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On the seven non-isomorphic solutions of the fifteen schoolgirl problem

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ABSTRACT

In this paper we give a simple and effective tool to analyze a given Kirkman triple system of order 15 and determine which of the seven well-known non-isomorphic KTS(15)s it is isomorphic to.
Our technique refines and improves the *lacing* of distinct parallel classes introduced by F. N. Cole, by means of the notion of *residual triple* defined by G. Falcone and the present author in a previous paper.
Unlike Cole's original lacing scheme, our algorithm allows one to distinguish two KTS(15)s also in the harder case where the two systems have the same underlying Steiner triple system. In the special case where the common STS is #19, an alternative method is given by means of the 1-factorizations of the complete graph K_8 associated to the two KTSs.
Moreover, we present three new visual solutions to the schoolgirl problem, and we catalogue most of the classical (or interesting) solutions in the literature in terms of what KTS(15)s they are isomorphic to.
This paper provides background on a classical topic, while shedding new light on the problem as well.

1. Introduction

The *fifteen schoolgirl problem* is one of the most important, celebrated and fascinating problems in combinatorics and recreational mathematics. It was proposed by Thomas Penyngton Kirkman in 1850 as a challenge in a popular annual publication [35, p. 48], and from the very beginning to the present day it has always intrigued both professional and amateur mathematicians, as well as puzzle lovers. The problem is to find a weekly schedule for fifteen girls walking out daily in five rows of three, in such a way that no two girls shall walk in the same row more than once (equivalently, any girl shall walk exactly once in the same row with each of the other girls).

The first published solution, due to Arthur Cayley, appeared in June 1850 [13], immediately followed by Kirkman's own solution in August 1850 [36] (replicated in [37, p. 260] and [38, p. 48]). The latter solution (as Kirkman himself revealed in [36, p. 170]) was implicit in the landmark and pioneering paper [34], which appeared three years earlier, where Kirkman ingeniously combined a Fano plane with a Room square of side 7 (which he later called a "curious arrangement" in [37, p. 261]). In fact, Cayley became aware of Kirkman's work when he was asked to referee the 1847 paper [34], about which he was enthusiastic [20, p. 143]. This success kept up during the second half of the century, when the schoolgirl problem and its variants were regularly given solutions both in scholarly journals and in recreational publications [26].

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In order to rephrase the problem in the modern terminology of combinatorial design theory, we need some preliminary definitions (see, e.g., [4,7,15,17,62]). A *Steiner triple system* of order v, denoted STS(v), is a pair $(\mathcal{V}, \mathcal{B})$, where $\mathcal{V}$ is a set of v elements (*points*), and $\mathcal{B}$ is a collection of (unordered) triples (*blocks*) of elements of $\mathcal{V}$, with the property that each (unordered) pair of points occurs as a subset of precisely one triple in $\mathcal{B}$. When v is a multiple of 3, a *parallel class* is a subcollection of $v/3$ mutually disjoint triples in $\mathcal{B}$ that partitions the point-set $\mathcal{V}$. When the entire collection $\mathcal{B}$ of triples can in turn be partitioned into parallel classes, such a partition is called a *resolution* (or *parallelism*) of the STS, and the STS is said to be *resolvable*. If $(\mathcal{V}, \mathcal{B})$ is an STS(v) and $\mathcal{R}$ is a resolution of it, then $(\mathcal{V}, \mathcal{B}, \mathcal{R})$ is a *Kirkman triple system* of order v, denoted KTS(v), and $(\mathcal{V}, \mathcal{B})$ is its *underlying* STS. In this abstract setting, the schoolgirl problem amounts to asking whether there exists a KTS of order 15 (note that a resolution consists of seven parallel classes, each containing five triples). In fact, the theory of resolvable designs stems precisely from Kirkman's schoolgirl problem.

An *isomorphism* from an STS $(\mathcal{V}_1, \mathcal{B}_1)$ to an STS $(\mathcal{V}_2, \mathcal{B}_2)$ is a one-to-one map π from $\mathcal{V}_1$ onto $\mathcal{V}_2$ that preserves triples: more precisely, $T = \{x, y, z\} \in \mathcal{B}_1$ if and only if $\pi(T) = \{\pi(x), \pi(y), \pi(z)\} \in \mathcal{B}_2$. An isomorphism from a KTS $(\mathcal{V}_1, \mathcal{B}_1, \mathcal{R}_1)$ to a KTS $(\mathcal{V}_2, \mathcal{B}_2, \mathcal{R}_2)$ is required, in addition, to preserve parallel classes: for any parallel class $\mathcal{C}$ in $\mathcal{R}_1$, the set $\{\pi(T) \mid T \in \mathcal{C}\}$ is a parallel class in $\mathcal{R}_2$. An *automorphism* is an isomorphism from an STS/KTS to itself.

A KTS(v) $(\mathcal{V}, \mathcal{B}, \mathcal{R})$ is *cyclic* if it admits an automorphism α consisting of a single cycle of length v such that α preserves both the set of triples $\mathcal{B}$ and the resolution $\mathcal{R}$ [46]. For instance, there exists no cyclic KTS(v) for $v = 15$ [46]. It is worth noting, however, that the word "cyclic" was used with a different meaning in the early literature on Kirkman's problem in the second half of the XIX century [2,5,6,52]; more precisely, a solution to the schoolgirl problem was said to be "cyclic" if, in modern terms, it was 2-*rotational*, that is, with an automorphism having of a single fixed point and two cycles of length 7, inducing a cyclic permutation of the seven parallel classes [3] (see also Example 5 in the final Appendix of the present paper). We will use ourselves the word cyclic in this earlier sense throughout the rest of the paper.

The distinction between KTSs and resolvable STSs is that there can exist non-isomorphic KTSs that share the same underlying STS. Of the eighty non-isomorphic STS(15)s [67], exactly four are resolvable [18] (cf. [15, p. 66], [17, p. 370]). Moreover, three of these four STSs underlie two non-isomorphic KTSs, whereas the fourth STS underlies a unique KTS, which leads to an overall number of seven non-isomorphic KTSs of order 15. The seven solutions are given here in Table 1, using the numbering of the underlying STSs as in [42] and [15, p. 67] (where, in the latter case, the solutions that are numbered 15a and 15b should be instead 19a and 19b, respectively. See also [61, Appendix, pp. 389-390]).

It must be said, in this respect, that a much more difficult problem than finding a KTS(15) is determining whether two given KTSs of order 15 are isomorphic or not. In fact, Kirkman himself erroneously thought at first that his arrangement was "the only possible solution" up to permutation [38], and Wesley S.B. Woolhouse, who was the first to raise the isomorphism issue, initially thought that all solutions were necessarily cyclic [68,69]. In 1860, Benjamin Peirce proved that there were three possible cyclic systems [52], corresponding to the solutions by Kirkman [36] and Cayley [13] and to one of Anstice's solutions [5]. In 1881 eleven solutions of the schoolgirl problem were published [12], but it was only in 1917 [47] and 1922 [18] that it was proved that only seven of them were non-isomorphic, precisely those given in 1862 and 1863 by Woolhouse [69,70] (an alternative proof, using graph theory, was given in [57]). Nevertheless, as far as we are aware, the literature lacks a description of the seven solutions that is easily understandable to the general reader.

One wishes to find simple and effective tools to establish whether two given KTS(15)s are isomorphic or not, and, possibly, determine which of the seven types they belong to. The first possibility is that the two systems do not have the same underlying STS. This can be established, for instance, by considering the two KTSs just as Steiner triple systems and computing, for each of them, some isomorphism-invariant STS parameter, such as the order of the automorphism group, the number of parallel classes, the number of Pasch configurations, or the number of Fano planes (see, e.g., [42] and [15, Table 1.29, p. 32]). Each of these parameters identifies one of the four resolvable STS(15)s uniquely, with the only exception of the number of Fano planes, which is equal to 1 for both systems #19 and #61.

However, a simpler and more effective tool to distinguish two KTSs of order 15, with two distinct underlying STSs, is using the notion of *lacing* of parallel classes, introduced by Frank Nelson Cole in [18] (although already suggested in [69], in the case of cyclic systems). We say that two distinct parallel classes of a KTS(15) are *laced in the mode* (α) if there exist two triples in one class and two triples in the other class, such that the four triples are mutually disjoint. Otherwise, there exists only one other possible lacing, in which case we say that the two parallel classes are *laced in the mode* (β). For instance, in the KTS numbered 1a (in Table 1) the triples abc, ehm, dko, fjl are mutually disjoint, hence the parallel classes Monday and Tuesday are laced in the mode (α). We wish to mention that there exists an alternative proof, by Alexander Rosa, that there exist only two possible lacings: the block-intersection graph of two distinct parallel classes in a KTS(15) is a bipartite cubic graph of order 10, and there exist exactly two such graphs, up to isomorphism [57].

As we mentioned above, the interlacing scheme of distinct parallel classes allows one to identify the underlying STS of a given KTS(15) [18] and, therefore, to distinguish two KTS(15)s with distinct underlying STSs. Indeed, in the systems 1a and 1b any two distinct parallel classes have only the lacing (α). In the systems 7a and 7b (in Table 1) the parallel class Monday is in lacing (α) with all the other parallel classes, whereas each of the latter has two (α) lacings and four (β) lacings. In the systems 19a and 19b (in Table 1) the parallel classes Friday, Saturday, and Sunday have the lacing (α) with each other, whereas all the other lacings are of type (β). Finally, in the system 61 the lacings of distinct parallel classes are all of type (β).

In the case where two KTS(15)s have the same underlying STS (up to isomorphism), the interlacing scheme of distinct parallel classes is the same for the two systems, hence it is no longer sufficient to distinguish them, nor can the two systems

Table 1
The seven solutions of Kirkman's schoolgirl problem.

#	Mon	Tue	Wed	Thu	Fri	Sat	Sun
	abc	ahi	ajk	ade	afg	alm	ano
	djn	beg	bmo	bln	bhj	bik	bdf
1a	ehm	cmn	cef	cij	clo	cdg	chk
	fio	dko	dhl	fkm	dim	ejo	eil
	gkl	fjl	gin	gho	ekn	fhn	gjm
	abc	ahi	ajk	ade	afg	alm	ano
	djn	beg	bmo	bik	bln	bdf	bhj
1b	ehm	cmn	cef	clo	chk	cij	cdg
	fio	dko	dhl	fhn	dim	ekn	eil
	gkl	fjl	gin	gjm	ejo	gho	fkm
	abc	ahi	ajk	ade	afg	alm	ano
	djo	bdf	beg	bln	bmo	bik	bhj
7a	eim	clo	cmn	cij	chk	cdg	cef
	fkl	ekn	dhl	fhm	din	eho	dkm
	ghn	gjm	fio	gko	ejl	fjn	gil
	abc	ahi	ajk	ade	afg	alm	ano
	djo	bdf	beg	bmo	bln	bhj	bik
7b	eim	clo	cmn	chk	cij	cef	cdg
	fkl	ekn	dhl	fjn	dkm	din	ejl
	ghn	gjm	fio	gil	eho	gko	fhm
	ade	afg	alm	ano	abc	ahi	ajk
	bik	bhj	bdf	beg	dho	bmo	bln
19a	chl	cin	cko	cjm	ekn	cef	cdg
	fmn	dkm	eij	dil	fjl	djn	ehm
	gjo	elo	ghn	fhk	gim	gkl	fio
	ade	afg	alm	ano	abc	ahi	ajk
	bik	bhj	bdf	beg	djn	bmo	bln
19b	chl	cin	cko	cjm	ehm	cdg	cef
	fmn	dkm	eij	dil	fio	ekn	dho
	gjo	elo	ghn	fhk	gkl	fjl	gim
	abc	ade	afg	ahi	ajk	alm	ano
	dik	bil	bhj	beg	bmo	bkn	bdf
61	ejn	cjm	cio	cln	cef	cdg	chk
	flo	fhn	dmn	djo	dhl	eho	eim
	ghm	gko	ekl	fkm	gin	fij	gjl

be distinguished by their automorphism groups, which are also the same. However, in some cases the automorphisms can nonetheless be used to distinguish the two systems [18]. Indeed, the automorphisms of 1a (in Table 1) are transitive on all points except on the point i, which is fixed under all automorphisms, whereas the automorphisms of 1b are transitive in seven and in eight points. The automorphisms of 7a are transitive in three and in twelve points, whereas the automorphisms of 7b are transitive in three, in four, and in eight points. An interpretation of these facts will be given in Remark 2.6(7), in the light of our forthcoming results.

On the other hand, for both systems 19a and 19b the automorphisms are precisely the same as for the underlying STS [18]: in particular, a single permutation of the 15 points is a KTS-automorphism of 19a if and only if it is a KTS-automorphism of 19b. Therefore the two KTSs cannot be distinguished by considering the lacings of distinct parallel classes, nor by looking at the orbits of their automorphisms. To the best of our knowledge, no simple method to distinguish the two systems is available in the literature.

Alternatively, a "standard" idea to test the isomorphism of two KTS(15)s is to compare the corresponding bipartite graphs as follows (for an early mention of this technique, see [43, §3.12]). For each of the two KTS(15)s, one constructs the associated *pairwise balanced design* (PBD) (see, e.g., [62, Ch. 7]), by adding a new point x_i to every block in the ith parallel class, and then adding a block consisting of the seven new points. Such a PBD has one block of size seven, while all the remaining blocks have size four. Then one constructs the bipartite *point-block incidence graph* of the PBD. The vertices of this graph comprise the points and the blocks of the PBD. A point x is joined to a block B if and only if x is in the block B. Finally, one tests the two resulting bipartite graphs using a graph isomorphism program such as Brendan McKay's nauty program [44].

In Section 2 of this paper we give a simple and effective tool to establish, in all possible cases, whether two given KTS(15)s are isomorphic or not, independently of the underlying STSs, by determining for any KTS(15) the system in Table 1 isomorphic to it. Because of the previous considerations, our method is particularly significant in the special case where the underlying STS is #19 for both systems. Moreover, in the case where the underlying STS of a given KTS(15) is either #1 or #7, our algorithm is even simpler, and allows one to settle the isomorphism problem in a much faster and more effective way than with the automorphism method described above. In the former case (#1), the algorithm had already appeared in a paper by G. Falcone and the present author, which contained, in addition, a visual description of the two non-isomorphic arrangements of the projective lines of PG$(3,2)$, by combining the fifteen simplicial elements of a tetrahedron [28].

Our technique refines and improves Cole's lacing of parallel classes, by means of the notion of *residual triple* implicitly introduced in [28]. Unlike in Cole [18], our algorithm allows one to use the lacing scheme to distinguish two KTS(15)s also in the harder case where the two systems have the same underlying STS. In the special case where the common STS is #19, we also present an alternative method in terms of the 1-factorizations of the complete graph K_8 that are naturally associated to the two KTSs.

In Section 3 we test the effectiveness and simplicity of our method, and exhibit a remarkable solution to the schoolgirl problem for each of the seven isomorphism classes. In fact, we go through the most significant solutions in the mathematical literature from 1850 to the present day, and we catalogue them by means of our algorithm in Theorem 2.4, which provides a central point of reference for the whole discussion. Among these solutions we include new octahedron-based representations of systems 7a and 7b.

The final Appendix, "Systems 1a and 1b revisited", is devoted to some very significant models of the two KTS(15)s whose underlying STS is the point-line design of the projective geometry PG$(3,2)$. In particular, we improve the well-known solution by A. Frost [30], and we reinterpret, in the light of our lacing algorithm, the solutions given by J.I. Hall [32], by identifying PG$(3,2)$ and the complete 3-design on seven points, and R. Ehrmann [27], by regarding PG$(3,2)$ as the projective completion of AG$(3,2)$. Finally, we describe a new algebraic model of the cyclic solutions 1a and 1b, and present two new visual representations of systems 1a and 1b, based on the complete graph on six points and on the regular triangular bipyramid.

2. The main results

In this section we describe how to determine, for a given Kirkman triple system of order 15, which of the seven systems in Table 1 it is isomorphic to. In order to do so, we extend Cole's lacing scheme [18] by means of the notion of *residual triple*, which was implicitly introduced in [28].

