

ON PROBLEMS DRIVEN BY THE $(p(\cdot), q(\cdot))$ -LAPLACE OPERATOR

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ABSTRACT. The aim of this paper is to prove the existence of at least one nontrivial weak solution for equations involving the $(p(\cdot), q(\cdot))$ -Laplace operator. The approach is variational and based on the critical point theory.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with smooth boundary. We consider the problem with Dirichlet condition:

$$(1) \quad \begin{cases} -\Delta_{p(x)}u(x) - \Delta_{q(x)}u(x) = \mu h(x, u(x)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \mu > 0. \end{cases}$$

where $\Delta_{p(x)}u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ and $\Delta_{q(x)}u := \operatorname{div}(|\nabla u|^{q(x)-2}\nabla u)$ are the $p(\cdot)$ -Laplace and $q(\cdot)$ -Laplace operators. Let $p, q \in C(\overline{\Omega})$ be such that

$$\begin{aligned} 1 < q^- := \inf_{x \in \Omega} q(x) \leq q(x) \leq q^+ := \sup_{x \in \Omega} q(x) \\ < p^- := \inf_{x \in \Omega} p(x) \leq p(x) \leq p^+ := \sup_{x \in \Omega} p(x) < +\infty. \end{aligned}$$

The study of equations driven by the $p(x)$ -Laplace operator presents major difficulties than the classical ones driven by the p -Laplace operator. Indeed, the $p(x)$ -Laplace operator is inhomogeneous.

Here, the function $h(x, z)$ (reaction term) is Carathéodory (that is, for all $z \in \mathbb{R}$, $x \rightarrow h(x, z)$ is measurable and for a.a. $x \in \Omega$, $z \rightarrow h(x, z)$ is continuous).

Let $p^*(x) = \frac{np(x)}{n-p(x)}$ if $p(x) < n$ and $p^*(x) = +\infty$ if $p(x) \geq n$. The hypothesis on $h : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is as follows:

(h_1) there exist constants $b_1, b_2 \in [0, +\infty[$ and $\beta \in C(\overline{\Omega})$ with $1 < \beta(x) < p^*(x)$ for each $x \in \overline{\Omega}$, such that

$$|h(x, \xi)| \leq b_1 + b_2|\xi|^{\beta(x)-1} \quad \text{for all } x \in \Omega, \text{ all } \xi \in \mathbb{R}.$$

We give two easy examples of functions satisfying the hypothesis (h_1) :

- $h(\xi) = \xi^{s(x)-1} + \xi^{r(x)-1}$ for all $\xi \geq 0$, with $p, q, r, s \in C(\overline{\Omega})$ such that $1 < s(x) < q(x) < p(x) < r(x) < p^*(x)$.
- $h(\xi) = \xi^{p(x)-1} \ln(1 + \xi)$, for all $\xi \geq 0$, with $p \in C(\overline{\Omega})$ such that $1 < p(x) \leq p^*(x)$.

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The solution to (1) is understood in the weak sense, that is, a function $u \in W_0^{1,p(x)}(\Omega)$ such that

$$\int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla v dx + \int_{\Omega} |\nabla u|^{q(x)-2} \nabla u \nabla v dx = \mu \int_{\Omega} h(x, u) v dx,$$

for all $v \in W_0^{1,p(x)}(\Omega)$. Here, by $W_0^{1,p(x)}(\Omega)$, we denote the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$, where $W^{1,p(x)}(\Omega)$ is the generalized Lebesgue–Sobolev space considered in Section 2.

Existence and multiplicity theorems for boundary value problems involving the $p(\cdot)$ -Laplace and $(p(\cdot), q(\cdot))$ -Laplace operators were recently obtained by Bonanno-Chinnì [2], Gasiński-Papageorgiou [5], Papageorgiou-Vetro [9], Tan-Fang [10], Vetro [11], and Zhou [12].

The structure of problem (1) is variational and hence we approach it by using critical point theory. So, we wish to prove two results concerning the existence of at least one and three weak solutions. Precisely, we construct our proofs on two critical point theorems due to Bonanno [1, Theorem 2.3] and Bonanno-Marano [3, Theorem 3.6]. Here we recall the following statements.

