COMPARABILITY OF THE TOTAL BETTI NUMBERS OF TORIC IDEALS OF GRAPHS

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Abstract. The total Betti numbers of the toric ideal of a simple graph are, in general, highly sensitive to any small change of the graph. In this paper we look at some combinatorial operations that cause total Betti numbers to change in predictable ways. In particular, we focus on a procedure that preserves these invariants.

1. Introduction

Given a finite simple graph G on the vertex set $V = \{x_1, \dots, x_r\}$ with edge set $E = \{e_1, \dots, e_n\}$, the *toric ideal of* G, denoted I_G, is the kernel of the map $\varphi : \mathbb{K}[E] = \mathbb{K}[e_1, \dots, e_n] \to \mathbb{K}[x_1, \dots, x_r]$ given by $\varphi(e_i) = x_{i_1}x_{i_2}$ where $e_i = \{x_{i_1}, x_{i_2}\} \in E$, with $\mathbb{K}$ an algebraically closed field of characteristic zero. The standard graded $\mathbb{K}$-algebra $\mathbb{K}[E]/I_G$ will be denoted by $\mathbb{K}[G]$.

Properties of toric ideals of graphs have been extensively studied in recent years from several points of view, see [2, 3, 5, 6, 7, 8, 9, 10, 13, 15, 16] just to cite some of them. Many questions remain unanswered, for instance it is not yet known a characterization of the graphs G such that the algebra $\mathbb{K}[G]$ is Cohen-Macaulay, see Section 1.1 for the definition. More generally, it is important to explore the procedures which transform a graph G into a new graph G' such that the Betti numbers (see Section 1.1) of $\mathbb{K}[G']$ are predictable from those of $\mathbb{K}[G]$.

We start by recalling the relevant background on the homological invariants, graphs and their toric ideals.

1.1. **Notation: Homological Invariants.** Given an homogeneous ideal I in a polynomial ring R, the minimal graded free resolution of R/I has the form

$$0 \to \bigoplus_{j \in \mathbb{N}} R(-j)^{\beta_{p,j}(R/I)} \to \cdots \to \bigoplus_{j \in \mathbb{N}} R(-j)^{\beta_{1,j}(R/I)} \to R \to R/I \to 0$$

where we recall that $R(-j)$ denotes the ring R with its grading shifted by j, and $\beta_{i,j}(R/I) = \dim_{\mathbb{K}} \operatorname{Tor}_i^R(R/I, \mathbb{K})_j$ is called the i,j*-th graded Betti number* of R/I. The numbers $\beta_i(R/I) = \sum_j \beta_{i,j}(R/I)$ are called the i*-th total Betti numbers* of R/I. The algebra R/I is *Cohen-Macaulay* (CM for short) if and only if its depth equals its (Krull) dimension.

In this paper we investigate some operations on graphs which have a *good behavior* with respect to the total Betti numbers and the Cohen-Macaulay property. It is known that,

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see [10, Corollary 3.5], if H is a subgraph of G then $\beta_{ij}(\mathbb{K}[H]) \le \beta_{ij}(\mathbb{K}[G])$. However, as noticed for instance in [10, Example 5.4], this inequality does not ensure the persistence of the Cohen-Macaulay property from $\mathbb{K}[G]$ to $\mathbb{K}[H]$. Indeed, such phenomenon could happen if $\operatorname{codim} \mathbb{K}[H] < \operatorname{codim} \mathbb{K}[G]$.

1.2. **Notation: Graphs.** Let $G = (V, E)$ be a simple graph and $x \in V$. We denote by $\mathcal{N}(x) = \{y \in V \mid \{x, y\} \in E\}$, the *neighbors set* of x. We denote by $d(x) = |\mathcal{N}(x)|$.

A *walk* in a graph $G = (V, E)$ is a sequence of vertices $w = (x_0, x_1, x_2, \dots, x_n)$ where $\{x_i, x_{i+1}\} \in E$ for $i = 0, \dots, n-1$. The *length* of the walk $w = (x_0, x_1, x_2, \dots, x_n)$ is n and it is denoted by $|w|$. A walk w is even (odd) if n is even (odd), i.e., it consists of an even (odd) number of edges. A walk is closed if $x_0 = x_n$. Given a walk $w = (x_0, x_1, x_2, \dots, x_n)$ it will be useful to work with the edges $e_i = \{x_{i-1}, x_i\}$, so we will use the same terminology and we will also call *walk* the sequence $(e_1, \dots, e_n)$. With an abuse of notation we write $w = (e_1, \dots, e_n) = (x_0, x_1, x_2, \dots, x_n)$.

Let w, w' be two walks in G, if the last vertex in w is the fist vertex in w', then the symbol $w \| w'$ denotes the concatenation of the two walks.

We say that a walk $p = (x_0, x_1, x_2, \dots, x_n)$ is a *path* if $x_0 \neq x_n$ and $d(x_i) = 2$ for $i = 1, \dots, n-1$, i.e. the vertices $x_1, \dots, x_{n-1}$ don't have neighbors outside of the walk (be aware that in this paper "path"has a stronger meaning than the usual definition which only requires that a path is a walk in which every vertex appears at most once except the start and end points). For a path $p = (x_0, x_1, \dots, x_t)$ we denote by p^r the *reverse path* of p, that is $p^r = (x_t, \dots, x_1, x_0)$.

1.3. **Notation: Toric Ideals of Graphs.** We refer the reader to [12, Section 5.3] for a more exhaustive overview of the topic. Let $G = (V, E)$ be a simple graph. Given a list of edges of G, say $w = (e_1, e_2, \dots, e_n)$, we denote by e_w the monomial given by the product of all the edges in w, i.e.,

$$e_w = \prod_j e_j \in \mathbb{K}[E].$$

We associate to an even list of edges $w = (e_1, e_2, \dots, e_{2n})$ two sub-lists which consist of the odd and even entries

$$w^+ = (e_1, e_3, \dots, e_{2n-1}) \quad \text{and} \quad w^- = (e_2, e_4, \dots, e_{2n}).$$

Moreover, we set $f_w \in \mathbb{K}[W]$ to be the homogeneous binomial defined by

$$f_w = e_{w^+} - e_{w^-}.$$

A set of generators of the toric ideal I_G, (which also constitutes a universal Gröbner basis of I_G, see [18, Proposition 10.1.10]), corresponds to the primitive closed even walks in G. Recall that a closed even walk w in a graph G (and the correspondent binomial f_w) is said to be *primitive* if there is not another closed even walk v in G such that e_{v^+} divides e_{w^+} and e_{v^-} divides e_{w^-}.

1.4. **Results of the paper.** We now summarize the content of the paper.

In Section 2 we recall the definition of path contractions in a graph and we introduce the simple path contractions (see Definition 2.2). We prove some preliminary lemmas, and we enunciate the following two results which provide a relation between the path contraction and the total Betti numbers of a graph.