Definition 2.1. ([18]) Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two distinct parallel classes of a KTS(15). We say that $\mathcal{C}_1$ and $\mathcal{C}_2$ are *laced in the mode* (α) if there exist two triples in $\mathcal{C}_1$ and two triples in $\mathcal{C}_2$ such that the four triples are mutually disjoint. Otherwise, we say that $\mathcal{C}_1$ and $\mathcal{C}_2$ are *laced in the mode* (β).

Definition 2.2. Let $(\mathcal{V},\mathcal{B},\mathcal{R})$ be a Kirkman triple system of order 15, and let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two distinct parallel classes in $\mathcal{R}$ that are laced in the mode (α). Let T_1, T_2 (respectively, T_3, T_4) be the two triples in $\mathcal{C}_1$ (respectively, in $\mathcal{C}_2$) such that the four triples T_1, T_2, T_3, T_4 are mutually disjoint. We say that a triple T in $\mathcal{B}$ is the *residual triple* of the lacing of $\mathcal{C}_1$ and $\mathcal{C}_2$ if the set $\{T_1, T_2, T_3, T_4, T\}$ is a partition of the point-set $\mathcal{V}$.

Remarks 2.3. 1) If T is the residual triple of the lacing of $\mathcal{C}_1$ and $\mathcal{C}_2$, as in Definition 2.2, then there exists a parallel class $\mathcal{C}_3$ in $\mathcal{R}$, different from $\mathcal{C}_1$ and $\mathcal{C}_2$, such that $T \in \mathcal{C}_3$. Indeed, if T_1, T_2, T_3, T_4 are as in Definition 2.2, and if, for instance, T were in $\mathcal{C}_2$, then the triples T_1, T_3, T_4, T could not be mutually disjoint, else T_1 would intersect one of the two triples in $\mathcal{C}_2$ different from T_3, T_4, and T in two points, thereby contradicting the definition of Steiner triple system.

For instance, in the KTS numbered 1a (in Table 1), Monday and Tuesday are laced in the mode (α), and the corresponding residual triple is the triple gin, in Wednesday.

2) The STS(15) numbered as #19 contains a unique Fano plane $\mathcal{S}$ (see, e.g., [15, Table 1.29, p. 32]). If the 35 triples of the system are given as in Table 1 above, then the seven triples in $\mathcal{S}$ are precisely abc, ade, afg, bdf, beg, cdg, cef. Moreover, each parallel class in systems 19a and 19b consists of a (unique) triple in $\mathcal{S}$ and four triples of the form sxy, where s is in $\mathcal{S}$ and x, y are not in $\mathcal{S}$. This fact will be essential in the following Theorem 2.4.

3) In [28], where only systems 1a and 1b are considered, the fact that the four triples in Definitions 2.1 and 2.2 are pairwise disjoint is referred to as the *four skew triples property*, and the residual triples are called *unconsidered triples*, in that they do not belong to any set of four mutually disjoint triples in a lacing of type (α).

Theorem 2.4. *Let $(\mathcal{V},\mathcal{B},\mathcal{R})$ be a Kirkman triple system of order 15, with $\mathcal{R} = \{\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_7\}$. Then one, and only one, of the following four cases occurs.*

1. (a) *Any two distinct parallel classes in $\mathcal{R}$ are laced in the mode (α). In this case, the KTS is isomorphic to either system 1a or system 1b.*
   (b) *For any pair of distinct classes $\mathcal{C}_i, \mathcal{C}_j$ in $\mathcal{R}$, there exists a class $\mathcal{C}_k$ in $\mathcal{R}$, different from $\mathcal{C}_i$ and $\mathcal{C}_j$, such that the lacing of any two parallel classes in $\{\mathcal{C}_i, \mathcal{C}_j, \mathcal{C}_k\}$ has a residual triple in the third class.*
   (c) *The set of all residual triples of the lacings of distinct parallel classes consists of precisely seven triples. Moreover, this set consists of either the seven triples containing p, for some point p in $\mathcal{V}$, or the seven triples of a Fano plane. In the former case the KTS is isomorphic to system 1a, in the latter case it is isomorphic to system 1b.*

2. (a) *There exists a unique parallel class in $\mathcal{R}$, say $\mathcal{C}_1$, that is laced in the mode (α) with each of the other six classes in $\mathcal{R}$. Each of the latter classes has two (α) lacings and four (β) lacings. In this case, the KTS is isomorphic to either system 7a or system 7b.*
   (b) *Up to permutation of the classes $\mathcal{C}_2, \ldots, \mathcal{C}_7$, any two parallel classes in any of the three sets $\{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3\}$, $\{\mathcal{C}_1, \mathcal{C}_4, \mathcal{C}_5\}$, $\{\mathcal{C}_1, \mathcal{C}_6, \mathcal{C}_7\}$ are laced in the mode (α), with a residual triple in the third class of the set.*
   (c) *The set of all residual triples of the lacings of type (α) consists of precisely seven triples. Moreover, this set consists of either seven triples whose union is the point-set $\mathcal{V}$, or the seven triples of a Fano plane. In the former case the KTS is isomorphic to system 7a, in the latter case it is isomorphic to system 7b.*
   Alternatively, if $\mathcal{C}_i$ and $\mathcal{C}_j$ are any two classes laced in the mode (β), and if the two residual triples in $\mathcal{C}_i$ and $\mathcal{C}_j$ are disjoint (resp., intersect in one point), then the KTS is isomorphic to system 7a (resp., to system 7b).
3. (a) *There exist three distinct parallel classes in $\mathcal{R}$, say $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$, that are laced in the mode (α) with each other. Any other pair of distinct parallel classes in $\mathcal{R}$ is laced in the mode (β). In this case, the KTS is isomorphic to either system 19a or system 19b.*
   (b) *The lacing of any two parallel classes in $\{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3\}$ has a residual triple in the third class.*
   (c) *There exists a (unique) Fano plane $\mathcal{S}$ that is a subdesign of $(\mathcal{V}, \mathcal{B})$ and whose seven triples include the three residual triples of the lacings of type (α). Given a class $\mathcal{C}_i$ in $\{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3\}$, and a class $\mathcal{C}_j$ in $\{\mathcal{C}_4, \mathcal{C}_5, \mathcal{C}_6, \mathcal{C}_7\}$, let x, y be any two points, not in $\mathcal{S}$, lying in the same triple in $\mathcal{C}_j$, and let z, w be the two corresponding points, not in $\mathcal{S}$, such that $\mathcal{C}_i$ has a triple containing x, z and a triple containing y, w. If z, w are in the same triple in $\mathcal{C}_j$, then the KTS is isomorphic to system 19a, else it is isomorphic to system 19b.*
4. *Any two distinct parallel classes in $\mathcal{R}$ are laced in the mode (β). In this case, the KTS is isomorphic to system 61.*

Proof. The statements *1(a), 2(a), 3(a),* and *4* are in [18], whereas the statements *1(b)* and *1(c)* are proved in [28] (a somewhat similar argument, although not fully explicit, is given in [69, pp. 86-87], where the word "collating" is used instead of "lacing"). Thus we are left with the proofs of *2(b), 2(c), 3(b),* and *3(c).*

Let us first consider the case where *2(a)* holds. Then the KTS is isomorphic to either system 7a or system 7b in Table 1. The lacings of type (α) in 7a and in 7b are those listed in Tables 2 and 3, respectively.

Table 2
The lacings of type (α) in system 7a.

7a	Parallel classes		Four mutually disjoint triples				Residual triple	
	Mon	Tue	eim	ghn	bdf	clo	ajk	(in Wed)
	Mon	Wed	djo	fkl	beg	cmn	ahi	(in Tue)
	Tue	Wed	ekn	gjm	dhl	fio	abc	(in Mon)
	Mon	Thu	fkl	ghn	ade	cij	bmo	(in Fri)
	Mon	Fri	djo	eim	afg	chk	bln	(in Thu)
	Thu	Fri	fhm	gko	din	ejl	abc	(in Mon)
	Mon	Sat	djo	ghn	alm	bik	cef	(in Sun)
	Mon	Sun	eim	fkl	ano	bhj	cdg	(in Sat)
	Sat	Sun	eho	fjn	dkm	gil	abc	(in Mon)

Table 3
The lacings of type (α) in system 7b.

7b	Parallel classes		Four mutually disjoint triples				Residual triple	
	Mon	Tue	eim	ghn	bdf	clo	ajk	(in Wed)
	Mon	Wed	djo	fkl	beg	cmn	ahi	(in Tue)
	Tue	Wed	ekn	gjm	dhl	fio	abc	(in Mon)
	Mon	Thu	fkl	ghn	ade	bmo	cij	(in Fri)
	Mon	Fri	djo	eim	afg	bln	chk	(in Thu)
	Thu	Fri	fjn	gil	dkm	eho	abc	(in Mon)
	Mon	Sat	djo	ghn	alm	cef	bik	(in Sun)
	Mon	Sun	eim	fkl	ano	cdg	bhj	(in Sat)
	Sat	Sun	din	gko	ejl	fhm	abc	(in Mon)

Hence statement *2(b)* holds (with $\mathcal{C}_1 = \text{Monday}$). Moreover, in either case there is an overall number of seven residual triples. In the former case (system 7a), the union of the residual triples is the point-set $\mathcal{V} = \{\text{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o}\}$, whereas in the latter case (system 7b) the set $\{\text{abc, ahi, ajk, bhj, bik, chk, cij}\}$ of the residual triples is the block-set of a Fano plane. Also, any two residual triples belonging to two classes laced in the mode (β) are mutually disjoint in the former case, whereas they intersect in one point in the latter case. Therefore statement *2(c)* holds.

Let us finally consider the case where *3(a)* holds. Then the KTS is isomorphic to either system 19a or system 19b in Table 1. The lacings of type (α) in these two systems are those listed in Tables 4 and 5, respectively.

Hence statement *3(b)* holds, with $\{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3\} = \{\text{Friday, Saturday, Sunday}\}$. In either case, the three residual triples abc, cdg, cef belong to the set $\mathcal{S} = \{\text{abc, ade, afg, bdf, beg, cdg, cef}\}$, which is the block-set of the unique Fano plane of the underlying STS (see, e.g., [15, Table 1.29, p. 32]).

Table 4
The lacings of type (α) in system 19a.

19a	Parallel classes		Four mutually disjoint triples				Residual triple
	Fri	Sat	ekn	fjl	ahi	bmo	cdg (in Sun)
	Fri	Sun	dho	gim	ajk	bln	cef (in Sat)
	Sat	Sun	djn	gkl	ehm	fio	abc (in Fri)

Table 5
The lacings of type (α) in system 19b.

19b	Parallel classes		Four mutually disjoint triples				Residual triple
	Fri	Sat	djn	gkl	ahi	bmo	cef (in Sun)
	Fri	Sun	ehm	fio	ajk	bln	cdg (in Sat)
	Sat	Sun	ekn	fjl	dho	gim	abc (in Fri)

Let us consider the parallel classes Friday and Monday in system 19a (respectively, 19b) in Table 1. In either case, the points $x = i$, $y = k$ (which are not in $\mathcal{S}$) are in the same triple in Monday, whereas i, m (resp., i, o) and k, n (resp., k, l) are in the same triple in Friday, and m, n are in the same triple in Monday (resp., o, l are *not* in the same triple in Monday). The same happens for any other choice of the two points x, y, not in $\mathcal{S}$, in the same triple in Monday, and for any other choice of a pair of parallel classes in {Friday, Saturday, Sunday} × {Monday, Tuesday, Wednesday, Thursday}. Hence statement *3(c)* holds.

This completes the proof of the theorem. □

The next result gives an alternative method to distinguish two KTS(15)s in the harder case where their common underlying STS is #19. It is worth mentioning that the following characterization is interesting in its own right from a theoretical point of view and, moreover, its formulation appears to be simpler and more elegant than that in Theorem 2.4. However, as we will explain in Remark 2.6(5), for practical purposes the following method proves to be less effective than the algorithm given in Theorem 2.4.

Let $(\mathcal{V}, \mathcal{B}, \mathcal{R})$ be a KTS(15) isomorphic to either system 19a or system 19b, and let $\mathcal{P}$ $(\subseteq \mathcal{V})$ be the point-set of the unique Fano plane contained in the underlying STS (see, e.g., [15, Table 1.29, p. 32]). Let us also regard the eight points in $\mathcal{V} \setminus \mathcal{P}$ as the vertices of the complete graph K_8. One can construct a 1-factorization of the graph in a very simple and natural way. Each of the seven parallel classes of the KTS determines a 1-factor, which is obtained by removing from the class the (unique) triple in the Fano plane and the (unique) point in $\mathcal{P}$ in each of the remaining four triples. The seven resulting 1-factors form a 1-factorization of the graph, which is invariant, up to isomorphism, under the automorphisms of the KTS.

Our characterization will now follow from the complete invariant for the 1-factorizations of K_8 that is known as the "division invariant". Let us recall that three 1-factors of a 1-factorization are called a 3-*division* if the union of all three is a non-connected subgraph (equivalently, the three corresponding parallel classes in the KTS are of the form $\{Ax_1x_2, Bx_3x_4, \ldots\}$, $\{Cx_1x_3, Dx_2x_4, \ldots\}$, $\{Ex_1x_4, Fx_2x_3, \ldots\}$, with x_1, x_2, x_3, x_4 not in the Fano plane), whereas two 1-factors are called a *maximal* 2-*division* if their union is not connected and any additional 1-factor connects the resulting subgraph. It turns out that the number of 3-divisions and the number of maximal 2-divisions form together a complete invariant for the 1-factorizations of K_8. There are six 1-factorizations for K_8 and each has a different division structure [66, p. 91].

We now present the following result.

Proposition 2.5. *Let $(\mathcal{V}, \mathcal{B}, \mathcal{R})$ be a Kirkman triple system of order* 15 *isomorphic to either system 19a or system 19b, let $\mathcal{F}$ be the corresponding* 1*-factorization of the complete graph K_8, and let d_3 be the number of* 3*-divisions contained in $\mathcal{F}$. Then $(\mathcal{V}, \mathcal{B}, \mathcal{R})$ is isomorphic to system 19a (resp., 19b) if and only if $d_3 > 1$ (resp., $d_3 = 1$).*

Proof. If the blocks of the KTS are denoted as in Table 1 in the Introduction (see systems 19a and 19b), then {abc, ade, afg, bdf, beg, cdg, cef} is the block-set of the unique Fano plane contained in the underlying STS. Hence, for system 19a, the corresponding 1-factorization $\mathcal{F}$ is that given in Table 6.

Table 6
The 1-factorization for system 19a.

M	*hl*	*ik*	*jo*	*mn*
TU	*hj*	*in*	*km*	*lo*
W	*hn*	*ij*	*ko*	*lm*
TH	*hk*	*il*	*jm*	*no*
F	*ho*	*im*	*jl*	*kn*
SA	*hi*	*jn*	*kl*	*mo*
SU	*hm*	*io*	*jk*	*ln*

Then, by definition, M-TU-F and M-W-SU are two distinct 3-divisions of $\mathcal{F}$. For the sake of completeness, M-TH-SA, TU-W-SA, TU-TH-SU, W-TH-F, and F-SA-SU are also 3-divisions, hence $\mathcal{F}$ is isomorphic to the 1-factorization of K_8 that is usually denoted by $\mathcal{F}_1$ [66, p. 93].

Similarly, one finds, for system 19b, that $\mathcal{F}$ contains a unique 3-division, which is F-SA-SU (note that these are precisely the three parallel classes that are mutually laced in the mode (α); see Table 5). In passing, there exist also six maximal 2-divisions (corresponding to the six 2-subsets of the set {M,TU,W,TH}), hence $\mathcal{F}$ is isomorphic to the 1-factorization of K_8 that is usually denoted by $\mathcal{F}_4$.

This completes the proof of the proposition. □

Remarks 2.6. 1) It follows from Theorem 2.4 that in both systems 1a and 1b the seven parallel classes can be seen as the points of a Fano plane, whose blocks are precisely the sets $\{\mathcal{C}_i, \mathcal{C}_j, \mathcal{C}_k\}$ in property *1(b)* of the theorem.

