SEMILINEAR ROBIN PROBLEMS DRIVEN BY THE LAPLACIAN PLUS AN INDEFINITE POTENTIAL

CALOGERO VETRO

Abstract. We study a semilinear Robin problem driven by the Laplacian plus an indefinite potential. We consider the case where the reaction term f is a Carathéodory function exhibiting linear growth near $\pm\infty$. So, we establish the existence of at least two solutions, by using the Lyapunov-Schmidt reduction method together with variational tools.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we deal with the following semilinear Robin problem

$$\begin{cases} -\Delta u(z) + \xi(z)u(z) = f(z, u(z)) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} + \beta(z)u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

In this problem the potential function $\xi \in L^s(\Omega)$, with $s > N$, and it is sign changing. So, the linear part of (1) is indefinite. The reaction term $f(z, x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \to f(z, x)$ is measurable and for a.a. $z \in \Omega$, $x \to f(z, x)$ is continuous). We assume that $f(z, \cdot)$ exhibits linear growth near $\pm\infty$. In the boundary condition $\frac{\partial u}{\partial n}$ denotes the normal derivative of $u \in H^1(\Omega)$ on $\partial\Omega$ defined by extension of the linear map

$$C^1(\overline{\Omega}) \ni u \to \frac{\partial u}{\partial n} = (\nabla u, n)_{\mathbb{R}^N},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. For the boundary coefficient $\beta(\cdot)$ we assume that it belongs to $W^{1,\infty}(\partial\Omega)$ and $\beta(z) \geq 0$ for all $z \in \partial\Omega$. When $\beta \equiv 0$ we recover the Neumann problem.

Semilinear Robin problems were studied by Shi-Li [14] (with superlinear reaction), Qian-Li [15] (with zero potential), Zhang-Li-Xue [18] (with positive potential, thus coercive differential operator), Papageorgiou-Rădulescu [12] (with reaction admitting z-dependent zeros), D'Aguì-Marano-Papageorgiou [4] (problems with an asymmetric reaction), and Papageorgiou-Vetro-Vetro [13] (with reaction resonant both at zero and $\pm\infty$). Dirichlet and Neumann problems with indefinite and unbounded potential were also studied by Papageorgiou-Papalini [9], Kyritsi-Papageorgiou [7] (Dirichlet problems) and Gasiński-Papageorgiou [6], Papageorgiou-Rădulescu [10] (Neumann problems).

The setting and methods here are different from the aforementioned works and are based on the so-called "Lyapunov-Schmidt reduction method" originally developed by

Key words and phrases. Semilinear Robin problem, Carathéodory reaction, indefinite potential, Lyapunov-Schmidt reduction method.

2010 Mathematics Subject Classification: 35J20, 35J60.

Amann [1] and Castro-Lazer [2]. We prove the existence of at least two nontrivial solutions of problem (1).

2. Auxiliary Results and Hypotheses

In this section we present some auxiliary results and notions which we will need in the sequel. Let X be a Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). In the study of problem (1), we will use the Sobolev space $H^1(\Omega)$, the Banach space $C^1(\overline{\Omega})$ and the boundary Lebesgue space $L^q(\partial\Omega)$, $1 \leq q \leq \infty$. By $\|\cdot\|$ we denote the norm of the Sobolev space $H^1(\Omega)$, defined by

$$\|u\| = \left[\|u\|_2^2 + \|\nabla u\|_2^2\right]^{1/2} \quad \text{for all } u \in H^1(\Omega).$$

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using $\sigma(\cdot)$, one can define in the usual way the boundary Lebesgue spaces $L^q(\partial\Omega)$, with $1 \leq q \leq \infty$. The theory of Sobolev spaces implies that there exists a unique continuous linear map $\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$, known as the "trace map", such that

$$\gamma_0(u) = u|_{\partial\Omega} \text{ for all } u \in H^1(\Omega) \cap C(\overline{\Omega}).$$

Thus, we understand the trace map as representing the boundary values of a Sobolev function $u \in H^1(\Omega)$. Moreover, γ_0 is compact into $L^q(\partial\Omega)$ with $1 \leq q < \frac{2N-2}{N-2}$, if $N > 2$.

For the sake of notational simplicity, we decide to drop the use of the trace map γ_0. All restrictions of the Sobolev functions on $\partial\Omega$ are understood in the sense of traces. Our hypotheses on the data of problem (1), involve the spectrum $\sigma(-\Delta + \xi(z)I)$ of the differential operator $u \to -\Delta u + \xi(z)u$ with Robin boundary condition. So, we consider the following linear eigenvalue problem:

$$\begin{cases} -\Delta u(z) + \xi(z)u(z) = \widehat{\lambda} u(z) & \text{in } \Omega \\ \frac{\partial u}{\partial n} + \beta(z)u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Assume that $\xi \in L^s(\Omega)$ $(s > N)$ and let $\gamma : H^1(\Omega) \to \mathbb{R}$ be the C^2-functional defined by

$$\gamma(u) = \|\nabla u\|_2^2 + \int_\Omega \xi(z)u^2 dz + \int_{\partial\Omega} \beta(z)u^2 d\sigma \quad \text{for all } u \in H^1(\Omega).$$

The eigenvalue problem (2) has a smallest eigenvalue $\widehat{\lambda}_1 > -\infty$ given by

$$\widehat{\lambda}_1 = \inf\left[\frac{\gamma(u)}{\|u\|_2^2} : u \in H^1(\Omega), u \neq 0\right]. \tag{3}$$

Then we can find $\mu > 0$ such that

$$\gamma(u) + \mu\|u\|_2^2 \geq c_0\|u\|^2 \quad \text{for all } u \in H^1(\Omega), \text{ some } c_0 > 0 \quad \text{(see [4])}. \tag{4}$$

