DIVERGENT SEQUENCE OF NONTRIVIAL SOLUTIONS FOR SUPERLINEAR DOUBLE PHASE PROBLEMS

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Abstract. We consider a double phase (unbalanced growth) Dirichlet problem with a Carathéodory reaction $f(z,x)$ which is superlinear in x but without satisfying the AR-condition. Using the symmetric mountain pass theorem, we produce a whole sequence of distinct bounded solutions which diverge to infinity.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with a Lipschitz boundary $\partial\Omega$. In this paper we study the following double phase Dirichlet problem

$$\begin{cases} -\Delta_p^a u(z) - \Delta_q u(z) = f(z, u(z)) \quad \text{in } \Omega, \\ u\Big|_{\partial\Omega} = 0, \; 1 < q < p. \end{cases} \tag{1}$$

Given $a \in L^\infty(\Omega)$ with $a(z) \geq 0$ for a.a. $z \in \Omega$, by Δ_p^a we denote the weighted p-Laplace differential operator defined by

$$\Delta_p^a u = \text{div } (a(z)|\nabla u|^{p-2}\nabla u).$$

Equation (1) is driven by the sum of such a weighted p-Laplacian and of a q-Laplacian (no weight). So, we are dealing with a double phase problem. The energy functional for this differential operator is given by

$$J(u) = \int_\Omega \left[\frac{1}{p}a(z)|\nabla u|^p + \frac{1}{q}|\nabla u|^q\right] dz.$$

The integrand of this integral functional is

$$\widehat{\eta}(z,y) = \frac{1}{p}a(z)|x|^p + \frac{1}{q}|y|^q \quad \text{for all } z \in \Omega, \text{ all } y \in \mathbb{R}^N.$$

We do not assume that the weight function $a(\cdot)$ is bounded away from zero, that is, we do not require that $0 < \operatorname*{ess\,inf}_\Omega a$. So, the integrand $\widehat{\eta}(z,\cdot)$ exhibits unbalanced growth, namely we have

$$|y|^q \leq \widehat{\eta}(z,y) \leq c_0[1 + |y|^p] \quad \text{for a.a. } z \in \Omega, \text{ all } y \in \mathbb{R}^N, \text{ some } c_0 > 0.$$

Such integral functionals were first investigated by Marcellini [10, 11] and Zhikov [20, 21], in the context of problems of calculus of variations and of nonlinear elasticity theory.

These nonautonomous functionals are characterized by the fact that the energy density changes its ellipticity at different points of Ω, depending on whether $a(z) \geq \varepsilon > 0$ for any fixed $\varepsilon > 0$ or $a(z) = 0$. In physical terms, in the framework of nonlinear

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elasticity theory, the modulating coefficient dictates the geometry of composites made of two different materials with distinct power hardening exponents p and q. Moreover, from a mathematical viewpoint, these functionals are important in the study of the so-called "Lavrentiev phenomenon" (see Zhikov [21]).

In recent years the interest for double phase problems was revived and there have been efforts to develop a regularity theory for the solutions of such problems. These developments can be traced in the works of Baroni-Colombo-Mingione [1], Hästö-Ok [5], Marcellini [12, 13], Mingione-Rădulescu [14], Ragusa-Tachikawa [19] and the references therein. So far only local regularity results for minimizers are obtained and a global regularity theory (that is, regularity up to the boundary of Ω) analogous to the one existing for (p,q)-equations (balanced growth double phase problems) remains elusive. This removes from consideration many powerful tools and makes the study of unbalanced growth double phase problems more difficult.

In problem (1), the reaction (right hand side) is a Carathéodory function $f(z,x)$ (that is, for all $x \in \mathbb{R}$ $z \to f(z,x)$ is measurable and for a.a. $z \in \Omega$ $x \to f(z,x)$ is continuous), which is $(p-1)$-superlinear as $x \to \pm\infty$, but without satisfying the usual for superlinear problems Ambrosetti-Rabinowitz condition (the AR-condition for short, see Papageorgiou-Rădulescu-Repovš [16]. Imposing a symmetry condition on $f(z,\cdot)$ (we assume that $f(z,\cdot)$ is odd) and using the $\mathbb{Z}_2$-version of the mountain pass theorem (see Rabinowitz [18, Theorem 9.12, p. 55]), we show the existence of a whole sequence of distinct solutions which diverge to infinity.