If we refer to Table 1, then the blocks of the Fano plane are precisely M-TU-W, M-TH-SU, M-F-SA, TU-TH-SA, TU-F-SU, W-TH-F, W-SA-SU for system 1a, and M-TU-W, M-TH-F, M-SA-SU, TU-TH-SA, TU-F-SU, W-TH-SU, W-F-SA for system 1b.

Moreover, in the latter case (system 1b) the seven residual triples afg, ahi, ano, fhn, fio, gho, gin form the blocks of a Fano plane as well (cf. property *1(c)* in Theorem 2.4), and, interestingly enough, the map that sends a parallel class to the (unique) residual triple belonging to it is an isomorphism of the Fano plane of the seven parallel classes with the dual design of the Fano plane of the seven residual triples. For instance, M $\mapsto$ fio, TU $\mapsto$ ahi, W $\mapsto$ gin, whence M-TU-W $\mapsto$ i. Equivalently, three parallel classes in 1b are in the same block of the Fano plane if and only if the three residual triples that they contain have a common point.

Similarly, the proof of Proposition 2.5 shows that the parallel classes of system 19a form the points of a Fano plane whose blocks are M-TU-F, M-W-SU, M-TH-SA, TU-W-SA, TU-TH-SU, W-TH-F, F-SA-SU, corresponding to the seven 3-divisions. If $\mathcal{S}$ is the (unique) Fano plane contained in the underlying STS $(\mathcal{V}, \mathcal{B})$, then three parallel classes are in the same block if and only if the three triples in $\mathcal{S}$ contained in them have a common point in $\mathcal{V}$.

2) As a consequence of the previous Remark 1), it follows immediately that, given a KTS(15), it suffices to apply the lacing scheme to only three pairs of distinct parallel classes in order to determine whether the underlying STS is #1 and, in addition, whether in that case the system is isomorphic to system 1a or system 1b.

Indeed, if the lacing of two given parallel classes X, Y is of type (α) and the residual triple is in a class Z, then, given a fourth class U, if the lacing of X and U is of type (α), then the underlying STS is either #1 or #7 and the residual triple is necessarily in a fifth class V by properties *1(b)* and *2(b)* in Theorem 2.4. Now let us consider the lacing of Z and U. If this lacing is again of type (α), then the underlying STS is #1 and we may assume, by property *1(c)* in Theorem 2.4, that the residual triple in Z is, say, $\alpha\beta\gamma$ and the residual triple in V is $\alpha\delta\epsilon$. Now, by Remark 1) above, the third residual triple containing α, in system 1b, is precisely the residual triple of the lacing of Z and V (which is contained in a sixth parallel class W, different from X, Y, Z, U, V). Therefore, if the residual triple of the lacing of Z and U contains α, then the KTS is isomorphic to system 1a, else it is isomorphic to system 1b.

This observation will be repeatedly applied throughout the rest of the paper (especially in the final Appendix, devoted to systems 1a and 1b).

3) For a KTS(15) whose underlying STS is #7, and for which the distinguished parallel class $\mathcal{C}_1$ is known, it suffices to apply the lacing scheme to only two pairs of distinct parallel classes in order to determine whether the system is isomorphic to system 7a or system 7b. Indeed, given $k \neq 1$, consider first the lacing of $\mathcal{C}_1$ and $\mathcal{C}_k$, with residual triple, say, in $\mathcal{C}_i$. Let h be different from $1, i, k$, and consider the lacing of $\mathcal{C}_1$ and $\mathcal{C}_h$, with residual triple, say, in $\mathcal{C}_j$. If the two residual triples in $\mathcal{C}_i$ and $\mathcal{C}_j$ are disjoint (resp., intersect in one point), then the KTS is isomorphic to system 7a (resp., to system 7b) by property *2(c)* in Theorem 2.4.

4) More generally, given an explicit KTS(15), it is natural to ask how many applications of the lacing scheme are necessary in order to determine the isomorphism class of the system. One can show that, if no preliminary information on the KTS is known, then the number of applications that are needed is at most 3 (resp., 6, 9, 10) if the underlying STS is #1 (resp., #7, #61, #19).

Moreover, if the underlying STS is either #19 or #61, then it can take up to 9 applications only to determine the underlying STS, besides the further difficulty of distinguishing systems 19a and 19b (this depends on the fact that in the two latter systems there are only three lacings of type (α) out of 21).

On the other hand, if the first lacing comes out to be of type (α), then only four further applications are needed, at most, to settle the isomorphism problem, no matter whether the underlying STS is #1, #7, or #19.

Indeed, let us denote the seven parallel classes of a given KTS(15) by the days of the week: M, TU, W, TH, F, SA, SU. Whenever a lacing is of type (α), with residual triple in the class denoted by the day X, we denote such a triple by T_X. The idea is to begin the investigation by first considering the four lacings M-TU, W-TH, F-SA, and M-SU, in this order. If one of these lacings is of type (α), then the underlying STS is #1, or #7, or #19. If, instead, these four lacings are all of type (β), then the underlying STS is either #19 or #61, and one continues with the five lacings TU-SU, W-F, W-SA, TH-F, and TH-SA, in any order. If at least one of these five lacings is of type (α), say X-Y, with residual triple T_Z, then the STS is #19, and the residual triple T_Y of the lacing X-Z, together with T_Z, allows one to determine the unique Fano plane contained in the system, and hence the isomorphism class, by means of property *3(c)* in Theorem 2.4.

If, instead, all the nine lacings are of type (β), then the KTS is necessarily isomorphic to system 61, because the lacings have been chosen in such a way that each triple of mutually distinct parallel classes contains at least one pair of classes corresponding to one of the nine lacings (note that this cannot be accomplished with less than nine lacings). This proves, as claimed, that if the underlying STS is #61 (resp., #19), then at most nine (resp., ten) lacings are necessary to determine the isomorphism class of the KTS.

Let us now consider the case where the first lacing, M-TU, is of type (α), with residual triple, say, T_W. In this case, we consider the lacing M-TH. If this lacing is of type (α), with residual triple, say, T_F, then the system is isomorphic to 7a if $T_W \cap T_F = \emptyset$, else it is isomorphic to either 1a, 1b, or 7b. In the latter case, we consider W-TH: if this lacing is of type (β), then the system is isomorphic to 7b; if it is of type (α), with residual triple T_X, then the system is isomorphic to 1a (resp., 1b) if $T_W \cap T_F \cap T_X$ is non-empty (resp., empty).

If, instead, the lacing M-TH is of type (β), then we consider the lacing W-TH. If this lacing is of type (α), with residual triple, say, T_F, and if T_{TU} is the residual triple of the lacing M-W, then the KTS is isomorphic to system 7a (resp., 7b) if $T_F \cap T_{TU}$ is empty (resp., non-empty). If, instead, W-TH is of type (β), then we consider the lacing TU-TH: if this is of type (α), with residual triple, say, T_F, then the KTS is isomorphic to system 7a (resp., 7b) if $T_F \cap T_W$ is empty (resp., non-empty); if TU-TH is of type (β), then the underlying STS is #19, and it takes only one further lacing M-W, with residual triple T_{TU}, to determine the unique Fano plane contained in the system, and hence the isomorphism class, by means of property *3(c)* in Theorem 2.4. This shows that if the first lacing, M-TU, is of type (α), then at most four more lacings are needed to settle the isomorphism problem, as claimed.

If M-TU is of type (β), then the underlying STS is #7, or #19, or #61. In this case, one considers the lacings W-TH, F-SA, and M-SU, in this order. If they are all of type (β), then the STS is either #19 or #61, and one continues as above, else it is either #7 or #19. In the latter case, and if the STS is #7, one can show, by arguing as above, that the highest number of lacings required to determine the isomorphism class of the system is six, and that this upper bound is attained precisely in the case where M-TU and W-TH are both of type (β), whereas F-SA is of type (α). If T_X is the corresponding residual triple, and if Y is any class not in {F, SA, X}, then one further considers the lacings Y-X, Y-F, and Y-SA, in this order. The upper bound six is attained precisely in the case where Y-X and Y-F are both of type (β) and Y-SA is of type (α).

5) As we pointed out earlier, Proposition 2.5 is an interesting and elegant result from a theoretical point of view, but for practical purposes it is more convenient to resort to the algorithm in Theorem 2.4. Indeed, given an arbitrary KTS(15), one can apply Proposition 2.5 only if one already knows that the underlying STS is #19. On the other hand, as we explained in the previous Remark 4), in order to get this information one must apply the lacing scheme in Theorem 2.4 as many as nine times, and once it is ascertained that the underlying STS is #19, it suffices to consider just one extra lacing to determine whether the system is 19a or 19b, with no need of constructing and examining the 1-factorization of K_8.

6) One may think of extending Proposition 2.5 to the case where the underlying STS of the KTS is not necessarily #19. However, in the general case, the 1-factorization of K_8 is not a complete invariant for the isomorphism classes of KTS(15)s. Indeed, depending on the chosen Fano plane, it turns out that the 1-factorization $\mathcal{F}$ of K_8 associated with the KTS is either $\mathcal{F}_4$ or $\mathcal{F}_6$, either $\mathcal{F}_1$ or $\mathcal{F}_5$, $\mathcal{F}_4$, either $\mathcal{F}_1$ or $\mathcal{F}_5$, $\mathcal{F}_1$, $\mathcal{F}_4$, $\mathcal{F}_1$, respectively, according to whether the KTS(15) is 1a, 1b, 7a, 7b, 19a, 19b, 61. Therefore, with the only exception of the case where $\mathcal{F} = \mathcal{F}_6$, $\mathcal{F}$ does not determine the KTS(15) uniquely. The details will be worked out in a forthcoming paper.

7) We mentioned in the Introduction that the automorphisms of 1a (in Table 1) are transitive on all points except on the point i, which is fixed under all automorphisms, whereas the automorphisms of 1b are transitive in seven and in eight points. Needless to say, the point i is the common point of the seven residual triples in system 1a, whereas the seven points in the latter case are precisely the points of the Fano plane of the residual triples in system 1b.

The automorphisms of 7a are transitive in three and in twelve points, whereas the automorphisms of 7b are transitive in three, in four and in eight points. If we refer again to Table 1, then in either case the three points are a, b, c (see Tables 2 and 3 in the proof of Theorem 2.4), whereas the four points are h, i, j, k, which, together with a, b, c, form the Fano plane determined by the residual triples of system 7b. Note, in passing, that a, b, c are precisely the three common points of the three Fano planes contained in the underlying STS.

As for systems 19a and 19b, the three residual triples provide a simple method to find the points of the unique Fano plane contained in the underlying STS.

8) Note that the Fano plane {abc, ade, afg, bdf, beg, cdg, cef} is contained in all four STS(15)s in Table 1. In particular, each of the four resolvable STS(15)s contains at least one Fano plane, that is, a *maximum subsystem* of the STS (cf. [48]). Also, as shown in Theorem 2.4, Fano planes play a crucial role in the identification of the isomorphism class of a given KTS(15). This is, however, a special property of the case $v = 15$, which does not extend to a more general setting. For instance, the point-line design of the affine geometry AG(3, 3) is a resolvable STS(27) that does not contain any maximum subsystem, that is, any sub-STS(13). Indeed, if AG(3, 3) contained a sub-STS(13), then the latter system would be an *additive* design, in contradiction with the fact that the only additive Steiner triple systems are the point-line designs of AG(d, 3) and PG(d, 2) [11, Theorem 3.7]. Alternatively, it suffices to observe that any three non-collinear points in AG(3, 3) generate an STS(9), which cannot be contained in an STS(13).

The fact that any resolvable STS(15) contains (at least) one Fano plane $\mathcal{S}$ implies that all KTS(15)s share the same formal structure. Indeed, since $\binom{8}{2} = 28$, the 28 triples of the STS(15) not belonging to $\mathcal{S}$ are all of the form sxy, where s is in $\mathcal{S}$ and x, y are not in $\mathcal{S}$. Since any two distinct triples in a Fano plane intersect in one point, the seven triples in $\mathcal{S}$ are necessarily distributed in all seven parallel classes of the KTS(15). This finally implies that each parallel class contains one triple in $\mathcal{S}$ and four triples of the form sxy, with s in $\mathcal{S}$ and x, y not in $\mathcal{S}$. This implies, in turn, that there is a 1-factorization $\mathcal{F}$ of the graph K_8 that is naturally associated to the KTS(15) and to $\mathcal{S}$.

Cayley was the first one to conjecture that this was the structure of all KTS(15)s: "there is obviously a division of the 15 things into $(7+8)$ things, viz. the 35 triads are composed seven of them each of 3 out of the 7 things, and the remaining 28 each of 1 out of the 7 things, and 2 out of the 8 things. (...) I believe, but am not sure, that in all the solutions which have been given of the school-girl problem there is an 8 without 3, (...) that is, there are 8 things such that no triad of them occurs in the system" [14].

3. Examples

In this section we test the effectiveness and simplicity of our method by determining, for some given KTS(15)s, which of the systems in Table 1 they are isomorphic to. By doing so, we will exhibit (at least) one KTS(15) for each of the seven types.

It is worth noting that almost all the solutions of the schoolgirl problem in the literature are isomorphic to either system 1b or system 1a, that is, the first two published solutions [13,36]. In either case, the underlying STS is the point-line design of the projective geometry PG$(3,2)$, hence the two systems have the richest automorphism group, a fact which, together with the cyclic nature of the two solutions, perhaps made the solutions 1a and 1b arise in a more "natural" way (see, in this regard, the final Appendix below).

1) (**System 1b**) The first solution to the schoolgirl problem that appeared in print was given by Cayley in 1850 [13]. Here, in Table 7, we actually describe Cayley's more revealing construction in [14] (cf. [17, p. 6]).

Table 7
Cayley's solution 1b.

	a	b	c	d	e	f	g
abc				35	17	82	64
ade		62	84			15	37
afg		13	57	86	42		
bdf	47		16		38		25
beg	58		23	14		67	
cdg	12	78			56	34	
cef	36	45		27			18

The bottom-right 7×7 "minor" of Table 7 is a *Room square* of side 7, whereas the seven triples in the first column are the blocks of a Fano plane. In general, a Room square of side n, on the symbol set $S = \{1, \dots, n+1\}$, is an $n \times n$ array F such that: every cell of F either is empty or contains an unordered pair of symbols from S; each symbol of S occurs once in each row and column of F; every unordered pair of symbols occurs in precisely one cell of F. In the case of Table 7, the schoolgirls are the fifteen symbols a, b, c, d, e, f, g, 1, 2, 3, 4, 5, 6, 7, 8. Each of the seven bottom rows of the array gives a parallel class, by taking the triple in the first column together with the triples obtained by adjoining each pair of numbers to the letter that appears in the same column (in passing, any KTS(15) can be constructed in this way; see the above Remark 2.6(8)). Hence the solution is that given in Table 8.

Table 8
Cayley's solution 1b in explicit form.

Mon	Tue	Wed	Thu	Fri	Sat	Sun
abc	ade	afg	bdf	beg	cdg	cef
d35	b62	b13	a47	a58	a12	a36
e17	c84	c57	c16	c23	b78	b45
f82	f15	d86	e38	d14	e56	d27
g64	g37	e42	g25	f67	f34	g18

Now Monday and Tuesday are laced in the mode (α), with residual triple afg in Wednesday. Also, Monday and Thursday are laced in the mode (α), with residual triple beg in Friday. Finally, Wednesday and Thursday are laced in the mode (α), with residual triple cef in Sunday. It follows from Theorem 2.4 that the KTS is isomorphic to either system 1a or system 1b. As the three residual triples afg, beg, and cef do not have any point in common, we may finally conclude, again by Theorem 2.4 (see also Remark 2.6(2)), that Cayley's KTS is isomorphic to system 1b. Also, it can be immediately checked that the seven residual triples are precisely the blocks of the Fano plane in the leftmost column of Table 7. Note that one can obtain a KTS isomorphic to 1a by suitably rearranging the triples in Monday, Tuesday, and Wednesday.