Theorem 1 ([1]). *Let X be a real Banach space and let $A, B : X \rightarrow \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that $\inf_{u \in X} A(u) = A(0) = B(0) = 0$. Assume that there exist $r > 0$ and $\bar{u} \in X$, with $0 < A(\bar{u}) < r$, such that*

$$(j) \quad \delta = \frac{1}{r} \sup_{A(u) \leq r} B(u) < \frac{B(\bar{u})}{A(\bar{u})} = \varrho;$$

(jj) *for each $\mu \in]\varrho^{-1}, \delta^{-1}[$ the functional $J_\mu := A - \mu B$ satisfies (P.S.) $^{[r]}$ -condition.*

Then, for each $\mu \in]\varrho^{-1}, \delta^{-1}[$, there is $u_{0,\mu} \in A^{-1}(]0, r[)$ such that $J'_\mu(u_{0,\mu}) \equiv \vartheta_{X^}$ and $J_\mu(u_{0,\mu}) \leq J_\mu(u)$ for all $u \in A^{-1}(]0, r[)$.*

Theorem 2 ([3]). *Let X be a reflexive real Banach space and let $A : X \rightarrow \mathbb{R}$ be a coercive, continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on X^* , $B : X \rightarrow \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that $\inf_{u \in X} A(u) = A(0) = B(0) = 0$. Assume that there exist $r > 0$ and $\bar{u} \in X$, with $0 < r < A(\bar{u})$, such that*

$$(j) \quad \delta = \frac{1}{r} \sup_{A(u) \leq r} B(u) < \frac{B(\bar{u})}{A(\bar{u})} = \varrho;$$

(jj) *for each $\mu \in]\varrho^{-1}, \delta^{-1}[$ the functional $J_\mu := A - \mu B$ is coercive.*

Then, for each $\mu \in]\varrho^{-1}, \delta^{-1}[$, the functional $J_\mu := A - \mu B$ has at least three distinct critical points in X .

2. MATHEMATICAL BACKGROUND

Let X be a Banach space. By X^* we denote its topological dual and by $\langle \cdot, \cdot \rangle$ the duality brackets for the pair (X^*, X) . In the analysis of problem (1), we work with the spaces $L^{p(x)}(\Omega)$ (variable exponent Lebesgue space) and $W^{1,p(x)}(\Omega)$ (generalized Lebesgue–Sobolev space). We mention that both $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, endowed with the norms $\|u\|_{L^{p(x)}(\Omega)}$ and $\|u\|_{W^{1,p(x)}(\Omega)}$ (see below), are separable, reflexive and

uniformly convex Banach spaces (see [6]). So, we have

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \rightarrow \mathbb{R} : u \text{ is measurable and } \rho_p(u) := \int_{\Omega} |u(x)|^{p(x)} dx < +\infty \right\},$$

$$W^{1,p(x)}(\Omega) = \{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \}.$$

Next, we consider the norms:

$$\|u\|_{L^{p(x)}(\Omega)} = \left\{ \mu > 0 : \int_{\Omega} \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\},$$

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)}.$$

We recall that

$\|u\|_{L^{p(x)}(\Omega)} \leq C \|\nabla u\|_{L^{p(x)}(\Omega)}$ for all $u \in W_0^{1,p(x)}(\Omega)$, for a specific constant $C > 0$, (see [4, Theorem 8.2.18] and [6, Proposition 2.5(iii)]). So, the norms $\|u\|_{W^{1,p(x)}(\Omega)}$ and $\|\nabla u\|_{L^{p(x)}(\Omega)}$ are equivalent on $W_0^{1,p(x)}(\Omega)$. Therefore, we can use $\|\nabla u\|_{L^{p(x)}(\Omega)}$ instead of $\|u\|_{W^{1,p(x)}(\Omega)}$ so that we put

$$\|u\| = \|\nabla u\|_{L^{p(x)}(\Omega)} \quad \text{in } W_0^{1,p(x)}(\Omega).$$

The following result is a generalization of the classical Sobolev embedding theorem.

Proposition 1. [6, Proposition 2.5(ii)] *Assume that $p \in C(\overline{\Omega})$ with $p(x) > 1$ for each $x \in \overline{\Omega}$. If $\beta \in C(\overline{\Omega})$ and $1 < \beta(x) < p^*(x)$ for all $x \in \Omega$, then there exists a continuous and compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{\beta(x)}(\Omega)$.*

We will also use another basic theorem, linking $\|\cdot\|_{L^{p(x)}(\Omega)}$ to $\rho_p(\cdot)$ (see, for example, Theorem 1.3 of [7]).