Theorem 2.7 *Let G be a simple graph. Let G/p be an even path contraction of G. Then*

$$\beta_i(\mathbb{K}[G]) \geq \beta_i(\mathbb{K}[G/p]) \quad for\ any\ i \geq 0.$$

Theorem 2.9 *Let G be a simple graph. Let G/p be an even path contraction of G. If there is a path q containing p with $|q| \geq |p| + 2$, then*

$$\beta_i(\mathbb{K}[G]) = \beta_i(\mathbb{K}[G/p]) \quad for\ any\ i \geq 0.$$

We derive several corollaries and we discuss significant examples related to these results. The proof of Theorem 2.9 is postponed to Section 3 where the background knowledge on simplicial complexes is introduced.

In Section 4 we begin the study of edge contractions for a special class of graphs introduced in Definition 4.1. The main result of this section is the following.

Theorem 4.2 *Let $G = G_1 \overset{e}{-} G_2$ be a graph connected by the edge e. Then*

$$\beta_1(\mathbb{K}[G]) = \beta_1(\mathbb{K}[G/e]).$$

2. Path contractions in a graph and the total Betti numbers

We recall the well known definition from Graph Theory of contraction of an edge in a graph.

Definition 2.1. Let $G = (V, E)$ be a simple graph. Let $e = \{x_0, x_1\} \in E$ and y_0 a new vertex. Set $V' = (V \cup \{y_0\}) \setminus e$. Let $\chi : V \to V'$ be the function which maps every vertex in $V \setminus e$ into itself and both x_0 and x_1 into y_0. Consider the set $E' = \{\{\chi(x), \chi(x')\} \mid \{x, x'\} \in E \setminus \{e\}\}$, where eventual redundancies are ignored. The graph (V', E'), denoted by G/e, is said to be an *edge contraction* of G.

Let p be a path in G. A *path contraction* of G, denoted by G/p, is the graph obtained by contracting all the edges in p. We say that a path contraction is *even (odd)* if p is an even (odd) path.

The graph $G/p = (V', E')$ in the above definition is simple.

We also need the following definition.

Definition 2.2. Let $G = (V, E)$ be a simple graph. Let p be a path in G and x and x' be the first and the last vertices in p. We say that a path is *simple* if $\mathcal{N}(x) \cap \mathcal{N}(x') \subseteq p$ (that is, either $\mathcal{N}(x) \cap \mathcal{N}(x') = \varnothing$ or the common neighbors to x and x' are in p). We say that a path contraction G/p is *simple* if p is simple.

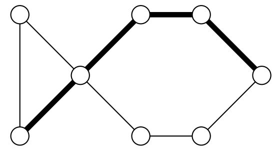
In particular, a path of length 2, $p = (x, y, x')$, is simple when $\mathcal{N}(x) \cap \mathcal{N}(x') = \{y\}$ and a path of length greater than 2 is simple when $\mathcal{N}(x) \cap \mathcal{N}(x') = \varnothing$.

In the next example we clarify the definitions.

Example 2.3. The walks in bold in Figure 1 are not paths.

The walks in bold in Figure 2 are paths. In particular, the one in Figure 2 (A) is a simple odd path and the one in Figure 2 (B) is a non-simple even path.

The graphs in Figure 3 are obtained by contracting the paths in Figure 2 (A) and Figure 2 (B). The black dots in the figures represent the new vertex of the graph where the path "collapses".



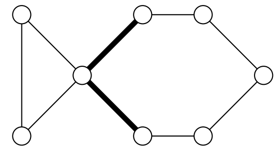
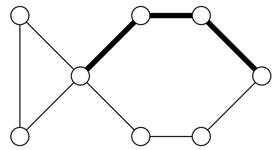


FIGURE 1



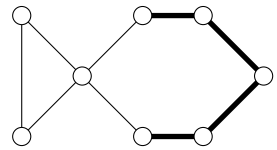
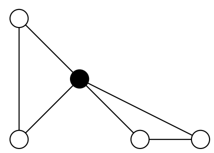


FIGURE 2



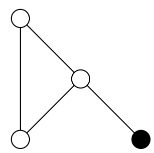


FIGURE 3

Remark 2.4. Note that an even simple path contraction maps even (odd) cycles of G into even (odd) cycles of G' and it does not create new cycles.

Lemma 2.5. Let $G = (V, E)$ be a simple graph and p be an even simple path in G. Then

$$\operatorname{codim} \mathbb{K}[G] = \operatorname{codim} \mathbb{K}[G/p].$$

Proof. First note that, from Remark 2.4, the graphs $G = (V, E)$ and $G/p = (V', E')$ are both either bipartite or not bipartite. Since $V' = (V \smallsetminus p) \cup \{y\}$, then, from [18, Corollary 10.1.21], in the non-bipartite case, we have

$$\operatorname{codim} \mathbb{K}[G] = |E| - |V| = |E| - |p| - |V| + |p| = |E'| - |V'| = \operatorname{codim} \mathbb{K}[G/p].$$

A similar computation can be made in the bipartite case. □

From the next result we will have that an even simple path contraction do not produce "new" primitive even closed walks. However, the next lemma does not require the assumption that the path is simple. For an even closed walk w and a path p in a graph G, we will write w/p with the meaning of Definition 2.1 considering w as a subgraph of G.

Lemma 2.6. Let G be a simple graph. Let p be an even path of G and let w be a primitive even closed walk in G. Then w/p is an even closed walk in G/p. On the other hand if v is an even closed walk in G/p then $v = w/p$ for some w an even closed walk in G.

Proof. Set $p = (x_0, \dots, x_{2t})$ and recall that we denote by p^r the path $(x_{2t}, \dots, x_0)$. Let $y \in V'$ be the new vertex and let $\chi : V \to V'$ be the function contracting the path p into y. The following cases may be distinguished

- w and p do not have edges in common. In this case $w/p = \chi(w)$;
- $w = p||w'$, where w' is an even walk connecting x_{2t} with x_0, also w' and p do not have edges in common. Then $w/p = \chi(w')$ is an even closed walk in G/p.
- $w = p||w'||p^r||w''$, where w', w'' are odd walks both connecting x_0 and x_{2t} and they have no edge in common with p. Then $w/p = \chi(w')||\chi(w'')$ is an even closed walk in G/p.

To prove the second part of the statement, let v be a cycle in G/p passing through $y = \chi(p)$, i.e., there are sub-paths of v of the type (x, y, x') where $x, x' \in \mathcal{N}(x_0) \cup \mathcal{N}(x_{2t+1})$. We explicitly construct a walk w in G, by replacing the y in the sequences (x, y, x') contained in v as follows

- if $x, x' \in \mathcal{N}(x_0)$ then we replace y by x_0;
- if $x, x' \in \mathcal{N}(x_{2t+1})$ then we replace y by x_{2t+1};
- if $x \in \mathcal{N}(x_0)$ and $x' \in \mathcal{N}(x_{2t})$ then we replace y by p;
- if $x' \in \mathcal{N}(x_0)$ and $x \in \mathcal{N}(x_{2t})$ then we replace y by p^r.

A straightforward calculation proves that the resulting walk w is closed and even in G and $w/p = v$. □

We are interested in how a simple even path contraction affects the total Betti numbers of a graph. We first show that performing this procedure the total Betti numbers can eventually only go down, then we look for cases where the equality holds. We prove a more general result which does not require that the path is simple.