Other "classical" examples of a KTS(15) isomorphic to system 1b are the solutions by W. Ahrens [1, p. 281], W. Spottiswoode [60], J. Horner [54], and W. Burnside [10], the first solution by A.C. Dixon [23], T.H. Gill's solution [58, p. 103], the first solution by E.J.F. Primrose [55], the solution by E. Brown and K.E. Mellinger [9, Table 2], the solution by J.P. Marceaux and A.R.P. Rau [41, Table 4] (where Kirkman's schoolgirls are seen in correspondence with the Lie-Clifford algebra of quantum spin pairs), and the second cyclic solution by B. Peirce [52, §31, p. 172] (also reported in [24, p. 18]), whose visual representation is given by means of a two-step rotating circle in [59, Figure iii, p. 200] (see also [31, Figure 51, p. 126], from *Scientific American*, May 1980), and where one of the two orbits of length 7 consists precisely of the points of the Fano plane of the seven residual triples. All this can be checked by the same method used above for Cayley's solution.

Another interesting visual example of a KTS(15) isomorphic to system 1b is the system denoted by ($\circlearrowleft$, $\circlearrowleft$) in [28], where the schoolgirls are represented as the fifteen simplicial elements of a tetrahedron, that is, the four vertices, the six edges, the four faces, and the whole tetrahedron.

2) (**System 1a**) Our second example is the 1850 system published by Kirkman [36] (replicated in [37, p. 260] and [38, p. 48]), who described it as "the neatest method of writing the solution of the problem". He also thought that this was the only possible solution up to permutation [38]. The fifteen schoolgirls are

$$a_1,\ a_2,\ a_3,\ b_1,\ b_2,\ b_3,\ c_1,\ c_2,\ c_3,\ d_1,\ d_2,\ d_3,\ e_1,\ e_2,\ e_3.$$

As a first parallel class we take

$$a_1a_2a_3,\ b_1b_2b_3,\ c_1c_2c_3,\ d_1d_2d_3,\ e_1e_2e_3.$$

Each of the other six classes contains three triples of the form $a_1x_iy_i$, $a_2x_jy_j$, $a_3u_kv_k$, where $\{x, y\}$ ranges over the six 2-subsets of $\{b, c, d, e\}$, $\{u, v\} = \{b, c, d, e\} \setminus \{x, y\}$, and $\{i, j, k\} = \{1,2,3\}$. In view of the choice of the first parallel class, the other two triples (in each of the other six classes) are necessarily of the form $x_ku_iv_j$ and $y_ku_jv_i$. Therefore each of the six classes has the form

$$a_1x_iy_i,\ a_2x_jy_j,\ a_3u_kv_k,\ x_ku_iv_j,\ y_ku_jv_i.$$

The six classes are uniquely determined by three choices of i, j, x, y, and u. Indeed, any such choice produces another class by just permuting $x \leftrightarrow u$ and $y \leftrightarrow v$. To get a KTS(15) it now suffices to make the following (cyclic) choice for the ordered quintuple (i, j, x, y, u): $(1, 2, b, c, d)$, $(2, 3, d, b, c)$, $(3, 1, c, d, b)$. This way we get precisely Kirkman's solution of the schoolgirl problem [36, p. 169] (up to changing the order of Saturday and Sunday), which we summarize in Table 9.

Table 9
Kirkman's solution 1a.

Mon	$a_1a_2a_3$	$b_1b_2b_3$	$c_1c_2c_3$	$d_1d_2d_3$	$e_1e_2e_3$
Tue	$a_1b_1c_1$	$a_2b_2c_2$	$a_3d_3e_3$	$b_3d_1e_2$	$c_3d_2e_1$
Wed	$a_1d_1e_1$	$a_2d_2e_2$	$a_3b_3c_3$	$d_3b_1c_2$	$e_3b_2c_1$
Thu	$a_1d_2b_2$	$a_2d_3b_3$	$a_3c_1e_1$	$d_1c_2e_3$	$b_1c_3e_2$
Fri	$a_1c_2e_2$	$a_2c_3e_3$	$a_3d_1b_1$	$c_1d_2b_3$	$e_1d_3b_2$
Sat	$a_1c_3d_3$	$a_2c_1d_1$	$a_3b_2e_2$	$c_2b_3e_1$	$d_2b_1e_3$
Sun	$a_1b_3e_3$	$a_2b_1e_1$	$a_3c_2d_2$	$b_2c_3d_1$	$e_2c_1d_3$

By construction, the permutation $(b_1\,c_3\,d_2)(c_1\,d_3\,b_2)(d_1\,b_3\,c_2)(e_3\,e_2\,e_1)(a_1)(a_2)(a_3)$ of the fifteen symbols is an automorphism of order 3 of the KTS, which induces the permutation (Tue Sat Thu)(Wed Sun Fri)(Mon) of the parallel classes.

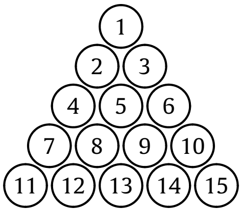
Now Monday and Tuesday are laced in the mode (α), with residual triple $a_3b_3c_3$ in Wednesday. Also, Monday and Saturday are laced in the mode (α), with residual triple $a_3c_2d_2$ in Sunday. Finally, Wednesday and Saturday are laced in the mode (α), with residual triple $a_3d_1b_1$ in Friday. It follows from Theorem 2.4 that the KTS is isomorphic to either system 1a or system 1b. As the three residual triples $a_3b_3c_3$, $a_3c_2d_2$, and $a_3d_1b_1$ have the point a_3 in common, we may finally conclude, by Remark 2.6(2), that the KTS is isomorphic to system 1a. Note that one can obtain a KTS isomorphic to 1b by suitably rearranging the triples in Monday, Tuesday, and Wednesday.

Further examples of a KTS(15) isomorphic to system 1a are R.R. Anstice's first solution [5, p. 280], the third cyclic solution by B. Peirce [52, §31, p. 172], the solutions by A. Frost [30] (also reported in [40, p. 184]), A.F.H. Mertelsmann [45], and H.E. Dudeney [25], the second solution by E.J.F. Primrose [55], the regular 14-gon model in [7, Figure 5.2, p. 28], the solution by B. Polster [53, Figure 8], and the tetrahedron-based model denoted by ($\circlearrowleft$, $\circlearrowright$) in [28].

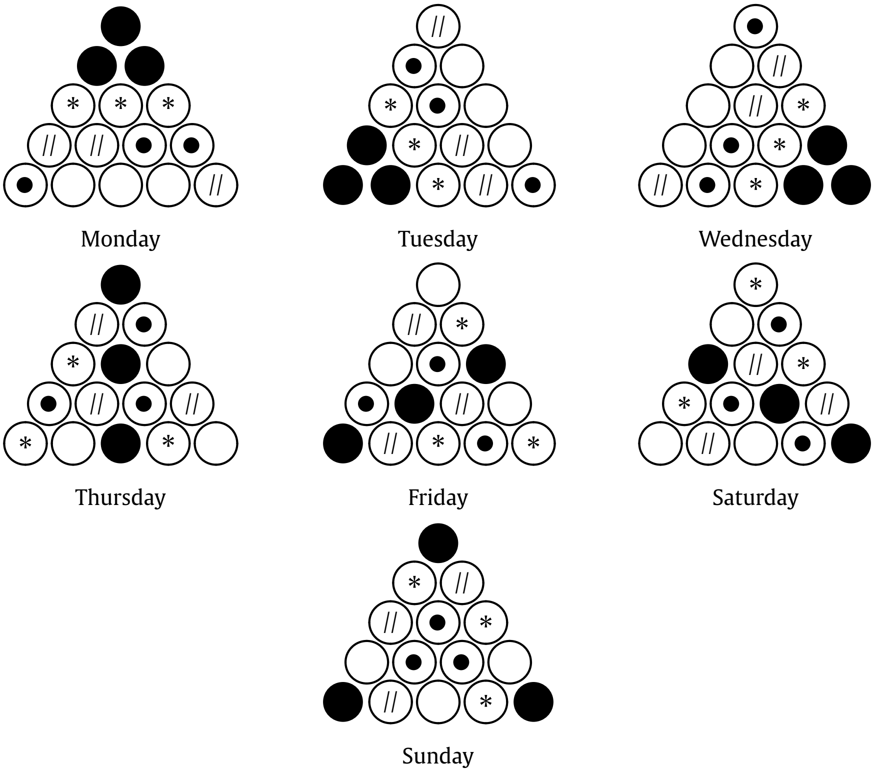
Interestingly enough, the joint solution by four authors in [38, p. 48], immediately after Kirkman's solution, is also isomorphic to system 1a. In a recent paper [8, §6.2], S. Bonvicini et al. constructed a model of system 1a which was designed to show that the KTS is 3-*pyramidal*, i.e., admitting an automorphism group acting sharply transitively on all but three points. A very elegant and highly symmetric planar model of system 1a was given by Ed Pegg Jr. [51], by representing the fifteen schoolgirls as the vertices of three concentric regular pentagons. The rotation by $2\pi/5$ around the common center is an order-5 automorphism of the underlying STS, but is not a KTS-automorphism (on the other hand, the automorphism group of system 1a has order 168 [18]).

Finally, whenever system 1a is constructed as a cyclic solution, the common point of the seven residual triples is precisely the fixed point of the order-7 permutation of the fifteen points that cyclically permutes the seven parallel classes.

3) (**Systems 7a and 7b**) In 2012 Kristýna Stodolová wrote a thesis on "Classic problems in combinatorics" [63], where she described the visual solutions of the schoolgirl problem given in [21] and [28] and, in addition, proposed a further elegant and symmetric visual solution, with no references. To this end, she arranged fifteen balls in the usual triangular pool-table configuration as follows.



The seven parallel classes are defined as follows, with the obvious interpretation of the symbols. For instance, the five triples in Monday are {1, 2, 3}, {4, 5, 6}, {7, 8, 15}, {9, 10, 11}, and {12, 13, 14}.

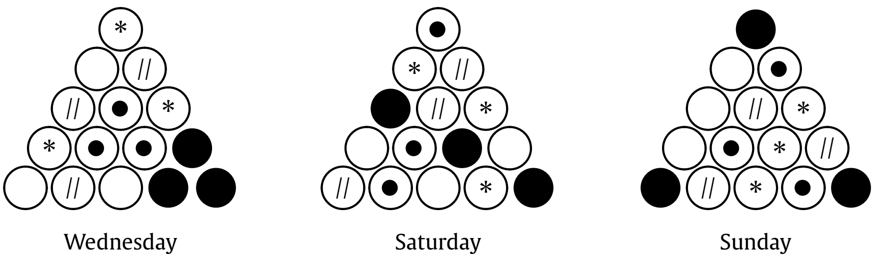


Note how the three configurations in each row are the orbit of the leftmost configuration under the 120-degree counterclockwise rotation

$$\phi = (1\ 11\ 15)(2\ 12\ 10)(3\ 7\ 14)(4\ 13\ 6)(5\ 8\ 9) \tag{1}$$

of the triangle around its center, and that the final parallel class (Sunday) is invariant under the same rotation. Also, Monday, Thursday, and Sunday are symmetric with respect to the vertical axis through the top ball.

An alternative choice (not reported in [63]) for Wednesday, Saturday, and Sunday is the following.



In either case, the resulting KTS(15) has the property that the only lacings in the mode (α) are those between any two parallel classes in any of the three sets {Mon, Thu, Sun}, {Tue, Fri, Sun}, {Wed, Sat, Sun}. It follows from Theorem 2.4 that the KTS is isomorphic to either system 7a or system 7b.

In the former case (with the first choice of the seven parallel classes), the seven residual triples of the lacings in the mode (α) are $\{1, 11, 15\}$, $\{1, 2, 3\}$, $\{1, 5, 13\}$, $\{11, 6, 8\}$, $\{11, 7, 12\}$, $\{15, 4, 9\}$, and $\{15, 10, 14\}$, whose union is the point-set $\{1, 2, \ldots, 15\}$, hence the KTS is isomorphic to system 7a by Theorem 2.4.

In the latter case, when we make the alternative choice for Wednesday, Saturday, and Sunday, the seven residual triples of the lacings in the mode (α) are $\{1, 4, 10\}$, $\{1, 9, 14\}$, $\{1, 11, 15\}$, $\{4, 9, 15\}$, $\{4, 11, 14\}$, $\{9, 10, 11\}$, and $\{10, 14, 15\}$, which form the blocks of a Fano plane. Hence, by Theorem 2.4, the KTS is isomorphic to system 7b.

According to [18], the full automorphism group of the system 7a described here is an order-24 group, generated by the order-3 rotation ϕ defined above in (1) and by the order-4 automorphism

$$\psi = (1)(11\ 15)(2\ 13\ 3\ 5)(4\ 8\ 14\ 7)(6\ 10\ 12\ 9).$$

The planar model in the present example is somehow the best possible one, up to isomorphism, to visualize the automorphism ϕ, whereas it does not seem to be particularly suitable to visualize ψ. Similarly, if we apply to the fifteen balls the permutation (1 5 13)(6 11 8)(4 15 9)(2 14)(3 12)(7 10), then we get an alternative visual solution for 7a, which is also invariant under rotations, and where the triples are seventeen equilateral triangles, six isosceles triangles, and twelve triples obtained from the base triples $\{1, 4, 11\}$, $\{1, 5, 13\}$, and $\{1, 8, 14\}$ under the six symmetries of the underlying equilateral triangle containing the fifteen balls (see [50]). This alternative planar representation of the underlying STS, which is obtained by means of a suitable *Pasch switch* from a visual model of the STS(15) #6 [50], is also not suited to visualize the order-4 automorphism.

We now propose a new spatial and highly symmetric representation of system 7a based on a regular octahedron, which is particularly suitable to reflect an order-4 automorphism of the KTS, and to visualize the three Fano planes of the underlying STS #7. The fifteen schoolgirls are represented as the six vertices, the eight faces, and the whole of a octahedron. Let us denote the six vertices by the numbers $1, 2, 3, 4, 5, 6$ as in Fig. 1. Each face is denoted by a triple of the form abc, where a, b, c are the three vertices belonging to that face. Also, the letter O denotes the whole of the octahedron.

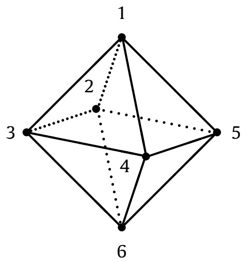


Fig. 1. The octahedron.

The underlying STS(15) is defined as follows. If f_1, f_2 are two disjoint faces, then $\{f_1, f_2, \mathrm{O}\}$ is a triple. If f_1, f_2 are two faces with only one common vertex x, with x not in $\{1, 6\}$, and y is the only vertex not on the two faces, then $\{f_1, f_2, y\}$ is a triple. If f_1, f_2 are two faces with only one common vertex x, with x in $\{1, 6\}$, then $\{f_1, f_2, y\}$ is a triple, where $y = 1$ (resp., $y = 6$) if the two parallel edges of the two faces are parallel to the edge 23 (resp., 34). If f_1, f_2 are two faces with two common vertices x, y, only one of which, say x, is in $\{1, 6\}$, then $\{f_1, f_2, y\}$ is a triple. If f_1, f_2 are two faces with two common vertices x, y, neither of which is in $\{1, 6\}$, then $\{f_1, f_2, z\}$ is a triple, where $z = 1$ (resp., $z = 6$) if the edge xy is parallel to the edge 34 (resp., 23). The remaining seven triples are $\{1, 2, 3\}$, $\{1, 4, 5\}$, $\{2, 5, 6\}$, $\{3, 4, 6\}$, $\{\mathrm{O}, 1, 6\}$, $\{\mathrm{O}, 2, 4\}$, $\{\mathrm{O}, 3, 5\}$ (which form the block-set of a Fano plane). If we define

$$\chi = (\mathrm{O})(1\ 6)(2\ 3\ 4\ 5)(123\ 134\ 145\ 125)(236\ 346\ 456\ 256), \tag{2}$$

then χ is an order-4 automorphism of the STS(15), which fixes O, interchanges 1 and 6, and rotates counterclockwise, by 90 degrees, the twelve remaining points of the STS. A KTS(15) can now be defined by taking the orbits under χ of the three base parallel classes

$\{2, 5, 6\}$	$\{\mathrm{O}, 145, 236\}$	$\{1, 125, 256\}$	$\{3, 123, 134\}$	$\{4, 346, 456\}$
$\{\mathrm{O}, 2, 4\}$	$\{1, 236, 456\}$	$\{3, 145, 256\}$	$\{5, 123, 346\}$	$\{6, 134, 125\}$
$\{\mathrm{O}, 1, 6\}$	$\{2, 145, 346\}$	$\{3, 125, 456\}$	$\{4, 123, 256\}$	$\{5, 134, 236\}$.