Theorem 3. *Let $u \in L^{p(x)}(\Omega)$. Then, the following relations hold:*

- (i) $\|u\|_{L^{p(x)}(\Omega)} < 1$ ($= 1$, > 1) $\Leftrightarrow \rho_p(u) < 1$ ($= 1$, > 1);
- (ii) if $\|u\|_{L^{p(x)}(\Omega)} > 1$, then $\|u\|_{L^{p(x)}(\Omega)}^{p^-} \leq \rho_p(u) \leq \|u\|_{L^{p(x)}(\Omega)}^{p^+}$;
- (iii) if $\|u\|_{L^{p(x)}(\Omega)} < 1$, then $\|u\|_{L^{p(x)}(\Omega)}^{p^+} \leq \rho_p(u) \leq \|u\|_{L^{p(x)}(\Omega)}^{p^-}$.

Next, let $H : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$H(x, t) = \int_0^t h(x, \xi) d\xi \quad \text{for all } t \in \mathbb{R}, x \in \Omega,$$

and consider the functional $B : W_0^{1,p(x)}(\Omega) \rightarrow \mathbb{R}$ defined by

$$B(u) = \int_{\Omega} H(x, u(x)) dx, \quad \text{for all } u \in W_0^{1,p(x)}(\Omega).$$

By the hypothesis (h_1) on the reaction term, we have $B \in C^1(W_0^{1,p(x)}(\Omega), \mathbb{R})$.

Remark 1. Proposition 1 ensures that B admits a compact Gâteaux derivative given by

$$\langle B'(u), v \rangle = \int_{\Omega} h(x, u(x)) v(x) dx, \quad \text{for all } u, v \in W_0^{1,p(x)}(\Omega).$$

Now, let $A_1, A_2, A : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ be the C^1 -functionals defined by

$$A_1(u) = \int_{\Omega} \frac{1}{p(x)} |\nabla u(x)|^{p(x)} dx, \quad A_2(u) = \int_{\Omega} \frac{1}{q(x)} |\nabla u(x)|^{q(x)} dx$$

and

$$A(u) = A_1(u) + A_2(u),$$

for all $u \in W_0^{1,p(x)}(\Omega)$. We note that A_1, A_2 and A are convex, sequentially weakly lower semi-continuous with Gâteaux derivatives $A'_1, A'_2, A' : W_0^{1,p(x)}(\Omega) \rightarrow (W_0^{1,p(x)}(\Omega))^*$ given by

$$\langle A'_1(u), v \rangle = \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla v dx, \quad \langle A'_2(u), v \rangle = \int_{\Omega} |\nabla u|^{q(x)-2} \nabla u \nabla v dx$$

and

$$\langle A'(u), v \rangle = \langle A'_1(u), v \rangle + \langle A'_2(u), v \rangle,$$

for all $u, v \in W_0^{1,p(x)}(\Omega)$.

Remark 2 (see [6], Theorem 3.1(ii)). A'_1, A'_2 are mappings of type (S_+) , that is, if $u_n \xrightarrow{w} u$ in $W_0^{1,p(x)}(\Omega)$ and $\limsup_{n \rightarrow +\infty} \langle A'_i(u_n), u_n - u \rangle \leq 0$, then $u_n \rightarrow u$ in $W_0^{1,p(x)}(\Omega)$, $i = 1, 2$. Of course, A' is a mapping of type (S_+) too.

Remark 3 (see [6], Theorem 3.1(i)). $A'_1, A'_2 : W_0^{1,p(x)}(\Omega) \rightarrow (W_0^{1,p(x)}(\Omega))^*$ are continuous strictly monotone mappings. Consequently, $A' : W_0^{1,p(x)}(\Omega) \rightarrow (W_0^{1,p(x)}(\Omega))^*$ is a continuous strictly monotone mapping (hence maximal monotone too). The continuity is obvious. We prove that A' is strictly monotone. In fact, if $(u, u^*), (v, v^*) \in Gr(A')$, then there exist $z^*, w^* \in (W_0^{1,p(x)}(\Omega))^*$ such that $(u, z^*), (v, w^*) \in Gr(A'_1)$ and $(u, u^* - z^*), (v, v^* - w^*) \in Gr(A'_2)$. So, we have

$$\begin{aligned} \langle u^* - v^*, u - v \rangle &= \langle z^* - w^* + (u^* - z^*) - (v^* - w^*), u - v \rangle \\ &= \langle z^* - w^*, u - v \rangle + \langle (u^* - z^*) - (v^* - w^*), u - v \rangle \geq 0, \end{aligned}$$

since

$$\langle z^* - w^*, u - v \rangle, \langle (u^* - z^*) - (v^* - w^*), u - v \rangle \geq 0.$$