Theorem 2.7. Let G be a simple graph. Let G/p be an even path contraction of G. Then

$$\beta_i(\mathbb{K}[G]) \geq \beta_i(\mathbb{K}[G/p]) \quad \textit{for any } i \geq 0.$$

Proof. Without loss of generality (since an even path is a concatenation of paths of length 2) we assume $p = (e_1, e_2) = (x_1, x_2, x_3)$. Let $\ell = e_2 - e_1 \in \mathbb{K}[E]$ be a linear form, and let $\mathfrak{a} = (e_3, \dots, e_N) \subseteq \mathbb{K}[E]$ be the ideal generated by all the variables except the two in p. We claim that

$$(\mathbb{K}[G]/\ell\mathbb{K}[G])_{\mathfrak{a}} \cong \mathbb{K}[G/p],$$

i.e., the edge ring of G/p is isomorphic to the localization at $\mathfrak{a}$ of the quotient of $\mathbb{K}[G]$ by ℓ.

The inequality in the statement is a direct consequence of the claim. In fact, the linear form ℓ is regular in $\mathbb{K}[G]$ and the ideal I_G defines a variety with positive dimension in $\mathbb{P}^{N-1}$. Thus, the operation of taking the quotient by ℓ preserves the total Betti numbers (from the geometrical point of view it is a proper hyperplane section), and the localization can only make them go down. Indeed, localization is an exact functor and a minimal free resolution remains exact upon localization at $\mathfrak{a}$. However, after the localization the resolution is not necessarily minimal anymore since some of the maps may have entries not belonging to $\mathfrak{a}$.

In order to prove the claim, we define $\mathcal{G}$ to be a set of binomial generators for I_G. The elements in $\mathcal{G}$ correspond, from [18, Proposition 10.1.10], to the closed primitive even walks of G. Notice that every closed primitive even walk of G which includes either e_1 or e_2 must include both; moreover, see [18, Proposition 10.1.8], it only might include e_1 and e_2 at most two times. Thus, there is a natural partition of the set of closed primitive even walks of G, that is $W_0 \cup W_1 \cup W_2$, where the elements in W_0 do not involve e_1 and e_2; the elements in W_1 involve only once e_1 and e_2; the elements in W_2 involve twice e_1 and e_2. The partition of the closed primitive even walks of G is automatically inherited by $\mathcal{G}$. We set $\mathcal{G} = \mathcal{G}_0 \cup \mathcal{G}_1 \cup \mathcal{G}_2$, where the set $\mathcal{G}_i$ contains the binomials corresponding to the cycles in W_i.

We have the following isomorphisms

$$\mathbb{K}[G]/\ell\mathbb{K}[G] \cong K[E]/I_G + (e_2 - e_1) \cong \frac{\mathbb{K}[E]/(e_2 - e_1)}{I_G + (e_2 - e_1)/(e_2 - e_1)}.$$

Morover,

$$\frac{\mathbb{K}[E]/(e_2 - e_1)}{I_G + (e_2 - e_1)/(e_2 - e_1)} \cong \mathbb{K}[E \smallsetminus \{e_2\}]/I',$$

where I' is minimally generated by the binomials in $\mathcal{G}' = \mathcal{G}_0 \cup \mathcal{G}_1' \cup \mathcal{G}_2'$ where the set $\mathcal{G}_i'$, for $i = 1, 2$, consists of binomials of the type $(f_w)'$ with $w \in W_i$. Here we denote by f' the image in $\mathbb{K}[E \smallsetminus \{e_2\}] \cong \mathbb{K}[E]/(e_2 - e_1)$ of a form in f in $\mathbb{K}[E]$. Note that e_1^i divides $(f_w)'$ for $w \in W_i$ and $i = 1, 2$.

Now, we define $\mathfrak{a}' = (e_3, \dots, e_N) \subseteq \mathbb{K}[E \smallsetminus \{e_2\}]$. We have the following isomorphisms

$$(\mathbb{K}[G]/\ell\mathbb{K}[G])_{\mathfrak{a}} \cong (\mathbb{K}[E \smallsetminus \{e_2\}]/I')_{\mathfrak{a}'}.$$

Localizing by $\mathfrak{a}'$, the form e_1 becomes a unit, so we further have

$$(\mathbb{K}[E \smallsetminus \{e_2\}]/I')_{\mathfrak{a}'} \cong \mathbb{K}(e_1)[E \smallsetminus \{e_1, e_2\}]/I'',$$

where $\mathbb{K}(e_1)$ is the field extension of $\mathbb{K}$ by e_1 and I'' is the ideal in $\mathbb{K}(e_1)[E \smallsetminus \{e_1, e_2\}]$ generated by binomials in $\mathcal{G}'' = \mathcal{G}_0 \cup \mathcal{G}_1'' \cup \mathcal{G}_2''$ where the set $\mathcal{G}_i''$, for $i = 1, 2$, consists of binomials of the type $(f_w)''/e_1^i$ with $w \in W_i$. Here we denote by f'' the image in $\mathbb{K}(e_1)[E \smallsetminus \{e_1, e_2\}]$ of a form in f' in $\mathbb{K}[E \smallsetminus \{e_2\}]$.

Finally note that, by construction and Lemma 2.6, the ideal I'' is isomorphic to the toric ideal of G/p. □

Corollary 2.8. Let G be a simple graph. Let p be an even simple path of G. If $\mathbb{K}[G/p]$ is not Cohen-Macaulay then $\mathbb{K}[G]$ is not Cohen-Macaulay.

Proof. It follows from Lemma 2.5 and Theorem 2.7. □

We will show in Examples 2.13 and 2.14 that the converse of Corollary 2.8 is false, i.e., for an even simple path p of G, having $\mathbb{K}[G]$ not Cohen-Macaulay does not necessarily imply that $\mathbb{K}[G/p]$ is not Cohen-Macaulay.

It would be interesting to find extra hypotheses in Theorem 2.7 to guarantee the equality of the total Betti number. In this direction we have the following result. A combinatorial proof of it is given in Section 3.

Theorem 2.9. Let G be a simple graph. Let G/p be an even path contraction of G. If there is a path q containing p with $|q| \geq |p| + 2$, then

$$\beta_i(\mathbb{K}[G]) = \beta_i(\mathbb{K}[G/p]) \quad \textit{for any } i \geq 0.$$

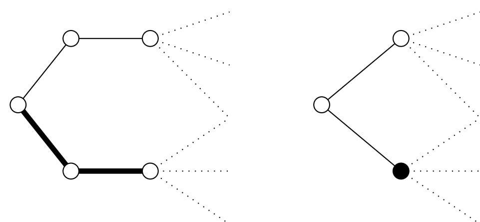
Note that the assumptions of the theorem force the path p to be simple.

The next corollary collects immediate consequences of Theorem 2.9.

Corollary 2.10. Let G be a simple graph. Let p be an even simple path of G such that there exists a path q in G containing p with $|q| \geq |p| + 2$. Then

(i) $\mathbb{K}[G]$ is Cohen-Macaulay if and only if $\mathbb{K}[G/p]$ is Cohen-Macaulay;
(ii) $\mathbb{K}[G]$ is Gorenstein if and only if $\mathbb{K}[G/p]$ is Gorenstein;

Remark 2.11. The graph G in Figure 4 (A) consists of a path q of length at least 4 and a dashed part that could be any simple graph. The path p consists of any two consecutive vertices of q. The graph in Figure 4 (B) is G/p obtained by contracting p. From Theorem 2.9 the two graphs in Figure 4 have the same total Betti numbers.

