# THREE SOLUTIONS TO MIXED BOUNDARY VALUE PROBLEM DRIVEN BY $p(z)$-LAPLACE OPERATOR

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Abstract. We prove the existence of at least three weak solutions to a mixed Dirichlet-Neumann boundary value problem for equations driven by the $p(z)$-Laplace operator in the principal part. Our approach is variational and use three critical points theorems.

## 1. Introduction

Let $M \subset \mathbb{R}^N$ $(N \geq 3)$ be an open bounded domain with smooth boundary. In this article we consider the following mixed Dirichlet-Neumann boundary value problem driven by the $p(z)$-Laplace operator:

$$(P_{\xi,\mu}) \qquad \begin{cases} -\text{div}\left(|\nabla u(z)|^{p(z)-2}\nabla u(z)\right) + a(z)|u(z)|^{p(z)-2}u(z) = \xi\, g(z,u(z)) & \text{in } M, \\ u = 0 & \text{on } M_1, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \mu h(\gamma(u)) & \text{on } M_2, \end{cases}$$

where $p \in C(\overline{M})$ is a function with some regularities satisfying

$$N < p^- := \inf_{z\in M} p(z) \leq p(z) \leq p^+ := \sup_{z\in M} p(z) < +\infty,$$

$M_1$, $M_2$ are smooth $(N-1)$-dimensional submanifolds of $\partial M$ and $\Gamma$ is a smooth $(N-2)$-dimensional submanifolds of $\partial M$ with $M_1 \cap M_2 = \emptyset$, $\overline{M}_1 \cup \overline{M}_2 = \partial M$, $\overline{M}_1 \cap \overline{M}_2 = \Gamma$, $a \in L^\infty(M)$ with $a_0 := \text{ess}\inf_{z\in M} a(z) > 0$ is the potential function, $g : M \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (that is, for all $z \in \mathbb{R}$, $z \to g(z,y)$ is measurable and for a.a. $z \in M$, $y \to g(z,y)$ is continuous), $h : \mathbb{R} \to \mathbb{R}$ is a nonnegative continuous function, $\gamma : W^{1,p}(M) \to L^p(\partial M)$ is the trace operator, $\xi > 0$ and $\mu \geq 0$ are real parameters, and $\nu$ is the outer unit normal to $\partial M$.

We recall that the $p(z)$-Laplace operator drives processes of physical interest, as stated in Diening-Harjulehto-Hästö-Růžička [7]. Existence and multiplicity results for problems involving the $p(z)$-Laplace operator were obtained by Papageorgiou-Vetro [12], Rodrigues [13], Vetro [14] (Dirichlet condition), Deng-Wang [6], Heidarkhani-Afrouzi-Hadjian [10], Pan-Afrouzi-Li [11] (Neumann condition). In [12] the authors consider a $(p(z), q(z))$-equation with reaction $g$ which depends on the solution but does not satisfy the Ambrosetti-Rabinowitz condition. Their approach is variational based on critical point theory together with Morse theory (critical groups). These

*Key words and phrases.* Dirichlet-Neumann boundary value problem; $p(z)$-Laplace operator; variable exponent Sobolev space.

*2010 Mathematics Subject Classification:* 35J20, 35J25.

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tools are used also in [13]. This time, the author studies a nonlinear eigenvalue problem for $p(z)$-Laplacian-like operator, originated from a capillary phenomena. The reaction is superlinear at infinity and the author proves the existence of infinite many pairs of solutions. [14] also considers a problem driven by the $p(z)$-Laplacian-like operator. There, the reaction satisfies a sub-critical growth condition and the results deal with the existence of one and three solutions (via critical point theory). In [10, 11] the authors prove the existence of three solutions for $p(z)$-Laplace problems with potential function. In particular, [10] considers small perturbations of nonhomogeneous Neumann condition. A similar problem is discussed in [6], where the authors use sub-supersolution method and strong comparison principle. For mixed boundary value problems there are the recent works of Barletta-Livrea-Papageorgiou [1], Bonanno-D'Aguì-Sciammetta [2] (for the constant $p$-Laplace operator).