Superlinear double phase problems were studied recently by Gasiński-Papageorgiou [2], Gasiński-Winkert [3], Kim-Kim-Oh-Zeng [7], Leonardi-Papageorgiou [8], Liu-Dai [9], Papageorgiou-Vetro-Vetro [17]. However none of the aforementioned works addresses the question of existence of an infinity of distinct nontrivial solutions.

2. Mathematical Background - Hypotheses

The unbalanced growth of the energy density for the differential operator leads to a functional framework that requires the use of generalized Orlicz spaces. A comprehensive presentation of the theory of these spaces can be found in the book of Harjulehto-Hästö [4].

Recall that by $C^{0,1}(\overline{\Omega})$ we denote the space of Lipschitz continuous functions from $\overline{\Omega}$ into $\mathbb{R}$. Our hypotheses on the weight $a(\cdot)$ are the following:

H_0: $a \in C^{0,1}(\overline{\Omega}) \setminus \{0\}$, $a(z) \geq 0$ for all $z \in \overline{\Omega}$, $1 < q < p$, $q < N$ and $\dfrac{p}{q} < 1 + \dfrac{1}{N}$.

Remark 1. *The last inequality in H_0 says that the two exponents p, q can not be far apart. This condition implies that $p < q^* = \dfrac{Nq}{N-q}$ and this leads to useful embeddings of some relevant function spaces. The condition appears in almost all double phase works.*

By $L^0(\Omega)$ we denote the space of all measurable functions $u : \Omega \to \mathbb{R}$. We identify two such functions which differ only on a Lebesgue-null subset of Ω. Let $\eta(z,t)$ be the continuous integrand defined by

$$\eta(z,t) = a(z)t^p + t^q \quad \text{for all } z \in \Omega, \text{ all } t \geq 0.$$

The generalized Orlicz space $L^\eta(\Omega)$ is defined by

$$L^\eta(\Omega) = \{u \in L^0(\Omega) : \rho_\eta(u) < \infty\},$$

where $\rho_\eta(\cdot)$ is the modular function defined by

$$\rho_\eta(u) = \int_\Omega \eta(z, |u|)dz.$$

This is continuous, convex, hence weakly lower semicontinuous too. The space $L^\eta(\Omega)$ is equipped with the so-called Luxemburg norm which is defined as follows

$$\|u\|_\eta = \inf\left\{\lambda > 0 : \rho_\eta\left(\frac{u}{\lambda}\right) \leq 1\right\}.$$

With this norm, $L^\eta(\Omega)$ becomes a Banach space which is separable and reflexive (in fact uniformly convex).

Using $L^\eta(\Omega)$ we can define the corresponding generalized Sobolev-Orlicz space by

$$W^{1,\eta}(\Omega) = \{u \in L^\eta(\Omega) : |\nabla u| \in L^\eta(\Omega)\},$$

with ∇u being the weak gradient of u. We equip this space with the following norm

$$\|u\|_{1,\eta} = \|u\|_\eta + \|\nabla u\|_\eta \quad \text{for all } u \in W^{1,\eta}(\Omega),$$

with $\|\nabla u\|_\eta = \| \, |\nabla u| \, \|_\eta$.

Consider the space $C_c^\infty(\Omega)$ of all $C^\infty(\Omega)$ functions which have compact support in Ω. We define

$$W_0^{1,\eta}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{1,\eta}}.$$

Since $a \in C^{0,1}(\overline{\Omega})$, the Poincaré inequality holds, namely there exists $c = c(\Omega) > 0$ such that

$$\|u\|_\eta \leq c\,\|\nabla u\|_\eta \quad \text{for all } u \in W_0^{1,\eta}(\Omega).$$

Therefore on $W_0^{1,\eta}(\Omega)$ we can consider the following equivalent norm

$$\|u\| = \|\nabla u\|_\eta \quad \text{for all } u \in W_0^{1,\eta}(\Omega).$$

Both spaces $W^{1,\eta}(\Omega)$ and $W_0^{1,\eta}(\Omega)$ are Banach spaces which are separable and reflexive (in fact uniformly convex).