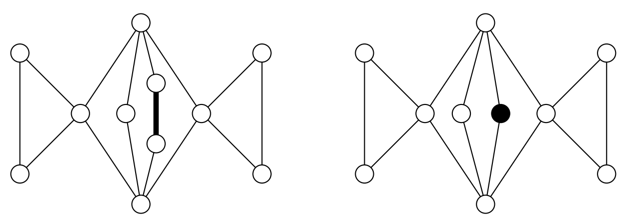


(A) (B)

FIGURE 4. Contraction of a path of length two contained in a simple path of length four.

Does an analogous of Theorem 2.9 exist with odd contractions? In the next example we show that the answer to such question is in general negative. Indeed, we give an example of an odd simple path contraction which leads to non comparable Betti numbers.

Example 2.12. Consider the graphs G and G' in Figure 5. Note that G' has a path q of length 3 and G' is obtained by G by contracting an edge in q.



(A) G (B) G'

FIGURE 5. The graph G' in Example 2.12 is an edge contraction of G.

A computation with CoCoA shows that the total Betti numbers are not comparable,

$i:$	0	1	2	3	4
$\beta_i(\mathbb{K}[G])$	1	8	18	16	5
$\beta_i(\mathbb{K}[G'])$	1	9	16	9	1

.

It is also natural to ask if an analogous of Theorem 2.9 holds with an even contraction and a weaker assumption on q. For instance if either $|q| = |p| + 1$ (see Figure 6) or p is a maximal path and $q = p$ (see Figure 7). In the next examples we show that in these cases the equality does not necessarily hold anymore. These examples also show that an even contraction may produce a Cohen-Macaulay graph from a non Cohen-Macaulay one.

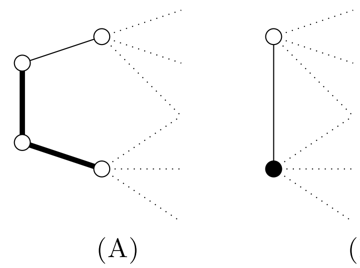


FIGURE 6. Contraction of a path of length two in a simple path of length three.

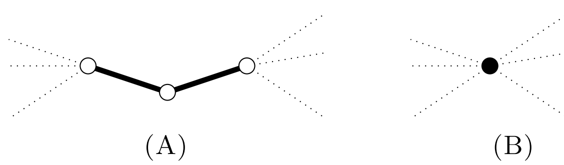


FIGURE 7. Contraction of a simple path of length two

Example 2.13. Let $G = (V, E)$ be the graph with vertex set $V = \{x_1, \dots, x_{10}\}$ and edges

$$E = \begin{array}{l} \{\{x_1,x_2\},\{x_2,x_3\},\{x_1,x_3\},\{x_4,x_5\},\{x_5,x_6\},\{x_4,x_6\}\}\cup \\ \{\{x_1,x_7\},\{x_7,x_8\},\{x_8,x_4\},\{x_1,x_9\},\{x_9,x_{10}\},\{x_{10},x_4\}\}. \end{array}$$

Set $q = (x_1, x_9, x_{10}, x_4)$ and $p = (x_1, x_9, x_{10})$, then p is a simple even path contained in a path of legth $|p| + 1$, but one can check that the Betti numbers strictly decrease after the contraction. Indeed, $\mathbb{K}[G]$ is not Cohen-Macaulay but $\mathbb{K}[G/p]$ is Cohen-Macaulay.

i	0	1	2	3
$\beta_i(\mathbb{K}[G])$	1	4	4	1
$\beta_i(\mathbb{K}[G/p])$	1	2	1	

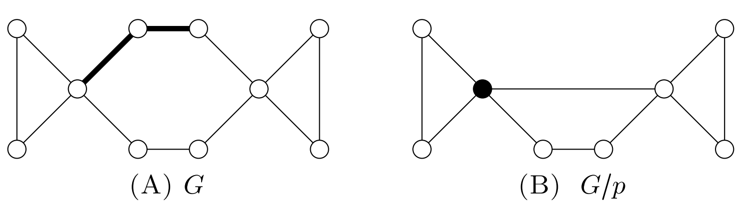


FIGURE 8. The graphs in Example 2.13.

With a slight modification of the graph in Example 2.13 we show that contracting a maximal path may the total Betti numbers go down.

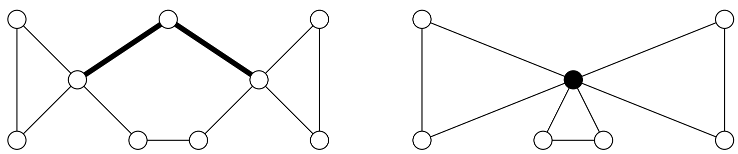
Example 2.14. Let $G' = (V', E')$ be the graph with vertex set $V' = \{x_1, \dots, x_9\}$ and edges

$$\begin{aligned} E' = \; & \{\{x_1,x_2\},\{x_2,x_3\},\{x_1,x_3\},\{x_4,x_5\},\{x_5,x_6\},\{x_4,x_6\}\} \cup \\ & \{\{x_1,x_7\},\{x_7,x_8\},\{x_8,x_4\},\{x_1,x_9\},\{x_9,x_4\}. \end{aligned}$$

Note that G' is isomorphic to the graph obtained from the graph G in Example 2.13 by contracting the edge $\{x_9, x_{10}\}$, $G' \cong G/\{x_9, x_{10}\}$.

Set $p' = (x_1, x_9, x_4)$, thus p' is a maximal simple even path. Again one can check that the Betti numbers strictly decrease after the contraction. Indeed $\mathbb{K}[G']$ is not Cohen-Macaulay but $\mathbb{K}[G'/p']$ has this property.

i	0	1	2	3
$\beta_i(\mathbb{K}[G'])$	1	4	4	1
$\beta_i(\mathbb{K}[G'/p'])$	1	3	2	



(A) G' (B) G'/p'

FIGURE 9. The graphs in Example 2.14.

Theorem 2.7 does not hold if we consider an even walk instead of even path. In fact, a contraction of a walk in a graph could produce new primitive odd cycles which make the total Betti numbers increase. This phenomenon is illustrated in the next example.