Here, we give two existence theorems of three weak solutions to problem $(P_{\xi,\mu})$ (that is, mixed boundary value problem with variable exponent version of the $p$-Laplace operator), by using a variational approach and critical point theorems. Here the reaction $g : M \times \mathbb{R} \to \mathbb{R}$ is $L^1$-Carathéodory (that is, $g$ is Carathéodory and for any $s > 0$ there exists $l_s \in L^1(M)$ with $|g(z, y)| \leq l_s(z)$, for a.e. $z \in M$ and for all $|y| \leq s$). So, the three critical points results of Bonanno-Marano [4] and Bonanno-Candito [3] apply to energy functionals associated to problem $(P_{\xi,\mu})$.

## 2. Mathematical background

Let $(E, E^*)$ be a Banach topological pair. Here, we use the variable exponent Lebesgue spaces $L^{p(z)}(M)$, $L^{p(z)}(\partial M)$, and the generalized Lebesgue-Sobolev space $W^{1,p(z)}(M)$. These spaces (referring to the norms below) are separable, reflexive and uniformly convex Banach spaces (see Fan-Zhang [8]). Precisely, we have

$$L^{p(z)}(M) = \left\{ u : M \to \mathbb{R} \ : \ \text{u is measurable and } \int_M |u(z)|^{p(z)} dz < +\infty \right\},$$

with the norm

$$\|u\|_{L^{p(z)}(M)} := \inf \left\{ \xi > 0 : \int_M \left| \frac{u(z)}{\xi} \right|^{p(z)} dz \leq 1 \right\} \quad \text{(i.e., Luxemburg norm).}$$

On the other hand, we have

$$L^{p(z)}(\partial M) = \left\{ u : \partial M \to \mathbb{R} \ : \ \text{u is measurable and } \int_{\partial M} |u(z)|^{p(z)} d\sigma < +\infty \right\},$$

where $\sigma$ is the surface measure on $\partial M$. This time, we consider the norm

$$\|u\|_{L^{p(z)}(\partial M)} := \inf \left\{ \xi > 0 : \int_M \left| \frac{u(z)}{\xi} \right|^{p(z)} d\sigma \leq 1 \right\}.$$

Also, the generalized Lebesgue-Sobolev space $W^{1,p(z)}(M)$ is defined as

$$W^{1,p(z)}(M) := \{u \in L^{p(z)}(M) : |\nabla u| \in L^{p(z)}(M)\},$$

and we take the norm

$$\|u\|_{W^{1,p(z)}(M)} = \|u\|_{L^{p(z)}(M)} + \| \, |\nabla u| \, \|_{L^{p(z)}(M)},$$

which is equivalent to the norm

$$\|u\| := \inf \left\{ \xi > 0 : \int_M \left( a(z) \left| \frac{u(z)}{\xi} \right|^{p(z)} + \left| \frac{\nabla u(z)}{\xi} \right|^{p(z)} \right) dz \leq 1 \right\},$$

(for details we refer to D'Aguì-Sciammetta [5]). So, we work with the norm $\|u\|$ instead of $\|u\|_{W^{1,p(z)}(M)}$ on $W^{1,p(z)}(M)$. In proving our theorems, we make use of the following result, which links $\|u\|$ to $\rho(u) = \int_M \left( a(z) \, |u(z)|^{p(z)} + |\nabla u(z)|^{p(z)} \right) dz$ (see Fan-Zhao [9]).

**Theorem 1.** *If $u \in W^{1,p(z)}(M)$, one has*

(i) $\|u\| < 1 \ (= 1, \ > 1) \Leftrightarrow \rho(u) < 1 \ (= 1, \ > 1)$;

(ii) *if* $\|u\| > 1$, *then* $\|u\|^{p^-} \leq \rho(u) \leq \|u\|^{p^+}$;

(iii) *if* $\|u\| < 1$, *then* $\|u\|^{p^+} \leq \rho(u) \leq \|u\|^{p^-}$.

For notational simplicity, by $E$ we denote the set

$$E = W^{1,p}_{0,M_1}(M) = \{u \in W^{1,p}(M) : u_{|M_1} = 0\},$$

where we consider the norm $\|u\|$.

We recall that a function $u \in E$ satisfying

$$\int_M |\nabla u(z)|^{p(z)-2} \nabla u(z) \nabla v(z) dz + \int_M a(z) |u(z)|^{p(z)-2} u(z) v(z) dz$$
$$= \xi \int_M g(z, u(z)) v(z) \, dz + \mu \int_{M_2} h(\gamma(u(z)) \gamma(v(z)) \, dz,$$

for all $v \in E$, means a weak solution of problem $(P_\xi, \mu)$.