If we use (4) and the spectral theorem for compact self-adjoint operators, we produce the spectrum of (2), which consists of a sequence $\{\widehat{\lambda}_k\}_{k\geq 1}$ of eigenvalues such that $\widehat{\lambda}_k \to +\infty$ as $k \to +\infty$. By $E(\widehat{\lambda}_k)$ we denote the eigenspace corresponding to the eigenvalue $\widehat{\lambda}_k$. We have

$$E(\widehat{\lambda}_k) \subseteq C^1(\overline{\Omega}) \quad \text{(see Wang [17])}$$

and it has the unique continuation property (the UCP for short), that is, if $u \in E(\widehat{\lambda}_k)$ and $u(z) = 0$ for all z in a set of positive measure, then $u = 0$ (see Motreanu-Motreanu-Papageorgiou [8]). If $\overline{H}_m = \bigoplus_{k=1}^{m} E(\widehat{\lambda}_k)$ and $\widehat{H}_{m+1} = \overline{H}_m^{\perp} = \overline{\bigoplus_{k\geq m+1} E(\widehat{\lambda}_k)}$, then $\overline{H}_m$ is finite dimensional and we have the following orthogonal direct sum decomposition

$$H'(\Omega) = \overline{H}_m \oplus \widehat{H}_{m+1}.$$

The higher eigenvalues $\{\widehat{\lambda}_m\}_{m\geq 2}$ have the following variational characterizations:

$$\begin{aligned} \widehat{\lambda}_m &= \inf\left[\frac{\gamma(u)}{\|u\|_2^2} : u \in \widehat{H}_m, u \neq 0\right] \\ &= \sup\left[\frac{\gamma(u)}{\|u\|_2^2} : u \in \overline{H}_m, u \neq 0\right], \quad m \geq 2. \end{aligned} \tag{5}$$

In both (3) and (5) the infimum (and for (5) also the supremum) is realized on the corresponding eigenspace. Now, $\widehat{\lambda}_1 \in \mathbb{R}$ is simple and has eigenfunctions of constant sign. In fact, if $\widehat{u}_1$ denotes the L^2-normalized (that is, $\|\widehat{u}_1\|_2 = 1$) positive eigenfunction corresponding to $\widehat{\lambda}_1$, then $\widehat{u}_1(z) > 0$ for all $z \in \overline{\Omega}$. We point out that all the other eigenvalues have nodal (that is, sign changing) eigenfunctions.

Using (3) and (5) and the UCP of the eigenspaces, we get the following proposition.

Proposition 1. *The following assertions hold:*

(a) *If $\eta \in L^\infty(\Omega)$, $\eta(z) \leq \widehat{\lambda}_k$ for a.a. $z \in \Omega$ and the inequality is strict on a set of positive measure, then there exists $c_1 > 0$ such that*

$$\gamma(u) - \int_\Omega \eta(z)u^2 dz \geq c_1\|u\|^2 \quad \textit{for all } u \in \widehat{H}_k.$$

(b) *If $\eta \in L^\infty(\Omega)$, $\eta(z) \leq \widehat{\lambda}_k$ for a.a. $z \in \Omega$ and the inequality is strict on a set of positive measure, then there exists $c_2 > 0$ such that*

$$\gamma(u) - \int_\Omega \eta(z)u^2 dz \leq -c_2\|u\|^2 \quad \textit{for all } u \in \overline{H}_k.$$

Moreover, we denote

$$\begin{aligned} m_0 &= \min\{m \in \mathbb{N} : \widehat{\lambda}_m > 0\} \quad (\widehat{\lambda}_{m_0} = \text{smallest positive eigenvalue}), \\ k_0 &= \max\{m \in \mathbb{N} : \widehat{\lambda}_k < 0\} \quad (\widehat{\lambda}_{k_0} = \text{biggest negative eigenvalue}). \end{aligned}$$

If there are no negative eigenvalues, we set $\widehat{\lambda}_{k_0} = -\infty$.

The hypotheses on the data of (1) are the following:

$H(\xi)$: $\xi \in L^s(\Omega)$ with $s > N$.
$H(\beta)$: $\beta \in W^{1,\infty}(\partial\Omega)$ with $\beta(z) \geq 0$ for all $z \in \partial\Omega$.

Remark 1. When $\beta \equiv 0$ we recover the Neumann problem.

$H(f)$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$, and

(i) $|f(z, x)| \leq a(z)(1 + |x|)$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)_+$;

(ii) if $F(z,x)=\int_0^x f(z,s)ds$, then there exists a measurable set $D\subseteq\Omega$ with $|D|_N>0$ (where $|\cdot|_N$ denote the Lebesgue measure on $\mathbb{R}^N$) and $\vartheta\in L^1(\Omega)$ such that

$$F(z,x)\to+\infty \text{ for a.a. } z\in D \text{ as } x\to\pm\infty,$$
$$F(z,x)\geq\vartheta(z) \text{ for a.a. } z\in\Omega, \text{ all } x\in\mathbb{R};$$

(iii) there exists $\eta\in L^\infty(\Omega)_+$ such that

$$\eta(z)\leq\widehat{\lambda}_{m_0} \text{ for a.a. } z\in\Omega,\ \eta\not\equiv\widehat{\lambda}_{m_0},$$
$$(f(z,x)-f(z,x'))(x-x')\leq\eta(z)(x-x')^2 \text{ for a.a. } z\in\Omega, \text{ all } x,x'\in\mathbb{R};$$

(iv) there exist $\delta>0$ and $\eta_0\in L^\infty(\Omega)$ with $\eta_0(z)\leq 0$ for a.a. $z\in\Omega$, $\eta_0\not\equiv 0$ and

$$\frac{\widehat{\lambda}_{k_0}}{2}x^2\leq F(z,x)\leq\frac{\eta_0(z)}{2}x^2 \text{ for a.a. } z\in\Omega, \text{ all } |x|\leq\delta.$$

Remark 2. If $\widehat{\lambda}_{k_0}=-\infty$, then there is no lower bound for $F(z,\cdot)$ on $[-\delta,\delta]$.