Moreover, from $\langle z^* - w^*, u - v \rangle > 0$ for $u \neq v$ it follows

$$\langle u^* - v^*, u - v \rangle > 0 \quad \text{if } u \neq v.$$

Remark 4. The mapping $A' : W_0^{1,p(x)}(\Omega) \rightarrow (W_0^{1,p(x)}(\Omega))^*$ has a continuous inverse $(A')^{-1} : (W_0^{1,p(x)}(\Omega))^* \rightarrow W_0^{1,p(x)}(\Omega)$.

Indeed, from Remark 3 it follows that A' is injective and maximal monotone. Since A' is also coercive, then A' is surjective (see Theorem 2.55 of Motreanu-Motreanu-Papageorgiou [8], p. 33). This ensures that A' has an inverse mapping $(A')^{-1} : (W_0^{1,p(x)}(\Omega))^* \rightarrow W_0^{1,p(x)}(\Omega)$.

Now, we prove that $(A')^{-1}$ is continuous. Let $s_n, s \in (W_0^{1,p(x)}(\Omega))^*$ with $s_n \rightarrow s$. For every $n \in \mathbb{N}$, there exists (a unique) $u_n \in W_0^{1,p(x)}(\Omega)$ such that $A'(u_n) = s_n$. Also, there exists $u \in W_0^{1,p(x)}(\Omega)$ such that $A'(u) = s$. Since A' is coercive and $A'(u_n) \rightarrow A'(u)$, we get that $\{u_n\} \subset W_0^{1,p(x)}(\Omega)$ is bounded. So, by passing to a subsequence if necessary, we can assume that there exists $u_0 \in W_0^{1,p(x)}(\Omega)$ such that $u_n \xrightarrow{w} u_0$. Since $s_n \rightarrow s$, we have

$$\lim_{n \rightarrow +\infty} \langle A'(u_n) - A'(u_0), u_n - u_0 \rangle = \lim_{n \rightarrow +\infty} \langle s_n, u_n - u_0 \rangle = 0,$$

$\Rightarrow u_n \rightarrow u_0$ (as A' is of type (S_+) , see Remark 2).

The continuity of A' implies that $s_n = A'(u_n) \rightarrow A'(u_0) = s = A'(u)$. The strictly monotonicity of A' implies that $u = u_0$. So, we conclude that $u_n \rightarrow u$, and hence $(A')^{-1}$ is continuous.

We consider the functional $J_\mu : W_0^{1,p(x)}(\Omega) \rightarrow \mathbb{R}$ given as $J_\mu(u) = A(u) - \mu B(u)$ for all $u \in W_0^{1,p(x)}(\Omega)$. We note that

$$\inf_{u \in W_0^{1,p(x)}(\Omega)} A(u) = A(0) = B(0) = 0.$$

Since the concept of $(PS)^{[r]}$ -condition appears in the statement of Theorem 1, we recall it.

Definition 1. Let (X, X^*) be a Banach dual pair. We say that $J_\mu : X \rightarrow \mathbb{R}$ has the Palais-Smale cut off upper at r property, for $r \in]-\infty, +\infty]$ fixed, if every $\{u_n\} \subset X$ such that:

- (j) $\{J_\mu(u_n)\}$ is bounded;
- (jj) $\|J'_\mu(u_n)\|_{X^*} \rightarrow 0$ as $n \rightarrow +\infty$;
- (jjj) $A(u_n) < r$,

admits a convergent subsequence.