Example 2.15. Let $G = (V, E)$ be the graph with vertex set $V = \{x_1, \dots, x_{11}\}$ and edge set

$$\begin{aligned} E = \; & \{\{x_1,x_2\},\{x_2,x_3\},\{x_3,x_4\},\{x_4,x_5\},\{x_5,x_6\},\{x_1,x_6\},\{x_2,x_5\}\} \cup \\ & \{\{x_1,x_7\},\{x_1,x_8\},\{x_7,x_9\},\{x_8,x_9\},\{x_9,x_{10}\},\{x_9,x_{11}\},\{x_{10},x_{11}\}\} \end{aligned}$$

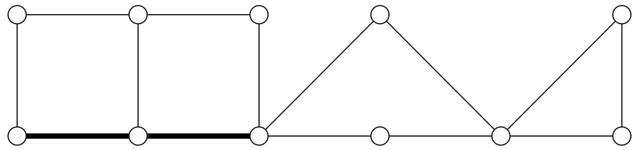
The contraction of the walk $p = (x_1, x_2, x_3)$ gives the graph $G' = G/p$ whose edge set is

$$\begin{aligned} E' = \; & \{\{x_1,x_4\},\{x_4,x_5\},\{x_5,x_6\},\{x_1,x_6\},\{x_1,x_5\}\} \cup \\ & \{\{x_1,x_7\},\{x_1,x_8\},\{x_7,x_9\},\{x_8,x_9\},\{x_9,x_{10}\},\{x_9,x_{11}\},\{x_{10},x_{11}\}\} \end{aligned}$$

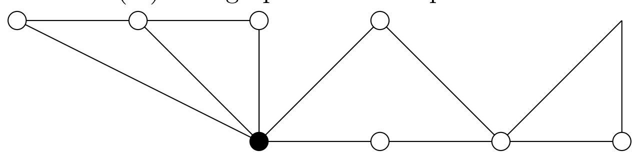
These graphs are pictured in Figure 10. A computation shows that the total Betti numbers of G' are bigger than those of G, precisely we get

$i:$	0	1	2	3	4
$\beta_i(\mathbb{K}[G])$	1	3	3	1	
$\beta_i(\mathbb{K}[G'])$	1	8	15	10	2

.



(A) The graph G in Example 2.15



(B) The graph G' in Example 2.15

FIGURE 10. G' is a walk contraction of G.

3. Proof of Theorem 2.9

In this section we prove Theorem 2.9 by using combinatorial tools. We begin by introducing some terminology and results about simplicial complexes. For what is not included in this section we refer to [11], see in particular [11, Section 1.5.1] for the background and [11, Section 5.1.4] for the results on simplicial homology. Then, we prove some preliminary results in order to apply the formula in [4, Theorem 2.1] which allows us to compute the total Betti numbers of the toric ideal of a graph.

3.1. **Simplicial Complexes.**

A *simplicial complex* Δ on $[n]$ is a finite collection of subsets of $[n]$ with the property that $A \in \Delta$ implies $B \in \Delta$ for any $B \subseteq A$. We assume n large enough and we do not require that $\{i\} \in \Delta$ for all $i \in [n]$.

Let Δ be a simplicial complex on $[n]$ and let $A \subseteq [n]$ be a non-empty set, we call the *cone* of Δ over A, and we denote by $A * \Delta$, the simplicial complex whose facets are $A \cup F$, where F is a facet of Δ. It is well known that cones are acyclic, i.e., their reduced simplicial homology groups are trivial in all dimensions, see for instance [17].

We need the following technical Lemma. It is an immediate consequence of the well known reduced Mayer–Vietoris exact sequence, see [11, Proposition 5.1.8]. We include a short proof for the convenience of the reader.

Lemma 3.1. Let Δ_1, Δ_2 be two acyclic simplicial complexes on $[n]$. Set $\Delta = \Delta_1 \cup \Delta_2$. Then

$$\tilde{H}_i(\Delta;\mathbb{K}) \cong \tilde{H}_{i-1}(\Delta_1 \cap \Delta_2;\mathbb{K}) \qquad \forall i > 0.$$

Proof. The statement follows from the long exact sequence,

$$\to \tilde{H}_i(\Delta_1;\mathbb{K}) \oplus \tilde{H}_i(\Delta_2;\mathbb{K}) \to \tilde{H}_i(\Delta;\mathbb{K}) \to \tilde{H}_{i-1}(\Delta_1 \cap \Delta_2;\mathbb{K}) \to \tilde{H}_{i-1}(\Delta_1;\mathbb{K}) \oplus \tilde{H}_{i-1}(\Delta_2;\mathbb{K}) \to$$

since, by hypothesis, $\tilde{H}_i(\Delta_1;\mathbb{K}) = \tilde{H}_i(\Delta_2;\mathbb{K}) = H_{i-1}(\Delta_1;\mathbb{K}) = \tilde{H}_{i-1}(\Delta_2;\mathbb{K}) = 0$. □

The following results are corollaries of Lemma 3.1. The symbol $\Delta|_U$ denotes the restriction of a simplicial complex Δ on $[n]$ to a subset U of $[n]$, see [11, Section 8.1.1].

Corollary 3.2. Let Δ_1, Δ_2 be simplicial complex on $[n]$. Let $A, B, A', B' \subseteq [n]$ be non-empty subsets such that $A \cap B = A' \cap B' = \varnothing$ and

$$\Delta_2|_{A\cup A'} = \Delta_1|_{B\cup B'} = \{\varnothing\}.$$

Then

$$\tilde{H}_i\big((A * \Delta_1) \cup (B * \Delta_2); \mathbb{K}\big) \cong \tilde{H}_i\big((A' * \Delta_1) \cup (B' * \Delta_2); \mathbb{K}\big) \qquad \forall i > 0.$$

Proof. Since we have

$$(A * \Delta_1) \cap (B * \Delta_2) = (A' * \Delta_1) \cap (B' * \Delta_2) = \Delta_1 \cap \Delta_2,$$

it follows from Lemma 3.1 and from the well known fact that cones are acyclic. □

In the next corollary we assume $B = B'$ and we make the other assumptions weaker.

Corollary 3.3. Let Δ_1, Δ_2 be simplicial complexes on $[n]$. Let A, A', B be non-empty subsets of $[n]$ such that $A \cap B = A' \cap B = \varnothing$ and

$$\Delta_2|_{A\cup A'} = \{\varnothing\}.$$

Then

$$\tilde{H}_i\left((A * \Delta_1) \cup (B * \Delta_2); \mathbb{K}\right) \cong \tilde{H}_i\left((A' * \Delta_1) \cup (B * \Delta_2); \mathbb{K}\right) \qquad \forall i > 0.$$

Proof. It follows from Lemma 3.1 since

$$(A * \Delta_1) \cap (B * \Delta_2) = (A' * \Delta_1) \cap (B * \Delta_2) = \Delta_1 \cap (B * \Delta_2).$$

□

Proposition 3.4. Let, $\Delta_1, \Delta_2, \Delta_3$ be simplicial complexes on $[n]$. For any A, B, non-empty subsets of $[n]$, consider the following simplicial complex

$$\Delta_{A,B} = (A * \Delta_1) \cup (B * \Delta_2) \cup ((A \cup B) * \Delta_3).$$

If $A, B, A', B' \subseteq [n]$ are non-empty subsets satisfying $A \cap B = A' \cap B' = \varnothing$ and

$$\Delta_2|_{A\cup A'} = \Delta_1|_{B\cup B'} = \{\varnothing\}.$$

Then

$$\tilde{H}_i(\Delta_{A,B}; \mathbb{K}) \cong \tilde{H}_i(\Delta_{A',B'}; \mathbb{K}) \qquad \forall i > 0.$$

Proof. Let $A, B \subseteq [n]$ be non-empty disjoint subsets such that $(\Delta_1 \cup \Delta_2 \cup \Delta_3)|_{A\cup B} = \{\varnothing\}$. Note that $\Delta_{A,B}$ can be written as follow

$$\Delta_{A,B} = (A * \Delta_1) \cup (B * (\Delta_2 \cup (A * \Delta_3))).$$

So, by Lemma 3.1, we get

$$\tilde{H}_i(\Delta_{A,B}; \mathbb{K}) \cong \tilde{H}_{i-1}\left(\Delta_1 \cap (\Delta_2 \cup (A * \Delta_3)); \mathbb{K}\right) \qquad \forall i > 0. \tag{1}$$

That means that the reduced homology groups of $\Delta_{A,B}$ does not explicitly depend on B.