H_0: $0\in\sigma(-\Delta+\xi(z)I)$.

Remark 3. This means that $H^0\neq\{0\}$, where $H^0=E(0)$ is the eigenspace corresponding to the eigenvalue 0.

The energy (Euler) functional of problem (1) is defined by

$$\varphi(u)=\frac{1}{2}\gamma(u)-\int_\Omega F(z,u)dz \quad \text{for all } u\in H^1(\Omega).$$

Evidently $\varphi\in C^1(H^1(\Omega))$. Recall that

$$H^1(\Omega)=Y\oplus H_+$$

with $Y=H_-\oplus H^0$, where $H_-=\oplus_{i=1}^{k_0}E(\widehat{\lambda}_i)$ and $H_+=\overline{\oplus_{i\geq m_0}E(\widehat{\lambda}_i)}=V$.

So, every $u\in H^1(\Omega)$ admits a unique sum decomposition

$$u=\overline{u}+u^0+\widehat{u}$$

with $\overline{u}\in H_-$, $u^0\in H^0$, $\widehat{u}\in H_+=V$.

Proposition 2. *If hypotheses $H(\xi)$, $H(\beta)$, $H(f)$ hold, then there exists a continuous map $\tau:Y\to V$ such that*

$$\varphi(y+\tau(y))=\inf\{\varphi(y+v):v\in V\} \quad \textit{for all } y\in Y.$$

Proof. Fix $y\in Y$ and consider the C^1-functional $\varphi_y:H^1(\Omega)\to\mathbb{R}$ defined by

$$\varphi_y(u)=\varphi(y+u) \quad \text{for all } u\in H^1(\Omega).$$

Consider the embedding map $i_V:V\to H^1(\Omega)$ and set

$$\widetilde{\varphi}_y=\varphi_y\circ i_V.$$

Clearly $\widetilde{\varphi}_y$ is C^1 and from the chain rule we have

$$\widetilde{\varphi}_y'=p_{V^*}\circ\varphi_y' \tag{6}$$

with p_{V^*} being the orthogonal projection of $H^1(\Omega)^*$ onto V^*. Let $\langle\cdot,\cdot\rangle_V$ denote the duality brackets for the pair (V^*,V) and define $A:H^1(\Omega)\to H^1(\Omega)^*$ by $\langle A(u),h\rangle=\int_\Omega(\nabla u,\nabla h)_{\mathbb{R}^N}dz$ for all $u,h\in H^1(\Omega)$. We have

$$\langle\widetilde{\varphi}_y'(v)-\widetilde{\varphi}_y'(\widehat{v}),v-\widehat{v}\rangle_V$$

$$
\begin{aligned}
&= \langle \varphi_y'(v) - \varphi_y'(\widehat{v}), v - \widehat{v}\rangle \quad \text{(see (6))}\\
&= \gamma(v-\widehat{v}) - \int_\Omega (f(z,v) - f(z,\widehat{v}))(v-\widehat{v})dz\\
&\geq \gamma(v-\widehat{v}) - \int_\Omega \eta(z)(v-\widehat{v})^2 dz \quad \text{(see hypothesis } H(f)(iii))\\
&\geq c_1\|v-\widehat{v}\|^2 \quad \text{for some } c_1>0 \text{ (see Proposition 1)} \qquad (7)
\end{aligned}
$$

$$\Rightarrow \quad \widetilde{\varphi}_y'(\cdot) \text{ is strongly monotone, therefore } \widetilde{\varphi}_y(\cdot) \text{ is strictly convex.}$$

Also, we have

$$
\begin{aligned}
\langle \widetilde{\varphi}_y'(v), v\rangle_V &= \langle \widetilde{\varphi}_y'(v) - \widetilde{\varphi}_y'(0), v\rangle_V + \langle \widetilde{\varphi}_y'(0), v\rangle_V\\
&\geq c_1\|v\|^2 - c_2\|v\| \quad \text{for some } c_2>0 \text{ (see (7))}, \qquad (8)
\end{aligned}
$$

$$\Rightarrow \quad \widetilde{\varphi}_y'(\cdot) \text{ is coercive.} \qquad (9)$$

The monotonicity and continuity of $\widetilde{\varphi}_y'(\cdot)$, imply that

$$\widetilde{\varphi}_y'(\cdot) \text{ is maximal monotone.} \qquad (10)$$

Then (9), (10) and Corollary 3.2.31, p. 319, of Gasiński-Papageorgiou [5], imply that

$$\widetilde{\varphi}_y'(\cdot) \text{ is surjective.}$$

Therefore, we can find $v_0 \in V$ such that

$$\widetilde{\varphi}_y'(v_0) = 0. \qquad (11)$$

Moreover, the strong monotonicity property of $\widetilde{\varphi}_y'$ (see (7)) implies that $v_0 \in V$ in (11) is unique (in fact v_0 is the unique minimizer of the strictly convex functional $\widetilde{\varphi}_y = \varphi_y\big|_V$). Now let $\tau : Y \to V$ be the map which to each $y \in Y$ assigns this unique solution v_0, that is, $\tau(y) = v_0$. Then from (6), (11) and the previous discussion we have

$$p_{V^*}\varphi'(y+\tau(y)) = 0, \quad \varphi(y+\tau(y)) = \inf\{\varphi(y+v) : v \in V\}. \qquad (12)$$