The norm $\|\cdot\|$ and the modular function $\rho_\eta(\cdot)$ are closely related.

Proposition 1. *If $u \in W_0^{1,\eta}(\Omega)$, $u \neq 0$, then:*

(a) $\|u\| = \lambda \Leftrightarrow \rho_\eta\left(\frac{\nabla u}{\lambda}\right) = 1$;
(b) $\|u\| < 1$ *(resp.* $= 1, > 1$*)* $\Leftrightarrow \rho_\eta(\nabla u) < 1$ *(resp.* $= 1, > 1$*)*;
(c) $\|u\| \leq 1 \Rightarrow \|u\|^p \leq \rho_\eta(\nabla u) \leq \|u\|^q$;
(d) $\|u\| > 1 \Rightarrow \|u\|^q \leq \rho_\eta(\nabla u) \leq \|u\|^p$;
(e) $\|u_n\| \to 0$ *(resp.* $\to \infty$*)* $\Leftrightarrow \rho_\eta(\nabla u_n) \to 0$ *(resp.* $\to \infty$*)*.

The following embeddings are important in our study of (1).

Proposition 2. *We have the following:*

(a) $L^\eta(\Omega) \hookrightarrow L^r(\Omega)$ *continuously for all* $r \in [1, q]$;
(b) $L^p(\Omega) \hookrightarrow L^\eta(\Omega)$ *continuously;*
(c) $W_0^{1,\eta}(\Omega) \hookrightarrow L^r(\Omega)$ *continuously if* $r \in [1, q^*]$ *and compactly if* $r \in [1, q^*)$.

Consider the nonlinear operator $V : W_0^{1,\eta}(\Omega) \to W_0^{1,\eta}(\Omega)^*$ defined by

$$\langle V(u), h\rangle = \int_\Omega [a(z)|\nabla u|^{p-2} + |\nabla u|^{q-2}](\nabla u, \nabla h)_{\mathbb{R}^N} dz \text{ for all } u, h \in W_0^{1,\eta}(\Omega).$$

This operator has the following properties (see Liu-Dai [9]).

Proposition 3. *The operator $V(\cdot)$ is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (thus maximal monotone too) and of type $(S)_+$ (that is, if $u_n \xrightarrow{w} u$ in $W_0^{1,\eta}(\Omega)$ and $\limsup_{n\to\infty}\langle V(u_n), u_n - u\rangle \le 0$, then $u_n \to u$ in $W_0^{1,\eta}(\Omega)$).*

By $\widehat{\lambda}_1(q)$ we denote the first eigenvalue of $(-\Delta_q, W_0^{1,q}(\Omega))$. We know that $\widehat{\lambda}_1(q) > 0$, it is simple, isolated and has the following variational characterization

$$\widehat{\lambda}_1(q) = \inf\left[\frac{\|\nabla u\|_q^q}{\|u\|_q^q} \,:\, u \in W_0^{1,q}(\Omega),\, u \neq 0\right]. \tag{2}$$

The infimum in (2) is realized on the corresponding one-dimensional eigenspace, the elements of which have constant sign.

The hypotheses on the reaction $f(z,x)$ are the following:

H_1: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for a.a. $z \in \Omega$ $f(z,0) = 0$, $f(z,\cdot)$ is odd and

- (i) $|f(z,x)| \le a(z)[1 + |x|^{r-1}]$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)$ and $p < r < q^* = \frac{Nq}{N-q}$;
- (ii) if $F(z,x) = \int_0^x f(z,s)ds$, then $\lim_{x\to\pm\infty} \frac{F(z,x)}{|x|^p} = +\infty$ uniformly for a.a. $z \in \Omega$;
- (iii) there exists $\tau \in \left((r-q)\frac{N}{q}, q^*\right)$ such that
$$0 < c_0 \le \liminf_{x\to\pm\infty} \frac{f(z,x)x - pF(z,x)}{|x|^\tau} \text{ uniformly for a.a. } z \in \Omega;$$
- (iv) there exists $\vartheta \in L^\infty(\Omega)$ such that
$$\vartheta(z) \le \widehat{\lambda}_1(q) \text{ for a.a. } z \in \Omega,\ \vartheta \not\equiv \widehat{\lambda}_1(q),$$
$$\limsup_{x\to 0} \frac{f(z,x)}{|x|^{q-2}x} \le \vartheta(z) \text{ uniformly for a.a. } z \in \Omega.$$