We mention the fact that $W^{1,p(z)}(M) \hookrightarrow W^{1,p^-}(M)$ continuously. Also, as $N < p^-$, $W^{1,p^-}(M) \hookrightarrow C_0(\overline{M})$ compactly, and hence $W^{1,p(z)}(M) \hookrightarrow C_0(\overline{M})$ compactly (so $E \hookrightarrow C_0(\overline{M})$ compactly). If we put

$$k = \sup_{u \in W^{1,p(z)}(M) \setminus \{0\}} \frac{\sup_{z \in M} |u(z)|}{\|u\|},$$

then

$$\|u\|_\infty \leq k \|u\|, \tag{1}$$

with $\|u\|_\infty$ to denote the usual norm in $L^\infty(M)$.

The quantity

$$k_b = 2^{\frac{p^- - 1}{p^-}} \max \left[ \left( \frac{1}{\|a\|_1} \right)^{\frac{1}{p^-}}, \frac{\text{diam}\,(\text{M})}{N^{\frac{1}{p^-}}} \left( \frac{p^- - 1}{p^- - N} \text{meas}\,(\text{M}) \right)^{\frac{p^- - 1}{p^-}} \frac{\|a\|_\infty}{\|a\|_1} \right],$$

where $M$ is convex, $\text{diam}\,(M)$ is the diameter of $M$, $\text{meas}\,(M)$ is the Lebesgue measure of $M$, satisfies the inequality $k_b(1 + \text{meas}\,(M)) \geq k$. This means that $k_b(1 + \text{meas}\,(M))$ is an upper bound of $k$ (see [5]).

Next, let $G : M \times \mathbb{R} \to \mathbb{R}$ be the function given as

$$G(z,t) = \int_0^t g(z,y)dy, \quad \text{for all } t \in \mathbb{R},\ z \in M,$$

and $H : \mathbb{R} \to \mathbb{R}$ be the function given as

$$H(t) = \int_0^t h(z)dz, \quad \text{for all } t \in \mathbb{R}.$$

Our approach is variational, which means that we study the critical points of the energy functional (say $I_\xi$) associated to problem $(P_\xi, \mu)$. So, we introduce the functional $B : E \to \mathbb{R}$ defined by

$$B(u) = \int_M G(z, u(z))\, dz + \frac{\mu}{\xi} \int_{\Gamma_2} H(\gamma(u(z)))\, d\sigma, \quad \text{for all } u \in E.$$

Clearly, $B \in C^1(E, \mathbb{R})$ and has a compact derivative given as

$$B'(u)(v) = \int_M g(z, u(z))v(z)\, dz + \frac{\mu}{\xi} \int_{\Gamma_2} h(\gamma(u(z)))\gamma(v(z))\, d\sigma, \quad \text{for all } u, v \in E.$$

Moreover, let $A : E \to \mathbb{R}$ be the functional given as

$$A(u) = \int_M \frac{1}{p(z)} \left[ |\nabla u(z)|^{p(z)} + a(z)|u(z)|^{p(z)} \right] dz, \quad \text{for all } u \in E,$$

with $A$ in $C^1(E, \mathbb{R})$. Note that $A$ is Gâteaux differentiable and sequentially weakly lower semicontinuous and its Gâteaux derivative $A' : E \to E^*$ is

$$A'(u)(v) = \int_M \left[ |\nabla u(z)|^{p(z)-2} \nabla u(z) \nabla v(z) + a(z)|u(z)|^{p(z)-2} u(z)v(z) \right] dz, \text{ for all } u, v \in E.$$

We recall the following property of $A'$ (see, for example, [13, Proposition 2.6]).

**Proposition 1.** *The functional $A' : E \to E^*$ is a strictly monotone and bounded homeomorphism.*

Finally, consider the functional $I_\xi : E \to \mathbb{R}$ defined by $I_\xi(u) = A(u) - \xi B(u)$ for all $u \in E$. We have

$$\inf_{u \in E} A(u) = A(0) = B(0) = 0.$$

We mention that the critical points of $I_\xi$ are the weak solutions of problem $(P_\xi, \mu)$.

## 3. Three weak solutions of Bonanno-Marano type

We establish a theorem producing three weak solutions to problem $(P_\xi, \mu)$. So, we use the following three critical point result of Bonanno-Marano [4, Theorem 3.6].