We need to show the continuity of $\tau(\cdot)$. Assume that $y_n \to y$ in Y. We have

$$
\begin{aligned}
0 &= \langle \widetilde{\varphi}_{y_n}'(\tau(y_n)), \tau(y_n)\rangle_V \quad \text{(see (12) and (6))}\\
&\geq c_1\|\tau(y_n)\|^2 - c_2\|\tau(y_n)\| \quad \text{(see (8))}\\
\Rightarrow &\quad \{\tau(y_n)\}_{n\geq 1} \subseteq V \text{ is bounded.}
\end{aligned}
$$

So, we may assume that

$$\tau(y_n) \xrightarrow{w} \widetilde{v} \text{ in } H^1(\Omega).$$

The Sobolev embedding theorem and the compactness of the trace map imply that φ is sequentially weakly lower semicontinuous. Hence

$$\varphi(y+\widetilde{v}) \leq \liminf_{n\to+\infty} \varphi(y_n + \tau(y_n)). \qquad (13)$$

From (12) we have

$$
\begin{aligned}
&\varphi(y_n+\tau(y_n)) \leq \varphi(y_n+v) \quad \text{for all } v\in V, \text{ all } n \in \mathbb{N},\\
\Rightarrow\quad &\varphi(y+\widetilde{v}) \leq \lim_{n\to+\infty}\varphi(y_n+v) = \varphi(y+v) \quad \text{for all } v \in V \text{ (see (13))},\\
\Rightarrow\quad &\varphi(y+\widetilde{v}) = \inf\{\varphi(y+v) : v\in V\},\\
\Rightarrow\quad &\widetilde{v} = \tau(y).
\end{aligned}
$$

From the Urysohn criterion for the convergence of sequences, we have for the original sequence that

$$\tau(y_n) \xrightarrow{w} \tau(y) \text{ in } H^1(\Omega) \text{ and } \tau(y_n) \to \tau(y) \text{ in } L^2(\Omega) \text{ and in } L^2(\partial\Omega). \tag{14}$$

For all $n \in \mathbb{N}$, we have

$$\begin{aligned}
& 0 = \langle \varphi'(y_n + \tau(y_n)), \tau(y_n) - \tau(y) \rangle \quad \text{(see (12))} \\
\Rightarrow \quad & \langle A(y_n + \tau(y_n)), \tau(y_n) - \tau(y) \rangle + \int_\Omega \xi(z)(y_n + \tau(y_n))(\tau(y_n) - \tau(y))dz \\
& + \int_{\partial\Omega} \beta(z)(y_n + \tau(y_n))(\tau(y_n) - \tau(y))d\sigma = 0, \\
\Rightarrow \quad & \limsup_{n\to+\infty} \langle A(y_n + \tau(y_n)), \tau(y_n) - \tau(y) \rangle \le 0, \\
\Rightarrow \quad & \|\nabla(y_n + \tau(y_n))\|_2 \to \|\nabla(y + \tau(y))\|_2 \quad \text{(recall } A \text{ is monotone)}, \\
\Rightarrow \quad & y_n + \tau(y_n) \to y + \tau(y) \text{ in } H^1(\Omega) \text{ (by the Kadec-Klee property)}, \\
\Rightarrow \quad & \tau(y_n) \to \tau(y) \text{ in } H^1(\Omega), \\
\Rightarrow \quad & \tau(\cdot) \text{ is continuous.}
\end{aligned}$$

□

Let $\widehat{\varphi}(y) = \varphi(y + \tau(y))$ for all $y \in Y$.

Proposition 3. *If hypotheses $H(\xi)$, $H(\beta)$, $H(f)$ hold, then $\widehat{\varphi} \in C^1(Y, \mathbb{R})$ and $\widehat{\varphi}'(y) = p_{Y^*}\varphi'(y + \tau(y))$.*

Proof. Let $y, h \in Y$ and $t > 0$. We have

$$\begin{aligned}
& \frac{1}{t}[\widehat{\varphi}(y + th) - \widehat{\varphi}(y)] \\
& \le \frac{1}{t}[\varphi(y + th + \tau(y)) - \varphi(y + \tau(y))], \\
\Rightarrow \quad & \limsup_{t\to 0^+} \frac{1}{t}[\widehat{\varphi}(y + th) - \widehat{\varphi}(y)] \le \langle \varphi'(y + \tau(y)), h \rangle.
\end{aligned} \tag{15}$$

Also, we have

$$\begin{aligned}
& \frac{1}{t}[\widehat{\varphi}(y + th) - \widehat{\varphi}(y)] \\
& \ge \frac{1}{t}[\varphi(y + th + \tau(y + th)) - \varphi(y + \tau(y + th))], \\
\Rightarrow \quad & \liminf_{t\to 0^+} \frac{1}{t}[\widehat{\varphi}(y + th) - \widehat{\varphi}(y)] \ge \langle \varphi'(y + \tau(y)), h \rangle. \\
& \text{(recall } \tau(\cdot) \text{ is continuous, see Proposition 2).}
\end{aligned} \tag{16}$$

From (15) and (16) it follows that

$$\lim_{t\to 0^+} \frac{1}{t}[\widehat{\varphi}(y + th) - \widehat{\varphi}(y)] = \langle \varphi'(y + \tau(y), h \rangle \quad \text{for all } h \in Y. \tag{17}$$