3. MAIN RESULTS

Firstly, we prove a theorem producing at least one nontrivial weak solution for the problem (1). We impose the growth condition (h_1) on the reaction term. Define the function $\zeta : \overline{\Omega} \rightarrow \mathbb{R}$ by $\zeta(x) = d(x, \partial\overline{\Omega})$, where d is the Euclidean distance. Let $x_0 \in \Omega$ be a point of maximum for ζ and let $\theta := \zeta(x_0)$, then $\mathcal{B}(x_0, \theta) = \{x \in \mathbb{R}^n : d(x_0, x) < \theta\} \subset \Omega$. For notational convenience, we fix $\eta \in]1, +\infty[$ and put $\eta_\theta = 1/\eta$ and $\omega_\theta = \frac{\eta}{(\eta-1)\theta}$. Clearly, $(1 - \eta_\theta)\theta\omega_\theta = 1$. Also, for $c > 0$ and $\kappa \in C(\overline{\Omega})$ with $1 < \kappa^-$, we put

$$[c]^\kappa := \max\{c^{\kappa^-}, c^{\kappa^+}\} \quad \text{and} \quad [c]_\kappa := \min\{c^{\kappa^-}, c^{\kappa^+}\}.$$

We impose the following condition on $H : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$:

$$(h_2) \quad \limsup_{t \rightarrow 0^+} \frac{\inf_{x \in \Omega} H(x, t)}{t^{q^-}} = +\infty.$$

Example 1. Let $\beta \in C(\overline{\Omega})$ with $1 < \beta(x) < p^*(x)$ for each $x \in \Omega$, and $g : \Omega \rightarrow \mathbb{R}$ such that $g \in L^1(\Omega)$ with $0 < a \leq g(x) \leq b_1 < +\infty$ for all $x \in \Omega$. Consider the function $h : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ given as

$$h(x, t) = \begin{cases} g(x) + ct^{\beta(x)-1} & t > 0, \\ g(x) & t \leq 0. \end{cases}$$

Clearly h satisfies (h_1) and H satisfies (h_2) .

Now, put $\mu^* = \left(b_1\chi_1(p^+)^{1/p^-} + \frac{b_2}{\beta^-}[\chi_\beta]^\beta(p^+)^{\beta^+/p^-}\right)^{-1}$, with χ_1 and χ_β denoting the constants of the compact embeddings $W_0^{1,p(x)}(\Omega) \hookrightarrow L^1(\Omega)$ and $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{\beta(x)}(\Omega)$. So, we prove our first theorem.

Theorem 4. *For each $\mu \in]0, \mu^*[$, problem (1) has at least one nontrivial weak solution, provided that (h_1) and (h_2) hold.*

Proof. We consider the Banach space $X := W_0^{1,p(x)}(\Omega)$ and work with A, B defined above (see Section 2). We wish to apply Theorem 1, so we check its hypotheses in the case $r = 1$. Supported by the fact that $A, B \in C^1(X, \mathbb{R})$ and the compactness of B' , we point out that J_μ satisfies the $(P.S.)^{[r]}$ -condition for all $r > 0$ (see Bonanno [1]). This means that Theorem 1(jj) holds. Let $m := \frac{2\pi^{n/2}}{n\Gamma(n/2)}$ be the measure of the unit ball of \mathbb{R}^n , where Γ is the Gamma function. Now, fixed $\mu \in]0, \mu^*[$, by using the hypothesis (h_2) , we have

$$(2) \quad 0 < \zeta_\mu < \min \left\{ 1, \left(\frac{q^-}{m\theta^n(1-\eta^n)([\omega_\theta]^p + [\omega_\theta]^q)} \right)^{1/q^-} \right\}$$

so that

$$(3) \quad \frac{q^- \eta_\theta^n \inf_{x \in \Omega} H(x, \zeta_\mu)}{(1 - \eta_\theta^n)([\omega_\theta]^p + [\omega_\theta]^q)(\zeta_\mu)^{q^-}} > \frac{1}{\mu}.$$

If $|\cdot|$ denotes the Euclidean norm on \mathbb{R}^n , define $u_\mu : \Omega \rightarrow \mathbb{R}$ by

$$u_\mu(x) = \begin{cases} 0 & x \in \Omega \setminus \mathcal{B}(x_0, \theta), \\ \zeta_\mu & x \in \mathcal{B}(x_0, \eta_\theta \theta), \\ \zeta_\mu \omega_\theta(\theta - |x - x_0|) & x \in \mathcal{B}(x_0, \theta) \setminus \mathcal{B}(x_0, \eta_\theta \theta). \end{cases}$$