**Theorem 2.** *Let $(E, E^*)$ be a Banach topological pair with $E$ reflexive. Let $A : E \to \mathbb{R}$ be a coercive, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative has a continuous inverse on $E^*$, $B : E \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that $A(0) = B(0) = 0$. Assume that there exist $r > 0$ and $w \in E$, with $0 < r < A(w)$, such that*

(i) $\sigma = \frac{1}{r} \sup_{A(u) \le r} B(u) < \frac{B(w)}{A(w)} = \rho$;

(ii) *for each* $\xi \in \left]\frac{1}{\rho}, \frac{1}{\sigma}\right[$, $I_\xi := A - \xi B$ *is coercive.*

*Then, for each* $\xi \in \left]\frac{1}{\rho}, \frac{1}{\sigma}\right[$, $I_\xi$ *has at least three distinct critical points in* $E$.

Here, we define $\eta : \overline{M} \to \mathbb{R}$ by $\eta(z) = \rho(z, \partial\overline{M})$, where $\rho$ means the Euclidean distance. Let $D = \eta(z_0)$ with $z_0 \in M$ point of maximum for $\eta$ so that $\mathcal{B}(z_0, D) = \{z \in \mathbb{R}^N : \rho(z_0, z) < D\} \subset M$. Fixed $s \in ]1, +\infty[$, we set $s_D = s^{-1}$ and $\kappa_D = \frac{s}{(s-1)D}$ (note that $(1 - s_D)D\kappa_D = 1$). For each $\alpha \ge 1$, we consider a function $w_\alpha : M \to \mathbb{R}$ defined by

$$w_\alpha(z) = \begin{cases} 0 & z \in M \setminus \mathcal{B}(z_0, D), \\ \alpha & z \in \mathcal{B}(z_0, s_D D), \\ \alpha\kappa_D(D - |z - z_0|) & z \in \mathcal{B}(z_0, D) \setminus \mathcal{B}(z_0, s_D D). \end{cases} \tag{2}$$

Now, we set $\alpha := d \ge 1$ (so we fix $w_d : M \to \mathbb{R}$) and $c \ge k$ with

$$A(w_d)p^+ \int_M \max_{|y| \le c} G(z, y)dz < \left(\frac{c}{k}\right)^{p^-} \int_{\mathcal{B}(z_0, s_D D)} G(z, d)dz.$$

Also, we choose

$$\xi \in \Omega := \left] \frac{A(w_d)}{\int_{\mathcal{B}(x_0, s_D D)} G(z, d)dz}, \frac{(\frac{c}{k})^{p^-}}{p^+ \int_M \max_{|y| \le c} G(z, y)dz} \right[$$

so that

$$\delta := \min\left\{ \frac{c^{p^-} - \xi p^+ k^{p^-} \int_M \max_{|y| \le c} G(z, y)dz}{p^+ k^{p^-} \sigma(M_2) \max_{|y| \le c} H(y)}, \frac{1}{2k^{p^-} p^+ \sigma(M_2) \limsup_{|y| \to +\infty} \frac{H(y)}{|y|^{p^-}}} \right\}, \tag{3}$$

with $\sigma(M_2) := \int_{M_2} d\sigma$ and, as usual, we take $r/0 = +\infty$.

Our first result is the following proposition, where we use the hypothesis:

($h$) $h : \mathbb{R} \to \mathbb{R}$ satisfies

$$\limsup_{|z| \to +\infty} \frac{H(z)}{|z|^{p^-}} < +\infty.$$

Recall that $h$ is continuous, too.

**Proposition 2.** *If* ($h$) *holds, then we can find* $\delta > 0$ *as in* (3) *such that, for each* $\mu \in [0, \delta]$, *the functional* $I_\xi(u) = A(u) - \xi B(u)$, *for all* $u \in E$ ($\xi \in \Omega$) *is coercive whenever*

$$\limsup_{|z| \to +\infty} \frac{\sup_{y \in M} G(z, y)}{|y|^{p^-}} < \frac{\int_M \max_{|y| \le c} G(z, y)dz}{2c^{p^-} \mathrm{meas}\,(M)}.$$