The resulting KTS is given in Table 10. It can be readily seen that the KTS is isomorphic to system 7a by Theorem 2.4 (alternatively, one can find an explicit isomorphism between the system in Table 10 and the above system 7a with point-set $\{1, 2, \ldots, 15\}$). The seven residual triples are $\{\mathrm{O}, 1, 6\}$, $\{\mathrm{O}, 2, 4\}$, $\{\mathrm{O}, 3, 5\}$, and the four triples of the form $\{f_1, f_2, z\}$.

Table 10
The octahedron-based system 7a.

Mon	{2, 5, 6}	{O, 145, 236}	{1, 125, 256}	{3, 123, 134}	{4, 346, 456}
Tue	{3, 2, 1}	{O, 125, 346}	{6, 123, 236}	{4, 134, 145}	{5, 456, 256}
Wed	{4, 3, 6}	{O, 123, 456}	{1, 134, 346}	{5, 145, 125}	{2, 256, 236}
Thu	{5, 4, 1}	{O, 134, 256}	{6, 145, 456}	{2, 125, 123}	{3, 236, 346}
Fri	{O, 2, 4}	{1, 236, 456}	{3, 145, 256}	{5, 123, 346}	{6, 134, 125}
Sat	{O, 3, 5}	{6, 346, 256}	{4, 125, 236}	{2, 134, 456}	{1, 145, 123}
Sun	{O, 1, 6}	{2, 145, 346}	{3, 125, 456}	{4, 123, 256}	{5, 134, 236}

By construction, χ is an order-4 automorphism of the system, which induces on the parallel classes the order-4 permutation

$$(\text{Mon Tue Wed Thu})(\text{Fri Sat})(\text{Sun}).$$

Also, χ fixes the Fano plane consisting of O and the six vertices, and interchanges the other two Fano planes of the STS, that is, {O, 1, 6, 125, 134, 256, 346} and {O, 1, 6, 123, 145, 236, 456}.

By suitably reshuffling the triples in Fri, Sat, and Sun, one finds a KTS(15) isomorphic to system 7b. Again, χ is an automorphism of the system, inducing on the parallel classes the same order-4 permutation as for system 7a.

Moreover, by simply observing that the dual Platonic solid of the octahedron is the cube, with the vertices and the faces interchanged, the two previous arrangements of systems 7a and 7b can be immediately represented on a cube, by taking as the fifteen schoolgirls the six faces and the eight vertices of the cube, and the whole of the cube.

In Fig. 2 the eight vertices of the cube are represented by the same eight triples as the faces of the octahedron, whereas each face of the cube is represented by the unique symbol belonging to the four vertices of the face. For instance, the face at the base of the cube is represented by the symbol 6. If we represent the whole of the cube by the letter O, then the two previous arrangements for system 7a (Table 10) and 7b can be visualized on the cube. Also, the order-4 automorphism χ in (2) fixes the cube, interchanges the two horizontal faces and rotates counterclockwise, by 90 degrees, the twelve remaining elements of the cube.

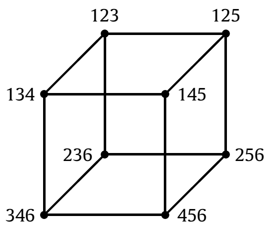


Fig. 2. The cube for an alternative representation of systems 7a and 7b.

Note that in this representation of the STS(15) #7, by means of the eight vertices, the six faces, and the whole of a cube, 25 triples out of 35 coincide with 25 triples of the STS(15) #19 defined in the next example with the same point-set (in passing, the common 25 triples are not contained in five parallel classes, else they would determine a unique KTS(15)).

We are not aware of any other visual models of systems 7a and 7b. Another KTS(15) isomorphic to system 7b is the second solution by A. C. Dixon [23].

4) (**Systems 19a and 19b**) In 1897 Ellery W. Davis, a former doctoral student of James J. Sylvester, gave a visual solution to the schoolgirl problem, where the fifteen schoolgirls were represented as the eight vertices, the six faces, and the whole of a cube [21].

Let us denote the eight vertices by the numbers $1, 2, \dots, 8$, as in Fig. 3. Each face is denoted by a quadruple of the form $abcd$, where a, b, c, d are the four vertices belonging to that face. For instance, 1234 is the face at the base of the cube in Fig. 3. Also, the letter C denotes the whole of the cube.

The first four parallel classes are defined as follows. Each class contains a triple of the type $\{C, v, w\}$, where v is a vertex in the set {2, 4, 5, 7} and w is the opposite vertex, a triple consisting of the three faces containing the vertex v, and three triples of the type $\{f, x, y\}$, where f is one of the three remaining faces, and x, y are two adjacent vertices belonging to f and different from w. There are two possible ways of taking the four classes, depending on whether {1256, 1, 2} or {1256, 1, 5} is chosen as one of the triples. Either choice determines uniquely the first four parallel classes. If we choose, for instance, {1256, 1, 2} to be one of the triples, then the four classes are determined as in Table 11.

The remaining three classes are defined as follows. Each class contains a triple consisting of C and two opposite faces, and four triples of the type $\{f, x, y\}$, where f is one of the remaining four faces, and x, y are two non-adjacent vertices belonging to the face opposite to f. If the first four classes are taken as in Table 11, then there are two possible ways of taking the remaining three classes, which are shown in Tables 12 and 13.

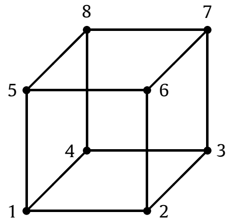


Fig. 3. Davis' cube.

Table 11
The first choice of the first four parallel classes.

Mon	C, 7, 1	2367, 3478, 5678	1234, 2, 3	1256, 5, 6	1458, 4, 8
Tue	C, 2, 8	1234, 1256, 2367	1458, 1, 5	3478, 3, 4	5678, 6, 7
Wed	C, 5, 3	1256, 1458, 5678	1234, 1, 4	2367, 2, 6	3478, 7, 8
Thu	C, 4, 6	1234, 1458, 3478	1256, 1, 2	2367, 3, 7	5678, 5, 8

Table 12
Friday, Saturday, and Sunday in system 19a.

Fri	C, 1234, 5678	1256, 4, 7	1458, 3, 6	2367, 1, 8	3478, 2, 5
Sat	C, 1256, 3478	1234, 6, 8	1458, 2, 7	2367, 4, 5	5678, 1, 3
Sun	C, 1458, 2367	1234, 5, 7	1256, 3, 8	3478, 1, 6	5678, 2, 4

Table 13
Friday, Saturday, and Sunday in system 19b.

Fri	C, 1234, 5678	1256, 3, 8	1458, 2, 7	2367, 4, 5	3478, 1, 6
Sat	C, 1256, 3478	1234, 5, 7	1458, 3, 6	2367, 1, 8	5678, 2, 4
Sun	C, 1458, 2367	1234, 6, 8	1256, 4, 7	3478, 2, 5	5678, 1, 3

In either case, the resulting KTS(15) has the property that the classes Friday, Saturday, and Sunday are laced in the mode (α) with each other, whereas all the other lacings are in the mode (β). It follows from Theorem 2.4 that the KTS is isomorphic to either system 19a or system 19b.

Also, in either case, the residual triples of the three lacings of type (α) are $\{C, 1234, 5678\}$, $\{C, 1256, 3478\}$, and $\{C, 1458, 2367\}$, which belong to the block-set of the Fano plane whose points are C and the six faces of the cube. Now the points 4 and 6 are in the same triple in Thursday, whereas, according to Table 12 (resp., Table 13) Friday has a triple containing 4 and 7 and a triple containing 6 and 3 (resp., a triple containing 4 and 5 and a triple containing 6 and 1). Since 7 and 3 are in the same triple in Thursday (resp., 5 and 1 are *not* in the same triple in Thursday), it follows from Theorem 2.4 that the KTS is isomorphic to system 19a (resp., to system 19b).

If at the beginning one takes $\{1256, 1, 5\}$, instead, to be one of the triples, then a second arrangement of the first four parallel classes is uniquely determined. This can be matched, in turn, with two possible choices of the remaining three classes, thereby producing again two KTSs isomorphic to systems 19a and 19b.

Note that, in addition to the points, the triples, and the parallel classes, this geometric model allows one to visualize all the automorphisms of the two systems as well. Indeed, in either case the automorphism group of the system is the (order-12) tetrahedral group [18], and it is easy to check that by construction, together with the identity, the three order-2 rotations around the midpoints of two opposite faces, and the eight order-3 rotations around the diagonals through two opposite vertices are all automorphisms of the two systems. For this reason, we believe that this model is the best visual solution to the schoolgirl problem.

Needless to say, Davis' solutions can also be visualized on an octahedron, because of the duality between the cube and the octahedron.

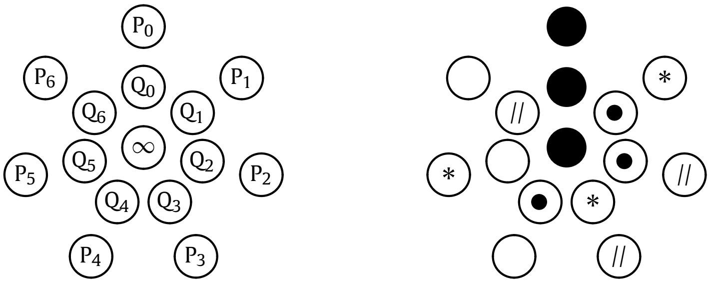
We are not aware of any other solution isomorphic to either 19a or 19b in the literature (with the exception, of course, of those provided by those authors who gave all seven solutions [18,47,57,69,70]).

5) (**System 61**) In this final example we describe a visual solution to the schoolgirl problem, which was inspired by the two-step rotating circle in [59, Figure ii, p. 200], which, in turn, was derived from Anstice's cyclic solution in [5, p. 285].

In 1852 Anstice published the first *cyclic* solutions to the schoolgirl problem. More precisely, they were KTS(15)s having an automorphism of order 7, with one fixed point and two orbits of length 7 (in modern terms, a KTS with this property is called 2-*rotational*). The same use of the term "cyclic" is found in [52]. One can easily check, by applying Theorem 2.4, that Anstice's first solution [5, p. 280] is isomorphic to system 1a, whereas the second solution [5, p. 285] is isomorphic to system 61. In the postscript of his paper [5, p. 291], Anstice shows that there exist “three distinct species of combinations of triads” of 15 symbols, but does not exhibit the arrangement 1b explicitly.

Kirkman considered Anstice’s solutions the “first properly mathematical solutions”, which revealed “the theory of the solution” of his puzzle [39]. Since 1850, Kirkman had been looking for the theoretical aspect hidden behind his puzzle: “The question has yet to be mathematically treated: I do not feel satisfied with knowing how to form thirty-five triads, which are found on *trial*, but not certainly proved before trial, to be capable of the required arrangement” [36].

In the present visual solution, unlike in [59], the fifteen schoolgirls are represented by labelling the seven vertices of an outer regular 7-gon by $P_0, \dots, P_6$, the seven vertices of an inner regular 7-gon by $Q_0, \dots, Q_6$, and the central point by the symbol ∞. In the following picture, on the right, we describe only the “base” parallel class, by representing each triple by three marks of the same kind. The remaining six parallel classes are obtained by the non-trivial rotations around the central point that leave the set of vertices invariant. Equivalently, the parallel classes are the orbits of the base parallel class under the automorphism defined by $P_n \mapsto P_{n+1}$, $Q_n \mapsto Q_{n+1}$ (mod 7), and $\infty \mapsto \infty$.



More precisely, the resulting KTS(15) is given in Table 14 (which is essentially the same as in [3, Example 1.1]).

Table 14
The (cyclic) system 61.

Mon	$P_0Q_0\infty$	$Q_1Q_2Q_4$	$P_1Q_3P_5$	$P_2P_3Q_6$	$P_4Q_5P_6$
Tue	$P_1Q_1\infty$	$Q_2Q_3Q_5$	$P_2Q_4P_6$	$P_3P_4Q_0$	$P_5Q_6P_0$
Wed	$P_2Q_2\infty$	$Q_3Q_4Q_6$	$P_3Q_5P_0$	$P_4P_5Q_1$	$P_6Q_0P_1$
Thu	$P_3Q_3\infty$	$Q_4Q_5Q_0$	$P_4Q_6P_1$	$P_5P_6Q_2$	$P_0Q_1P_2$
Fri	$P_4Q_4\infty$	$Q_5Q_6Q_1$	$P_5Q_0P_2$	$P_6P_0Q_3$	$P_1Q_2P_3$
Sat	$P_5Q_5\infty$	$Q_6Q_0Q_2$	$P_6Q_1P_3$	$P_0P_1Q_4$	$P_2Q_3P_4$
Sun	$P_6Q_6\infty$	$Q_0Q_1Q_3$	$P_0Q_2P_4$	$P_1P_2Q_5$	$P_3Q_4P_5$

As Monday and Tuesday are laced in the mode (β), the KTS is not isomorphic to system 1a nor to system 1b by Theorem 2.4. On the other hand, systems 7a, 7b, 19a, and 19b do not have an automorphism of order 7 (see, for instance, [18] and [61, Appendix]), whence the KTS is isomorphic to system 61.

Alternatively, a direct proof can be given, in view of Remark 2.6(4) in Section 2, by showing that there exist nine suitable lacings of distinct parallel classes of type (β).

Note that the labelling and the arrangement of the fifteen points help us not only to highlight the cyclicity of the solution, but also to get a more immediate understanding of some other properties of the system. For instance, the vertices $Q_0, \dots, Q_6$ of the inner 7-gon are precisely the points of the unique Fano plane contained in the underlying STS (see, e.g., [15, Table 1.29, p. 32]). Also, the full order-21 automorphism group of the system is generated by the order-7 permutation $(P_0P_1P_2P_3P_4P_5P_6)(Q_0Q_1Q_2Q_3Q_4Q_5Q_6)(\infty)$ (that is, the clockwise rotation of the 7-gons that generates the parallel classes) and the order-3 permutation $(P_1\,P_4\,P_2)(Q_1\,Q_4\,Q_2)(P_3\,P_5\,P_6)(Q_3\,Q_5\,Q_6)(P_0)(Q_0)(\infty)$ (see [18]). Finally, the fact that the system generated under the above rotation by the *base block* $\{Q_1, Q_2, Q_4\}$ is a Fano plane depends precisely on the fact that every non-zero element of the group $\mathbb{Z}/7\mathbb{Z}$ can be written in a unique way as a difference $x - y$ (mod 7), with x, y in $\{1, 2, 4\}$ (accordingly, the set $\{1, 2, 4\}$ is called a (cyclic) (7,3,1)-*difference set* [15, §VI.18], that is, a (cyclic) (7,3,1)-*difference family* with only one set [15, §VI.16]).

It is worth mentioning that, by applying to the special case $q = 7$ the well-known construction by Ray-Chaudhuri and Wilson of a KTS($2q + 1$), for a prime power $q \equiv 1$ (mod 6), one obtains a KTS(15) with point-set $(\mathbb{F}_7 \times \{1, 2\}) \cup \{\infty\}$, whose seven parallel classes are derived by developing modulo 7 (in the first coordinate) the base parallel class $\{(0, 1), (0, 2), \infty\}$, $\{(1, 1), (3, 1), (2, 2)\}$, $\{(2, 1), (6, 1), (4, 2)\}$, $\{(4, 1), (5, 1), (1, 2)\}$, $\{(6, 2), (5, 2), (3, 2)\}$ ([56]; see also [16, 14.5.21, p. 592] and [17, Theorem 19.10]). Arguing as above, one immediately finds that the KTS is isomorphic to system 61. Alternatively, one can easily find an explicit isomorphism with the KTS described in Table 14. In particular, one finds that the Fano plane contained in the system is that generated modulo 7 by the base block $\{(6, 2), (5, 2), (3, 2)\}$ (equivalently, $\{6, 5, 3\}$ is a (cyclic) (7,3,1)-difference set).