Remark 2. *Hypotheses $H_1(ii), (iii)$ imply that*

$$\lim_{x\to\pm\infty} \frac{f(z,x)}{|x|^{p-2}x} = +\infty \text{ uniformly for a.a. } z \in \Omega.$$

So, $f(z,\cdot)$ is $(p-1)$-superlinear as $x \to \pm\infty$. However, we do not use the usual in such cases Ambrosetti-Rabinowitz condition (the AR-condition for short). Instead we use the weaker condition $H_1(iii)$ which allows also superlinear nonlinearities with slower growth as $x \to \pm\infty$ which fail to satisfy the AR-condition. For example consider the following function

$$f(z,x) = \begin{cases} \vartheta(z)|x|^{q-2}x & \text{if } |x| \le 1,\\ |x|^{p-2}x\ln|x| + \vartheta(z)|x|^{s-2}x & \text{if } 1 < |x|, \end{cases}$$

with $\vartheta \in L^\infty(\Omega)$, $\vartheta(z) \le \widehat{\lambda}_1(q)$ for a.a. $z \in \Omega$, $\vartheta \not\equiv \widehat{\lambda}_1(q)$, $1 < s \le p$. This function satisfies hypotheses H_1, but $f(z,\cdot)$ fails to satisfy the AR-condition.

3. Infinitely Many Solutions

In what follows

$$\eta(z,t) = a(z)t^p + t^q \quad \text{for all } z \in \Omega, \text{ all } t \ge 0$$

and

$$\rho_a(\nabla u) = \int_\Omega a(z)|\nabla u|^p dz \quad \text{for all } u \in W_0^{1,\eta}(\Omega).$$

Let $\varphi : W_0^{1,\eta}(\Omega) \to \mathbb{R}$ be the energy functional for problem (1) defined by

$$\varphi(u) = \frac{1}{p}\rho_a(\nabla u) + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega F(z,u)dz \quad \text{for all } u \in W_0^{1,\eta}(\Omega).$$

We know that $\varphi \in C^1(W_0^{1,\eta}(\Omega))$ and $\varphi(\cdot)$ is even. Next we consider the following compactness condition which allows a suitable minimax characterization of critical values of C^1-functionals (see [16], p. 366).

Definition 1. We say that $\varphi \in C^1(W_0^{1,\eta}(\Omega))$ satisfies the C-condition, if every sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\eta}(\Omega)$ such that

(a) $\{\varphi(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded;

(b) $(1 + \|u_n\|)\varphi'(u_n) \to 0$ in $W_0^{1,\eta}(\Omega)^*$ as $n \to \infty$,

admits a strongly convergent subsequence.

Proposition 4. *If hypotheses H_0, H_1 hold, then $\varphi(\cdot)$ satisfies the C-condition.*

Proof. We consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\eta}(\Omega)$ such that

$$|\varphi(u_n)| \le c_1 \quad \text{for some } c_1 > 0, \text{ all } n \in \mathbb{N}, \tag{3}$$

$$(1 + \|u_n\|)\varphi'(u_n) \to 0 \text{ in } W_0^{1,\eta}(\Omega)^* \text{ as } n \to \infty. \tag{4}$$

From (4) we have

$$\left|\langle V(u_n), h\rangle - \int_\Omega f(z,u_n)hdz\right| \le \frac{\varepsilon_n\|h\|}{1 + \|u_n\|} \tag{5}$$

$$\text{for all } h \in W_0^{1,\eta}(\Omega), \text{ with } \varepsilon_n \to 0^+.$$

In (5) we use the test function $h = u_n \in W_0^{1,\eta}(\Omega)$ and obtain

$$-\rho_a(\nabla u_n) - \|\nabla u_n\|_q^q + \int_\Omega f(z,u_n)u_n dz \le \varepsilon_n \quad \text{for all } n \in \mathbb{N}. \tag{6}$$

On the other hand from (3), we have

$$\rho_a(\nabla u_n) + \frac{p}{q}\|\nabla u_n\|_q^q - \int_\Omega pF(z,u_n)dz \le pc_1 \quad \text{for all } n \in \mathbb{N}. \tag{7}$$