*Proof.* Suppose

$$\limsup_{|y|\to+\infty} \frac{\sup_{z\in M} G(z,y)}{|y|^{p^-}} > 0,$$

so that we can find $l > 0$ with

$$\limsup_{|y|\to+\infty} \frac{\sup_{z\in M} G(z,y)}{|y|^{p^-}} < l < \frac{\int_M \max_{|y|\le c} G(z,y)dz}{2c^{p^-}\operatorname{meas}(M)},$$

$$\Rightarrow \quad G(z,y) \le l|y|^{p^-} + C_l, \quad \text{for each } y\in\mathbb{R} \text{ and } z\in M \text{ (for some } C_l > 0).$$

Since $\left(\frac{c}{k}\right)^{p^-} > \xi p^+ \int_M \max_{|y|\le c} G(z,y)dz$, we have

$$\begin{aligned}
\xi\int_M G(z,u(z))dz &\le \xi l\int_M |u(z)|^{p^-}dz + \xi C_l \operatorname{meas}(M)\\
&< \frac{(\frac{c}{k})^{p^-}}{p^+\int_M \max_{|y|\le c} G(z,y)dz}\left(l\int_M |u(z)|^{p^-}dz + C_l\operatorname{meas}(M)\right)\\
&\le \frac{(\frac{c}{k})^{p^-}\operatorname{meas}(M)}{p^+\int_M \max_{|y|\le c} G(z,y)dz}(lk^{p^-}\|u\|^{p^-} + C_l) \quad \text{for all } u\in E \text{ (by (1))}.
\end{aligned} \tag{4}$$

So, as $\delta > \mu$, we get

$$1 > 2\mu k^{p^-}p^+\sigma(M_2)\limsup_{|y|\to+\infty}\frac{H(y)}{|y|^{p^-}},$$

$$\Rightarrow \quad H(y) \le \frac{|y|^{p^-}}{2\mu k^{p^-}p^+\sigma(M_2)} + C_\mu, \quad \text{for all } y\in\mathbb{R} \text{ (for some } C_\mu > 0).$$

By (1), we obtain

$$\begin{aligned}
\int_{M_2} H(\gamma(u(z)))d\sigma &\le \frac{1}{2\mu k^{p^-}p^+\sigma(M_2)}\int_{M_2}|u(z)|^{p^-}dz + C_\mu\,\sigma(M_2)\\
&\le \frac{1}{2\mu p^+}\|u\|^{p^-} + C_\mu\sigma(M_2), \quad \text{for all } u\in E.
\end{aligned} \tag{5}$$

If $\|u\| \ge 1$, (4) and (5) lead to

$$\begin{aligned}
I_\xi(u) &= A(u) - \xi B(u)\\
&\ge \frac{1}{p^+}\|u\|^{p^-} - \frac{(\frac{c}{k})^{p^-}\operatorname{meas}(M)}{p^+\int_M \max_{|y|\le c} G(z,y)dz}(lk^{p^-}\|u\|^{p^-} + C_l) - \frac{1}{2p^+}\|u\|^{p^-} - \mu C_\mu\sigma(M_2)\\
&= \frac{1}{p^+}\left(\frac{1}{2} - \frac{c^{p^-}\operatorname{meas}(M)}{\int_M \max_{|y|\le c} G(z,y)dz}l\right)\|u\|^{p^-} - \frac{(\frac{c}{k})^{p^-}\operatorname{meas}(M)}{p^+\int_M \max_{|y|\le c} G(z,y)dz} - \mu C_\mu\sigma(M_2).
\end{aligned}$$

By the choice of $l$, we get

$$\begin{aligned}&\frac{1}{2}-\frac{c^{p^-}\operatorname{meas}(M)}{\int_M \max_{|y|\le c} G(z,y)dz} l>0\\ \Rightarrow\quad & I_\xi \text{ is coercive.}\end{aligned}$$

On the other hand, if

$$\limsup_{|y|\to+\infty}\frac{\sup_{z\in M} G(z,y)}{|y|^{p^-}}\le 0,$$

we can find a positive constant $C$ with $G(z,y)\le C$ for all $y\in\mathbb{R}$ and $z\in M$. So, following the same lines as above, we deduce that

$$\begin{aligned}&I_\xi(u)\ge \frac{1}{2p^+}\|u\|^{p^-}-\frac{(\frac{c}{k})^{p^-}\operatorname{meas}(M)C}{p^+\int_M a\max_{|y|\le c} G(z,y)dz}-\mu C_\mu\sigma(M_2)\\ \Rightarrow\quad & I_\xi \text{ is (again) coercive.}\end{aligned}$$

□

We are ready to establish the existence of three weak solutions. To this aim we suppose that there are $d\ge 1$ and $c\ge k$ with