Note that, for a prime $p \equiv 1 \pmod 6$, the construction of a (2-rotational) KTS($2p+1$) had been given by Anstice himself [5,6], making use of primitive roots and difference families for the first time in the history of block designs, and constructing infinitely many cyclic Room squares (see also [2]).

Another example of a KTS(15) isomorphic to system 61 is reported in [64, p. 95] (where no other solutions are given). Probably the most important appearance of system 61 in the literature is related to *Sylvester's problem* of the 15 schoolgirls: can the 35×13 (unordered) triples of elements of a 15-set be grouped into 13 different KTS(15)s? Denniston's 1974 solution [22] (see also [4, Example 7.3.2] and [15, Example 2.71, p. 66]) contains 13 KTS(15)s all isomorphic to system 61 (and which form the basis of the musical score *Kirkman's Ladies* by the composer Tom Johnson [33]). As of 2023, it is not known whether there exist other non-isomorphic solutions to Sylvester's problem.

4. Appendix. Systems 1a and 1b revisited

1) (**Frost's solution**) In 1871 A. Frost [30] published an interesting solution to the schoolgirl problem, based on the observation that if the fifteen schoolgirls are denoted by $p, a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, e_1, e_2, f_1, f_2, g_1, g_2$, and if the seven letters $a, b, \dots, g$ are the points of a Fano plane, then the seven parallel classes can be constructed as follows. Each letter x in $\{a, b, \dots, g\}$ determines a parallel class containing the triple px_1x_2 and four triples of the form $u_iv_jw_k$, where uvw ranges over the four blocks of the Fano plane not containing x (in modern terms, such a configuration of four triples is called *quadrilateral* or *Pasch configuration*).

After this elegant and promising premise, however, the various arrangements of the subscripts i, j, k are found after an excessively long and involved search, which takes two full pages of the article. Moreover, in the final solution, the ordered triple (i, j, k) takes up all the eight possible values, with no symmetry nor apparent logic. The same thing happens in the account given in [24, p. 15].

Here we describe a faster and more effective way to obtain a solution which is consistent with Frost's requirements and which, moreover, is particularly symmetric, cyclic, and simple. Our construction is inspired by one of Anstice's cyclic solutions [5, p. 280] (isomorphic to system 1a), and is based on the fact that one of the orbits of the automorphism of order 7 consists of the seven points of a Fano plane.

Let us denote the fifteen schoolgirls by $p, a_1, a_2, \dots, g_1, g_2$ as above, and let us choose $abc, bdf, cfe, dcg, ead, fga, geb$ as the blocks of a Fano plane (which Frost calls the "fundamental triads"). If we take

$$pa_1a_2 \quad b_1d_1f_1 \quad d_2c_1g_2 \quad c_2f_2e_1 \quad g_1e_2b_2$$

as the base parallel class, then the other six parallel classes will be produced by the action of the cyclic group of order 7 generated by the permutation

$$(a\ b\ d\ c\ f\ g\ e),$$

which is also an automorphism of the Fano plane of fundamental triads. The resulting KTS will be that given in Table 15.

Table 15
A "Frost-type" realization of system 1a.

Mon	pa_1a_2	$b_1d_1f_1$	$d_2c_1g_2$	$c_2f_2e_1$	$g_1e_2b_2$
Tue	pb_1b_2	$d_1c_1g_1$	$c_2f_1e_2$	$f_2g_2a_1$	$e_1a_2d_2$
Wed	pd_1d_2	$c_1f_1e_1$	$f_2g_1a_2$	$g_2e_2b_1$	$a_1b_2c_2$
Thu	pc_1c_2	$f_1g_1a_1$	$g_2e_1b_2$	$e_2a_2d_1$	$b_1d_2f_2$
Fri	pf_1f_2	$g_1e_1b_1$	$e_2a_1d_2$	$a_2b_2c_1$	$d_1c_2g_2$
Sat	pg_1g_2	$e_1a_1d_1$	$a_2b_1c_2$	$b_2d_2f_1$	$c_1f_2e_2$
Sun	pe_1e_2	$a_1b_1c_1$	$b_2d_1f_2$	$d_2c_2g_1$	$f_1g_2a_2$

Note that each of the fundamental triads appears four times (once in each of the four rightmost columns), with subscripts 111, 212, 221, 122. An equivalent and perhaps more illuminating way of describing the system is given in Table 16, where the seven parallel classes are given by the seven rightmost columns, and where each cell of the grid represents the ordered subscripts to be given to the fundamental triad in the same row.

By applying Theorem 2.4, it can be readily seen that the KTS is isomorphic to system 1a (just like Frost's original solution), and that the residual triples are the seven triples of the form px_1x_2. By replacing 122, 212, 221 in Table 16 by 212, 221, 122, respectively, the KTS becomes isomorphic to system 1b, and the residual triples are the triples $u_1v_1w_1$, where uvw ranges over the seven fundamental triads of the Fano plane.

2) (**PG(3, 2) as the complete 3-design on seven points**) We now consider the fascinating model of PG(3, 2) by Ascher Wagner [65] (whose resolutions were characterized by Jonathan I. Hall [32]), and we revisit it in the light of the algorithm in Theorem 2.4. It is probably the model of PG(3, 2) that displays in the simplest and most direct way the duality of the projective space (see also [28, §5], where the mutual duality of systems 1a and 1b, with respect to the canonical duality of the projective space, is described algebraically, and illustrated by means of the fifteen simplicial elements of a tetrahedron).

Table 16
An equivalent description of system 1a in Table 15.

	pa_1a_2	pb_1b_2	pd_1d_2	pc_1c_2	pf_1f_2	pg_1g_2	pe_1e_2
bdf	111			122		221	212
dcg	212	111			122		221
cfe	221	212	111			122	
fga		221	212	111			122
geb	122		221	212	111		
ead		122		221	212	111	
abc			122		221	212	111

Let $X = \{1, 2, 3, 4, 5, 6, 7\}$. There exist precisely 30 distinct Fano planes with point-set X (this had already been noticed by Woolhouse [70] in 1863). The action of the alternating group A_7 on X induces a natural action on the 30 Fano planes, with two orbits of 15 planes each. Let us denote the two orbits by $\mathcal{P}$ ("points") and $\mathcal{H}$ ("planes"), where $\mathcal{P}$ (resp., $\mathcal{H}$) is the orbit containing the Fano plane whose blocks are obtained by developing (mod 7) the base block $(1, 2, 4)$ (resp., $(1, 3, 4)$). Finally, let us call "lines" the 35 unordered triples of elements of X (that is, the elements of the set $\binom{X}{3}$).

One can define an incidence structure on $\mathcal{P} \cup \binom{X}{3} \cup \mathcal{H}$ as follows. If $l \in \binom{X}{3}$ and F is a Fano plane in $\mathcal{P} \cup \mathcal{H}$, then l and F are incident if and only if the triple l is a block of F. If $F_1 \in \mathcal{P}$ and $F_2 \in \mathcal{H}$, then F_1 and F_2 are incident if and only if the intersection $F_1 \cap F_2$ of the two Fano planes contains at least one "line" from $\binom{X}{3}$.

The incidence structure on $\mathcal{P} \cup \binom{X}{3} \cup \mathcal{H}$ is isomorphic to the incidence structure of points, lines and planes of the projective geometry PG(3, 2). Also, one can show that two given "lines" l_1, l_2 in $\binom{X}{3}$ satisfy $|l_1 \cap l_2| = 1$ if and only if they are incident to a unique common "point" and to a unique common "plane". If this is not the case, then l_1 and l_2 are incident to no common "point" and to no common "plane". Therefore, in this model, the five "lines" of a parallel class of PG(3, 2) correspond to five triples of $\binom{X}{3}$ with pairwise intersections never of cardinality 1.

J.I. Hall [32] showed that a parallel class of PG(3, 2) either consists of the five triples in $\binom{X}{3}$ containing a given (unordered) pair (i, j) in $\binom{X}{2}$, or consists of a given (unordered) triple (a, b, c) in $\binom{X}{3}$, together with the four triples in $\binom{X}{3}$ that are disjoint from it. In the former case, the parallel class is denoted by the symbol $\langle \infty, i, j \rangle$, whereas in the latter case it is denoted by the symbol $\langle a, b, c \rangle$.

Furthermore, Hall proved that seven parallel classes form a resolution of PG(3, 2) if and only if their symbols form the blocks of a Fano plane whose point-set is a 7-subset of the set $\{\infty\} \cup X = \{\infty, 1, 2, 3, 4, 5, 6, 7\}$. In particular, all this yields an elementary and immediate proof of the fact that PG(3, 2) has 56 distinct parallel classes and 240 distinct resolutions (this was already known to Woolhouse [68,69], and was later proved by Conwell [19] by using Galois geometry).

Among these resolutions, 30 have all their seven symbols in $\binom{X}{3}$, whereas the remaining 210 have three parallel classes with symbols of the type $\langle \infty, i, j \rangle$, and four parallel classes with symbols of the type $\langle a, b, c \rangle$. In the former case, there is a one-to-one correspondence between the 30 resolutions and the 30 Fano planes in $\mathcal{P} \cup \mathcal{H}$. In either case, we will apply Theorem 2.4 to determine whether a given resolution is isomorphic to system 1a or 1b, in terms of its seven symbols.

Let us start by enumerating the fifteen "points" in $\mathcal{P}$, by writing explicitly their corresponding Fano planes.

$$
\begin{aligned}
P_1 &= \{(1,2,3), (1,4,5), (1,6,7), (2,4,7), (2,5,6), (3,4,6), (3,5,7)\} \\
P_2 &= \{(1,2,3), (1,4,6), (1,5,7), (2,4,5), (2,6,7), (3,4,7), (3,5,6)\} \\
P_3 &= \{(1,2,3), (1,4,7), (1,5,6), (2,4,6), (2,5,7), (3,4,5), (3,6,7)\} \\
P_4 &= \{(1,2,4), (1,3,5), (1,6,7), (2,3,6), (2,5,7), (3,4,7), (4,5,6)\} \\
P_5 &= \{(1,2,4), (1,3,6), (1,5,7), (2,3,7), (2,5,6), (3,4,5), (4,6,7)\} \\
P_6 &= \{(1,2,4), (1,3,7), (1,5,6), (2,3,5), (2,6,7), (3,4,6), (4,5,7)\} \\
P_7 &= \{(1,2,5), (1,3,4), (1,6,7), (2,3,7), (2,4,6), (3,5,6), (4,5,7)\} \\
P_8 &= \{(1,2,5), (1,3,6), (1,4,7), (2,3,4), (2,6,7), (3,5,7), (4,5,6)\} \\
P_9 &= \{(1,2,5), (1,3,7), (1,4,6), (2,3,6), (2,4,7), (3,4,5), (5,6,7)\} \\
P_{10} &= \{(1,2,6), (1,3,4), (1,5,7), (2,3,5), (2,4,7), (3,6,7), (4,5,6)\} \\
P_{11} &= \{(1,2,6), (1,3,5), (1,4,7), (2,3,7), (2,4,5), (3,4,6), (5,6,7)\} \\
P_{12} &= \{(1,2,6), (1,3,7), (1,4,5), (2,3,4), (2,5,7), (3,5,6), (4,6,7)\} \\
P_{13} &= \{(1,2,7), (1,3,4), (1,5,6), (2,3,6), (2,4,5), (3,5,7), (4,6,7)\} \\
P_{14} &= \{(1,2,7), (1,3,5), (1,4,6), (2,3,4), (2,5,6), (3,6,7), (4,5,7)\} \\
P_{15} &= \{(1,2,7), (1,3,6), (1,4,5), (2,3,5), (2,4,6), (3,4,7), (5,6,7)\}.
\end{aligned}
$$

Let us first describe explicitly a resolution of PG(3, 2) whose seven symbols are in $\binom{X}{3}$. Let us consider, for instance, the case where the resolution is associated with the "point" $P_1 \in \mathcal{P}$ above. We represent in Table 17 each projective "line" as a triple in $\binom{X}{3}$ and also as the corresponding triple of "points" in $\mathcal{P}$ that are incident with it.

It is immediate that the lacing of two parallel classes is of type (α) if there exist two "lines" in one class and two "lines" in the other class (as triples in $\binom{X}{3}$), such that the four triples contain a common pair of elements of X, and that the corresponding residual triple is the fifth triple in $\binom{X}{3}$ containing that common pair. For instance, the residual triple

Table 17
The system 1a associated with the Fano plane $P_1 \in \mathcal{P}$.

Mon $\langle 1,2,3\rangle$	(1,2,3) $P_1P_2P_3$	(4,5,6) $P_4P_8P_{10}$	(4,5,7) $P_6P_7P_{14}$	(4,6,7) $P_5P_{12}P_{13}$	(5,6,7) $P_9P_{11}P_{15}$
Tue $\langle 1,4,5\rangle$	(1,4,5) $P_1P_{12}P_{15}$	(2,3,6) $P_4P_9P_{13}$	(2,3,7) $P_5P_7P_{11}$	(2,6,7) $P_2P_6P_8$	(3,6,7) $P_3P_{10}P_{14}$
Wed $\langle 1,6,7\rangle$	(1,6,7) $P_1P_4P_7$	(2,3,4) $P_8P_{12}P_{14}$	(2,3,5) $P_6P_{10}P_{15}$	(2,4,5) $P_2P_{11}P_{13}$	(3,4,5) $P_3P_5P_9$
Thu $\langle 2,4,7\rangle$	(2,4,7) $P_1P_9P_{10}$	(1,3,5) $P_4P_{11}P_{14}$	(1,3,6) $P_5P_8P_{15}$	(1,5,6) $P_3P_6P_{13}$	(3,5,6) $P_2P_7P_{12}$
Fri $\langle 2,5,6\rangle$	(2,5,6) $P_1P_5P_{14}$	(1,3,4) $P_7P_{10}P_{13}$	(1,3,7) $P_6P_9P_{12}$	(1,4,7) $P_3P_8P_{11}$	(3,4,7) $P_2P_4P_{15}$
Sat $\langle 3,4,6\rangle$	(3,4,6) $P_1P_6P_{11}$	(1,2,5) $P_7P_8P_9$	(1,2,7) $P_{13}P_{14}P_{15}$	(1,5,7) $P_2P_5P_{10}$	(2,5,7) $P_3P_4P_{12}$
Sun $\langle 3,5,7\rangle$	(3,5,7) $P_1P_8P_{13}$	(1,2,4) $P_4P_5P_6$	(1,2,6) $P_{10}P_{11}P_{12}$	(1,4,6) $P_2P_9P_{14}$	(2,4,6) $P_3P_7P_{15}$

of the lacing of Monday and Tuesday is the triple $(1,6,7)$ in Wednesday. The complete set of residual triples coincides precisely with the block-set of the Fano plane P_1, that is, with all the “lines” that are incident with the “point” $P_1 \in \mathcal{P}$. The second column of Table 17 contains all the residual triples and shows clearly that these are precisely the triples of “points” containing the common “point” P_1. Hence the resolution is isomorphic to system 1a.

Similarly, for a resolution of $PG(3,2)$ whose seven symbols are the blocks of a Fano plane F in $\mathcal{H}$, the seven residual triples are again the seven triples in F, which represent the seven “lines” of a “plane” in $PG(3,2)$.

By applying Theorem 2.4, we can conclude that a resolution of $PG(3,2)$, whose seven symbols are in $\binom{X}{3}$, is isomorphic to system 1a (resp., 1b) if the seven symbols are the blocks of a Fano plane in $\mathcal{P}$ (resp., in $\mathcal{H}$).