But we can also write $\Delta_{A,B}$ as

$$\Delta_{A,B} = (A * (\Delta_1 \cup (B * \Delta_3))) \cup (B * \Delta_2) \tag{2}$$

so, by Lemma 3.1 we have

$$\tilde{H}_i(\Delta_{A,B};\mathbb{K}) \cong \tilde{H}_{i-1}\left(\Delta_2 \cap (\Delta_1 \cup (B * \Delta_3));\mathbb{K}\right) \qquad \forall i > 0.$$

To conclude the proof it is enough to take a non-empty subset $U \subseteq [n]$ such that U is disjoint from A, B, A', B' (we can choose $U = \{j\}$ for some $j \in [n] \smallsetminus (A \cup B \cup A' \cup B')$, this is possible since we assumed at the beginning of the subsection that n is large enough) and to note that, for equations 1 and 2, we have

$$\tilde{H}_i(\Delta_{A,B};\mathbb{K}) \cong \tilde{H}_i(\Delta_{A,U};\mathbb{K}) \cong \tilde{H}_i(\Delta_{A',U};\mathbb{K}) \cong \tilde{H}_i(\Delta_{A',B'};\mathbb{K}).$$

□

3.2. **Betti numbers of toric ideals via simplicial complexes.**
Simplicial complexes are involved in the computation of the Betti numbers of a toric algebra in [4, Theorem 2.1]. Here we recall the notation and we include a short exposition of such result.

Let $G = (V, E)$ be a graph on the vertex set $V = \{x_1, \dots, x_r\}$ with edges $E = \{e_1, \dots, e_n\}$. Consider the semigroups $\mathbb{N}[V] = \langle x_1, \dots, x_r\rangle_{\mathbb{N}}$ and $\mathbb{N}[E] = \langle e_1, \dots, e_n\rangle_{\mathbb{N}}$ and call $\mathcal{V} = [x_1, \dots, x_r]$ and $\mathcal{E} = [e_1, \dots, e_n]$ the standard basis of $\mathbb{N}[V]$ and $\mathbb{N}[E]$.

Consider the linear map

$$\varphi_G : \mathbb{N}[E] \to \mathbb{N}[V]$$

induced by

$$\varphi_G(e_j) = x_{j_1} + x_{j_2} \qquad \text{where} \quad e_j = \{x_{j_1}, x_{j_2}\}.$$

The incidence matrix $M_G \in \mathbb{N}^{r,n}$ of the graph G is the matrix associated to the linear map φ_G with respect to the basis $[e_1, \dots, e_n]$ and $[x_1, \dots, x_r]$.

We have the following commutative square

$$\begin{array}{ccc} \mathbb{N}[E] & \xrightarrow{\varphi_G} & \mathbb{N}[V] \\ \downarrow^{\pi_{\mathcal{E}}} & & \downarrow^{\pi_{\mathcal{V}}} \\ \mathbb{N}^n & \xrightarrow{M_G} & \mathbb{N}^r \end{array}$$

where $\pi_{\mathcal{V}}$ and $\pi_{\mathcal{E}}$ are the canonical isomorphisms defined by

$$\pi_{\mathcal{V}}(a_1x_1 + a_2x_2 + \dots + a_rx_r) = (a_1, a_2, \dots, a_r) \quad \text{and} \quad \pi_{\mathcal{E}}(b_1e_1 + b_2e_2 + \dots + b_ne_n) = (b_1, b_2, \dots, b_n).$$

For $s \in \mathbb{N}[V]$ we set for short $\overline{s} = \pi_{\mathcal{V}}(s)$; and for $F \subseteq [n]$ we denote $e_F = \sum_{i\in F} e_i$; we define the simplicial complex

$$\Delta_s^G = \Delta_{\overline{s}}^G = \{F \subseteq [n] \mid s - \varphi_G(e_F) \in Im(\varphi_G)\}.$$

It is immediate to note that if $s \notin Im(\varphi_G)$ then $\Delta_s^G = \emptyset$.

Remark 3.5. Note that given $s \in Im(\varphi_G)$ and F a facet of Δ_s^G, then

$$s \in \langle \varphi_G(e_i) | \ i \in F\rangle_{\mathbb{N}}.$$

Indeed, $s - \varphi_G(e_F) \in Im(\varphi_G)$, so if $s \notin \langle \varphi_G(e_i) | \ i \in F\rangle_{\mathbb{N}}$, then there exists a $j \in [n] \smallsetminus F$ such that $s - \varphi_G(e_F) - \varphi_G(e_j) = s - \varphi_G(e_{F\cup\{j\}}) \in Im(\varphi_G)$, contradicting the maximality of F.

Thus

$$s = \varphi_G(\sum_{i\in F} a_ie_i) \quad \text{for some positive integers } a_i.$$

We denote by $\beta_{i,\overline{s}}(\mathbb{K}[G])$, the i-th multigraded Betti number of $\mathbb{K}[G]$ in degree $\overline{s}$. Briales, Campillo, Marijuán, and Pisón in [4, Theorem 2.1] proved the following result (we write it in the above terminology). Let G be a finite simple graph. If $s \in \mathbb{N}[V]$, then

$$\beta_{j+1,\overline{s}}(\mathbb{K}[G]) = \dim_{\mathbb{K}} \tilde{H}_j(\Delta_s^G;\mathbb{K}). \tag{3}$$

What follow are preparatory lemmas for the proof of Theorem 2.9.

Lemma 3.6. Let $G = (V,E)$ be a graph and $(e_1,e_2) = (x_1,x_2,x_3)$ be a path of G.
Let $s = w_1x_1 + w_2x_2 + \cdots + w_rx_r \in Im(\varphi_G)$ be such that $w_1 < w_2$. Then $\Delta_{\overline{s}}^G$ is acyclic.

Proof. Let F be a facet of $\Delta_{\overline{s}}^G$. We claim that $e_2 \in F$. Indeed, by Remark 3.5, for some positive integers $a_1,\dots,a_n$ we have

$$s = a_1\varphi_G(e_1) + a_2\varphi_G(e_2) + \cdots + a_n\varphi_G(e_n), \text{ where } \ a_j = 0 \ \text{ if } \ j \notin F.$$

Then

$$w_1x_1 + w_2x_2 + \cdots + w_rx_r = a_1(x_1+x_2) + a_2(x_2+x_3) + \cdots$$

and x_2 does not appear in the rest of the sum. Thus $w_2 = a_1 + a_2$ and $a_1 \le w_1$. This implies

$$a_2 = w_2 - a_1 \ge w_2 - w_1 > 0$$

so, e_2 belongs to the facet F. Therefore $\Delta_{\overline{s}}^G$ is a cone and it only has trivial reduced homology groups. □

Lemma 3.7. Let $G = (V,E)$ be a graph. Let $(e_1,e_2,e_3) = (x_1,x_2,x_3,x_4)$ be a path of G. Let $s = w_1x_1 + w_2x_2 + w_3x_3 + \cdots + w_rx_r \in Im(\varphi_G)$ be such that $\Delta_{\overline{s}}^G$ is not acyclic. Then $w_2 = w_3$.

