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TABLE OF CONTENTS

TWG03: Algebraic Thinking 460

TWG10: Diversity and Mathematics Education: Social, Cultural and Political Challenges 1657

TWG11: Algorithmics 1890

TWG20: Mathematics teacher knowledge, beliefs and identity 3493

TWG26: Mathematics in the context of STEM education 4535

ADDENDUM

Preface: CERME 12 in virtual Bolzano

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The Conferences of ERME (European society for Research in Mathematics Education) have been held roughly biannually since 1998. The core of the conference is the *thematic working groups* (TWG) in which new, related research studies are discussed, based on papers which participants read in advance. While this format has remained almost unchanged since 1998, the scale and scope of CERME have developed considerably over the years: from 120 participants and 7 TWG at CERME 1, to 915 participants and 27 TWG at CERME 12. Moreover, CERME has gone from being a mainly regional congress (with only a few participants from other continents) to being a truly global event in mathematics education research, known for fostering high quality scientific communication, cooperation and collaboration. At CERME12, no less than 48 nations were represented (Table 1).

At the same time, it is evident that CERME12 was a very special – and historically difficult – congress to organize. It was first scheduled for February 3-7, 2021. In May 2020, the ERME board announced its decision to postpone the congress by one year, due to the then roaring outbreak of the COVID-19 pandemic. Indeed, large parts of the world – and most of Europe – continued to experience lockdowns and restrictions that would have made the scheduled congress impossible. Instead, an online *Pre-CERME12 event* was organized in February 2021, hosted by the Institute of Education at University College London, and made possible by the tireless efforts of the team led by Jeremy Hodgen and Eirini Geraniou (also chair resp. co-chair of the IPC of CERME12). The Pre-CERME12 event allowed the 27 TWGs to meet and prepare for conference, now postponed to 2022.

The biannual General Assembly of ERME was also held during this event. We warmly thank Susanne Prediger for her service as President of ERME from 2017 to 2021! Her leadership also contributed crucially to the organisation of CERME12, and thus to the results presented in these proceedings.

During the summer and fall of 2021, we all continued to plan for CERME12 as an onsite event in Bolzano, Italy. The YESS summer school was held near Bolzano in August, with great success. More than 700 papers and posters were submitted for CERME12 in September. But in November 2021, new and unknown variants of the virus appeared. Their alarming spread forced us to reconsider the situation. Finally, the LOC, the IPC and the ERME board jointly decided that CERME12 would be held as an online congress, as announced in a mail sent to all members of ERME on December 1st:

> It is with great sadness that we must communicate a decision which is forced upon us by the current developments of the COVID epidemic in Europe, and which has been taken by the ERME board in full agreement with us: CERME12 will be organized by the Bolzano team as an online conference, on the same dates as originally foreseen. For a long time we hoped for the much desired possibility of having the first CERME in three years as a normal, face to face event. Organizing a virtual CERME – which we will strive to hold as much "CERME spirit" as possible – will be a very demanding task, both in terms of finding good technical solutions, and in terms of organizing the programme and preparing the many TWG teams in a good way.

Germany	209	Turkey	15	Iceland	3
Norway	85	Ireland	13	Lithuania	3
Italy	81	Czech Rep.	11	New Zealand	3
Spain	62	Slovakia	11	China	2
Sweden	62	Chile	9	Colombia	2
USA	41	Brazil	8	Hong Kong	2
UK	38	Croatia	8	Poland	2
Israel	31	Finland	6	Algeria	1
Netherlands	26	South Africa	6	Egypt	1
Denmark	24	Switzerland	6	Faroe Islands	1
France	21	Australia	5	Malta	1
Austria	20	Malawi	5	Romania	1
Canada	19	Mexico	5	Russia	1
Greece	19	Belgium	4	Thailand	1
Portugal	17	Japan	4	Tunisia	1
Hungary	15	Cyprus	3	Ukraine	1

Table 1: The success of CERME12 in numbers – 915 participants from 48 countries

Indeed, it took a unique *tour de force* for all organisers to prepare – in just two months – an online version of CERME, based as it is on group work and interaction, rather than on one-way presentations (which are relatively easy to transmit online). These effort was crowned by the best success the new conditions could possibly allow: an online congress with more participants than ever, with virtually no technical problems, and not least with a high level of participant satisfaction.

In the history of ERME, CERME12 will be remembered as a scientific highlight during the long and hard pandemic. First of all, that is due to the plenary speakers and panelists, and to the contributors of papers and posters. Your efforts shine through the quality of the scientific texts offered by these proceedings. ERME, as a society of scholars, was not stopped – hardly delayed – by the pandemic, thanks to your ingenuity and unfailing determination to do and share first class research.

The realization of CERME12 was made possible also by the many people who organised the congress, under the difficult conditions alluded to above, namely:

- The Local Organizing committee, led by Giorgio Bolondi and Federica Ferretti, and all of the Bolzano team, including also the technicians who made the online congress run smoothly;
- The International Programme committee, led by Jeremy Hodgen and Eirini Geraniou;
- The leader teams of all 27 Thematic Working Groups.

