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A Unifying Approach to Weyl Type Theorems for Banach Space Operators

Pietro Aiena, Jesús R. Guillén and Pedro Peña

Abstract. Weyl type theorems have been proved for a considerably large number of classes of operators. In this paper, by introducing the class of quasi totally hereditarily normaloid operators, we obtain a theoretical and general framework from which Weyl type theorems may be promptly established for many of these classes of operators. This framework also entails Weyl type theorems for perturbations $f(T + K)$, where K is algebraic and commutes with T, and f is an analytic function, defined on an open neighborhood of the spectrum of $T + K$, such that f is non constant on each of the components of its domain.

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Keywords. Totally hereditarily normaloid operators, polaroid operators, Weyl type theorems.

1. Introduction

Weyl type theorems have been studied in the last two decades by several authors and most of them have essentially proved that such theorems hold for special classes of operators. Many times the arguments used, to prove Weyl type theorems for each one of these classes of operators, are rather similar. In this paper we show that it is possible to bring back up these theorems from some general common ideas. Actually, we determine a very useful and unique theoretical framework, from which we can deduce that Weyl type theorems hold for all these classes of operators. This framework is created by introducing the class of *quasi totally hereditarily normaloid* operators and by proving that these operators are *hereditarily polaroid*. Many classes of operators T on Hilbert spaces are quasi totally hereditarily normaloid, and this fact, together with SVEP, permits to us to extend all Weyl type theorems to the perturbations $f(T + K)$, where K is! algebraic and commutes with

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T, f is an analytic function, defined on an open neighborhood of the spectrum of $T+K$, such that f is nonconstant on each of the components of its domain. Consequently, our results subsume and extend many results existing in literature.

2. Totally Hereditarily Normaloid Operators

A bounded linear operator $T \in L(X)$, defined on a complex infinite dimensional Banach space X, is said to be *normaloid* if $\|T\| = r(T)$, $r(T)$ the spectral radius of T. An operator $T \in L(X)$ is said to be *hereditarily normaloid*, $T \in \mathcal{HN}$, if the restriction $T|M$ of T, to any closed T-invariant subspace M, is normaloid. Finally, $T \in L(X)$ is said to be *totally hereditarily normaloid*, $T \in \mathcal{THN}$, if $T \in \mathcal{HN}$ and every invertible restriction $T|M$ has a normaloid inverse. Totally hereditarily operators were introduced in [22], and have since investigated in [18], and [19], for establishing Weyl type theorems.

Remark 2.1. It is rather simple to see that if $T \in L(X)$ is $\mathcal{THN}$ and M is a T-invariant closed subspace of X then the restriction $T|M$ is also $\mathcal{THN}$.

In the sequel we list examples of $\mathcal{THN}$-operators:

(i) *Paranormal operators* on Banach spaces are $\mathcal{THN}$-operators, where $T \in L(X)$ is said to be paranormal if

$$\|Tx\| \leq \|T^2x\|\|x\| \quad \text{for all } x \in X,$$

see [22] or [2] for details. Also p-quasi-hyponormal operators are $\mathcal{T}HN$-operators, where an operator $T \in L(H)$, H a separable infinite dimensional Hilbert space, is said to be *p-quasi-hyponormal*, for some $0 < p \leq 1$, if

$$T^*(|T|^{2p} - |T^*|^{2p}T \geq 0,$$

where $|T| := (T^*T)^{1/2}$. Indeed, every p-quasi-hyponormal is paranormal, see [23]. Another subclass of paranormal operators on Hilbert spaces is given by the the A class of operators introduced by Furuta et al. [26], where $T \in L(H)$ is said to be *a class A* operator if $|T|^2 \leq |T^2|$.

(ii) An operator $T \in L(H)$, H a Hilbert space, is called *quasi *-paranormal* if

$$\|T^*Tx\|^2 \leq \|T^3x\|\|Tx\| \quad \text{for all unit vectors } x \in H$$

Every quasi *-paranormal operator is totally hereditarily normaloid, see [35]. The class of quasi *-paranormal contains the class of all *-paranormal operators, i.e. the class of $T \in L(H)$ for which

$$\|T^*x\|^2 \leq \|T^2x\| \quad \text{for all unit vectors } x \in H,$$

see [34] for details. Every *quasi hyponormal operator* is quasi *-paranormal, see [34].

(iii) A bounded operator $T \in L(H)$, H a separable Hilbert space, is said to be *k-quasi-*-class A operator* if

$$T^{*k}|T^2|T^k \geq T^{*k}|T^{*2}||T^*$$

Every k-quasi-*-class A operator is totally hereditarily normaloid, see [32]. For $k = 1$ we obtain the class of all quasi-*-class A operators, which is included in the class of all quasi *-paranormal operators.