Proof. It follows from Lemma 3.6 since $(e_2,e_1) = (x_3,x_2,x_1)$ and $(e_2,e_3) = (x_2,x_3,x_4)$ are a paths of G. □

We are now in the position to prove Theorem 2.9. We recall the statement.
Theorem 2.9. *Let G be a simple graph. Let G/p be an even path contraction of G. If there is a path q containing p with $|q| \ge |p| + 2$, then*

$$\beta_i(\mathbb{K}[G]) = \beta_i(\mathbb{K}[G/p]) \quad for\ any\ i \ge 0.$$

Proof of Theorem 2.9. We claim that for each $s \in Im(\varphi_G)$ with Δ_s^G not acyclic, there is $s' \in Im(\varphi_{G/p})$ such that $\tilde{H}_i(\Delta_s^G;\mathbb{K}) \cong \tilde{H}_i(\Delta_{s'}^{G/p};\mathbb{K})$, for all $i > 0$. Hence the claim implies $\beta_i(\mathbb{K}[G]) \le \beta_i(\mathbb{K}[G/p])$ for any $i \ge 1$ which, together with Theorem 2.7, concludes the proof.

To show the claim, set $G' = G/p$ and let $s \in Im(\varphi_G)$ be such that Δ_s^G is not acyclic. It is not restrictive to prove the claim for $q = (x_1,x_2,x_3,x_4,x_5) = (e_1,e_2,e_3,e_4)$ and $p = (x_2,x_3,x_4) = (e_2,e_3)$, since the general statement follows by iterating such case (even choosing $p = (e_1,e_2)$ would result in a graph G/p isomorphic to $G/(e_2,e_3)$).

So, we write $s = w_1x_1+w_2x_2+w_3x_3+w_4x_4+w_5x_5+\bar{x} \in Im(\varphi_G)$ where $\bar{x}$ is a linear combination of the vertices with index greater than 5.

Note that, since we assume that Δ_s^G is not acyclic, from Lemma 3.7, we have $w_2 = w_3 = w_4 = w$ and, from Lemma 3.6, we have $w \le w_1$ and $w \le w_5$. Also, we assume $w > 0$, since the case $w = 0$ is trivial.

Hence the elements in $\varphi_G^{-1}(s)$ must be of type $ae_1 + (w-a)e_2 + ae_3 + (w-a)e_4 + \bar{e}$ for some $0 \le a \le w$ and where $\bar{e}$ is a linear combination of the edges in E with index greater than 4.

Thus we note that, in correspondence to the values $a = w$, $a = 0$ and $0 < a < w$, we can write

$$\Delta_s^G = \{1,3\} * \Delta_1 \cup \{2,4\} * \Delta_2 \cup \{1,2,3,4\} * \Delta_3$$

for some simplicial complexes $\Delta_1, \Delta_2, \Delta_3$ on $\{5,\dots,n\}$.

Let $\chi : V \to V'$ be the function as in Definition 2.1, that is, $\chi(x_i) = x_i$ if $i \neq 2,3,4$ and, say, $\chi(x_2) = \chi(x_3) = \chi(x_4) = y \in [n] \setminus V$. Then the image of q under χ is a simple path in G', precisely $(x_1, y, x_5) = (e_h, e_k)$, where $e_h = \{x_1, y\}, e_k = \{y, x_5\} \in E' \setminus E$.

Then, in correspondence to the above $s \in Im(\varphi_G)$ we consider $s' = w_1x_1 + wy + w_5x_5 + \bar{x} \in \mathbb{N}[V']$ and we note that $s' \in Im(\varphi_{G'})$. Indeed, for each $ae_1+(w-a)e_2+ae_3+(w-a)e_4+\bar{e} \in \varphi_G^{-1}(s)$ we have $\varphi_{G'}(ae_h + (w-a)e_k + \bar{e}) = s'$, with the same $\bar{e}$ as in s. So, in correspondence to the values $a = w$, $a = 0$ and $0 < a < w$, we get

$$\Delta_{s'}^{G'} = \{h\} * \Delta_1 \cup \{k\} * \Delta_2 \cup \{h,k\} * \Delta_3,$$

with $\Delta_1, \Delta_2, \Delta_3$ as above. Then, from Proposition 3.4 we are done. □

4. Edge contraction of special graphs

As shown in Example 2.12, in a graph G *an edge contraction*, i.e., the contraction of a single edge e, does not always produces comparable Betti numbers between $\mathbb{K}[G]$ and $\mathbb{K}[G/e]$. However, a particular class of graphs, introduced in the next definition, deserves a further investigation.

Definition 4.1. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple connected graphs such that the vertices sets are disjoint, i.e., $V_1 \cap V_2 = \emptyset$. Let $x \in V_1, y \in V_2$ be two vertices and consider $e = \{x, y\}$ an edge which connects G_1 and G_2. Then, we say that the graph $G = (V_1 \cup V_2, E_1 \cup E_2 \cup \{e\})$ is *connected by the edge* e and we write $G = G_1 \overset{e}{-} G_2$.

We show that the contraction of the edge e in a graph G connected by e preserves the number of minimal generators of $\mathbb{K}[G]$.

Theorem 4.2. Let $G = G_1 \overset{e}{-} G_2$ be a graph connected by the edge e. Then

$$\beta_1(\mathbb{K}[G]) = \beta_1(\mathbb{K}[G/e]).$$

Proof. Let $w = e\|w_1\|e\|w_2$ be a primitive even closed walk in G where w_1 and w_2 are odd closed walk in G_1 and G_2 respectively. Let denote by w_i^+ (resp. w_i^-) the list of the odd (resp. even) entries in w_i, for $i = 1,2$. Thus $w^+ = e\|w_1^-\|e\|w_2^-$ and $w^- = w_1^+\|w_2^+$. Furthermore, $w/e = w_1\|w_2$ is a closed even walk in G/e where $(w/e)^+ = w_1^+\|w_2^-$ and $(w/e)^- = w_1^-\|w_2^+$.

With this notation, the binomial corresponding to w/e is then $f_{w/e} = e_{w_1^+}e_{w_2^-} - e_{w_1^-}e_{w_2^+}$. Assume it is not primitive. By definition there exists in $I_{G/e}$ a binomial $f_v = e_{v^+} - e_{v^-}$ such that $e_{v^+} | e_{w_1^+}e_{w_2^-}$ and $e_{v^-} | e_{w_1^-}e_{w_2^+}$. We now construct a binomial in order to show that $f_w = e^2 e_{w_1^-}e_{w_2^-} - e_{w_1^+}e_{w_2^+}$ is not primitive, a contradiction.