(6) $$A(w_d)>\left(\frac{c}{k}\right)^{p^-},$$

where $w_d : M\to\mathbb{R}$ is given as in (2), satisfying

($S_1$) $p^+A(w_d)\int_M \max_{|y|\le c} G(z,y)dz<\left(\frac{c}{k}\right)^{p^-}\int_{\mathcal{B}(z_0,s_D D)} G(z,d)dz$;

($S_2$) $\limsup_{|y|\to+\infty}\dfrac{\sup_{z\in M} G(z,y)}{|y|^{p^-}}<\dfrac{\int_M \max_{|y|\le c} G(z,y)dz}{2c^{p^-}\operatorname{meas}(M)}$;

($S_3$) $G(z,y)>0$ for all $z\in M$, $y\in[0,d]$.

**Theorem 3.** *If $(h)$, $(S_1)$-$(S_3)$ hold, then we can find $\delta>0$ as in (3) such that, for each $\mu\in[0,\delta]$, problem $(P_{\xi,\mu})$ has at least three weak solutions in $E$ $(\xi\in\Omega)$.*

*Proof.* We set

$$r:=\frac{1}{p^+}\left(\frac{c}{k}\right)^{p^-},$$

so that, by (6), we have

$$A(w_d)>\left(\frac{c}{k}\right)^{p^-}>r.$$

By Theorem 1, for all $u\in E$ such that $u\in A^{-1}(]-\infty,r])$, we obtain

$$\begin{aligned}&\min\{\|u\|^{p^+},\|u\|^{p^-}\}\le rp^+,\\ \Rightarrow\quad & \|u\|\le\max\left\{\left(p^+r\right)^{\frac{1}{p^+}},\left(p^+r\right)^{\frac{1}{p^-}}\right\}=\frac{c}{k},\\ \Rightarrow\quad & \max_{z\in M}|u(z)|\le k\|u\|\le c\quad \text{(by (1))}.\end{aligned}$$

Also, we have

$$B(w_d) = \int_M G(z, w_d(z))\, dz + \frac{\mu}{\xi} \int_{M_2} H(\gamma(w_d(z)))\, d\sigma.$$

So, we deduce that

$$\begin{aligned}\frac{1}{r} \sup_{A(u)\le r} B(u) &\le \frac{\int_M \max_{|y|\le c} G(z,y)dz + \frac{\mu}{\xi}\int_{M_2} \max_{|y|\le c} H(y)d\sigma}{\frac{1}{p^+}\left(\frac{c}{k}\right)^{p^-}} \\ &= p^+ \left(\frac{k}{c}\right)^{p^-} \left[\int_M \max_{|y|\le c} G(z,y)dz + \frac{\mu}{\xi}\sigma(M_2) \max_{|y|\le c} H(y)\right].\end{aligned}$$

Now, if $\max_{|y|\le c} H(y) = 0$, we have

$$\frac{1}{r} \sup_{A(u)\le r} B(u) < \frac{1}{\xi},$$

and if $\max_{|y|\le c} H(y) > 0$, it turns out to be true as

$$\mu < \frac{c^{p^-} - \xi p^+ k^{p^-} \int_M \max_{|y|\le c} G(z,y)dz}{p^+ k^{p^-} \sigma(M_2) \max_{|y|\le c} H(y)}.$$

By $(S_3)$ we get

$$\begin{aligned} & B(w_d) \ge \int_{\mathcal{B}(z_0, s_D D)} G(z,d)dz \\ \Rightarrow\quad & \frac{B(w_d)}{A(w_d)} \ge \frac{\int_{\mathcal{B}(z_0,s_D D)} G(z,d)dz}{A(w_d)} > \frac{1}{\xi} \\ \Rightarrow\quad & \frac{B(w_d)}{A(w_d)} > \frac{1}{r}\sup_{A(u)\le r} B(u), \\ \Rightarrow\quad & \text{Theorem 2}(i) \text{ is true.}\end{aligned}$$

By Proposition 2, we know that Theorem 2$(ii)$ holds true. Since all the regularity hypotheses of Theorem 2 on $A$ and $B$ are true, then Theorem 2 gives us the existence of at least three critical points of $I_\xi$, which are three weak solutions of $(P_{\xi,\mu})$. □

## 4. Three weak solutions of Bonanno-Candito type

In this section, we do not use hypothesis $(h)$ in establishing the existence of three weak solutions. Here, we assume that $g$ and $h$ are nonnegative. We apply the following three critical points result of Bonanno-Candito [3, Theorem 3.3].