It is evident that

$$T \in L(X) \text{ quasi-nilpotent normaloid} \Rightarrow T = 0,$$

Let $\mathcal{H}_{nc}(\sigma(T))$ denote the set of all analytic functions, defined on an open neighborhood of $\sigma(T)$, such that f is non constant on each of the components of its domain. Define, by the classical functional calculus, $f(T)$ for every $f \in \mathcal{H}_{nc}(\sigma(T))$.

Let $\mathcal{C}$ be any class of operators. We say that T is an *analytically* $\mathcal{C}$-operator if there exists some analytic function $f \in \mathcal{H}_{nc}(\sigma(T))$ such that $f(T) \in \mathcal{C}$.

Lemma 2.2. *The property of being analytically $\mathcal{C}$ is translation invariant.*

Proof. We have to show that

$$T \text{ analytically } \mathcal{C} \text{ and } \lambda_0 \in \mathbb{C} \Rightarrow \lambda_0 I - T \text{ analytically } \mathcal{C}.$$

Suppose that $f(T) \in \mathcal{C}$ for some $f \in \mathcal{H}_{nc}(\sigma(T))$. Let $\lambda_0 \in \mathbb{C}$ arbitrary and set $g(\mu) := f(\lambda_0 - \mu)$. Then g is analytic and

$$g(\lambda_0 I - T) = f(\lambda_0 I - (\lambda_0 I - T)) = f(T),$$

thus $\lambda_0 I - T$ is analytically $\mathcal{C}$. □

Recall that an invertible operator $T \in L(X)$ is said to be *doubly power-bounded* if $\sup\{\|T^n\| : n \in \mathbb{Z}\} < \infty$.

Theorem 2.3. *Suppose that $T \in L(X)$ is quasi-nilpotent. If T is an analytically $\mathcal{THN}$ operator, then T is nilpotent.*

Proof. Let $T \in L(X)$ and suppose that $f(T)$ is a $\mathcal{THN}$-operator for some $f \in \mathcal{H}_{nc}(\sigma(T))$. From the spectral mapping theorem we have

$$\sigma(f(T)) = f(\sigma(T)) = \{f(0)\}.$$

We claim that $f(T) = f(0)I$. To see this, let us consider the two possibilities: $f(0) = 0$ or $f(0) \neq 0$.

If $f(0) = 0$ then $f(T)$ is quasi-nilpotent and $f(T)$ is normaloid, and hence $f(T) = 0$. The equality $f(T) = f(0)I$ then trivially holds.

Suppose the other case $f(0) \neq 0$, and set $f_1(T) := \frac{1}{f(0)} f(T)$. Clearly, $\sigma(f_1(T)) = \{1\}$ and $\|f_1(T)\| = 1$. Further, $f_1(T)$ is invertible and is $\mathcal{THN}$. This easily implies that its inverse $f_1(T)^{-1}$ has norm 1. The operator $f_1(T)$ is then doubly power-bounded and, by a classical theorem due to Gelfand (see [30, Teorem 1.5.14] for an elegant proof), it then follows that $f_1(T) = I$, and consequently $f(T) = f(0)I$, as claimed.

Now, let $g(\lambda) := f(0) - f(\lambda)$. Clearly, $g(0) = 0$, and g may have only a finite number of zeros in $\sigma(T)$. Let $\{0, \lambda_1, \ldots, \lambda_n\}$ be the set of all zeros of g, where $\lambda_i \neq \lambda_j$, for all $i \neq j$, and λ_i has multiplicity $n_i \in \mathbb{N}$. We have

$$g(\lambda) = \mu\lambda^m \prod_{i=1}^{n} (\lambda_i I - T)^{n_i} h(\lambda),$$

where $h(\lambda)$ has no zeros in $\sigma(T)$. From the equality $g(T) = f(0)I - f(T) = 0$ it then follows that

$$0 = g(T) = \mu T^m \prod_{i=1}^{n} (\lambda_i I - T)^{n_i} h(T) \quad \text{with } \lambda_i \neq 0,$$

where all the operators $\lambda_i I - T$ and $h(T)$ are invertible. This, obviously, implies that $T^m = 0$, i.e. T is nilpotent. ☐

Two classical quantities associated with a linear operator T are the *ascent* $p := p(T)$, defined as the smallest non-negative integer p (if it does exist) such that $\ker T^p = \ker T^{p+1}$, and the *descent* $q := q(T)$, defined as the smallest non-negative integer q (if it does exists) such that $T^q(X) = T^{q+1}(X)$. It is well-known that if $p(\lambda I - T)$ and $q(\lambda I - T)$ are both finite then $p(\lambda I - T) = q(\lambda I - T)$ and λ is a pole of the the function resolvent $\lambda \to (\lambda I - T)^{-1}$, in particular λ is an isolated point of the spectrum $\sigma(T)$, see Proposition 38.3 and Proposition 50.2 of Heuser [28].