We add (6) and (7) and since $q < p$, we obtain

$$\int_\Omega [f(z,u_n)u_n - pF(z,u_n)]dz \le c_2 \quad \text{for some } c_2 > 0, \text{ all } n \in \mathbb{N}. \tag{8}$$

Hypotheses $H_1(i), (iii)$ imply that we can find $c_3, c_4 > 0$ such that

$$c_3|x|^\tau - c_4 \le f(z,x)x - pF(z,x) \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{9}$$

Using (9) in (8), we obtain that

$$\{u_n\}_{n\in\mathbb{N}} \subseteq L^\tau(\Omega) \text{ is bounded.} \tag{10}$$

From hypothesis $H_1(iii)$ we see that we can always assume that $1 < \tau < r < q^*$. So, we can find $t \in (0,1)$ such that

$$\frac{1}{r} = \frac{1-t}{\tau} + \frac{t}{q^*}. \tag{11}$$

The interpolation inequality (see Hu-Papageorgiou [6], p. 82) implies that

$$\begin{aligned} & \|u_n\|_r \leq \|u_n\|_\tau^{1-t}\|u_n\|_{q^*}^t \quad \text{for all } n \in \mathbb{N}, \\ \Rightarrow \quad & \|u_n\|_r^r \leq c_5\|u_n\|^{tr} \quad \text{for some } c_5 > 0, \text{ all } n \in \mathbb{N} \\ & \text{(see (10) and Proposition 2).} \end{aligned} \tag{12}$$

From (5) using $h = u_n \in W_0^{1,\eta}(\Omega)$, we have

$$\rho_a(\nabla u_n) + \|\nabla u_n\|_q^q \leq \varepsilon_n + \int_\Omega f(z, u_n)u_n dz \quad \text{for all } n \in \mathbb{N}.$$

We want to show that $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\eta}(\Omega)$ is bounded. So, we may assume that $\|u_n\| \geq 1$ for all $n \in \mathbb{N}$. Then since $\rho_a(\nabla u_n)+\|\nabla u_n\|_q^q = \rho_\eta(\nabla u_n)$ and using Proposition 1, we have

$$\begin{aligned} \|u_n\|^q &\leq \int_\Omega f(z, u_n)u_n dz \\ &\leq c_6\left[1 + \|u_n\|_r^r\right] \quad \text{for some } c_6 > 0, \text{ all } n \in \mathbb{N} \\ &\text{(see hypothesis } H_1(i)) \\ &\leq c_7\left[1 + \|u_n\|^{tr}\right] \quad \text{for some } c_7 > 0, \text{ all } n \in \mathbb{N} \text{ (see (12)).} \end{aligned} \tag{13}$$

From (11) we have

$$tr = \frac{q^*(r-\tau)}{q^*-\tau} < q \quad \text{(see hypothesis } H_1(iii)).$$

So, from (13), we infer that

$$\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\eta}(\Omega) \text{ is bounded.}$$

On account of the reflexivity of $W_0^{1,\eta}(\Omega)$, we may assume that

$$\begin{aligned} & u_n \xrightarrow{w} u \text{ in } W_0^{1,\eta}(\Omega),\ u_n \to u \text{ in } L^r(\Omega) \\ & \text{(since } r < q^*, \text{ see Proposition 2).} \end{aligned} \tag{14}$$

In (5) we choose $h = (u_n - u) \in W_0^{1,\eta}(\Omega)$, pass to the limit as $n \to \infty$ and use (14). We obtain

$$\begin{aligned} & \lim_{n\to\infty} \langle V(u_n), u_n - u\rangle = 0, \\ \Rightarrow \quad & u_n \to u \text{ in } W_0^{1,\eta}(\Omega) \text{ (see Proposition 3),} \\ \Rightarrow \quad & \varphi(\cdot) \text{ satisfies the } C\text{-condition.} \end{aligned}$$

□

Our aim is to apply the symmetric mountain pass theorem of Rabinowitz [18] (Theorem 9.12, p. 55); see also Corollary 5.6.21, p. 439, of [16]. To this end we will need the following result.