In a similar fashion, we show that

$$\lim_{t\to 0^-} \frac{1}{t}[\widehat{\varphi}(y + th) - \widehat{\varphi}(y)] = \langle \varphi'(y + \tau(y), h \rangle \quad \text{for all } h \in Y. \tag{18}$$

From (17) and (18) it follows that

$$\widehat{\varphi} \in C^1(Y,\mathbb{R}) \text{ and } \widehat{\varphi}'(y) = p_{Y^*}\varphi'(y+\tau(y)).$$

□

Proposition 4. *For every $\varepsilon > 0$, we can find $r_\varepsilon > 0$ such that*

$$\left|\left\{z \in \Omega : |u^0(z)| < r_\varepsilon \|u^0\|\right\}\right|_N < \varepsilon \quad \textit{for all } u^0 \in H^0.$$

Proof. We proceed by contradiction. So, suppose that the proposition is not true. Then we can find $\varepsilon > 0$ and $\{u_n^0\}_{n\in\mathbb{N}} \subseteq H^0$ such that

$$\left|\left\{z \in \Omega : |u_n^0(z)| < \frac{1}{n}\|u_n^0\|\right\}\right|_N \geq \varepsilon \quad \text{for all } n \in \mathbb{N}. \tag{19}$$

Let $y_n^0 = \dfrac{u_n^0}{\|u_n^0\|}$, $n \in \mathbb{N}$. Then $y_n^0 \in H^0$, $\|y_n^0\| = 1$ for all $n \in \mathbb{N}$. Exploiting the finite dimensionality of H^0 and by passing to a subsequence if necessary, we may assume that

$$y_n^0 \to y^0 \in H^0 \text{ in } H^1(\Omega), \text{ hence } \|y^0\| = 1. \tag{20}$$

If

$$E_n = \left\{z \in \Omega : |u_n^0(z)| < \frac{1}{n}\|u_n^0\|\right\} = \left\{z \in \Omega : |y_n^0(z)| < \frac{1}{n}\right\},$$
$$E = \left\{z \in \Omega : |y^0(z)| = 0\right\},$$

then from (20) we have

$$\limsup_{n\to+\infty} E_n \subseteq E,$$
$$\Rightarrow \quad \varepsilon \leq \limsup_{n\to+\infty} |E_n|_N \leq |E|_N.$$

But $y^0 \in H^0$ and by the UCP we have

$$y(z) \neq 0 \quad \text{for a.a. } z \in \Omega \text{ (see (20))},$$
$$\Rightarrow \quad |E|_N = 0, \text{ a contradiction.}$$

□

Using this proposition, we can establish the following useful property of the functional $\widehat{\varphi}$.

Proposition 5. *If hypotheses $H(\xi)$, $H(\beta)$, $H(f)$, H_0 hold, then $\widehat{\varphi}(y) \to -\infty$ as $\|y\| \to +\infty$ (that is, $\widehat{\varphi}$ is anticoercive).*

Proof. Again we argue indirectly. So, suppose we can find $M_1 > 0$ and $\{y_n\}_{n\geq 1} \subseteq Y$ such that

$$\|y_n\| \to +\infty \text{ and } -M_1 \leq \widehat{\varphi}(y_n) \quad \text{for all } n \in \mathbb{N}. \tag{21}$$

From Tang-Wu [16, Lemmata 2 and 3], given $\varepsilon > 0$, we can find $D_\varepsilon \subseteq D$ measurable, $g \in C(\mathbb{R},\mathbb{R})$, $g \geq 0$, subadditive and $\zeta \in L^1(D)_+$ such that

$$|D_\varepsilon|_N > 0, \quad |D \setminus D_\varepsilon|_N < \varepsilon, \tag{22}$$
$$F(z,x) \geq g(x) - \zeta(z) \text{ for a.a. } z \in D_\varepsilon, \text{ all } x \in \mathbb{R}, \tag{23}$$
$$g(\cdot) \text{ is coercive (that is } g(x) \to +\infty \text{ as } x \to \pm\infty), \tag{24}$$
$$|g(x)| \leq 4 + |x| \text{ for all } x \in \mathbb{R}. \tag{25}$$

Recall $Y = H_- \oplus H^0$ and $y_n \in Y$ for all $n \in \mathbb{N}$. So, we can write in a unique way

$$y_n = \overline{y}_n + y_n^0 \quad \text{with } \overline{y}_n \in H_-,\ y_n^0 \in H^0.$$

Then from (21) and the orthogonality of the component spaces, we have

$$\begin{aligned}
-M_1 \le \widehat{\varphi}(y_n) &\le \varphi(y_n) \\
&\le \gamma(\overline{y}_n) - \int_\Omega F(z, y_n)dz \\
&\le -c_3\|\overline{y}_n\|^2 - \int_\Omega F(z, y_n)dz \quad \text{for some } c_3 > 0 \\
&\qquad (\text{since } \overline{y}_n \in H_- \text{ and } \dim H_- < +\infty) \\
&\le -c_3\|\overline{y}_n\|^2 - \int_{D_\varepsilon} g(y_n)dz + \int_{D_\varepsilon} \zeta(z)dz + \|\vartheta\|_1 \\
&\qquad (\text{see (23) and hypothesis } H(f)(ii)) \\
&\le -c_3\|\overline{y}_n\|^2 + c_4 \quad \text{for some } c_4 > 0 \quad (\text{recall } g \ge 0),
\end{aligned}$$

(26) $$\Rightarrow \quad \{\overline{y}_n\}_{n\ge1} \subseteq H_- \text{ is bounded.}$$

We have

$$\|y_n\| \le \|\overline{y}_n\| + \|y_n^0\| \quad \text{for all } n \in \mathbb{N}.$$