For all your tireless and unselfish work during the three years between CERME 11 and CERME 12, the community owes you immense and extraordinary gratitude.

And the story goes on: ERME invites all interested researchers to CERME 13 (Budapest , Hungary, July 2023), and after that, to CERME 14 to be held in *real* Bolzano in February 2025.

Introduction to the Proceedings of the Twelfth Congress of the European Society for Research in Mathematics Education (CERME12)

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About CERME12

The Eleventh Congress of European Research in Mathematics Education (CERME 12) took place virtually, hosted by the Free University of Bozen-Bolzano, Italy, from 2nd to 6th of February 2022, after a year's delay due to the pandemic. Jeremy Hodgen (UK) and Eirini Geraniou (UK) were chair and co-chair of the International Programme Committee (IPC), which comprised Giorgio Bolondi (Italy), Jason Cooper (Israel), Ana Donevska-Todorova (Germany / North-Macedonia), Çiğdem Haser (Finland / Turkey), Uffe Thomas Jankvist (Denmark), Leander Kempen (Germany), Esther Levenson (Israel), Nuria Planas (Spain) and Michiel Veldhuis (The Netherlands). Giorgio Bolondi and Federica Ferretti were chair and co-chair, respectively, of the Local Organizing Committee (LOC).

CERME12 hosted 27 Thematic Working Groups, listed in the table below. The TWGs 11 and 27 were new TWGs, created following a call launched just after CERME11, and a selection process involving the CERME12 IPC and the ERME board. They have both been very successful. Nine of the TWGs received so many submissions that they had to be split in two – more precisely the TWGs 01, 03, 05, 09, 14, 16, 18, 19 and 20. In the end, CERME12 had 27 TWG leaders and 110 TWG co-leaders.

Thematic Working Group	Leader	Co-Leaders
TWG1: Argumentation and Proof	Andreas Moutsios-Rentzos (Greece)	Orly Buchbinder (USA); Jenny Christine Cramer (Germany); Nicolas Leon (YR) until Aug 2021; and from Sep 2021: Viviane Durand-Guerrier (France); David A. Reid (Norway); Mei Yang

		(British Indian Ocean Territory/UK) YR
TWG2: Arithmetic and Number Systems	Elisabeth Rathgeb-Schnierer (Germany)	Judy Sayers (UK); Beatrice Vargas Dorneles (Brazil) until Sep 2021; Pernille Bødtker Sunde (Denmark) from Sep 2021; Renata Carvalho (Portugal) YR
TWG3: Algebraic Thinking	Dave Hewitt (UK)	Maria Chimoni (Cyprus); Cecilia Kilhamn (Sweden); Luis Radford (Canada) from Sep 2021; Jorunn Reinhardtsen (Norway) YR
TWG4: Geometry Teaching and Learning	Michela Maschietto (Italy)	Alik Palatnik (Israel); Lina Brunheira (Portugal); Chrysi Papadaki (Germany) YR
TWG5: Probability and Statistics Education	Caterina Primi (Italy)	Sibel Kazak (Turkey); Aisling Leavy (Ireland); Orlando Rafael Gonzalez (Thailand); Daniel Frischemeier (Germany) YR
TWG6: Applications and Modelling	Berta Barquero (Spain)	Susana Carreira (Portugal); Jonas Bergman Ärlebäck (Sweden); Katrin Vorhölter (Germany); Gilbert Greefrath (Germany) from Sep 2021; Britta Eyrich Jessen (Denmark) YR
TWG7: Adult Mathematics Education	Kees Hoogland (The Netherlands)	Javíer Díez-Palomar (Spain); Fiona Faulkner (Ireland); Beth Kelly (UK) YR
TWG8: Affect and the Teaching and Learning of Mathematics	Stanislaw Schukajilow (Germany)	Inés Mª Gómez-Chacón (Spain); Çiğdem Haser (Finland); Peter Liljedahl (Canada); Chiara Andrà (Italy); Hanna Viitala (Sweden) YR
TWG9: Mathematics and Language	Jenni Ingram (UK)	Kirstin Erath (Germany); Aurélie Chesnais (France); Ingólfur Gíslason (Iceland) YR