Moreover, the map $\langle a,b,c\rangle \mapsto (a,b,c)$ can be easily interpreted in the light of the previous Remark 2.6(1). Also, the automorphisms of a resolution of this kind are of the type $(a,b,c) \mapsto (\varphi(a),\varphi(b),\varphi(c))$ and $\langle a,b,c\rangle \mapsto \langle \varphi(a),\varphi(b),\varphi(c)\rangle$, where φ is an automorphism of the underlying Fano plane in $\mathcal{P}\cup\mathcal{H}$. In passing, this gives a direct combinatorial proof of the fact that the group of automorphisms of both systems 1a and 1b is isomorphic to the group of automorphisms of the Fano plane.

In particular, an automorphism of the KTS induces a cyclic permutation of the parallel classes if and only if the automorphism of the underlying Fano plane is a cyclic permutation of the seven points in X. For instance, for the resolution given in Table 17, associated with the “point” $P_1 \in \mathcal{P}$, the cyclic permutation $\sigma = (1\,2\,4\,3\,7\,6\,5)$ can be seen, at the same time, as an automorphism of the Fano plane P_1 and as an automorphism of the whole KTS, which fixes the “point” P_1 and induces a cyclic permutation of the parallel classes as the orbit of the class $\langle 1,2,3\rangle$.

Finally, in the case of a resolution of $PG(3,2)$, whose seven symbols are the blocks of a Fano plane whose point-set contains ∞, let F be the Fano plane in $\mathcal{P}\cup\mathcal{H}$ obtained by replacing ∞ with the element of X that does not appear in the seven symbols. Arguing as above, one can easily show that the seven residual triples form the blocks of a Fano plane in $\mathcal{P}$ (resp., in $\mathcal{H}$) if F is in $\mathcal{H}$ (resp., in $\mathcal{P}$). Therefore, the resolution is isomorphic to system 1a if F is in $\mathcal{H}$, and is isomorphic to system 1b if F is in $\mathcal{P}$.

3) (**A finite-geometry cyclic solution**) We already pointed out in Section 3, and in the two previous examples in this Appendix, that there exist cyclic solutions to the schoolgirl problem that are isomorphic to either system 1a or system 1b. Their visual representations can be easily obtained by applying the same construction with two concentric regular 7-gons as in Example 5 in Section 3, with just different choices of the base parallel class (see also [59, Figure iii, p. 200], [31, Figure 51, p. 126] and [7, Figure 5.2, p. 28]).

An alternative algebraic description of the cyclic solutions 1a and 1b can be given as follows, where we essentially regard $PG(3,2)$ as the derived design at $(0,0,0,0)$ of $AG_2(4,2)$, that is, of the point-plane design of the affine geometry $AG(4,2)$.

In the classical model of $PG(3,2)$, the points are the fifteen non-zero elements of the 4-dimensional vector space $GF(2)^4$, and the projective lines are all the unordered triples of points summing up to zero in (the additive group of) the vector space (from this point of view, the point-line design of $PG(3,2)$ is an example of *additive* block design [11]). We may also represent the fifteen points, up to isomorphism, as the non-zero elements of the 2-dimensional vector space $GF(4)^2$, where $GF(4) = \{0,1,\alpha,\alpha^2\}$ is the (unique) field with four elements and characteristic 2, with operations $1+\alpha=\alpha^2, 1+\alpha^2=\alpha, \alpha+\alpha^2=1, \alpha\alpha^2=1$.

We may now choose as the base parallel class the set of the five triples of points in $GF(4)^2$, obtained by removing $(0,0)$ from the five lines through the origin in the affine plane $AG(2,4)$ (the five triples are the five columns of the following array).

$$\begin{array}{lllll} (1,1) & (1,0) & (0,1) & (1,\alpha) & (\alpha,1) \\ (\alpha,\alpha) & (\alpha,0) & (0,\alpha) & (\alpha,\alpha^2) & (\alpha^2,\alpha) \\ (\alpha^2,\alpha^2) & (\alpha^2,0) & (0,\alpha^2) & (\alpha^2,1) & (1,\alpha^2). \end{array}$$

Next, we rewrite the base parallel class by representing again the 15 points in $GF(2)^4$, via the standard identification $0 \mapsto (0,0)$, $1 \mapsto (1,0)$, $\alpha \mapsto (0,1)$, $\alpha^2 \mapsto (1,1)$. Finally, we consider the orbit of the base parallel class under the action of the order-7 linear transformation on $GF(2)^4$ defined on the canonical basis by $(1,0,0,0) \mapsto (1,1,1,1)$, $(0,1,0,0) \mapsto (1,0,0,1)$, $(0,0,1,0) \mapsto (0,1,0,0)$, $(0,0,0,1) \mapsto (0,1,1,1)$. What one gets, in Table 18, is a (cyclic) resolution of PG(3, 2), where for simplicity we write every element of $GF(2)^4$ in the form *abcd*. Note, in passing, that any STS-automorphism of PG(3, 2) is necessarily induced by a linear map on $GF(2)^4$ by [29, Theorem 3.1].

Table 18
A cyclic resolution of PG(3, 2) isomorphic to system 1b.

	1010	1000	0010	1001	0110
Mon	0101	0100	0001	0111	1101
	1111	1100	0011	1110	1011
	1011	1111	0100	1000	1101
Tue	1110	1001	0111	1010	0001
	0101	0110	0011	0010	1100
	1100	0101	1001	1111	0001
Wed	0010	1000	1010	1011	0111
	1110	1101	0011	0100	0110
	0110	1110	1000	0101	0111
Thu	0100	1111	1011	1100	1010
	0010	0001	0011	1001	1101
	1101	0010	1111	1110	1010
Fri	1001	0101	1100	0110	1011
	0100	0111	0011	1000	0001
	0001	0100	0101	0010	1011
Sat	1000	1110	0110	1101	1100
	1001	1010	0011	1111	0111
	0111	1001	1110	0100	1100
Sun	1111	0010	1101	0001	0110
	1000	1011	0011	0101	1010

By arguing as in the Example 1 in Section 3, one can immediately show that the KTS is isomorphic to system 1b. Also, the seven residual triples are precisely the blocks of the Fano plane consisting of the seven rightmost triples in Table 18. The construction of this (cyclic) KTS is a crucial tool in the proof that $AG_2(4,2)$ decomposes into seven disjoint isomorphic copies of the affine plane of order four [49, §4].

Similarly, if one considers the orbit of the same base parallel class under the action of the order-7 linear transformation on $GF(2)^4$, defined on the canonical basis by $(1,0,0,0) \mapsto (0,0,0,1)$, $(0,1,0,0) \mapsto (1,0,0,0)$, $(0,0,1,0) \mapsto (0,1,0,0)$, $(0,0,0,1) \mapsto (1,0,1,1)$, then one gets a (cyclic) resolution of PG(3, 2) isomorphic to system 1a, whose seven residual triples are all the projective lines containing the point $(0,1,1,1)$, which is also the (unique) fixed point of the linear transformation.

4) (**Completing** AG(**3**, **2**) **to** PG(**3**, **2**)) We now describe the visual solution to the schoolgirl problem (essentially contained in [27]) that probably best reflects the projective nature of the underlying STS PG(3, 2), in order to interpret it in the light of Theorem 2.4.

Let the fifteen schoolgirls be denoted by $0, x, y, z, xy, xz, yz, xyz, X, Y, Z, XY, XZ, YZ, XYZ$, and let the first eight of them label in a standard way the vertices of a cube, as illustrated in Fig. 4.

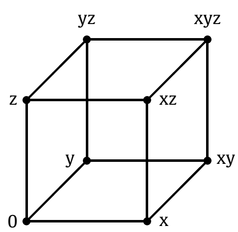


Fig. 4. A visual representation of the affine geometry AG(3, 2).

The 35 triples of the STS are defined as follows (note the similarity with Cayley's Example 1 in Section 3). The first 28 triples are precisely those of the type $\{0, a, A\}$, $\{0, ab, AB\}$, $\{0, abc, ABC\}$, $\{a, b, AB\}$, $\{a, ab, B\}$, $\{a, bc, ABC\}$, $\{a, abc, BC\}$, $\{ab, ac, BC\}$, and $\{ab, abc, C\}$, whereas the remaining seven triples are those of the type $\{A, B, AB\}$, $\{AB, AC, BC\}$, and $\{A, ABC, BC\}$. Note that the seven triples of the latter type determine a Fano plane (recall Remark 2.6(8) in Section 2).

In order to construct the seven parallel classes, we partition the (unordered) pairs of distinct vertices of the cube into three classes. A pair $\{v, w\}$ is of type (A) if v and w are adjacent vertices, that is, if they are the extreme points of an edge of the cube. A pair $\{v, w\}$ is of type (D) if v and w lie on the same face of the cube but are not adjacent, that is, if they are the extreme points of one of the two diagonals of a face of the cube. A pair $\{v, w\}$ is of type (O) if v and w are opposite vertices of the cube.

The solution to the schoolgirl problem is completely determined by the choice of just two pairs of distinct vertices. This initial choice partitions the 28 pairs of distinct vertices of the cube into seven classes consisting of four pairs each, where each class is in turn a partition of the eight vertices of the cube.

Let $\{v, w\}$ and $\{t, u\}$ be any two disjoint pairs of type (D), under the only condition that they do not lie on the same face of the cube, nor on two opposite faces. Up to rotation, we may assume that $\{v, w\} = \{xz, yz\}$, and either $\{t, u\} = \{x, z\}$ or $\{t, u\} = \{y, z\}$. We complete these two pairs with the only two possible pairs of type (A), such that the four pairs partition the vertices of the cube.

The second partition of the vertices of the cube is constructed as follows. The first two pairs are the two pairs of type (D) that lie on the faces opposite to those of $\{v, w\}$ and $\{t, u\}$, but are not parallel to $\{v, w\}$ and $\{t, u\}$. In view of the initial assumption, the first pair is $\{0, xy\}$ and the second pair is either $\{y, xyz\}$ or $\{x, xyz\}$, respectively. We complete the partition by adding, in a unique possible way, a pair of type (O) and a pair of type (A).

The next four partitions are obtained from the first two by applying, to each of them, the two order-3 rotations of the cube around the axis through the vertices 0 and xyz. Finally, the seventh partition contains the remaining four pairs of vertices that were not already considered in the previous six partitions.

In Fig. 5 we illustrate the first, second, and seventh partition, corresponding to the initial choice $\{v, w\} = \{xz, yz\}$ and $\{t, u\} = \{x, z\}$.

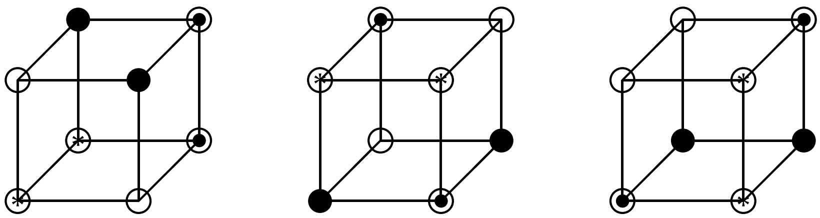


Fig. 5. The three basic partitions of the eight vertices.

We now construct the seven parallel classes of the KTS as follows. For each of the seven partitions of the eight vertices of the cube, we replace each of its four pairs by the unique triple of the STS containing that pair, and we complete the four triples thus obtained by adding the unique triple that is needed to get a partition of the fifteen schoolgirls.

Hence we obtain the resolution in Table 19, where Monday (resp., Thursday) is obtained from the first (resp., second) partition in Fig. 5, whereas Tuesday and Wednesday (resp., Friday and Saturday) are obtained from the two partitions constructed by rotation of the first (resp., second) partition. Finally, Sunday is obtained from the third partition in Fig. 5.

Table 19
The system 1b obtained by completing AG(**3**, **2**) to PG(**3**, **2**).

Mon	0, y, Y	x, z, XZ	xz, yz, XY	xy, xyz, Z	X, YZ, XYZ
Tue	0, z, Z	x, y, XY	xy, xz, YZ	yz, xyz, X	Y, XZ, XYZ
Wed	0, x, X	y, z, YZ	xy, yz, XZ	xz, xyz, Y	Z, XY, XYZ
Thu	0, xy, XY	z, xz, X	y, xyz, XZ	x, yz, XYZ	Y, Z, YZ
Fri	0, yz, YZ	x, xy, Y	z, xyz, XY	y, xz, XYZ	X, Z, XZ
Sat	0, xz, XZ	y, yz, Z	x, xyz, YZ	z, xy, XYZ	X, Y, XY
Sun	0, xyz, XYZ	x, xz, Z	y, xy, X	z, yz, Y	XY, XZ, YZ

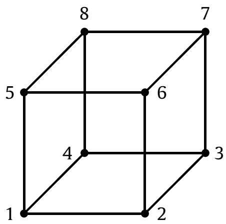
By arguing as in the previous Example 1 in Section 3, one immediately finds that the resulting KTS is isomorphic to system 1b. Also, the seven residual triples are precisely those in the rightmost column of Table 19, that is, they are the blocks of the Fano plane whose points are all written in capital letters. Moreover, the system admits by construction an order-3 automorphism induced by the permutations $(x\ y\ z)$ and $(X\ Y\ Z)$. The same conclusions hold in the case of the alternative initial choice $\{v, w\} = \{xz, yz\}$ and $\{t, u\} = \{y, z\}$.

Rephrased in different terms, the initial eight-point structure, whose points are the vertices of the cube, can be interpreted as the 3-dimensional affine space AG(3, 2) over GF(2), and the points X, Y, Z, XY, XZ, YZ, XYZ are the points at infinity for the parallel classes of the affine space (for instance, X is the point at infinity for the parallel class $\{0, x\}, \{y, xy\}, \{z, xz\}, \{yz, xyz\}$). By completing each of the 28 affine lines with the corresponding point at infinity, one finally gets a 3-dimensional projective space over GF(2) with three points per line, whose remaining 7 lines are those of a projective plane over GF(2) at infinity.

This proves again that the underlying STS(15) in this geometric construction is the point-line design of PG(3, 2), and shows that the seven residual triples of the KTS(15) are precisely the seven projective lines at infinity. Also, the seven partitions of the vertices of the cube form a 1-factorization of the complete graph K_8 (isomorphic to that which is usually denoted by $\mathcal{F}_1$), which is precisely the 1-factorization of K_8 associated to the KTS(15) in Table 19 with respect to the Fano plane at infinity. Our construction was inspired by a similar construction in [27], where the KTS(15) is also isomorphic to system 1b, although the seven residual triples are not the seven projective lines at infinity.

To get a KTS(15) isomorphic to system 1a it suffices to replace the first partition in Fig. 5 by the partition obtained by replacing each vertex of the cube with the opposite vertex, that is, by taking $\{0, z\}, \{x, y\}, \{xy, yz\}, \{xz, xyz\}$. The second and third partition in Fig. 5 are left unchanged. The resulting KTS(15) is isomorphic to system 1a, and the seven residual triples are precisely the seven projective lines containing the point at infinity XYZ.

We finally note that Table 19 allows one to get a nice visual representation of system 1b by means of the vertices, the edges, the faces of a cube, and the whole of a cube, maybe even more "naturally" than in Examples 3 and 4 in Section 3 (a similar representation, although not fully explicit, is given in [41]). Let us denote the eight vertices of the cube by the numbers $1, 2, \ldots, 8$, as in Fig. 3, which we repeat here for the convenience of the reader.



Each edge is denoted by a pair of the form vw, where v, w are the two vertices of that edge, and each face is denoted by a quadruple of the form $abcd$, where a, b, c, d are the four vertices belonging to that face. Also, we denote by C the whole of the cube. The fifteen schoolgirls are the eight vertices of the cube, C, and the six equivalence classes obtained by regarding as equivalent any two parallel edges and any two parallel faces (that is, any two edges and any two faces with the same point at infinity). For instance, the edges 12, 34, 56, 78 are all mutually equivalent and are the elements of one of the six equivalence classes. In what follows, we will always denote an equivalence class by any of its representatives.