From (21) and (26) it follows that

(27) $$\|y_n^0\| \to +\infty.$$

We fix $\delta > 0$. Proposition 4 implies that we can find $r_\delta > 0$ such that

(28) $$\left|\left\{z \in \Omega : |u^0(z)| < r_\delta\|u^0\|\right\}\right|_N < \delta \quad \text{for all } u^0 \in H^0 = E(0).$$

We set

$$E_n = \left\{z \in \Omega : |y_n^0(z)| \ge r_\delta\|y_n^0\|\right\} \quad \text{for all } n \in \mathbb{N}.$$

From (28) it follows that

(29) $$|\Omega \setminus E_n|_N < \delta \quad \text{for all } n \in \mathbb{N}.$$

Recall that H_- is finite dimensional. So, we can find $M_2 > 0$ such that

(30) $$\|\overline{y}_n\|_\infty \le M_2 \quad \text{for all } n \in \mathbb{N} \text{ (see (26)).}$$

Because of (24), given any $k > 0$, we can find $M_3 = M_3(k) > 0$ such that

(31) $$g(x) \ge k \quad \text{for all } |x| \ge M_3.$$

Let

$$C_n = \{z \in \Omega : |y_n(z)| \ge M_3\} \quad \text{for all } n \in \mathbb{N}.$$

From (30) and since $y_n = y_n^0 + \overline{y}_n$, we have

$$\begin{aligned}
|y_n(z)| &\ge |y_n^0(z)| - |\overline{y}_n(z)| \\
&\ge r_\delta\|y_n^0\| - M_2 \quad \text{for a.a. } z \in E_n, \text{ all } n \in \mathbb{N}, \\
&\Rightarrow \quad |y_n(z)| \ge M_3 \quad \text{for all } n \ge n_0 \text{ (see (27)),}
\end{aligned}$$

(32) $$\Rightarrow \quad E_n \subseteq C_n \quad \text{for all } n \ge n_0.$$

We have

$$\int_{D_\varepsilon} g(y_n)dz = \int_{D_\varepsilon \cap C_n} g(y_n)dz + \int_{D_\varepsilon \setminus C_n} g(y_n)dz$$

$$\geq \int_{D_\varepsilon \cap E_n} g(y_n) dz \quad \text{(recall that } g \geq 0 \text{ and see (31), (32))}$$
$$\geq k|D_\varepsilon \cap E_n|_N \quad \text{for all } n \geq n_0. \tag{33}$$

Note that

$$\begin{aligned} |D_\varepsilon \cap E_n|_N &= |D_\varepsilon|_N - |D_\varepsilon \setminus E_n|_N \\ &\geq |D_\varepsilon|_N - |\Omega \setminus E_n|_N \\ &\geq |D_\varepsilon|_N - \delta \quad \text{(see (29))}. \end{aligned}$$

Choosing $\delta > 0$ small we have

$$|D_\varepsilon \cap E_n|_N > 0 \quad \text{for all } n \geq n_0. \tag{34}$$

Returning to (33) and using (34) and the fact that $k > 0$ is arbitrary, we conclude that

$$\int_{D_\varepsilon} g(y_n) dz \to +\infty. \tag{35}$$

Recall that

$$\begin{aligned} & \widehat{\varphi}(y_n) \leq -c_3 \|\overline{y}_n\|^2 - \int_{D_\varepsilon} g(y_n) dz + c_5 \quad \text{for some } c_5 > 0, \text{ all } n \in \mathbb{N} \\ \Rightarrow \quad & \widehat{\varphi}(y_n) \to -\infty \quad \text{as } n \to +\infty \text{ (by (35))}, \end{aligned}$$

a contradiction to (21). This proves the anticoercivity of $\widehat{\varphi}$. □

3. Multiplicity Theorem

In this section, we prove the existence of two solutions for problem (1).

Let $\psi = -\widehat{\varphi}$. Then ψ is coercive (see Proposition 5) and so we can state the following result.

Proposition 6. *If hypotheses $H(\xi)$, $H(\beta)$, $H(f)$, H_0 hold, then ψ has local linking at $u = 0$ with respect to $H_- \oplus H^0$.*

Proof. Let $u^0 \in H^0$. Exploiting the orthogonality of the component spaces, we have

$$\begin{aligned} \psi(u^0) &= -\widehat{\varphi}(u^0) = -\varphi(u^0 + \tau(u^0)) \\ &= -\frac{1}{2}\gamma(u^0 + \tau(u^0)) + \int_\Omega F(z, u^0 + \tau(u^0)) dz. \end{aligned} \tag{36}$$

Hypotheses $H(f)(i),(iv)$ imply that given $r \in (2, 2^*)$, we can find $c_6 = c_6(r) > 0$ such that

$$F(z, x) \leq \frac{\eta_0(z)}{2} x^2 + c_6 |x|^r \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{37}$$

Using (37) in (36) we obtain

$$\begin{aligned} \psi(u^0) &\leq -\frac{1}{2}\gamma(u^0 + \tau(u^0)) + \frac{1}{2}\int_\Omega \eta_0(z)(u^0 + \tau(u^0))^2 dz + c_7 \|u^0 + \tau(u^0)\|^r \\ &\qquad \text{for some } c_7 > 0, \\ &\leq -c_8 \|u^0 + \tau(u^0)\|^2 + c_7 \|u^0 + \tau(u^0)\|^r \quad \text{for some } c_8 > 0. \end{aligned} \tag{38}$$