A bounded operator $T \in L(X)$ defined on a Banach space is said to be *polaroid* if every isolated point of the spectrum $\sigma(T)$ is a pole of the resolvent. Polaroid operators have been studied in recent papers in relation with Weyl type theorems, see [3,6,20,21]. Note that by Theorem 2.2 of [6], $T \in L(X)$ is polaroid if and only if there exists $p := p(\lambda I - T) \in \mathbb{N}$ such that

$$H_0(\lambda I - T) = \ker (\lambda I - T)^p \quad \text{for all } \lambda \in \operatorname{iso} \sigma(T), \tag{1}$$

where $\operatorname{iso} \sigma(T)$ denotes the set of all isolated points of $\sigma(T)$.

The following result has been proved in [2, Theorem 2.4].

Theorem 2.4. *For an operator $T \in L(X)$ the following statements are equivalent:*

(i) T is polaroid;
(ii) there exists $f \in \mathcal{H}_{nc}(\sigma(T))$ such that $f(T)$ is polaroid.
(iii) $f(T)$ is polaroid for every $f \in \mathcal{H}_{nc}(\sigma(T))$;

Two important subspaces in local spectral theory and Fredholm theory are defined in the sequel. The *quasi-nilpotent part* of an operator $T \in L(X)$ is the set

$$H_0(T) := \left\{ x \in X : \lim_{n \to \infty} \|T^n x\|^{\frac{1}{n}} = 0 \right\}.$$

Clearly, $\ker T^n \subseteq H_0(T)$ for every $n \in \mathbb{N}$. If $T \in L(X)$, the *analytic core* $K(T)$ is the set of all $x \in X$ such that there exists a constant $c > 0$ and a sequence of elements $x_n \in X$ such that $x_0 = x, Tx_n = x_{n-1}$, and $\|x_n\| \le c^n \|x\|$ for all $n \in \mathbb{N}$.

An operator $T \in L(X)$ is said to have *the single valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open neighborhood U of λ_0, the only analytic function $f : U \to X$ which satisfies the equation $(\lambda I - T) f(\lambda) = 0$ for all $\lambda \in U$ is the function $f \equiv 0$. The operator T is said to have SVEP if it has SVEP at every $\lambda \in \mathbb{C}$. It follows from the identity theorem for analytic functions that T has SVEP at every point of the boundary of the spectrum. In particular, T and its dual T^* have SVEP at every isolated point of $\sigma(T)$. We also have (see [1, Theorem 3.8])

$$p(\lambda I - T) < \infty \Rightarrow T \text{ has SVEP at } \lambda, \tag{2}$$

and dually

$$q(\lambda I - T) < \infty \Rightarrow T^* \text{ has SVEP at } \lambda. \tag{3}$$

Moreover,

$$H_0(\lambda I - T) \text{ closed} \Rightarrow T \text{ has SVEP at } \lambda. \tag{4}$$

It is known that all the operators listed in the examples (i)–(iv) have SVEP.

3. Quasi-$\mathcal{THN}$ Operators

In this section we extend the results of the previous section to a class of operators which properly contain the class $\mathcal{T}HN$.

Definition 3.1. An operator $T \in L(X)$, X a Banach space, is said to be k-quasi totally hereditarily normaloid, k a nonnegative integer, if the restriction $T|\overline{T^k(X)}$ is $\mathcal{THN}$.

Evidently, every $\mathcal{THN}$-operator is quasi-$\mathcal{THN}$, and if $T^k(X)$ is dense in X then a quasi-$\mathcal{THN}$ operator T is $\mathcal{THN}$. In the sequel by $\overline{Y}$ we denote the closure of $Y \subseteq X$.

Lemma 3.2. *If $T \in L(X)$ is quasi-$\mathcal{THN}$ and M is a closed T-invariant subspace of X, then $T|M$ is quasi-$\mathcal{THN}$.*

Proof. Let k a nonnegative integer such that $T_k := T|\overline{T^k(X)}$ is $\mathcal{THN}$. Let T_M denote the restriction $T|M$. Clearly, $\overline{{T_M}^k(M)} \subseteq \overline{T^k(X)}$, so $\overline{{T_M}^k(M)}$ is T_k-invariant subspace of $\overline{T^k(X)}$. By Remark 2.1 it then follows that $T_M|\overline{{T_M}^k(M)} = T_k|\overline{{T_M}^k(M)}$ is $\mathcal{THN}$. □