We have

$$\begin{aligned} A(u_\mu) &= \int_\Omega \frac{1}{p(x)} |\nabla u(x)|^{p(x)} dx + \int_\Omega \frac{1}{q(x)} |\nabla u(x)|^{q(x)} dx \\ &\leq \frac{1}{p^-} (\zeta_\mu)^{p^-} m [\omega_\theta]^p (1 - \eta_\theta^n) \theta^n + \frac{1}{q^-} (\zeta_\mu)^{q^-} m [\omega_\theta]^q (1 - \eta_\theta^n) \theta^n \\ &\leq \frac{1}{q^-} (\zeta_\mu)^{q^-} m (1 - \eta_\theta^n) \theta^n ([\omega_\theta]^p + [\omega_\theta]^q), \\ \Rightarrow A(u_\mu) &\leq \frac{1}{q^-} (\zeta_\mu)^{q^-} m (1 - \eta_\theta^n) \theta^n ([\omega_\theta]^p + [\omega_\theta]^q) < 1 \quad (\text{by (2)}). \end{aligned}$$

Also, we get

$$\begin{aligned} B(u_\mu) &\geq \int_{\mathcal{B}(x_0, \eta_\theta \theta)} H(x, u_\mu) dx \geq \inf_{x \in \Omega} H(x, \zeta_\mu) m \eta_\theta^n \theta^n \quad (\text{by (3)}), \\ \Rightarrow \frac{B(u_\mu)}{A(u_\mu)} &\geq \frac{q^- \eta_\theta^n \inf_{x \in \Omega} H(x, \zeta_\mu)}{(1 - \eta_\theta^n)([\omega_\theta]^p + [\omega_\theta]^q)(\zeta_\mu)^{q^-}} > \frac{1}{\mu}. \end{aligned}$$

For each $u \in A^{-1}(]-\infty, 1])$, by using Theorem 3, we have

$$\begin{aligned} \|\nabla u\|_{L^{p(x)}(\Omega)} &\leq [\rho_p(|\nabla u(x)|)]^{1/p} \leq \left[p^+ \int_\Omega \frac{|\nabla u(x)|^{p(x)}}{p(x)} dx \right]^{1/p} \\ &\leq [p^+]^{1/p} = (p^+)^{1/p^-}, \\ (4) \quad \Rightarrow \|u\| &= \|\nabla u\|_{L^{p(x)}(\Omega)} \leq (p^+)^{1/p^-}. \end{aligned}$$

By Proposition 1 and Theorem 3, we have

$$(5) \quad \int_{\Omega} |u(x)|^{\beta(x)} dx = \rho_{\beta}(u) \leq [\|u\|_{L^{\beta(x)}(\Omega)}]^{\beta} \leq [\chi_{\beta} \|u\|]^{\beta} \quad \text{for all } u \in X.$$

By hypothesis (h_1) , (4), (5) and the compact embedding $X \hookrightarrow L^1(\Omega)$, for each $u \in A^{-1}(]-\infty, 1])$ we get

$$\begin{aligned} B(u) &\leq b_1 \int_{\Omega} |u(x)| dx + \frac{b_2}{\beta^-} \int_{\Omega} |u(x)|^{\beta(x)} dx \leq b_1 \chi_1 \|u\| + \frac{b_2}{\beta^-} [\chi_{\beta}]^{\beta} [\|u\|]^{\beta} \\ &\leq b_1 \chi_1 (p^+)^{1/p^-} + \frac{b_2}{\beta^-} [\chi_{\beta}]^{\beta} (p^+)^{\beta^+/p^-}, \\ \Rightarrow \sup_{A(u) \leq 1} B(u) &\leq b_1 \chi_1 (p^+)^{1/p^-} + \frac{b_2}{\beta^-} [\chi_{\beta}]^{\beta} (p^+)^{\beta^+/p^-} = \frac{1}{\mu^*} < \frac{1}{\mu}, \\ \Rightarrow \sup_{A(u) \leq 1} B(u) &< \frac{1}{\mu} < \frac{B(u_{\mu})}{A(u_{\mu})}. \end{aligned}$$

So, Theorem 1(j) is satisfied. Since $\mu \in]\frac{A(u_{\mu})}{B(u_{\mu})}, \frac{1}{\sup_{A(u) \leq 1} B(u)}[$, we apply Theorem 1 to conclude the existence of a local minimum point v_{μ} for J_{μ} with $0 < A(v_{\mu}) < 1$. Thus, v_{μ} is a nontrivial weak solution of (1). \square