Proposition 5. *If hypotheses H_0, H_1 hold, then there exists $\rho > 0$ such that*

$$0 < \widehat{c} \leq \varphi(u) \text{ for all } u \in W_0^{1,\eta}(\Omega),\ \|u\| = \rho.$$

Proof. Hypotheses $H_1(i), (iv)$ imply that given $\varepsilon > 0$, we can find $c_\varepsilon > 0$ such that

$$F(z, x) \leq \frac{1}{q}[\vartheta(z) + \varepsilon]|x|^q + c_\varepsilon|x|^r \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{15}$$

Assume that $\|u\| \leq 1$. Then using (15), we have

$$\begin{aligned}\varphi(u) &\geq \frac{1}{p}\rho_a(\nabla u) + \frac{1}{q}\left[\|\nabla u\|_q^q - \int_\Omega \vartheta(z)|u|^q dz - \frac{\varepsilon}{\widehat{\lambda}_1(q)}\|\nabla u\|_q^q\right] - c_\varepsilon\|u\|_r^r \\ &\geq \frac{1}{p}\rho_a(\nabla u) + \frac{1}{q}\left[c^* - \frac{\varepsilon}{\widehat{\lambda}_1(q)}\right]\|\nabla u\|_q^q - \widehat{c}_\varepsilon\|u\|^r\end{aligned}$$

for some $c^*, \widehat{c}_\varepsilon > 0$ (see [15], Lemma 4.11).

We choose $\varepsilon \in (0, c^*\widehat{\lambda}_1(q))$ and obtain

$$\varphi(u) \geq c_8\|u\|^p - \widehat{c}_\varepsilon\|u\|^r \quad \text{for some } c_8 > 0 \tag{16}$$

(recall $\|u\| \leq 1$ and see Proposition 1).

Since $p < r$, from (16) we see that we can find $\rho \in (0,1)$ small such that

$$\varphi(u) \geq \widehat{c} > 0 \quad \text{for all } u \in W_0^{1,\eta}(\Omega),\ \|u\| = \rho.$$

□

Let $\mathcal{V}$ be a finite dimensional subspace of $W_0^{1,\eta}(\Omega)$.

Proposition 6. *If hypotheses H_0, H_1 hold, then the set $K = \{u \in \mathcal{V} : 0 \leq \varphi(u)\}$ is bounded.*

Proof. Let $u \in K$. We have

$$\frac{1}{p}\rho_a(\nabla u) + \frac{1}{q}\|\nabla u\|_q^q \geq \int_\Omega F(z,u)dz. \tag{17}$$

On account of hypotheses $H_1(i), (ii)$, given $M > 0$, we can find $c_M > 0$ such that

$$F(z,x) \geq \frac{M}{p}|x|^p - c_M \quad \text{for a.a. } z \in \Omega. \tag{18}$$

Using (18) in (17), we obtain

$$\frac{M}{p}\|u\|_p^p - \frac{1}{q}\left(\rho_a(\nabla u) + \|\nabla u\|_q^q\right) \leq c_M|\Omega|_N$$

with $|\cdot|_N$ denoting the Lebesgue measure on $\mathbb{R}^N$. Since $\mathcal{V}$ is finite dimensional, all norms are equivalent. So, we have

$$Mc_9\|u\|^p - \frac{1}{q}\rho_\eta(\nabla u) \leq c_M|\Omega|_N \quad \text{for some } c_9 > 0.$$

Without any loss of generality we assume that $\|u\| \geq 1$. Then, using Proposition 1, we have

$$\left[Mc_9 - \frac{1}{q}\right]\|u\|^p \leq c_M|\Omega|_N.$$

Since $M > 0$ is arbitrary, choosing $M > \frac{1}{qc_9}$, we see that

$$\begin{aligned}&\|u\|^p \leq c_{10} \quad \text{for some } c_{10} > 0, \text{ all } u \in K, \\ \Rightarrow\quad &K \subseteq W_0^{1,\eta}(\Omega) \text{ is bounded.}\end{aligned}$$

□

We look at Theorem 9.12, p. 55, of Rabinowitz [19] (the $\mathbb{Z}_2$-mountain pass theorem). We see that with Proposition 5, we satisfy condition I_1' of Theorem 9.12 of [19] and with Proposition 6, we satisfy condition I_2' of the same theorem. Therefore, we can use the symmetric mountain pass theorem and have the following multiplicity theorem. Note that by Theorem 3.1 of Gasiński-Winkert [3], the solutions are in $L^\infty(\Omega)$ and $\varphi(u_n) \leq \int_\Omega [f(z, u_n)u_n - F(z, u_n)]dz$.