We recall now some elementary algebraic facts. Suppose that $T \in L(X)$ and $X = M \oplus N$, with M and N closed subspace of X, M invariant under T. With respect to this decomposition of X it is known that T may be represented by a upper triangular operator matrix $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$, where $A \in L(M)$, $C \in L(N)$ and $B \in L(N, M)$. It is easily seen that for every $x = \begin{pmatrix} x \\ 0 \end{pmatrix} \in M$ we have $Tx = Ax$, so $A = T|M$. Let us consider now the case of operators T acting on a Hilbert space H, and suppose that $T^k(H)$ is not dense in H. In this case we can consider the nontrivial orthogonal decomposition

$$H = \overline{T^k(H)} \oplus \overline{T^k(H)}^{\perp}, \tag{5}$$

where $\overline{T^k(H)}^{\perp} = \ker(T^*)^k$, T^* the adjoint of T. Note that the subspace $\overline{T^k(H)}$ is T-invariant, since

$$T(\overline{T^k(H)}) \subseteq \overline{T(T^k(H))} = \overline{T^{k+1}(H)} \subseteq \overline{T^k(H)}.$$

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Thus we can represent, with respect the decomposition (5), T as an upper triangular operator matrix

$$\begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}, \tag{6}$$

where $T_1 = T|\overline{T^k(H)}$. Moreover, T_3 is nilpotent. Indeed, if $x \in \overline{T^k(X)}^{\perp}$, an easy computation yields $T^k x = T \begin{pmatrix} 0 \\ x \end{pmatrix} = {T_3}^k x$. Hence ${T_3}^k x = 0$, since $T^k x \in \overline{T^k(H)} \cup \overline{T^k(H)}^{\perp} = \{0\}$. Therefore we have:

Theorem 3.3. *Suppose that $T \in L(H)$ and $T^k(H)$ non dense in H. Then, according the decomposition (5), $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ is quasi-$\mathcal{T}HN$ if and only if T_1 is $\mathcal{THN}$. Furthermore,*

$$\sigma(T) = \sigma(T_1) \cup \sigma(T_3) = \sigma(T_1) \cup \{0\}.$$

Proof. The first assertion is clear, since $T_1 = T|\overline{T^k(H)}$. The second assertion follows from the following general result: if $T := \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is an upper triangular operator matrix acting on some direct sum of Banach spaces and $\sigma(A) \cap \sigma(B)$ has no interior points, then $\sigma(T) = \sigma(A) \cup \sigma(B)$; see [31]. □

Upper triangular operator matrices have been studied by many authors, see for instance [13,17,27,41]. In the sequel we give some examples of operators which are quasi totally hereditarily normaloid.

(iv) The class of quasi-paranormal operators may be extended as follows: $T \in L(H)$ is said to be (n,k)-*quasiparanormal* if

$$\|T^{k+1}x\| \le \|T^{1+n}(T^k x)\|^{\frac{1}{1+n}} \|T^k x\|^{\frac{n}{1+n}} \quad \text{for all } x \in H.$$

The class of $(1,k)$-quasiparanormal operators has been studied in [33]. The $(1,1)$-quasiparanormal operators has been studied in [39]. If $T^k(H)$ is not dense then, in the triangulation $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$, $T_1 = T|\overline{T^k(H)}$ is n-quasiparanormal, and hence $\mathcal{THN}$, see [40].

(v) An extension of class A operators is given by the class of all k-*quasiclass A operators*, where $T \in L(H)$, H a separable infinite dimensional Hilbert space, is said to be a k-quasiclass A operator if

$$T^{*k}(|T|^2 - |T|^2)T^k \ge 0.$$

Every k-quasiclass A operator is quasi-$\mathcal{THN}$. Indeed, if T has dense range then T is a class A operator and hence paranormal. If T does not have dense range then T with respect the decomposition $H = \overline{T^k(H)} \oplus \ker T^{*k}$ may be represented as a matrix $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$, where $T_1 := T|\overline{T^k(H)}$ is a class A operator, and hence $\mathcal{T}HN$, see [37].

As it has been observed in [24, Example 0.2], a quasi-class A operator (i.e. $k = 1$), need not to be normaloid. This shows that, in general, a quasi-$\mathcal{THN}$ operator is not normaloid, so the class of quasi-$\mathcal{THN}$ operators properly contains the class of $\mathcal{THN}$ operators.

(vi) An operator $T \in L(H)$, H a separable infinite dimensional Hilbert space, is said to be *k-quasi *-paranormal*, $k \in \mathbb{N}$, if

$$\|T^*T^kx\|^2 \leq \|T^{k+2}x\|\|T^kx\| \quad \text{for all unit vectors } x \in H.$$

This class of operators contains the class of all quasi- $*$-paranormal operators (which corresponds to the value $k = 1$). Every k-quasi *-paranormal operator is quasi-$\mathcal{THN}$. Indeed, if T^k has dense range then T is $*$-paranormal and hence $\mathcal{THN}$. If T^k does not have dense range then T may be decomposed, according the decomposition $H = \overline{T^k(H)} \oplus \ker T^{*k}$, as $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$, where $T_1 = T|\overline{T^k(H)}$ is $*$-paranormal, hence $\mathcal{THN}$, see [34, Lemma 2.1].