TWG10: Diversity and Mathematics Education: Social, Cultural and Political Challenges	Laura Black (UK)	Anette Bagger (Sweden); Anna Chronaki (Greece); Nina Bohlmann (Germany); Sabrina Bobsin Salazar (Brazil) YR
TWG11: Algorithmics	Christof Weber (Switzerland)	Janka Medova (Slovakia); Ulrich Kortenkamp (Germany); Simon Modeste (France); Piers Saunders (UK) YR until Oct 2021; Maryna Rafalska (France) from Oct 2021
TWG12: History in Mathematics Education	Renaud Chorlay (France)	Antonio M. Oller-Marcén (Spain); Jenneke Krüger (The Netherlands); Tanja Hamann (Germany) YR
TWG13: Early Years Mathematics	Bożena Maj-Tatsis (Poland)	Marianna Tzekaki (Greece); Esther Levenson (Israel); Martin Carlsen (Norway); Andrea Maffia (Italy) YR
TWG14: University Mathematics Education	Alejandro González-Martín (Canada)	Ghislaine Gueudet (France); Olov Viirman (Sweden); Athina Thoma (UK) YR; and from Sep 2021: Irene Biza (United Kingdom); Chris Rasmussen (United States); Ignasi Florensa (Spain) YR
TWG15: Teaching Mathematics with Technology and Other Resources	Alison Clark-Wilson (UK)	Ornella Robutti (Italy); Melih Turgut (Norway); Daniel Thurm (Germany) from Sep 2021; Gülay Bozkurt (Turkey) YR
TWG16: Learning Mathematics with Technology and Other Resources	Paul Drijvers (The Netherlands)	Florian Schacht (Germany); Nathalie Sinclair (Canada); Osama Swidan (Israel); Eleonora Faggiano (Italy) from Sep 2021; Seçil Yemen Karpuzcu (Turkey) YR
TWG17: Theoretical Perspectives and Approaches in Mathematics Education Research	Angelika Bikner-Ahsbahs (Germany)	Heather Johnson (USA); Anna Shvarts (The Netherlands); Abdel Seidouvy (Togo/Sweden) YR

TWG18: Mathematics Teacher Education and Professional Development	Janne Fauskanger (Norway)	Libuse Samkova (Czech Republic); Andreas Ebbelind (Sweden); Marita Eva Friesen (Germany) YR; and from Sep 2021: Tracy Helliwell (UK); Macarena Larrain (YR)
TWG19: Mathematics Teaching and Teacher Practice(s)	Reidar Mosvold (Norway)	Mark Hoover (USA); Siún Nic Mhuiri (Ireland); Edyta Nowinska (Poland/Germany); Helena Grundén (Sweden) YR
TWG20: Mathematics Teacher Knowledge, Beliefs and Identity	Fatma Aslan-Tutak (Turkey)	Miguel Montes (Spain); Francesca Martignone (Italy); Miguel Ribeiro (Brazil); Veronika Hubeňáková (Slovakia) YR
TWG21: Assessment in Mathematics Education	Paola Iannone (UK)	Francesca Morselli (Italy); Michal Ayalon (Israel); Michiel Veldhuis (The Netherlands); Gözde Kaplan Can (Turkey) YR
TWG22: Curricular Resources and Task Design in Mathematics Education	Shai Olsher (Israel)	Sebastian Rezat (Germany); Annalisa Cusi (Italy); Nataly Essonnier (France) YR
TWG23: Implementation of Research Findings in Mathematics Education	Mario Sánchez Aguilar (Mexico)	Boris Koichu (Israel); Morten Misfeldt (Denmark); Rikke Maagaard Gregersen (Denmark) YR until Aug 2021; Linda Marie Ahl (Sweden) YR from Sep 2021
TWG24: Representations in Mathematics Teaching and Learning	Anna Baccaglini-Frank (Italy)	Carla Finesilver (UK); Michal Tabach (Israel); Kate O'Brien (USA/UK) YR
TWG25: Inclusive Mathematics Education – Challenges for Students with Special Needs	Petra Scherer (Germany)	Hana Moraova (Czech Republic); Michael Gaidoschik (Italy); Helena Roos (Sweden) YR
TWG26: Mathematics in the Context of STEM Education	Behiye Ubuz (Turkey)	Michele Stephan (USA); Clelia Cascella (Italy/UK); Nelleke Den Braber (The Netherlands) YR

TWG27: The Professional Practices, Preparation and Support of Mathematics Teacher Educators	Ronnie Karsenty (Israel)	Stefan Zehetmeier (Austria); Hilda Borko (USA); Alf Coles (UK); Bettina Rösken-Winter (Germany); Birte Friedrich-Pöhler (Germany) YR

Editorial information

These proceedings are available as a complete volume online on the ERME website and each individual text is also available on the HAL open archive, where it can be found through keywords, title or author name. This has been the practice since CERME9, to increase the visibility of the huge work done in CERME conferences.

This volume begins with texts corresponding to the three plenary activities of CERME12: the plenary lecture by Susanne Prediger (Germany) on "Enhancing language for developing conceptual understanding: A research journey connecting different research approaches"; the plenary lecture by Jeppe Skott (Sweden / Norway) on "Conceptualizing individual-context relationships in teaching: Developments in research on teachers' knowledge, beliefs and identity"; and finally the panel discussion "Big Questions in Mathematics Education". This panel discussion was led by Anna Baccaglini-Frank (Italy), Ingi Højsted (Faroe Islands) and Janka Medova (Slovakia), chaired by Michiel Veldhuis (The Netherlands) and moderated by Eirini Geraniou (UK/Greece). The two plenary speakers, Susanne Prediger and Jeppe Skott, each gave a response to the panel discussion.