The 35 triples of the underlying STS(15) are defined as follows. If v, w are two distinct vertices of the cube, then they belong to the triple $\{v, w, \alpha\}$, where: $\alpha = vw$ if v, w are adjacent vertices; $\alpha = abcd$ if v, w are non-adjacent vertices lying on a face $abcd$; $\alpha = \mathrm{C}$ if v, w are opposite vertices of the cube. The seven remaining triples are defined as follows. If e_1, e_2 are two non-equivalent edges, then they belong to the triple $\{e_1, e_2, f\}$, where f is the face determined by e_1, e_2 (note that, up to equivalence, one may assume that e_1, e_2 lie on the same face). If e is an edge orthogonal to a face f, then $\{e, f, \mathrm{C}\}$ is a triple. Finally, if f_1, f_2, f_3 are three mutually non-equivalent faces, then $\{f_1, f_2, f_3\}$ is a triple.

We can now transform Table 19 into Table 20 by replacing: the vertices $0, x, xy, y, z, xz, xyz, yz$ in Fig. 4 by the corresponding vertices 1, 2, 3, 4, 5, 6, 7, 8, in this order, in Fig. 3; XYZ by C, and each "point" of the form A, or AB, with capital letters, by the corresponding edge vw, or the corresponding face $abcd$, respectively, according to the above definition of the 35 triples. By construction of Table 19, the order-3 permutation $(2\,4\,5)(3\,8\,6)(1)(7)$ of the vertices induces an order-3 rotation of the whole cube around the axis through the vertices 1 and 7, which induces an order-3 automorphism of the KTS in Table 20 and an order-3 permutation (M TU W)(TH F SA)(SU) of its parallel classes.

Table 20
The cube-based representation of system 1b.

Mon	1, 4, 14	2, 5, 1256	6, 8, 5678	3, 7, 37	56, 2367, C
Tue	1, 5, 15	4, 2, 1234	3, 6, 2367	8, 7, 87	23, 3478, C
Wed	1, 2, 12	5, 4, 1458	8, 3, 3478	6, 7, 67	48, 5678, C
Thu	1, 3, 1234	5, 6, 56	4, 7, 3478	2, 8, C	14, 15, 1458
Fri	1, 8, 1458	2, 3, 23	5, 7, 5678	4, 6, C	15, 12, 1256
Sat	1, 6, 1256	4, 8, 48	2, 7, 2367	5, 3, C	12, 14, 1234
Sun	1, 7, C	2, 6, 26	3, 4, 34	5, 8, 58	1234, 1458, 1256

5) (**A "cyclic" solution in $\mathbb{Z}/15\mathbb{Z}$**) The present example perfectly illustrates the difference between the two notions of cyclicity that appear in the literature on the fifteen schoolgirl problem (and, more generally, in the literature on Kirkman triple systems). It is known that there exists no cyclic KTS(15) in the sense of [46]. In other words, there exists no KTS(15)

with an automorphism consisting of a single cycle of length 15, preserving the triples and the parallel classes. Nevertheless, PG(3, 2) is a resolvable cyclic STS(15) (again in the sense of [46]). Indeed, if one takes $\mathbb{Z}/15\mathbb{Z}$ as the point-set of the design, then the 35 triples of PG(3, 2) can be obtained by developing (mod 15) the base blocks $\{0, 1, 4\}$, $\{0, 2, 8\}$, and $\{0, 5, 10\}$. In particular, the map $i \mapsto i + 1$ preserves the triples and is an STS-automorphism consisting of a single cycle of length 15. Also, the fifteen Fano planes are obtained as a single orbit under the action of $i \mapsto i + 1$ on the base plane with triples $\{0, 1, 4\}$, $\{0, 2, 8\}$, $\{0, 5, 10\}$, $\{1, 2, 5\}$, $\{1, 8, 10\}$, $\{2, 4, 10\}$, $\{4, 5, 8\}$. This map, however, does not preserve any resolution of PG(3, 2).

A possible resolution is given in Table 21.

Table 21
System 1a with points in $\mathbb{Z}/15\mathbb{Z}$.

Mon	{14,0,3}	{1,2,5}	{7,8,11}	{9,10,13}	{4,6,12}
Tue	{0,1,4}	{2,3,6}	{8,9,12}	{10,11,14}	{5,7,13}
Wed	{4,5,8}	{6,7,10}	{11,12,0}	{13,14,2}	{1,3,9}
Thu	{8,10,1}	{9,11,2}	{12,14,5}	{13,0,6}	{3,4,7}
Fri	{7,9,0}	{2,4,10}	{3,5,11}	{6,8,14}	{12,13,1}
Sat	{0,2,8}	{10,12,3}	{11,13,4}	{14,1,7}	{5,6,9}
Sun	{0,5,10}	{1,6,11}	{2,7,12}	{3,8,13}	{4,9,14}

By applying Theorem 2.4 (see also Remark 2.6(2)), it is immediate that the KTS is isomorphic to system 1a (and that the residual triples are the seven triples containing the point 5). A KTS isomorphic to system 1b can be easily obtained by suitably rearranging the classes Monday, Tuesday, and Thursday. As it was well-known in the 1850s ([2,5,6,52]), these two systems are 2-rotational. For instance, the permutation (0 2 7 8 12 11 9)(10 1 13 4 14 3 6)(5) preserves the 35 triples and induces a cyclic permutation (M TU W TH F SA SU) of the parallel classes of system 1a in Table 21. This reveals the "dual nature" of the system: the points and the triples are defined in $\mathbb{Z}/15\mathbb{Z}$, but the underlying structure of the resolution is the cyclic additive structure of $\mathbb{Z}/7\mathbb{Z}$ (as in [3, Example 1.1]).

6) (**A hexagon-based solution and a bipyramid-based solution**) We conclude this paper by proposing two new visual solutions to the schoolgirl problem, with PG(3, 2) as their underlying Steiner triple system. The first one is based on the observation that, since $15 = \binom{6}{2}$, the fifteen schoolgirls can be seen as the edges of the complete graph K_6 on six points, which, in turn, can be represented as line segments between pairs of distinct vertices of a regular hexagon. With this point-set, the 35 triples of the STS(15) correspond precisely to the twenty triangles and the fifteen 1-factors of the complete graph K_6.

As in all the previous examples, the solution has two different versions, isomorphic to systems 1a and 1b. The arrangement proposed in Fig. 6 is that isomorphic to system 1b, as it can be readily seen by arguing as in the Example 1 in Section 3. Also, the corresponding residual triples are precisely the seven leftmost triples in the figure.

If the hexagons in Fig. 6 are rotated counterclockwise (respectively, clockwise) by 60 degrees in Monday and Tuesday (respectively, Wednesday and Thursday), and are left unchanged in Friday, Saturday, and Sunday, then the resulting KTS is isomorphic to system 1a.

An interesting property of the solution in Fig. 6 is the fact that, unlike all the other visual representations of 1a and 1b that we are aware of, this arrangement allows one to visualize an automorphism of order 4. Indeed, if we denote the upper left vertex of the hexagon by 1, and we label consecutively the other vertices clockwise by 2, 3, 4, 5, 6, then the permutation (1 2 5 4)(3)(6) induces on the pairs ab of vertices an order-4 automorphism

$$\psi = (36)(15\,24)(12\,25\,45\,14)(13\,23\,35\,34)(16\,26\,56\,46) \tag{3}$$

of the KTS, which in turn induces the order-4 permutation

(SUN)(FRI SAT)(MON TUE WED THU)

of the parallel classes.

We note, in passing, that the equality $15 = \binom{6}{2}$ allows us also to transform the two hexagon-based models of systems 1a and 1b into two new tetrahedron-based models of the same systems, since the fifteen simplicial elements of the tetrahedron are in a natural one-to-one correspondence with the 2-subsets of the set $\{V, F, a, b, c, d\}$. Indeed, if we label the four vertices of the tetrahedron by a, b, c, d, then, for any i and for any $j \neq k$ in $\{a, b, c, d\}$, the pairs Vi, Fi, and jk represent the vertex i, the face opposite to the vertex i, and the edge with endpoints j and k, respectively, whereas VF represents the whole tetrahedron. The transformation from the hexagon to the tetrahedron can be obtained, for instance, by means of the map

$$1 \mapsto a \quad 2 \mapsto b \quad 3 \mapsto V \quad 4 \mapsto c \quad 5 \mapsto d \quad 6 \mapsto F, \tag{4}$$

where 1, 2, 3, 4, 5, 6 denote, as above, the six vertices of the hexagon. Under this identification, the points of the Fano plane of the seven residual triples in system 1b are precisely the six edges and the whole tetrahedron.

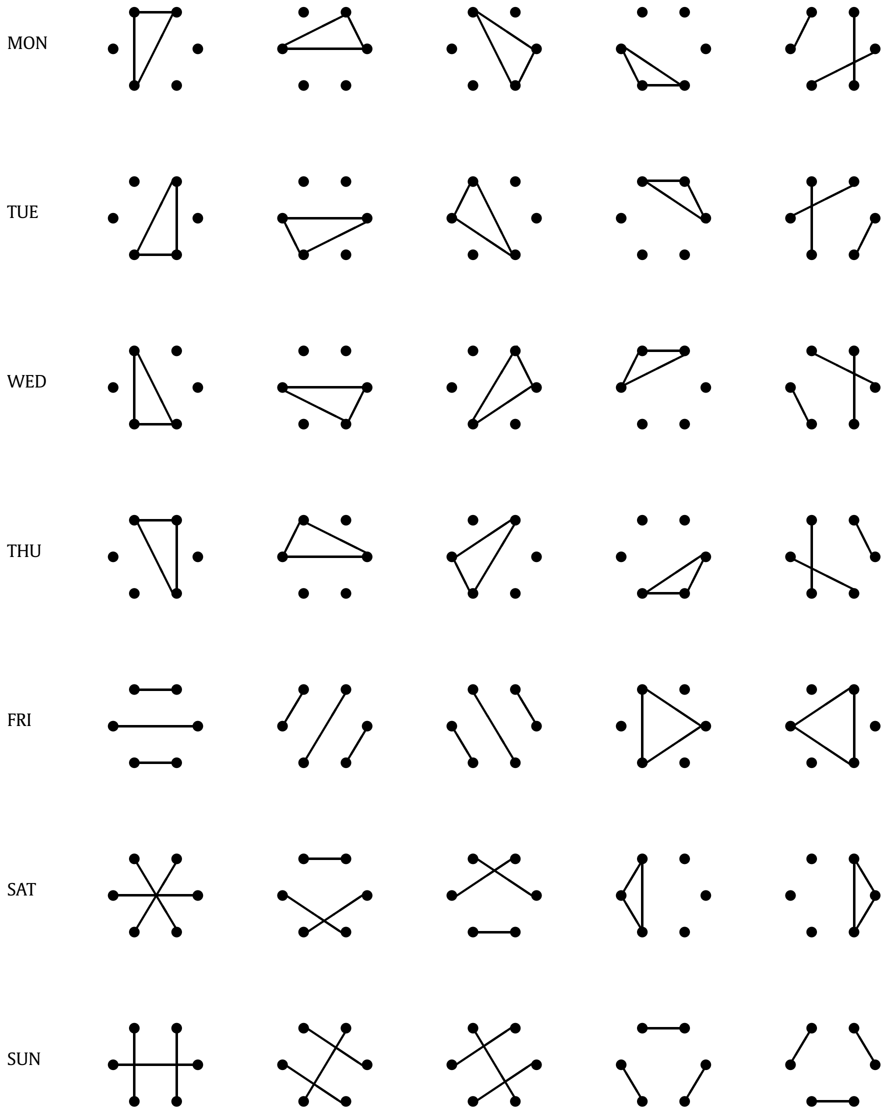


Fig. 6. The hexagon-based system 1b.

Unlike the two models in [28], which are particularly suited to visualize some automorphisms of order three (that is, the rotations of the tetrahedron that fix one vertex), the new tetrahedron-based model of system 1b (obtained from Fig. 6 by means of the map (4)) provides a nice visualization of the order-4 automorphism ψ defined above in (3), since ψ fixes the whole tetrahedron and has three orbits of length four, which include the set of the four vertices and the set of the four faces.

We finally propose a new spatial representation of systems 1a and 1b based on a regular triangular bipyramid, which is particularly suitable to reflect an order-3 automorphism. A regular triangular bipyramid is a composite solid made up of two tetrahedra sharing a common face. The fifteen schoolgirls are represented as the five vertices, the nine edges, and the whole of the bipyramid, which we can identify with the common face of the two tetrahedra. Let us denote the five vertices by the numbers $1, 2, 3, 4, 5$ as in Fig. 7. Each edge is denoted by a pair of the form ab, where a, b are the two vertices belonging to that edge. Also, the triple 234 denotes the whole of the bipyramid.

The underlying STS(15) is defined in a quite "natural" and symmetric way as follows. If e_1, e_2 are two disjoint edges, and v is the only vertex not belonging to either of the two edges, then $\{e_1, e_2, v\}$ is a triple. If e_1, e_2 are two edges lying on the same face (including also the hidden face 234), and e_3 is the third edge on that face, then $\{e_1, e_2, e_3\}$ is a triple. If e_1, e_2 are two edges with a common vertex and not lying on the same face, then $\{e_1, e_2, 234\}$ is a triple. If v_1, v_2 are two vertices on

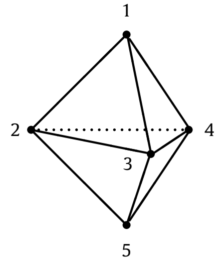


Fig. 7. The regular triangular bipyramid.

the same edge e, then $\{v_1, v_2, e\}$ is a triple. If v_1, v_2 are two vertices not on the same edge (that is, $\{v_1, v_2\} = \{1, 5\}$), then $\{v_1, v_2, 234\}$ is a triple. If v is a vertex in $\{2, 3, 4\}$, and $\{u, w\} = \{2, 3, 4\} \setminus \{v\}$, then $\{v, uw, 234\}$ is a triple. If we define

$$\phi = (2\ 3\ 4)(12\ 13\ 14)(23\ 34\ 24)(25\ 35\ 45)(1)(5)(234), \tag{5}$$

then ϕ is an order-3 automorphism of the STS, which fixes 1, 5, and 234, and rotates counterclockwise, by 120 degrees, the twelve remaining points of the STS. A KTS(15) can now be defined by taking the orbits under ϕ of the three base parallel classes

$$\begin{array}{lllll} \{12, 25, 234\} & \{3, 4, 34\} & \{2, 13, 45\} & \{1, 24, 35\} & \{5, 23, 14\} \\ \{2, 34, 234\} & \{1, 4, 14\} & \{3, 5, 35\} & \{12, 13, 23\} & \{24, 25, 45\} \\ \{1, 5, 234\} & \{23, 24, 34\} & \{2, 14, 35\} & \{3, 12, 45\} & \{4, 13, 25\}. \end{array}$$

The resulting KTS is given in Table 22.

Table 22
The system 1b based on a triangular bipyramid.

Mon	{12, 25, 234}	{3, 4, 34}	{2, 13, 45}	{1, 24, 35}	{5, 23, 14}
Tue	{13, 35, 234}	{4, 2, 42}	{3, 14, 25}	{1, 32, 45}	{5, 34, 12}
Wed	{14, 45, 234}	{2, 3, 23}	{4, 12, 35}	{1, 43, 25}	{5, 42, 13}
Thu	{2, 34, 234}	{1, 4, 14}	{3, 5, 35}	{12, 13, 23}	{24, 25, 45}
Fri	{3, 42, 234}	{1, 2, 12}	{4, 5, 45}	{13, 14, 34}	{32, 35, 25}
Sat	{4, 23, 234}	{1, 3, 13}	{2, 5, 25}	{14, 12, 42}	{43, 45, 35}
Sun	{1, 5, 234}	{23, 24, 34}	{2, 14, 35}	{3, 12, 45}	{4, 13, 25}

It can be readily seen that the KTS is isomorphic to system 1b by Theorem 2.4. Also, the seven residual triples are the blocks of the Fano plane constructed on the common face of the two tetrahedra. By construction, ϕ is an order-3 automorphism of the system, which induces on the parallel classes the order-3 permutation

(Mon Tue Wed)(Thu Fri Sat)(Sun).

By suitably reshuffling the triples in Thu, Fri, and Sat, one finds a KTS(15) isomorphic to system 1a, whose residual triples are the seven triples containing the common face 234. Again, ϕ is an automorphism of the system, inducing on the parallel classes the same order-3 permutation as for system 1b.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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