(vii) An extension of p-quasi-hyponormal operators is defined as follows: an operator $T \in L(H)$ is said to be (p,k)-*quasihyponormal* for some $0 < p \leq 1$ and $k \in \mathbb{N}$, if

$$T^{*k}|T^*|^{2p}T^k \leq T^{*k}|T|^{2p}T^k.$$

Every (p,k)-quasihyponormal operator T with respect to the decomposition $H = \overline{T^k(H)} \oplus \ker T^{*k}$, may be represented as a matrix $T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix}$, where $T_1 := T|\overline{T^k(H)}$ is k-hyponormal (hence paranormal) and consequently $\mathcal{THN}$, see [29].

The next result generalizes the result of Lemma 2.3.

Theorem 3.4. *Suppose that $T \in L(H)$, H a Hilbert space, is analytically quasi-$\mathcal{THN}$ and quasi-nilpotent. Then T is nilpotent.*

Proof. Suppose first that T is quasi-nilpotent and k-quasi $\mathcal{THN}$. If $T^k(H)$ is dense then T is $\mathcal{THN}$, so T is nilpotent by Theorem 2.3. Suppose that $T^k(H)$ is not dense and write $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$, where T_1 is $\mathcal{THN}$, ${T_3}^k = 0$, and $\sigma(T) = \sigma(T_1) \cup \{0\}$. Since $\sigma(T) = \{0\}$ and $\sigma(T_1)$ is not empty, we then have $\sigma(T_1) = \{0\}$, thus T_1 is a quasi-nilpotent $\mathcal{THN}$ operator and hence $T_1 = 0$. Therefore $T = \begin{pmatrix} 0 & T_2 \\ 0 & T_3 \end{pmatrix}$. An easy computation yields that

$$T^{k+1} = \begin{pmatrix} 0 & T_2 \\ 0 & T_3 \end{pmatrix}^{k+1} = \begin{pmatrix} 0 & T_2T_3^k \\ 0 & T_3^{k+1} \end{pmatrix}^{k+1} = 0,$$

so that T is nilpotent.

Finally, suppose that T is quasi-nilpotent and analytically k-quasi $\mathcal{THN}$. Let $h \in \mathcal{H}_{nc}(\sigma(T))$ be such that $h(T)$ is quasi-$\mathcal{THN}$. We claim that $h(T)$ is nilpotent. If $h(T)^k$ has dense range then $h(T)$ is $\mathcal{THN}$ and hence, by Lemma 2.3, $h(T)$ is nilpotent. Suppose that $h(T)^k$ has not dense range. Then with respect the decomposition $X = \overline{h(T)^k(H)} \oplus \overline{h(T)^k(H)}^{\perp}$, the operator $h(T)$ has a triangulation $h(T) = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$, such that $A = h(T)|\overline{h(T)^k(H)}$ is $\mathcal{THN}$ and

$$\sigma(h(T)) = \sigma(A) \cup \{0\}.$$

By the spectral mapping theorem we have

$$\sigma(h(T)) = h(\sigma(T)) = \{h(0)\}.$$

Consequently, $0 \in \{h(0)\}$, i.e. $h(0) = 0$, and therefore $h(T)$ is quasi-nilpotent. Since $h(T)$ is quasi-$\mathcal{THN}$, by the first part of proof it then follows that $h(T)$ is nilpotent. Now, $h(0) = 0$ so we can write

$$h(\lambda) = \mu\lambda^n \prod_{i=1}^{n} (\lambda_i I - T)^{n_i} g(\lambda),$$

where $g(\lambda)$ has no zeros in $\sigma(T)$ and $\lambda_i \neq 0$ are the other zeros of g with multiplicity n_i. Hence

$$h(T) = \mu T^n \prod_{i=1}^{n} (\lambda_i I - T)^{n_i} g(T),$$

where all $\lambda_i I - T$ and $g(T)$ are invertible. Since $h(T)$ is nilpotent then also T is nilpotent. □

Theorem 3.5. *If $T \in L(H)$ is an analytically quasi $\mathcal{THN}$ operator, then T is polaroid.*