After the plenaries, the reader will find 27 chapters corresponding to the work done in the TWGs of CERME12 (with combined introductions from all the split TWGs). These chapters follow a similar structure: they start with an introduction; then the long paper contributions (8-page papers) and the short poster contributions (2 pages) are presented – in alphabetical order by first author's name.

There are two kinds of introductions to the TWGs, according to the team's choice: short introductions (4 pages) presenting the contributions; or long introductions (8 pages), which propose, in addition, an analysis of the current research on the theme of the TWG, and perspectives for the future. TWGs 04, 06, 07, 09, 14, 15, 17, 18, 19, 22, 25 and 26 have chosen this form of long introduction.

The publication of these proceedings is the result of a collaborative work, involving the CERME12 IPC, the TWG leaders and co-leaders, the LOC chair and co-chair and the wider team at Bozen-Bolzano. Particular thanks are due to Katrin Lambacher. We warmly thank all these people for their involvement, and hope that this volume will contribute to the development of mathematics education research in Europe and beyond.

Problems with variation in teaching/learning Geometry: an example of Chinese Cultural Transposition

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This paper discusses some theoretical/methodological observation and some qualitative results coming from a Cultural Transposition experience, implemented in the Italian school context (grade 8), according to the methodology of variation, as one of the most significant problem solving approach in Chinese schools. The framework of the Cultural Transposition and the methodology of variation are presented as an important condition for "decentralizing" the didactic practice from a specific social and cultural context. We argue that looking at different teaching/learning mathematics strategies coming from East-Asia cultures can favor some cultural contaminations at school and allow students to a significant and unusual thought about "inclusion" and "diversity" in mathematics. Our variation problems are designed on 3D Geometry and are aimed to guide students in discovering the relationship between pyramid and cone areas/volumes.

Keywords: Cultural Transposition, Chinese problems with variation, Teaching Geometry.

Introduction

In a society dominated by cognitive zapping (Veen & van Staalduinen, 2009), which is constantly changing due to socio-cultural transformations, school is also changing. Learning and teaching trajectories are in fact, in many cases, varying in all countries. Looking at the new scenarios that the school is experiencing in recent years, the classroom realities that teachers and students observe are in fact nowadays changing, enriched by new approaches, new stimuli, new routines, new didactical processes that come from inside and outside social and cultural classroom contexts. These scenarios are in many cases complex and difficult to study for the mathematics education Communities. One pioneering work (Bishop, 1988) highlighted the importance of recognizing mathematics practices as social phenomena that are embedded in those cultures and those societies that generated them. D'Ambrosio (2010) pointed out that taking care about cultural and social issues into mathematical practices contributes to the understanding of cultures and the mathematics itself. Nowadays the awareness of taking into account cultural and historical contextualization in Math classroom teaching practices and the crucial assumption that culture permeates mathematics education practices is well known by all the mathematics education Communities. As Radford underlined, it has to be clear for teachers that the "configuration and the content of mathematical knowledge is properly and intimately defined by the culture in which it develops and in which it is subsumed". (Radford, 1997). Barton's research (Barton, 2008) reinforced the importance of a cultural perspective to study mathematics education phenomena, looking for example at different languages, and cultural manners. Agreeing to this perspective, we argue that culture gives students an opportunity to engage in mathematics focusing on concepts of inclusion, integration … to discover diversity as a resource! In some cases, the participation of many countries in international student competence tests (such as PISA) gives also the opportunity not only to compare different counties obtained results (Schleicher, 2019), but also to reflect on the educational practices, social, political and cultural environments that determined these performances. The last good results (as the previous ones) obtained by students from East Asia, especially those from Hong Kong and Shanghai, have prompted comparisons to reflect on the "reasons" for this educational success. It is well known (Spagnolo & Di Paola, 2010) that in these countries some learning approaches that compared to those of their Western peers produce different mathematical skills, in many cases more useful. According to Di Paola (2016) and Mellone et al. (2019; 2021) in these learning/teaching approaches there are many cultural factors linked to different cultural assumptions, educational practices and related cognitive styles (Bartolini Bussi & Martignone, 2013; Mellone et al., 2019). The same feeling is nowadays claimed by many teachers and researchers around the world: they in fact confirm how, in many cases, East Asian students, already educated in their own country, show more developed knowledge and skills in many mathematics subjects.