Next, we prove a multiplicity result for the problem (1). We impose the following conditions on $H : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$:

(h_3) there exist $c \in [0, +\infty[$ and $\gamma \in C(\bar{\Omega})$ with $1 < \gamma^- \leq \gamma^+ < p^-$ such that

$$H(x, t) \leq c(1 + |t|^{\gamma(x)}) \quad \text{for all } x \in \Omega, \text{ all } t \in \mathbb{R};$$

(h_4) $H(x, t) \geq 0$ for all $x \in \Omega$, all $t \in \mathbb{R}^+$;

(h_5) there exist $r > 0$ and $\zeta > 0$ with $r < \frac{1}{p^+}(\zeta)^{p^+} m(1 - \eta_{\theta}^n) \theta^n ([\omega_{\theta}]_p + [\omega_{\theta}]_q)$ such that

$$R := \frac{1}{r} \{b_1 \chi_1 (p^+)^{1/p^-} [r]^{1/p} + \frac{b_2}{\beta^-} [\chi_{\beta}]^{\beta} (p^+)^{\beta^+/p^-} [[r]^{1/p}]^{\beta}\} < \frac{q^- \eta_{\theta}^n \inf_{x \in \Omega} H(x, \zeta)}{(1 - \eta_{\theta}^n)([\omega_{\theta}]^p + [\omega_{\theta}]^q)(\zeta)^{q^-}}.$$

Example 2. Let $\gamma \in C(\bar{\Omega})$ with $1 < \gamma^- \leq \gamma^+ < p^-$ for each $x \in \Omega$, and $g : \Omega \rightarrow \mathbb{R}$ such that $g \in L^1(\Omega)$ with $0 < a \leq g(x) \leq b_1 < +\infty$ for all $x \in \Omega$. Consider the function $h : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ given as

$$h(x, t) = \begin{cases} g(x) + ct^{\gamma(x)-1} & t > 0, \\ 0 & t \leq 0. \end{cases}$$

Clearly h satisfies (h_1) and H satisfies (h_3) , (h_4) and (h_5) .

Now, we can have the existence theorem producing at least three weak solutions.

Theorem 5. For each $\mu \in \Theta := \left] \frac{(1 - \eta_{\theta}^n)([\omega_{\theta}]^p + [\omega_{\theta}]^q)(\zeta)^{q^-}}{q^- \eta_{\theta}^n \inf_{x \in \Omega} H(x, \zeta)}, \frac{1}{R} \right]$, problem (1) has at least three weak solutions, provided that (h_1) , (h_3) , (h_4) and (h_5) hold.

Proof. From same notation and similar arguments as in the proof of Theorem 4, we deduce that the requirements of Theorem 2 (see also Remark 4) are met.

For r, ζ satisfying (h_5) , we define $w : \Omega \rightarrow \mathbb{R}$ by

$$w(x) = \begin{cases} 0 & x \in \Omega \setminus \mathcal{B}(x_0, \theta), \\ \zeta & x \in \mathcal{B}(x_0, \eta_\theta \theta), \\ \zeta \omega_\theta (\theta - |x - x_0|) & x \in \mathcal{B}(x_0, \theta) \setminus \mathcal{B}(x_0, \eta_\theta \theta). \end{cases}$$

From (h_4) , in the lines of the proof of Theorem 4, we get

$$\frac{B(w)}{A(w)} \geq \frac{q^- \eta_\theta^n \inf_{x \in \Omega} H(x, \zeta)}{(1 - \eta_\theta^n)([\omega_\theta]^p + [\omega_\theta]^q)(\zeta)^{q^-}}.$$

Also, we have

$$A(w) \geq \frac{1}{p^+} (\zeta)^{p^+} m(1 - \eta_\theta^n) \theta^n ([\omega_\theta]_p + [\omega_\theta]_q).$$

Since $r < \frac{1}{p^+} (\zeta)^{p^+} m(1 - \eta_\theta^n) \theta^n ([\omega_\theta]_p + [\omega_\theta]_q)$, we get $r < A(w)$. So, we have

$$(6) \quad \int_{\Omega} |u(x)|^{\beta(x)} dx = \rho_\beta(u) \leq [\|u\|_{L^{\beta(x)}(\Omega)}]^\beta \leq [\chi_\beta \|u\|]^\beta \quad \text{for all } u \in X,$$