Proof. We show that for every isolated point λ of $\sigma(T)$ we have $p(\lambda I - T) = q(\lambda I - T) < \infty$. Let λ be an isolated point of $\sigma(T)$, and denote by P_λ denote the spectral projection associated with $\{\lambda\}$. Then $M := K(\lambda I - T) = \ker P_\lambda$ and $N := H_0(\lambda I - T) = P_\lambda(X)$, see [1, Theorem 3.74]. Therefore, $H = H_0(\lambda I - T) \oplus K(\lambda I - T)$. Furthermore, since $\sigma(T|N) = \{\lambda\}$, while $\sigma(T|M) = \sigma(T)\setminus\{\lambda\}$, so the restriction $\lambda I - T|N$ is quasi-nilpotent and $\lambda I - T|M$ is invertible. Since $\lambda I - T|N$ is analytically quasi $\mathcal{THN}$, then Lemma 3.4 implies that $\lambda I - T|N$ is nilpotent. In other worlds, $\lambda I - T$ is an operator of Kato Type, see [1, Chapter 1] for details and definitions.

Now, both T and the dual T^* have SVEP at λ, since λ is isolated in $\sigma(T) = \sigma(T^*)$, and this implies, by Theorem 3.16 and Theorem 3.17 of [1], that both $p(\lambda I - T)$ and $q(\lambda I - T)$ are finite. Therefore, λ is a pole of the resolvent. □

A bounded operator $T \in L(X)$ is said to be *hereditarily polaroid*, i.e. any restriction to an invariant closed subspace is polaroid. An example of polaroid operator which is not hereditarily polaroid may be found in [21, Example 2.6]. A very important class of hereditarily operators is the class of $\mathcal{H}(p)$ operators, where $T \in L(X)$ is said to belong to the class $\mathcal{H}(p)$ if there exists a natural $p := p(\lambda)$ such that:

$$H_0(\lambda I - T) = \ker(\lambda I - T)^p \quad \text{for all } \lambda \in \mathbb{C}. \tag{7}$$

The class $H(p)$ has been introduced by Oudghiri in [36]. Property $H(p)$ is satisfied by every generalized scalar operator, and in particular for p-hyponormal, log-hyponormal or M-hyponormal operators on Hilbert spaces, see [36]. Therefore, algebraically p-hyponormal or algebraically M-hyponormal operators are $H(p)$. From the implication (4) we see that every operator

T which belongs to the class $H(p)$ has SVEP. Moreover, from (1) it follows that every $H(p)$ operator T is polaroid. The restriction to closed invariant subspaces of any $\mathcal{H}(p)$ operator is also $\mathcal{H}(p)$, see [36], so every $\mathcal{H}(p)$ is hereditarily polaroid.

Note that a paranormal operator need not to be $\mathcal{H}(p)$, and hence a quasi $\mathcal{THN}$ operator in general is not $\mathcal{H}(p)$. However, we have the following result:

Theorem 3.6. *If $T \in L(H)$ is analytically quasi $\mathcal{THN}$, then T is hereditarily polaroid.*

Proof. Let $f \in \mathcal{H}_{nc}(\sigma(T))$ such that $f(T)$ is quasi $\mathcal{THN}$. If M is a closed T-invariant subspace of X, we know that $f(T)|M$ is quasi $\mathcal{THN}$, by Lemma 3.2, and $f(T)|M = f(T|M)$, so $f(T|M)$ is polaroid, by Theorem 3.5, and consequently, $T|M$ is polaroid, by Theorem 2.4. □

Corollary 3.7. *If $T \in L(H)$ is the direct sum $T = S \oplus N$, where S is $\mathcal{THN}$ and N is nilpotent, then T is hereditarily polaroid.*

Proof. If $T = S \oplus N$, where S is $\mathcal{THN}$ and N is nilpotent, then T is quasi $\mathcal{THN}$, since T admits a triangulation $T = \begin{pmatrix} S & 0 \\ 0 & N \end{pmatrix}$, with respect a suitable decomposition. □

4. Weyl Type Theorems for Analytically Quasi $\mathcal{THN}$ Operators

Denote by $\sigma_{\mathrm{a}}(T)$ the classical *approximate point spectrum*, and by $\sigma_{\mathrm{s}}(T)$ the *surjectivity spectrum*. These two spectra are dual one to each other, i.e., $\sigma_{\mathrm{a}}(T^*) = \sigma_{\mathrm{s}}(T)$ and $\sigma_{\mathrm{s}}(T^*) = \sigma_{\mathrm{a}}(T)$.

An operator $T \in L(X)$ is said to be *a-polaroid* if every $\lambda \in \operatorname{iso} \sigma_{\mathrm{a}}(T)$ is a pole of the resolvent of T. Obviously, every a-polaroid operator is polaroid.