Theoretical framework

Bartolini Bussi & Martignone (2013), studying didactical phenomena discussed in the previous paragraph, emphasized the possible correlation between mathematical knowledge and the cultural context, cultural beliefs, in which they are and have been inserted and in which, therefore, mathematical knowledge is elaborated, assimilated and transmitted. In recent years, some other research works, through qualitative and/or quantitative approaches (Bartolini Bussi et al. 2017; Di Paola 2016; Mellone et al., 2019) pointed out how the *cultural diversity* could become, in this sense, an opportunity in mathematics Education (Kaiser, 2018). Researchers and teachers, coming into contact with educational practices adopted in other cultural contexts, are able to *deconstruct* (Derrida, 1967) them, reconsidering the themes of educational intentionality defined as background of their educational practices. Mellone at al. (2019), inspired by the Skovsmose's (1994) approach, defined the framework *Cultural Transposition* as "a condition for decentralizing the didactic practice of a specific cultural context through contact with the didactic practices of different cultural contexts". *Cultural Transposition* (CT) is a perspective that can allow the meeting between different mathematics education school practices/approaches, coming out from different cultural contexts, and define a potential space for reflection and awareness, and also development for researchers, educators, teachers, students (Mellone et al., 2021). CT involves those who implement/observe math teaching practices, coming from other cultures, to a "*deconstruction*" process useful to a re-interpretation of their own thought and consequently a possibility to change/improve personal (cultural) beliefs, values, and didactical principles. As Derrida stated defining the deconstruction, "*an analysis of the different levels in which a culture is stratified*" (Derrida, 1967). According to this perspective it is possible to recognize, valorize and include possible differences coming from "other" cultural Communities, linked to different values, principles, beliefs systems, happens, for example, in schools Communities (Mellone et al. 2021). To "get in touch" with different educational practices, coming out from different social and cultural contexts, can help researcher and teachers not only to become more aware of their social and cultural paradigm in regards to the classroom teaching practices but also to deconstruct their thought decentralizing their cultural expectation and assumptions and rethinking mathematics educational practices (Bartolini Bussi et al, 2013) in terms of "inclusion" and significant us of possible social, cultural "diversities". Of course this "changing process" is complex and needs more and more opportunities for reflection and *contamination* (Bartolini Bussi et al., 2017; Jullien, 2006). From the school student's perspective, *diversity and difference* in learning mathematics strategies appears, in this sense, a great opportunity, a good chance for something that maybe should not be favored without this "revolution" of prospective and without this change in the cultural system of reference. This could permit us to look at diversity and difference in mathematics and in mathematics teaching within the realms of the cultural, the social and the political. Spagnolo & Di Paola (2010), a pioneer of this kind of subject, presented this approach as a continuous open dialogue between cultures, societies, histories … useful to *cross* the didactics of mathematics. In the last years several researches (e.g. Bartolini Bussi et al., 2013; Di Paola, 2016; Mellone et al., 2021; Spagnolo & Di Paola 2010) discussed different CT experiences in Western school contexts. Several of them look to Chinese culture and in particular the use of Chinese practices of "*problem with variation"* for an early approach to Algebra in Primary school. What is a rather new (Leung, 2003), research subject discussed in this paper, is the use of this methodology in teaching/learning mathematics in Middle school and, in particular, in teaching/learning Geometry.

"*Problems with variation*" is considered one of the most significant mathematics education tools used in Chinese Primary school (e.g. Bartolini Bussi et al. 2017; Fan et al., 2004; Mellone et al., 2019; Sun, 2011). In the last twenty years, many researchers underlined the importance of the variation approach as a necessary condition for deep learning (e.g. Marton & Booth, 1997; Sun, 2011) and in particular for the learning of mathematics. variation is typically expressed by Rowland (2008) as a practice in structured exercises which varies considerably from country to country and from text to text. Sun (2011) underlines the use of variation problems as a tool "to discern and compare the invariant features of the relationship among concepts, solutions and contexts, and provide opportunities for making connections, since comparison is considered the pre-condition to perceive the structures, dependencies, and relationships that may lead to mathematical abstraction." (p.107). In Chinese language it is *Bianshi* (變式,) where *bian* stands for "*changing*" and *shi* means "form.". Yakes and Star (2009) looked to variation as a critical means for comparing and developing flexibility for learning mathematics already from the first school years. In China, the variation approach to problem solving is linked to many disciplines and used in all school levels from the first grades. The issue of variations in problems perfectly reproduces one of a Chinese proverb: "*no clarification, without comparison*", and it is "*in contrast*" to the assumption used in many textbook in different countries: "to consolidate one topic, or skill, before moving on to another" (Rowland, 2008). According to Rowland, comparing this problem solving approach with the one used, in many cases, in the Italian school (textbooks and/or practice), it is in fact possible to point out a strong difference, in some cases, common in several Western countries and related to the use of isolated problems/exercises, organized in progressive steps, strongly partitioned between them and so not very useful to define a possible abstract thinking looking relationships between concepts, strategies, algorithms (Cai & Nie 2008; Spagnolo & Di Paola, 2010). These considerations can be useful to underline the significant opportunity to proceed with *contamination experiments* (Jullien, 2006) aimed to get in touch with different "good practices", coming from different cultural contexts and linked to different language; different historical tradition; ideologies; school systems; governance structures… In this sense, the *contamination* appears as an important condition for teachers to "decentralize" their own didactic practice of a specific cultural context to something more wild, different, and in many cases more useful for their own mathematics students and could be included. Bartolini Bussi et al. (2013), Di Paola et al. (2016), Mellone et al. (2021) discussed this approach in Italian schools for an early approach to Algebra in Primary school. As we already discussed, very few research papers refer instead to the inclusion and the use of "*problem with variation*" in teaching/learning mathematics in Western Middle school (grade 8) and, in particular, in Geometry. The CT discussed in this paper is aimed to look at a different teaching/learning mathematics practice, coming from East-Asia cultures, as an important opportunity for a significant and unusual (for the Western culture) 3D Geometry problem solving. In particular the proposed variation problems are aimed to guide Italian students in discovering, almost autonomously, the relationship between pyramids and cones area/volume. According to the declared aim, our lens was focused on students' reaction to this approach and, in particular, the research question to which we tried to answer was: *What kind of process do grade 8 students of Italian (Western) culture show in variation geometric problem solving?*