Theorem 1. *If hypotheses H_0, H_1 hold, then problem* (1) *has a sequence of distinct nontrivial solutions $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega)$ such that $\|u_n\| \to \infty$.*

References

[1] P. Baroni, M. Colombo, G. Mingione, *Regularity for general functionals with double phase*, Calc. Var. Partial Differential Equations **57**:62 (2018).

[2] L. Gasiński, N.S. Papageorgiou, *Constant sign and nodal solutions for superlinear double phase problems*, Adv. Calc. Var. **14** (2021), 613–626.

[3] L. Gasiński, P. Winkert, *Constant sign solutions for double phase problems with superlinear nonlinearity*, Nonlinear Anal. **195**:111739 (2020).

[4] P. Harjulehto, P. Hästö, *Orlicz Spaces and Generalized Orlicz Spaces*, Lecture Notes in Math., Vol. 2230, Springer, Cham, 2019.

[5] P. Hästö, J. Ok, *Maximal regularity for local minimizers of non-autonomous functionals*, J. Eur. Math. Soc. **24** (2022), 1285–1334.

[6] S. Hu, N.S. Papageorgiou, *Research Topics in Analysis, Volume I: Grounding Theory*, Birkhäuser, Cham, 2022.

[7] I.H. Kim, Y.-H. Kim, M.W. Oh, S. Zen, *Existence and multiplicity of solutions to concave-convex-type double-phase problems with variable exponent*, Nonlinear Anal. Real World Appl. **67**:103627 (2022).

[8] S. Leonardi, N.S. Papageorgiou, *Anisotropic Dirichlet double phase problems with competing nonlinearities*, Rev. Mat. Complut. (2022). https://doi.org/10.1007/s13163-022-00432-3

[9] W. Liu, G. Dai, *Existence and multiplicity results for double phase problem*, J. Differential Equations **265** (2018), 4311–4334.

[10] P. Marcellini, *Regularity and existence of solutions of elliptic equations with p, q-growth conditions*, J. Differential Equations **90** (1991), 1–30.

[11] P. Marcellini, *Regularity of minimizers of integrals of the calculus of variations with nonstandard growth conditions*, Arch. Ration. Mech. Anal. **105** (1989), 267–284.

[12] P. Marcellini, *Growth conditions and regularity for weak solutions to nonlinear elliptic pdes*, J. Math. Anal. Appl. **501**:124408 (2021).

[13] P. Marcellini, *Local Lipschitz continuity for p, q-PDEs with explicit u-dependence*, Nonlinear Anal. **226**:113066 (2023).

[14] G. Mingione, V.D. Rădulescu, *Recent developments in problems with nonstandard growth and nonuniform ellipticity*, J. Math. Anal. Appl. **501**:125197 (2021).

[15] D. Mugnai, N.S. Papageorgiou, *Resonant nonlinear Neumann problems with indefinite weight*, Ann. Sc. Norm. Super. Pisa **11** (2012), 729–788.

[16] N.S. Papageorgiou, V.D. Rădulescu, D.D. Repovš, *Nonlinear Analysis – Theory and Methods*, Springer Monographs in Mathematics, Springer, Cham, 2019.

[17] N.S. Papageorgiou, C. Vetro, F. Vetro, *Multiple solutions for parametric double phase Dirichlet problems*, Commun. Contemp. Math. **23**:2050006 (2021).

[18] P.H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, Cbms Regional Conference Series in Mathematics, Vol. 65, American Mathematical Society, Providence, 1986.

[19] M.A. Ragusa, A. Tachikawa, *Regularity for minimizers for functionals of double phase with variable exponents*, Adv. Nonlinear Anal. **9** (2020), 710–728.

[20] V.V. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory*, Math. USSR Izv. **29** (1987), 33–66.

[21] V.V. Zhikov, *On Lavrentiev's phenomenon*, Russ. J. Math. Phys., **3** (1995), 249–269.

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