**Theorem 4.** *Let $(E, E^*)$ be a Banach pair with $E$ reflexive. Let $A : E \to \mathbb{R}$ be a convex, coercive and continuously Gâteaux differentiable functional whose derivative has a continuous inverse on $E^*$, $B : E \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose derivative is compact with*

$$\inf_{u\in E} A(u) = A(0) = B(0) = 0.$$

*If there exist* $r_1, r_2 > 0$ *and* $w \in E$*, with* $4r_1 < 2A(w) < r_2$*, satisfying*

(i) $\displaystyle \frac{1}{r_1} \sup_{A(u)<r_1} B(u) < \frac{2}{3}\frac{B(w)}{A(w)}$;

(ii) $\displaystyle \frac{1}{r_2} \sup_{A(u)<r_2} B(u) < \frac{1}{3}\frac{B(w)}{A(w)}$;

(iii) $\inf_{s\in[0,1]} B(su_1 + (1-s)u_2) \geq 0$*, for all* $u_1, u_2 \in E$*, with* $B(u_1) \geq 0$ *and* $B(u_2) \geq 0$*, which are local minima of* $I_\xi = A - \xi B$*, for each* $\xi \in \widehat{\Omega}$*, where*

$$\widehat{\Omega} := \left]\frac{3}{2}\frac{A(w)}{B(w)}, \min\left\{\frac{r_1}{\sup_{A(u)<r_1} B(u)}, \frac{\frac{r_2}{2}}{\sup_{A(u)<r_2} B(u)}\right\}\right[,$$

*then* $I_\xi$ *has at least three distinct critical points in* $A^{-1}(]-\infty, r_2[)$*.*

Next, we suppose that there are $d \geq 1$ and $c_1, c_2 > 0$, with $\min\{c_1, c_2\} \geq k$, such that

$$\frac{3}{2}\frac{A(w_d)}{\int_{\mathcal{B}(z_0,s_D D)} G(z,d)dz} < \min\left\{\frac{(\frac{c_1}{k})^{p^-}}{p^+\int_M G(z,c_1)dz}, \frac{(\frac{c_2}{k})^{p^-}}{2p^+\int_M G(z,c_2)dz}\right\},$$

where $w_d : M \to \mathbb{R}$ is given as in (2), satisfying

($S_1'$) $\frac{2}{p^+}(\frac{c_1}{k})^{p^-} < A(w_d) < \frac{1}{2p^+}(\frac{c_2}{k})^{p^-}$;

($S_2'$) $\displaystyle \max\left\{\frac{\int_M G(z,c_1)dz}{(\frac{c_1}{k})^{p^-}}, \frac{2\int_M G(z,c_2)dz}{(\frac{c_2}{k})^{p^-}}\right\} < \frac{2}{3}\frac{\int_{\mathcal{B}(z_0,s_D D)} G(z,d)dz}{\Phi(w_d)}$;

($S_3'$) $g(z,y) \geq 0$ for each $(z,y) \in M \times \mathbb{R}$.

Here, we consider

$$\xi \in \widetilde{\Omega} := \left[\frac{3}{2}\frac{A(w_d)}{\int_{\mathcal{B}(z_0,s_D D)} G(z,d)dz}, \frac{1}{p^+}\min\left\{\frac{(\frac{c_1}{k})^{p^-}}{\int_M G(z,c_1)dz}, \frac{(\frac{c_2}{k})^{p^-}}{2\int_M G(z,c_2)dz}\right\}\right],$$

so that

$$\delta^* := \min\left\{\frac{(\frac{c_1}{k})^{p^-} - \xi p^+\int_M G(z,c_1)dz}{p^+\sigma(M_2)H(c_1)}, \frac{(\frac{c_2}{k})^{p^-} - 2\xi p^+\int_M G(z,c_2)dz}{2p^+\sigma(M_2)H(c_2)}\right\}. \tag{7}$$

**Theorem 5.** *If* ($S_1'$)-($S_3'$) *hold, then we can find* $\delta^* > 0$ *as in* (7) *such that, for each* $\mu \in [0, \delta^*[$*, problem* $(P_\xi, \mu)$ $(\xi \in \widetilde{\Omega})$ *has at least three distinct weak solutions* $u_*, u^*, \widetilde{u}$*, whose values range is the interval* $[0, c_2[$*.*