Methodology

In this section we discuss our CT in teaching/learning Geometry, implemented in the Italian school context (four classrooms of grade 8), according to the methodology of variation. It is important to underline that our intention was not an attempt to *translate*, or even worse *import* a Chinese practice/teaching strategy into the Italian culture and more in general in the Western one. On the contrary, the educational path we analyze, developed by the author of this paper and the group of teachers and researchers with whom we have collaborated for some years, is aimed to create a real CT, with an interesting inclusion of the different Chinese model into the Italian didactical practice in grade 8. With the aim to reply to the research question expressed before, our CT was particularly focused to help students to transfer their mathematics knowledge from a well-known context to another apparently different context, using the variation approach. The two contexts refer to 3D Geometry and in particular to the concepts of pyramid, cone and their area and volume. Authors of this paper were involved in all phases of the education path, the design, the implementation of it with school teachers and the analysis of the collected data. The grade 8 students (around 100) were engaged for almost 20 hours. All of them were conscious about experimenting with a pedagogical method from another culture; during the CT path they explored it, initially by themselves, after being guided by the Authors of this paper.

According to the declared aim and what the literature discusses about educational practices based on the same research subject (e.g. Bartolini Bussi e al., 2017, Mellone et. al., 2019), the sections of the CT path referred to the proposed mathematical subject were designed following this frame:

1. Pre/Post-questionnaire about 3D solids knowledge (pyramid and cone and their area/volumes),
2. Single students resolution of different variation problems about pyramid and cone solids.
3. Students interview about their own variation problem solutions (difficulties and mistakes personally implemented CT).

Before to start the first phase, with the aim to propose useful tasks in all research phases of the implemented educational path, we provided a structured and very fine *a-priori* analysis of Chinese and Italian textbooks (schemes, images, writing, task …) and of methodological implicit assumptions that variation approach could have had on the involved students about the inclusion of this didactic methodologies in the one commonly adopted by Italian teachers. All research phases were video and audio recorded. These collected data were qualitatively and quantitatively analyzed (with cluster and implicative analysis). Researchers and teachers studied these data step by step during all CT path phases in order to eventually redefine and redesign the path frames. According to the contamination theory, protocols and interviews were examined with a specific focus on possible interesting used of different sings (words, pictures, arrows, colors, …) useful to describe students approach to variation and, in particular, to underline possible difficulties or readiness to define relationship and among concepts, solutions and contexts. Cluster and implicative analysis gave us the possibility to put in evidence possible stable behaviors in the analyzed students sample. In this paper we don't present these data; we are referring only to some qualitative findings.

An example of an implemented variation problem task (the first triplet) is shown in Figure 1. Starting from a problem concerning only the pyramid solid and its area (first problem on the left), we asked the students to solve two more problems on the cone solid (regarding its area and volume). According the variation problems theory and the contamination one, discussed before, the defined triplet was structured in order to favor the possibility to find, independently, the relationships (similarities and differences) between the two proposed texts and the images of geometric solids, without an explicit "presentation" by the teacher of the second one (about the cone).

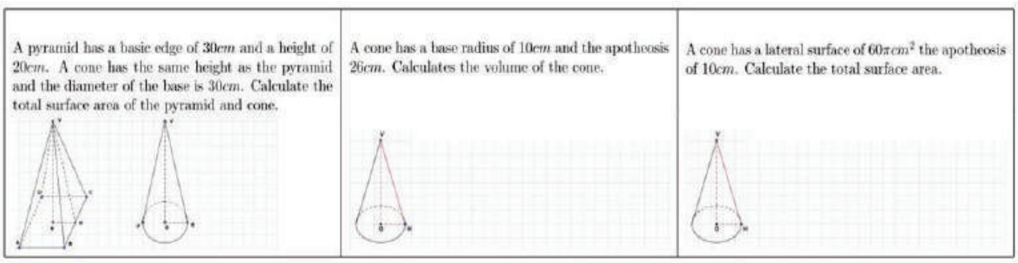

Figure 1. Our CT: an example of a first triplet of variation problem

As we also mentioned before, all students knew the concept of pyramid and its area and volume calculation and properties; none of them knew (we formally tested it) the cone solid. Students, to solve this task needed to autonomously "transfer" (using variation approach) knowledge and skills acquired on the pyramid solid, to the cone. In the three problems the graphical representation has been also inserted in order to help students to better focus on the relationships and possible links in area and volume calculation. This choice retraces what Chinese textbooks and Chinese teachers commonly propose in classrooms.