For every $\widehat{u} \in H_+$ we have

$$\begin{aligned}
\varphi(\widehat{u}) &= \frac{1}{2}\gamma(\widehat{u}) - \int_\Omega F(z,\widehat{u})dz \\
&\geq \frac{1}{2}\gamma(\widehat{u}) - \frac{1}{2}\int_\Omega \eta(z)\widehat{u}^2 dz \quad \text{(see hypothesis } H(f)(iii)) \\
&\geq c_9\|\widehat{u}\|^2 \quad \text{for some } c_9 > 0 \text{ (see Proposition 1)}, \qquad (39)\\
\Rightarrow \quad & \inf_{H_+} \varphi = 0. \qquad (40)
\end{aligned}$$

We have

$$\begin{aligned}
& \widehat{\varphi}(0) = \varphi(0 + \tau(0)) = \varphi(\tau(0)) = 0 \quad \text{(see (40))}, \\
\Rightarrow \quad & \tau(0) = 0 \quad \text{(see (39))}.
\end{aligned}$$

So, from (38) and the continuity of $\tau(\cdot)$ (see Proposition 2) it follows that we can find $\rho_1 > 0$ such that

$$\psi(u^0) \leq 0 \quad \text{for all } u^0 \in H^0,\ \|u^0\| \leq \rho_1. \tag{41}$$

Next let $\overline{u} \in H_-$. Then

$$\begin{aligned}
\psi(\overline{u}) = -\widehat{\varphi}(\overline{u}) \geq -\varphi(\overline{u}) & \\
&= -\frac{1}{2}\gamma(\overline{u}) + \int_\Omega F(z,\overline{u})dz \\
&\geq -\frac{\widehat{\lambda}_{k_0}}{2}\|u\|_2^2 + \int_\Omega F(z,\overline{u})dz. \qquad (42)
\end{aligned}$$

Since H_- is finite dimensional, all norms are equivalent. So, we can find $\rho_2 > 0$ such that

$$\|\overline{u}\| \leq \rho_2 \quad \Rightarrow \quad |\overline{u}(z)| \leq \delta \quad \text{for a.a. } z \in \Omega.$$

Therefore from hypothesis $H(f)(iv)$ we have

$$F(z,\overline{u}(z)) \geq \frac{\widehat{\lambda}_{k_0}}{2}\overline{u}(z)^2 \quad \text{for a.a. } z \in \Omega.$$

Using this in (42) we obtain

$$\psi(\overline{u}) \geq 0 \quad \text{for all } \overline{u} \in H_-,\ \|\overline{u}\| \leq \rho_2. \tag{43}$$

From (41) and (43) we conclude that ψ has local linking at $u = 0$ with respect to $H_- \oplus H^0$. ☐

Let $K_\varphi := \{u \in Y : \varphi'(u) = 0\}$.

Proposition 7. *If hypotheses $H(\xi)$, $H(\beta)$, $H(f)$, H_0 hold, then $y \in K_{\widehat{\varphi}}$ if and only if $y + \tau(y) \in K_\varphi$.*

Proof. $\Leftarrow$: is immediate from Proposition 3.

$\Rightarrow$: Let $y \in K_{\widehat{\varphi}}$. Then

$$\begin{aligned}
& \widehat{\varphi}'(y) = 0, \\
\Rightarrow \quad & p_{Y^*}\varphi'(y + \tau(y)) = 0 \quad \text{(see Proposition 3)}, \qquad (44)\\
\Rightarrow \quad & \varphi'(y + \tau(y)) \in H_+^* \quad \text{(recall } H^1(\Omega)^* = Y^* \oplus H_+^*).
\end{aligned}$$

From (12) we have

$$\begin{aligned}
&\varphi'(y+\tau(y)) \in Y^* \\
\Rightarrow\quad & \varphi'(y+\tau(y)) = 0 \quad \text{(see (44))}, \\
\Rightarrow\quad & y+\tau(y) \in K_\varphi.
\end{aligned}$$

□

Now we are ready for the multiplicity theorem which gives two nontrivial solutions for problem (1).

Theorem 1. *If hypotheses $H(\xi)$, $H(\beta)$, $H(f)$, H_0 hold, then problem* (1) *has at least two nontrivial solutions $u_0, \widehat{u} \in C^1(\overline{\Omega})$, $u_0 \neq \widehat{u}$.*

Proof. Note that $\psi = -\widehat{\varphi}$ is coercive (see Proposition 5), hence

$$\psi \text{ satisfies the PS-condition and its bounded below, } \psi(0) = 0.$$

In addition from Proposition 6 we have that

$$\psi \text{ has local linking at } u = 0 \text{ with respect to } H_- \oplus H^0.$$

If $\inf_Y \psi = 0$, then all $u^0 \in H^0$ with $\|u^0\| \leq \rho = \min\{\rho_1, \rho_2\}$ are critical points of ψ (see (41)). Hence by Proposition 7 we have an infinity of solutions.