Recall that an operator $T \in L(X)$ is said to be *Weyl* ($T \in W(X)$), if T is *Fredholm* (i.e. $\alpha(T) := \dim \ker T$ and $\beta(T) := \operatorname{codim} T(X)$ are both finite) and the *index* $\operatorname{ind} T := \alpha(T) - \beta(T) = 0$. The *Weyl spectrum* of $T \in L(X)$ is defined by

$$\sigma_{\mathrm{w}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W(X)\}.$$

An operator $T \in L(X)$ is said to be *Browder* ($T \in B(X)$), if T is *Fredholm* and $p(T) = q(T) < \infty$. The *Browder spectrum* of $T \in L(X)$ is defined by

$$\sigma_{\mathrm{b}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin B(X)\}.$$

Following Coburn [15], we say that *Weyl's theorem holds* for $T \in L(X)$ (in symbol, (W)) if

$$\sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \pi_{00}(T), \tag{8}$$

where

$$\pi_{00}(T) := \{\lambda \in \operatorname{iso} \sigma(T) : 0 < \alpha(\lambda I - T) < \infty\}.$$

Note that T satisfies (W) if and only if T satisfies *Browder's theorem*, (i.e., $\sigma_{\rm b}(T) = \sigma_{\rm w}(T)$) and $\pi_{00}(T) = p_{00}(T)$, where $p_{00}(T) := \sigma(T)\setminus\sigma_{\rm b}(T)$, see for instance [5, Theorem 3.3].

The concept of Fredholm operators has been generalized in the following way [11]: for every $T \in L(X)$ and a nonnegative integer n let us denote by $T_{[n]}$ the restriction of T to $T^n(X)$ viewed as a map from the space $T^n(X)$ into itself (we set $T_{[0]} = T$). $T \in L(X)$ is said to be *B-Fredholm* if for some integer $n \geq 0$ the range $T^n(X)$ is closed and $T_{[n]}$ is a Fredholm operator. In this case $T_{[m]}$ is a Fredholm operator for all $m \geq n$ [11]. This enables one to define the index of a Fredholm as $\text{ind}\, T = \text{ind}\, T_{[n]}$. A bounded operator $T \in L(X)$ is said to be *B-Weyl* ($T \in BW(X)$) if for some integer $n \geq 0$ $T^n(X)$ is closed and $T_{[n]}$ is Weyl. The *B-Weyl spectrum* $\sigma_{\rm bw}(T)$ is defined

$$\sigma_{\rm bw}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin BW(X)\}.$$

Another version of Weyl's theorem has been introduced by Berkani and Koliha ([12] as follows: $T \in L(X)$ is said to verify *generalized Weyl's theorem*, (in symbol (gW)) if

$$\sigma(T)\setminus\sigma_{\rm bw}(T) = E(T), \tag{9}$$

where

$$E(T) := \{\lambda \in \text{iso}\,\sigma(T) : 0 < \alpha(\lambda I - T)\}.$$

Note that (gW) holds for T if and only if T satisfies *generalized Browder's theorem* (or, equivalently, Browder's theorem, see [9]) and $E(T) = \Pi(T)$, where $\Pi(T)$ is the set of all poles of the resolvent of T, see [7, Theorem 3.13]. Note that generalized Weyl's theorem entails Weyl's theorem.

The following result shows that in presence of SVEP the polaroid condition entails Weyl type theorems.

Theorem 4.1. *Let $T \in L(X)$ be polaroid and suppose that either T or T^* has SVEP. Then both T and T^* satisfy generalized Weyl's theorem.*

Proof. If T is polaroid also T^* is polaroid, and Weyl's theorem and generalized Weyl's theorem for T, or T^*, are equivalent, see [3, Theorem 3.7]. The assertion then follows from [3, Theorem 3.3]. □

Remark 4.2. In the case of a Hilbert space operator $T \in L(H)$ it is more appropriated to consider the Hilbert adjoint T' instead of the dual T^*. Note that T^* satisfies (gW) if and only if T' does. This easily follows from the well known equalities, $\sigma_{\rm w}(T') = \overline{\sigma_{\rm w}(T^*)}$, where $\overline{E}$ is the conjugate of $E \subseteq \mathbb{C}$, $\sigma_{\rm b}(T') = \overline{\sigma_{\rm b}(T^*)}$, $E(T') = \overline{E(T^*)}$, and $\Pi(T') = \overline{\Pi(T^*)}$. Furthermore, T^* satisfies SVEP if and only if T' satisfies SVEP, so, in the statement of Theorem 4.1, T^* may be replaced by the Hilbert adjoint T'.

We have already seen that quasi $\mathcal{THN}$ operators are polaroid, so, in order to apply Theorem 4.1 to these operators, it has a certain interest to know whenever these operators have SVEP.