Empirical findings and first conclusion

In order to give an overview of the obtained results and related discussion, in this section, we briefly present some finding, come from the analysis of students protocols referred to the phase 2 of the implemented CT path and some data coming from a qualitative analysis of the same two students interview (phase 3 of the CT) about their own variation problem solutions.

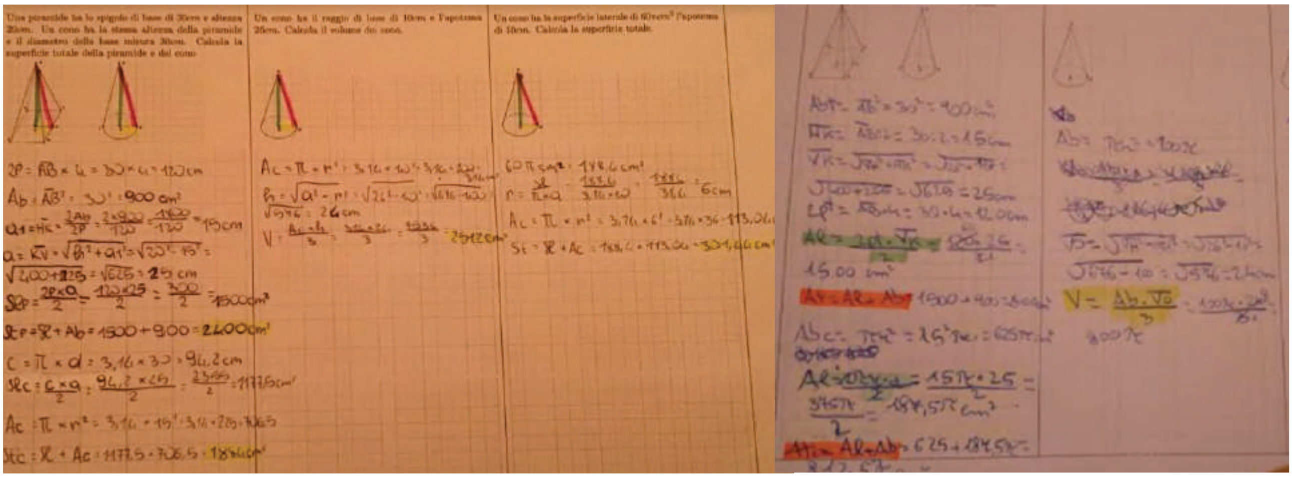

Figure 2. Maria's and Valentina's variation problem protocols

Maria's and Valentina's protocols appear interesting for replying to our research question. In both students sheets it is, in fact, possible to observe an noteworthy and autonomous use of the colors in the proposed variation problems. Maria used colors to highlight analogies and relationships between part of solid. A solid that she knows and a solid that she doesn't know. Valentina did the same but taking into account the solving strategies and the formulas, instead of the graphic solid representation. Thanks to the Chinese variation approach, she autonomously discovers a relationship between two used formulas. In both cases (we found similar approaches in many student protocols) the opportunity to be *contaminated* by the Chinese variation methodology (Mellone et al., 2019, 2021) guided them in discovering in the problem solving activities something new as the relationship between pyramids and cones area/volume. Their approach to variation problems and their "surprise" to the possibility to autonomously learn "new mathematics' ' (Maria used these terms during her interview), clearly emerged also during their subsequent interview. The following part of the dialogue between Maria and her teacher, appears interesting:

Teacher: Could you better explain to me what you mean?

Maria: I highlighted with the same color what behaves in the same way.

…

Maria: Here I have something that I know and something unknown into the same problem. I tried to look, to search for analogies and relationships. It's a interesting problem. I like it.

Teacher: What did you find?

Maria: All are right triangles, the first two are just the same ... same numbers! So I highlighted them the same way.

…

Maria: Prof., could we do the same argumentation between Prisms and Cylinders? We studied them… We could find a general "rule", we could use less memory and more variation. Problem solving can become more "simple".

Teacher: You can try, …

Maria: It is great. I didn't know anything before today about it. We have to look also to other countries (smiling) …

As Maria, almost all students benefited from the Chinese contamination of variation problems. they declared that working with this methodology, they became more aware of their knowledge and they were able to construct step by step a general solving approach to this kind of problems (the same finding in Cai & Nie, 2008). Of course a few of them (Giovanni, Marco and Francesca speeches are some examples) didn't immediately reach up to this expertise.

Giovanni: We haven't studied the cone solid, the teacher was wrong … that's why I only solved the pyramid problem.

Marco: I don't know the cone and I don't remember the pyramid, sorry.

Francesca: We have not studied the cone, but the teacher gave us a task with the pyramid and the cone; we don't know anything about the cone ... I tried to use what I knew about the pyramid but I couldn't find the way.. why did the teacher put the pyramid next to the cone? I will try to think about it. We didn't do this same kind of problems before

As we said before, this "changing process" is complex and long (Bartolini Bussi et al., 2014); Giovanni, Marco and Francesca need more and more opportunities for reflection and *contamination*.