(by Proposition 1 and Theorem 3). For each $u \in A^{-1}([-\infty, r])$, thanks to Theorem 3 we get

$$\begin{aligned} \|\nabla u\|_{L^{p(x)}(\Omega)} &\leq [\rho_p(|\nabla u(x)|)]^{1/p} \leq \left[p^+ \int_{\Omega} \frac{|\nabla u(x)|^{p(x)}}{p(x)} dx \right]^{1/p} \\ &\leq [p^+ A(u)]^{1/p} \leq [p^+ r]^{1/p} = (p^+)^{1/p^-} [r]^{1/p}, \\ (7) \quad &\Rightarrow \|u\| = \|\nabla u\|_{L^{p(x)}(\Omega)} \leq (p^+)^{1/p^-} [r]^{1/p}. \end{aligned}$$

Now, (h_1) , (6), (7) and the compact embedding $X \hookrightarrow L^1(\Omega)$, for each $u \in A^{-1}([-\infty, r])$, give us

$$\begin{aligned} B(u) &\leq b_1 \int_{\Omega} |u(x)| dx + \frac{b_2}{\beta^-} \int_{\Omega} |u(x)|^{\beta(x)} dx \leq b_1 \chi_1 \|u\| + \frac{b_2}{\beta^-} [\chi_\beta]^\beta [\|u\|]^\beta \\ &\leq b_1 \chi_1 (p^+)^{1/p^-} [r]^{1/p} + \frac{b_2}{\beta^-} [\chi_\beta]^\beta (p^+)^{\beta^+/p^-} [[r]^{1/p}]^\beta, \\ &\Rightarrow \frac{1}{r} \sup_{A(u) \leq r} B(u) \leq \frac{1}{r} \left\{ b_1 \chi_1 (p^+)^{1/p^-} [r]^{1/p} + \frac{b_2}{\beta^-} [\chi_\beta]^\beta (p^+)^{\beta^+/p^-} [[r]^{1/p}]^\beta \right\}, \\ &\Rightarrow \frac{1}{r} \sup_{A(u) \leq r} B(u) < \frac{B(w)}{A(w)}, \\ &\Rightarrow \text{Theorem 2(i) is true.} \end{aligned}$$

We wish to show that $J_\mu := A - \mu B$, for each $\mu > 0$, is coercive. We point out that

$$(8) \quad \int_{\Omega} |u(x)|^{\gamma(x)} dx = \rho_\gamma(u) \leq [\|u\|_{L^{\gamma(x)}(\Omega)}]^\gamma \leq [\chi_\gamma \|u\|]^\gamma \quad \text{for all } u \in X,$$

(by Proposition 1 and Theorem 3), where χ_γ is the constant for the compact embedding $X \hookrightarrow L^{\gamma(x)}(\Omega)$. From (h_3) and (8), for each $u \in X$ with $\|u\| \geq \max\{1, \chi_\gamma^{-1}\}$ we have

$$\begin{aligned} B(u) &= \int_{\Omega} H(x, u(x)) dx \leq \int_{\Omega} c(1 + |u(x)|^{\gamma(x)}) dx \\ &\leq c\{|\Omega| + [\chi_\gamma \|u\|]^\gamma\} = c\{|\Omega| + [\chi_\gamma]^\gamma \|u\|^{\gamma^+}\}, \quad |\Omega| \text{ is the Lebesgue measure of } \Omega. \end{aligned}$$

So, we have

$$\begin{aligned} J_\mu(u) &\geq \int_{\Omega} \frac{1}{p(x)} |\nabla u(x)|^{p(x)} dx + \int_{\Omega} \frac{1}{q(x)} |\nabla u(x)|^{q(x)} dx - \mu c\{|\Omega| + [\chi_\gamma]^\gamma \|u\|^{\gamma^+}\} \\ &\geq \frac{1}{p^+} \|u\|^{p^-} - \mu c\{|\Omega| + [\chi_\gamma]^\gamma \|u\|^{\gamma^+}\}, \\ &\Rightarrow J_\mu \text{ is coercive.} \end{aligned}$$

By Theorem 2, $\Theta \subset \left[\frac{A(w)}{B(w)}, \frac{r}{\sup_{A(u) \leq r} B(u)} \right]$ implies that J_μ (for each $\mu \in \Theta$) has at least three critical points, which are weak solutions of (1). \square

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