Theorem 4.3. *Suppose that $T \in L(X)$ admits, with respect to the decomposition $X = M \oplus N$, the representation $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$, where T_3 is nilpotent. Then T has SVEP if and only if T_1 has SVEP.*

Proof. Suppose that T_1 has SVEP. Fix arbitrarily $\lambda_0 \in \mathbb{C}$ and let $f : U \to X$ be an analytic function defined on open disc U centered at λ_0 such that $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in U$. Set $f(\lambda) := f_1(\lambda) \oplus f_2(\lambda)$ on $X = M \oplus N$. Then we can write

$$0 = (\lambda I - T)f(\lambda) = \begin{pmatrix} \lambda I - T_1 & -T_2 \\ 0 & -\lambda I - T_3 \end{pmatrix} \begin{pmatrix} f_1(\lambda) \\ f_2(\lambda) \end{pmatrix}$$

$$= \begin{pmatrix} (\lambda I - T_1)f_1(\lambda) - T_2 f_2(\lambda) \\ (\lambda I - T_3)f_2(\lambda) \end{pmatrix}.$$

Then $(\lambda I - T_3)f_2(\lambda) = 0$ and $(\lambda I - T_1)f_1(\lambda) - T_2 f_2(\lambda) = 0$. Since a nilpotent operator has SVEP then $f_2(\lambda) = 0$, and consequently $(\lambda I - T_1)f_1(\lambda) = 0$. But T_1 has SVEP at λ_0, so $f_1(\lambda) = 0$ and hence $f(\lambda) = 0$ on U. Thus, T has SVEP at λ_0. Since λ_0 is arbitrary then T has SVEP.

Conversely, suppose that T has SVEP. Since T_1 is the restriction of T to M and the SVEP from T is inherited by the restriction to closed invariant subspaces, then T_1 has SVEP. □

Every $\mathcal{HN}$ operator which is hereditarily polaroid has SVEP, see [14, Lemma 3.1], so, by Theorem 3.6, we have:

Corollary 4.4. *Every quasi $\mathcal{THN}$ operator $T \in L(H)$ has SVEP.*

Recall that a bounded operator $K \in L(X)$ is said to be *algebraic* if there exists a non-constant polynomial h such that $h(K) = 0$. Trivially, every nilpotent operator is algebraic and it is well-known that if $K^n(X)$ has finite dimension for some $n \in \mathbb{N}$ then K is algebraic. In [4] it is shown that if T is hereditarily polaroid and has SVEP, and K is an algebraic operator which commutes with T then $T + K$ is polaroid and $T^* + K^*$ is a-polaroid.

The following perturbation result has been proved in [4, Theorem 3.12].

Theorem 4.5. *Suppose that $T \in L(X)$ and $K \in L(X)$ an algebraic operator commuting with $T \in L(X)$. If $T \in L(X)$, or T^*, has SVEP and T, or T^*, is hereditarily polaroid, then $f(T+K)$ and $f(T^*+K^*)$ satisfies (gW) for every $f \in \mathcal{H}_{nc}(\sigma(T+K))$.*

Observe that in the case of Hilbert space operators

$$T^* + K^* \text{ is } a\text{-polaroid} \Leftrightarrow T' + K' \text{ is } a\text{-polaroid},$$

see Theorem [3, Theorem 2.3].

Theorem 4.6. *Let $T \in L(H)$ be an analytically quasi $\mathcal{THN}$ operator on a Hilbert space H, and let $K \in L(H)$ be an algebraic operator commuting with T. Then both $f(T+K)$ and $f(T'+K')$ satisfies (gW) for every $f \in \mathcal{H}_{nc}(\sigma(T+K))$.*

Proof. Suppose that $T \in L(H)$ is analytically quasi $\mathcal{THN}$, and let $f \in \mathcal{H}_{nc}(\sigma(T))$ be such that $f(T)$ is quasi $\mathcal{THN}$, Since T has SVEP then $f(T)$ has SVEP, by [1, Theorem 2.40]. Now, by Theorem 3.6 T is hereditarily polaroid, and hence, by Theorem 4.5, $T+K$ is polaroid and $T'+K'$ is a-polaroid (and hence polaroid). By Theorem 2.4 then $f(T+K)$ is polaroid. Moreover, $T+K$ has SVEP, by [8, Theorem 2.14] and hence $f(T+K)$ has SVEP, again by [1, Theorem 2.40]. The assertions then follows by Theorem 4.1. □

Theorem 4.6 gives to us a general framework and applies to all classes of operators (i)–(viii) considered in this paper (and much more!). Moreover, Theorem 4.6 considerably improves most the existing results in literature concerning Weyl type theorems for these classes of operators. Observe that, always in the situation of Theorem 4.6, the fact that $f(T+K)$ is polaroid entails that all Weyl type theorems [as property (gw) and a-Weyl's theorem] hold for $f(T'+K')$, see [3] for definitions and details, in particular Theorem 3.10.

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