If $\inf_Y \psi < 0$, then we apply the theorem of Brezis-Nirenberg [3] and have

$$\begin{aligned}
& y_0, \widehat{y} \in K_\psi = K_{\widehat{\varphi}}, \quad y_0 \neq \widehat{y}, \quad y_0, \widehat{y} \neq 0, \\
\Rightarrow\quad & u_0 = y_0 + \tau(y_0), \quad \widehat{u} = \widehat{y} + \tau(\widehat{y}) \in K_\varphi, \quad u_0 \neq \widehat{u}, \quad u_0, \widehat{u} \neq 0 \text{ (see Proposition 7)}.
\end{aligned}$$

Note that hypotheses $H(f)$ imply that

$$|f(z,x)| \leq c_{10}|x| \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ some } c_{10} > 0. \tag{45}$$

We have

$$\begin{aligned}
& \varphi'(u_0) = 0, \\
\Rightarrow\quad & \langle A(u_0), h\rangle + \int_\Omega \xi(z) u_0 h dz + \int_{\partial\Omega} \beta(z) u_0 h d\sigma = \int_\Omega f(z, u_0) h dz \quad \text{for all } h \in H^1(\Omega), \\
\Rightarrow\quad & -\Delta u_0(z) + \xi(z) u_0(z) = f(z, u_0(z)) \quad \text{for a.a. } z \in \Omega, \\
& \frac{\partial u_0}{\partial n} + \beta(z) u_0 = 0 \quad \text{on } \partial\Omega \quad \text{(see Papageorgiou-Rădulescu [11])}.
\end{aligned} \tag{46}$$

Let

$$d_0(z) = \begin{cases} \dfrac{f(z, u_0(z))}{u_0(z)} & \text{if } u_0(z) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $d_0 \in L^\infty(\Omega)$ (see (45)). From (46) we have

$$\begin{aligned}
& -\Delta u_0(z) = (d_0(z) - \xi(z)) u_0(z) \quad \text{for a.a. } z \in \Omega, \\
& \frac{\partial u_0}{\partial n} + \beta(z) u_0 = 0 \quad \text{on } \partial\Omega.
\end{aligned}$$

Note that $(d_0 - \xi)(\cdot) \in L^s(\Omega)$, $s > N$. Then using Lemmata 5.1 and 5.2 of Wang [17], we have

$$u_0 \in W^{2,s}(\Omega) \hookrightarrow C^{1,\alpha}(\overline{\Omega}) \quad \alpha = 1 - \frac{N}{s} > 0.$$

The embedding of $C^{1,\alpha}(\overline{\Omega})$ into $C^1(\overline{\Omega})$ implies $u_0 \in C^1(\overline{\Omega})$.
Similarly, we show $\widehat{u} \in C^1(\overline{\Omega})$. □

References

[1] H. Amann, *Saddle points and multiple solutions of differential equations*, Math. Z., **169** (1979), 127–166.

[2] A. Castro and A.C. Lazer, *Critical point theory and the number of solutions of a nonlinear Dirichlet problem*, Ann. Mat. Pura Appl., **120** (1979) 113–137.

[3] H. Brezis and L. Nirenberg, *Remarks on finding critcal points*, Commun. Pure Appl. Anal., **44** (1991), 939–963.

[4] G. D'Aguì, S.A. Marano and N.S. Papageorgiou, *Multiple solutions to a Robin problem with indefinite weight and asymmetric reaction*, J. Math. Anal. Appl., **433** (2016), 1821–1845.

[5] L. Gasiński and N.S. Papageorgiou, Nonlinear Analysis, Ser. Math. Anal. Appl. **9**, Chapman and Hall/CRC Press, Boca Raton, (2006).

[6] L. Gasiński and N.S. Papageorgiou, *Multiplicity of solutions for Neumann problems with an indefinite and unbounded potential*, Commun. Pure Appl. Anal., **12** (2013), no. 5, 1985–1999.

[7] S. Kyritsi and N.S. Papageorgiou, *Multiple solutions for superlinear Dirichlet problems with an indefinite potential*, Ann. Mat. Pura Appl., **192** (2013), 297–315.

[8] D. Motreanu, V. Motreanu and N.S. Papageorgiou, Topological and Variational Methods with Applications to Nonlinear Boundary Value Problems, Springer, New York, (2014).

[9] N.S. Papageorgiou and F. Papalini, *Seven solutions with sign information for sublinear equations with indefinite and unbounded potential and no symmetries*, Israel J. Math. **201** (2014), 761–796.

[10] N.S. Papageorgiou and V.D. Rădulescu, *Semilinear Neumann problems with indefinite and unbounded potential and crossing nonlinearity*, Contemp. Math., **595** (2013), 293–315.

[11] N.S. Papageorgiou and V.D. Rădulescu, *Multiple solutions with precise sign for nonlinear parametric Robin problems*, J. Differential Equations, **256** (2014), 2449–2479.

[12] N.S. Papageorgiou and V.D. Rădulescu, *Robin problems with indefinite, unbounded potential and reaction of arbitrary growth*, Rev. Mat. Complut. **29** (2016), 91–126.

[13] N.S. Papageorgiou, C. Vetro and F. Vetro, Pairs of solutions for Robin problems with an indefinite and unbounded potential, resonant at zero and infinity, Manuscripta Math. (2018). https://doi.org/10.1007/s00229-018-1044-x

[14] S. Shi and S. Li, *Existence of solutions for a class of semilinear elliptic equations with the Robin boundary value condition*, Nonlinear Anal., **71** (2009), 3292–3298.

[15] A. Qian and C. Li, *Infinitely many solutions for a Robin boundary value problem*, International Journal of Differential Equations, **2010**:548702 (2010), 9 pages.

[16] C.-L. Tang, X.-P. Wu, *Periodic solutions for second order systems with not uniformly coercive potential*, Journal of Mathematical Analysis and Applications, **259** (2001), 386–397.

[17] X.-J. Wang, *Neumann problems of semilinear elliptic equations involving critical Sobolev exponents*, J. Differential Equations, **93** (1991), 283–310.

[18] J. Zhang, S. Li, X. Xue, *Multiple solutions for a class of semilinear elliptic problems with Robin boundary condition*, Journal of Mathematical Analysis and Applications, **388** (2012), 435–442.

(C. Vetro) University of Palermo, Department of Mathematics and Computer Science, Via Archirafi 34, 90123, Palermo, Italy

Email address: calogero.vetro@unipa.it