*Proof.* For reader convenience we set $r_1 := \frac{1}{p^+}\left(\frac{c_1}{k}\right)^{p^-}$ and $r_2 := \frac{1}{p^+}\left(\frac{c_2}{k}\right)^{p^-}$. So, by ($S_1'$) we have $4r_1 < 2A(w_d) < r_2$. As $\delta^* > \mu$ and $H(z) \geq 0$ for $z > 0$, we obtain

$$\begin{aligned}\frac{1}{r_1}\sup_{A(u)\leq r_1} B(u) &\leq \sup_{A(u)\leq r_1} \frac{\int_M G(z,u(z))dz + \frac{\mu}{\xi}\int_{M_2} H(\gamma(u(z)))d\sigma}{\frac{1}{p^+}\left(\frac{c_1}{k}\right)^{p^-}} \\ &= p^+\left(\frac{k}{c_1}\right)^{p^-}\left[\int_M G(z,c_1)dz + \frac{\mu}{\xi}\sigma(M_2)H(c_1)\right]\end{aligned}$$

$$< \frac{1}{\xi} < \frac{2}{3}\frac{\int_{\mathcal{B}(z_0,s_D D)} G(z,d)dz}{A(w_d)} \leq \frac{2}{3}\frac{B(w_d)}{A(w_d)}.$$

Also, we have

$$\begin{aligned}
\frac{2}{r_2}\sup_{A(u)\leq r_2} B(u) &\leq 2\sup_{A(u)\leq r_2}\frac{\int_M G(z,u(z))dz + \frac{\mu}{\xi}\int_{M_2} H(\gamma(u(z)))d\sigma}{\frac{1}{p^+}\left(\frac{c_2}{k}\right)^{p^-}} \\
&= 2p^+\left(\frac{k}{c_2}\right)^{p^-}\left[\int_M G(z,c_2)dz + \frac{\mu}{\xi}\sigma(M_2)H(c_2)\right] \\
&< \frac{1}{\xi} < \frac{2}{3}\frac{\int_{\mathcal{B}(z_0,s_D D)} G(z,d)dz}{A(w_d)} \leq \frac{2}{3}\frac{B(w_d)}{A(w_d)} \\
\Rightarrow \quad \frac{1}{r_2}\sup_{A(u)\leq r_2} B(u) &< \frac{1}{3}\frac{B(w_d)}{A(w_d)}.
\end{aligned}$$

This means that Theorem 4($i$)-($ii$) hold true.

Next, consider two local minima of $I_\xi$, say $u_*$, $u^* \in E$. Clearly, $u_*$, $u^*$ are critical points of $I_\xi$, and hence weak solutions of $(P_{\xi,\mu})$. We have to show that $u_*$, $u^* \geq 0$. Let $w$ be a weak solution of $(P_{\xi,\mu})$ so that

$$\int_M |\nabla w|^{p(z)-2}|\nabla w||\nabla v|dz + \int_M a(z)|w|^{p(z)-2}|w||v|dz = \xi\int_M g(z,w)vdz + \mu\int_{M_2} h(\gamma(w)))\gamma(v)d\sigma$$

for all $v \in E$. So, if we choose $v = \min\{w,0\} = w^- \in E$, we get

$$\int_M |\nabla w^-|^{p(z)}dz + \int_M a(z)|w^-|^{p(z)}dz = \xi\int_M g(z,w)w^- dz + \mu\int_{M_2} h(\gamma(w)))\gamma(w^-)d\sigma \leq 0$$

(recall the sign assumptions on the data).

This leads to $\|w^-\| = 0$, which is absurd, and hence $u_*$, $u^*$ are nonnegative. So, we have

$$\begin{aligned}
& su_* + (1-s)u^* \geq 0 \quad \text{for all } s \in [0,1], \\
\Rightarrow \quad & B(su_* + (1-s)u^*) \geq 0 \quad \text{for all } s \in [0,1], \\
\Rightarrow \quad & \text{Theorem 4}(iii) \text{ is true.}
\end{aligned}$$

Since all the regularity hypotheses of Theorem 4 on $A$ and $B$ remain true, we conclude that $(P_{\xi,\mu})$ has at least three distinct weak solutions for each $\xi \in \widetilde{\Omega}$. □

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