According to Spagnolo & Di Paola (2010), Sun (2011) and Mellone et al. (2021) findings, our students approach underlines how the implemented CT path gave them a good chance to stimulate and improve a possible abstract thinking, finding relationships between concepts, strategies, algorithms in variation problems. The defined variation problems gave them an important opportunity to deepen abstract geometrical thinking - structure, strategies, relationship, concepts … - (Leung, 2003) than does typical isolated problems on the two investigated solids as, in many cases, happens in Italian (but also in other Western countries) school context. They discovered a "new", "diverse" culture and different related problems solving approaches that enrich their teaching practices and, according to our finding, their mathematics learning (Barton, 2008). Thanks to the CT path they had the opportunity to engage in mathematics through a cultural lens, helpful to discover and use diversities as resources for their future teaching. This gave us, as researchers, the chance to rethink our own cultural expectations and assumptions about possible important cultural, social and political issues in Mathematics Education research.

References

Bartolini Bussi, M.G., Martignone, F. (2013). Cultural issues in the communication of research on Mathematics Education. *For the Learning of Mathematics*, Vol. 33, pp. 2-8. Oxford.

Bartolini Bussi, M.G., Bertolini, C., Ramploud, A., Sun, X. (2017). Cultural CT of Chinese lesson study to Italy. An exploratory study on fractions in a fourth-grade classroom, International Journal for Lesson and Learning Studies, 6(4), 1-17.

Barton, B. (2008). Cultural and social aspects of mathematics education: Responding to Bishop's challenge. In *Critical Issues in Mathematics Education* (pp. 121-133). Springer, Boston, MA.

Bishop, A. J. (1988). *Mathematical enculturation. A cultural perspective on Mathematics education*. Dordrecht: Kluwer Academic Publishers.

Cai, J., Nie, B. (2008) Problem Solving in Chinese Mathematics Education: Research and Practice, *ZDM–Mathematics Education, 39*: 459-473

D'Ambrosio, U. (2010). *Ethnomathematics—Link between traditions and modernity*. Sense Publishers.

Derrida, J. (1967). *De la gramma Derrida tologie*. Paris, France: De Minuit.

Di Paola, B. (2016). Why Asian children outperform students from other countries? Linguistic and parental influences comparing Chinese and Italian children in Preschool Education. *International Electronic Journal of Mathematics Education*, 11(9), 3351-3359.

Fan, L., Wong, N., Cai, J., Li, S. (eds.) (2004). *How Chinese learn mathematics: perspectives from insiders*. World Scientific (Series on Mathematics Education; 1).

Jullien F. (2006). *Si parler va sans dire. Du logos et d'autres resourcess*, Paris, Edition du Seuil.

Kaiser, G., Forgasz, H., Graven, M., Kuzniak, A., Simmt, E., Xu, B. (2018). *Invited Lectures from the 13th International Congress on Mathematical Education*. Springer Nature.

Leung, A. (2003). Dynamic Geometry and The Theory of variation. *International Group for the Psychology of Mathematics Education*, *3*, 197-204.

Marton, F., Booth, S. (1997). *Learning and Awareness*. New Jersey, Lawrence Erlbaum Associates, INC, Publishers.

Mellone, M., Ramploud, A., Di Paola, B., Martignone, F. (2019). Cultural CT: Italian didactic experiences inspired by Chinese and Russian perspectives on whole number arithmetic. *ZDM–Mathematics Education,* 51(1), 199-212.

Mellone, M., Pacelli, T., Liljedahl, P. (2021). Cultural CT of a thinking classroom: to conceive possible unthoughts in mathematical problem solving activity. *ZDM–Mathematics Education*, 1-14.

Radford, L. (1997). On psychology, historical epistemology, and the teaching of mathematics: Towards a socio-cultural history of mathematics. *For the learning of mathematics*, *17*(1), 26-33.

Rowland, T. (2008). The purpose, design, and use of examples in the teaching of elementary mathematics. *Educational Studies in Mathematics*, 69, 149- 163.

Schleicher, A. (2019). PISA 2018: Insights and Interpretations. *OECD Publishing*.

Skovsmose, O. (1994). Towards a critical Mathematics education. *Educational studies in Mathematics*, 27(1), 35-57.

Spagnolo, F., Di Paola, B. (2010). *European and Chinese cognitive styles and their impact on teaching Mathematics*. Springer, Studies in Computational Intelligence, 277, 1-267.

Sun, X. (2011). "variation problems" and their roles in the topic of fraction division in Chinese mathematics textbook examples, *Educational Studies in Mathematics*: volume 76, Issue 1.

Veen, W., van Staalduinen, J. P. (2009). Homo zappiens and its impact on learning in higher education. *IADIS e-LEAR IG*, *1*, 11-18.

Zhou, F., Leung, K., Bond, M. H. (2009). Social axioms and achievement across cultures: The influence of reward for application and fate control. *Learning and Individual Differences*, *19*(3), 366-371.