FIG. 6. Histogram showing the statistical distribution of the efficiency η , for dimension of the hot reservoir qudits $\mu = 2$ (top panels) and 8 (bottom panels), for the same parameter values as in Fig. 5. Right panels show the $\eta > 0$ data in log-log scale (natural logarithms), with power law fits $p(\eta) \propto 1/\eta^\alpha$, with $\alpha = -2.25$ (top) and -1.97 (bottom).

IV. CONCLUSIONS

In this paper, we have characterized the working of the smallest size quantum medium, i.e. a single qubit, when operating either as a quantum battery or as a working fluid in a quantum Otto cycle. While we have considered a standard, thermal reservoir as a cold bath, we have modeled the action of the hot, non-equilibrium reservoir via random collision. This setup allows to extract work, with the mean, macroscopic efficiency equal to the one obtained with the hot reservoir being a thermal one. On the other hand, fluctuations due to the random nature of interactions play a very important role. In particular, the distribution of efficiencies does not afford finite moments of any order, so that the mean of efficiencies does not coincide with the macroscopic efficiency.

It is interesting to remark that random collisions, that is, operations which by definition do not require specific control of the unitary transformations, can be used to extract work. It would be interesting to explore such possibility in real, noisy intermediate-scale quantum hardware or in quantum annealers.

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Appendix A: Efficiency probability distribution

Assuming that the work $W = W_{\text{in}} + W_{\text{out}}$ and the input heat Q_{in} are independent Gaussian random variables, we can obtain an analytical expression the efficiency p.d.f. $p(\eta)$. Following [48, 49], we have

$$p(\eta) = \int dW dQ \delta\left(\eta - \frac{W}{Q}\right) p(W, Q) = \int_{-\infty}^{\infty} |Q| p(\eta Q, Q) dQ, \quad (\text{A1})$$

where $p(W, Q)$ is the joint probability distribution function for W and Q_{in} (to simplify writing, hereafter we set $Q \equiv Q_{\text{in}}$):

$$p(W, Q) = \frac{1}{2\pi\sigma_Q\sigma_w} \exp\left\{-\frac{1}{2}\left[\frac{(W - \mu_w)^2}{\sigma_w^2} + \frac{(Q - \mu_Q)^2}{\sigma_Q^2}\right]\right\}, \quad (\text{A2})$$

with μ_W , μ_Q and σ_W , σ_Q mean and standard deviation of the Gaussian distributions for W and Q . After straightforward integration we obtain

$$p(\eta) = \frac{d(\eta)b(\eta)}{2\sqrt{2\pi}\sigma_Q\sigma_w a(\eta)^3} \left[2 - \operatorname{erf}\left(-\frac{b(\eta)}{\sqrt{2}a(\eta)}\right) + \operatorname{erf}\left(\frac{b(\eta)}{\sqrt{2}a(\eta)}\right)\right] + \frac{e^{-\frac{\eta^2}{2}}}{\pi\sigma_Q\sigma_w a(\eta)^2}, \quad (\text{A3})$$

where we have introduced

$$a(\eta) = \sqrt{\frac{\eta^2}{\sigma_w^2} + \frac{1}{\sigma_Q^2}}, \quad b(\eta) = \frac{\mu_w \eta}{\sigma_w^2} + \frac{\mu_Q}{\sigma_Q^2}, \quad c = \frac{\mu_w^2}{\sigma_w^2} + \frac{\mu_Q^2}{\sigma_Q^2}, \quad d(\eta) = \exp\left\{\frac{b^2(\eta) - ca^2(\eta)}{2a^2(\eta)}\right\}. \quad (\text{A4})$$

This distribution has power-law tails: $p(\eta \rightarrow \pm\infty) \propto \eta^{-2}$.

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