If v is an even walk in G/e either in G_1 or in G_2 we are done since f_v will have the required property. Assume $v = v_1\|v_2$ where v_1 is an odd walk in G_1 and v_2 in G_2, and consider $\bar{v} = e\|v_1\|e\|v_2$, that is an even cycle in G. Then $\bar{v}^+ = e\|v_1^-\|e\|v_2^-$ (with the same meaning as above) and $\bar{v}^- = v_1^+\|v_2^+$ where $v^+ = v_1^+\|v_2^-$ and $v^- = v_1^-\|v_2^+$. Hence the corresponding binomial $f_{\bar{v}} = e^2 e_{v_1^-}e_{v_2^-} - e_{v_1^+}e_{v_2^+}$ exhibits the non primitiveness of f_w. □

It is natural to ask then when the total Betti numbers of a graph $G = G_1 \overset{e}{-} G_2$ connected by the edge e are preserved by the contraction of e. We have the following property.

Proposition 4.3. Let $G = G_1 \overset{e}{-} G_2$ be a graph connected by the edge e. If either G_1 or G_2 is bipartite then $\mathbb{K}[G] \cong \mathbb{K}[G/e] \cong \mathbb{K}[G_1] \otimes \mathbb{K}[G_2]$. So in particular $\beta_i(\mathbb{K}[G]) = \beta_i(\mathbb{K}[G/e])$ for any $i \geq 0$.

Proof. Let G_1 be a bipartite graph. Then each closed even walk of G containing e and edges in G_1 cannot be primitive. Indeed, an even walk w of G involving edges in G_1 and G_2 would be of type $w = w_1||e||w_2||e$. So both w_1 and w_2 must be even closed walks. Therefore w is certainly not a primitive walk.

Moreover, the primitive even walks of G are either contained in G_1 or in G_2. Then $I_G = I_{G_1} + I_{G_2}$ where the ideals I_{G_1}, I_{G_2} are generated by binomials in two different sets of variables. □

Remark 4.4. In the case of Proposition 4.3, in analogy to [7, Theorem 2.6], the graded Betti numbers of $\mathbb{K}[G]$ are then obtained by those of $\mathbb{K}[G_1]$and $\mathbb{K}[G_2]$ via the Künneth formula. See for instance [14, Theorem 3.4.(b)] for more details on the Künneth formula of graded modules.

It is not yet clear when the contraction of $e = \{x, y\}$ in a graph connected by e preserves the total Betti numbers. Also, we observe that the total Betti numbers of $G_1 \overset{e}{-} G_2$ depends on the vertices $x \in G_1$ and $y \in G_2$, i.e., on how G_1 and G_2 are connected by e. The next example shows three different cases where the total Betti numbers and the edge contraction of the graphs $G_1 \overset{e}{-} G_2$ depend on the choice of x and y.

Example 4.5. One can check that contracting the edge e in the graph pictured in Figure 11, the total Betti numbers do not change. The graph G/e is connected by the edge e'. The contraction of the path (e, e') pushes down some of the Betti numbers.

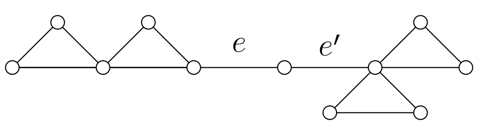


Figure 11

i	0	1	2	3
$\beta_i(\mathbb{K}[G])$	1	6	9	4
$\beta_i(\mathbb{K}[G/e])$	1	6	9	4
$\beta_i(\mathbb{K}[G/(e,e')])$	1	6	8	3

Contracting the edge e in the graph G in Figure 12, the total Betti numbers go down. The further contraction of e' does not modify them anymore.

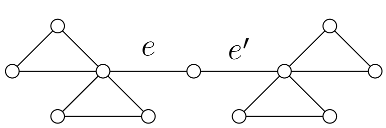


Figure 12

i	0	1	2	3
$\beta_i(\mathbb{K}[G])$	1	6	9	4
$\beta_i(\mathbb{K}[G/e])$	1	6	8	3
$\beta_i(\mathbb{K}[G/(e,e')])$	1	6	8	3

The contractions of e and e' in the graph G in Figure 13 do not change the total Betti numbers.

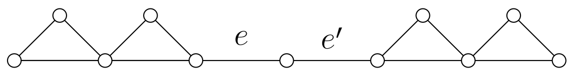


FIGURE 13

i	0	1	2	3
$\beta_i(\mathbb{K}[G])$	1	6	9	4
$\beta_i(\mathbb{K}[G/e])$	1	6	9	4
$\beta_i(\mathbb{K}[G/(e,e')])$	1	6	9	4

Example 4.5 suggests several additional questions.

Question 4.6. Let G be a graph connected by the edge e. Is $\beta_i(\mathbb{K}[G]) \geq \beta_i(\mathbb{K}[G/e])$?

Question 4.7. Let G be a graph connected by the edge e. What conditions guarantee $\beta_i(\mathbb{K}[G]) = \beta_i(\mathbb{K}[G/e])$?

Question 4.8. Let G be a graph connected by the edge e. Is $\mathbb{K}[G]$ Cohen-Macaulay if and only if $\mathbb{K}[G/e]$ is Cohen-Macaulay?

In particular it is interesting to look at a special class of graphs which includes the graph in Figure 13.

We call *sequence of n triangles*, see Figure 14, the graph $T_n = (V, R)$ with vertex set

$$V = \{x_1, \dots, x_{2n+1}\}$$

and edge set

$$E = \{\{x_{2i-1}, x_{2i}\}, \{x_{2i}, x_{2i+1}\}, \{x_{2i+1}, x_{2i-1}\} \mid 1 \leq i \leq n\}.$$

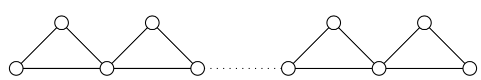


FIGURE 14. The graph T_n, a sequence of n triangles.

The graph T_1 is a cycle of length three, so its toric ideal is the zero ideal.

Remark 4.9. Computer experiments suggest the following properties.

(a) The algebra $\mathbb{K}[T_n]$ is Cohen-Macaulay of Krull dimension $n-1$, for any $n \geq 2$.
(b) Let T_m and T_n be two graphs sequence of triangles. Let $x \in V(T_m)$ and $y \in V(T_n)$ be vertices of degree 2 having a neighbor of degree 2. Set $e = \{x, y\}$. Then the algebra $\mathbb{K}[T_n \overset{e}{-} T_m]$ is Cohen-Macaulay, for any n and m.
(c) Let T_m and T_n be two graphs sequence of triangles. Let $x \in V(T_m)$ and $y \in V(T_n)$ be two vertices of degree 2 having a neighbor of degree 2. Set $e = \{x, y\}$. Then

$$\beta_i(\mathbb{K}[T_n \overset{e}{-} T_m]) = \beta_i(\mathbb{K}[T_n \overset{e}{-} T_m/e]) = \beta_i(\mathbb{K}[T_{n+m}]) = \beta_i(\mathbb{K}[in(T_{n+m})])$$

for any n and m, where $in(T_{n+m})$ denotes the initial ideal of T_{n+m} with respect to the lexicographic order induced by $e_1 > e_2 > \cdots > e_{3n}$.

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