CHESS THINKING AND CONFIGURAL CONCEPTS

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Abstract. The purpose of this work is to connect chess and mathematics education. First, we introduce the idea of configural concepts in chess thinking and then we outline a scheme to show the phases of chess reasoning and how to apply this idea to some conflictual situations. We conclude this work proposing two research problems in introducing chess in mathematical classroom activities.

1 INTRODUCTION

Inspired by Gestalt theory (Kohler, 1947) and the phenomenology of perception (M.M. Ponty, 1945), this article expands the idea of figural concepts (Fischbein, 1993) and introduces the concept of configural concepts to investigate chess thinking.

First we need to define chess objects and chess elements. We define *chess objects* the elements of the artifact "chess" – the pieces, the squares and the chessboard – which therefore possess rules and scopes.

Now, when a chess player moves a piece on the chessboard[1], he or she gives to the move a motivation linked to a judgment on the position. Thus, when a chess player moves, not only he or she "follows the rules", but he or she also deals with chess elements. We define *chess elements* the mental (i.e., personal-ideal) entities that a chess player uses and "builds" when he/she thinks about a move, a position or a variant with a specific aim.

A configural concept is made up of chess objects and their conceptual relationships. Its meaning comes from the hierarchical linkage of the conceptual relationships between the involved chess objects and from its position in the whole theoretical structure of the pieces in the chessboard.

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[1] Or when he/she thinks to move it on the chessboard

The use of a configural concept depends on the goal that an individual is pursuing in a chess game. It involves the identification of general structure of the game at a certain moment and the role of the configural concept therein. In terms of learning, configural concepts become noticed and valued through a process in which the student becomes conscious of the chess objects and their conceptual mutual relations. This process is what Radford calls objectification (Radford, 2010). The objectification of a configural concept requires the student to notice the chess objects' organization, often through a process mediated by artifacts, body, language, and signs (Radford, 2002). A configural concept embodies theoretical relationships that are objectified as the individual gain of a progressive awareness of their meanings, achievable by reasoned handling of elements during individual experience.

We have introduced the idea of configural concepts to analyze chess thinking for several reasons:

–   Their meaning depends on their placement in the structure of the position;
–   Their meaning depends on the configuration of its parts;
–   They are "dynamic" because they depend on visual, spatial and temporal features that are always evolving in a chess game;
–   They may have intersections between them;
–   Chess elements are a particular type of mental entities that cannot be reduced, neither to usual images, or to traditional concepts.

For example, let us consider the following chess elements: the isolated pawn and the material concept. Now we want to show that only the isolated pawn is a configural concept.

A pawn is isolated if there isn't any pawn in the adjacent columns, and this configuration allows to assign it some features. For example, there is no pawn that could defend it or that could attack a "blocking piece".

The material concept consists of the static comparison of value of black and white pieces present on chessboard. We say static because it does not depend neither on the position of pieces on the chessboard, nor on possible future moves.

The difference between these chess elements consists of their nature. In fact the material comparison depends on all the pieces present on chessboard, but not on their positions and on the relationships between them.

The concept of "isolated pawn" is a configural concepts because its definition depends by the configuration of pawns in adjacent columns; also, its meaning (and judgment of it) depends on its position and on the complete structure of the pieces present on chessboard.

An isolated pawn could be an advantage or a disadvantage, it could be not recognized as relevant for chess player analysis or could be fundamental for an entire variation.

However, what we think about "isolated pawn" is a result of cultural process in which previous generations of chess players highlighted some important configurations of the chessboard.[2]

In this article we focus our attention on process of objectification of configural concepts and we will refer to those sensations and feelings that do not allow the chess player to recognize the isolated pawn in some positions, or that in other positions do not allow the chess player to "see" another element out of the isolated pawn.

It is clear how difficult is the task to correctly "find" and "evaluate" chess configural concepts, their relationships and their intersections. For these reasons, we have studied the phenomenology of perception and the phenomenology of judgment in recognizing chess configural concepts.

In this work, we divide the chess player thinking in two phases: the intuitive thinking and the reasoned thinking.

The intuitive thinking consists of the considerations and analysis that the chess player makes intuitively. In this phase the player recognizes some configural concepts and organizes them into a structure that is the mental image of the position (Figure 1).

This structure is focused on matching this mental image (and therefore, the position) with a class of mental images, allowing the player to express a judgment and one or more ways to play that position ("This position usually has to be played in this way").

The "reasoned" thinking consists of all activities that allow the player to gain awareness of the position and find *the* or *a* "correct" move. In fact, by knowing how to play in similar positions, the player begins to analyze accurately some variants and to verify their validity.

We call this phase the "analysis phase".

If the solutions are negative during the analysis phase, or during the matching with the generalized mental image, the chess player must find something else and re–pose the main problem. In this way, he or she becomes aware of other configural concepts and try to re–organize the structure of the position. In according to Gestalt theory, we call this phase the "insight phase" (Figure 2).

In this phase it is built the factual generalizations (Radford, 2003) in chess (that we will cover in the following paragraphs). In fact, in reasoned thinking the player tries to evaluate the considerations deducted from the generalized mental image and the intuitive or insight thinking.

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[2] In line with Radford's conception of mathematical objects "More precisely, mathematical objects are fixed patterns of reflexive activity (in the explicit sense mentioned previously) incrusted in the ever-changing world of social practice mediated by artefacts." Radford (2006, p.9)

Player's expertise, through the success or failing of this structure, gives to some mental images a "width", composing a hierarchical set of generalized mental images.
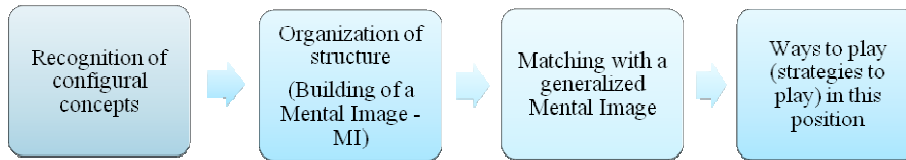

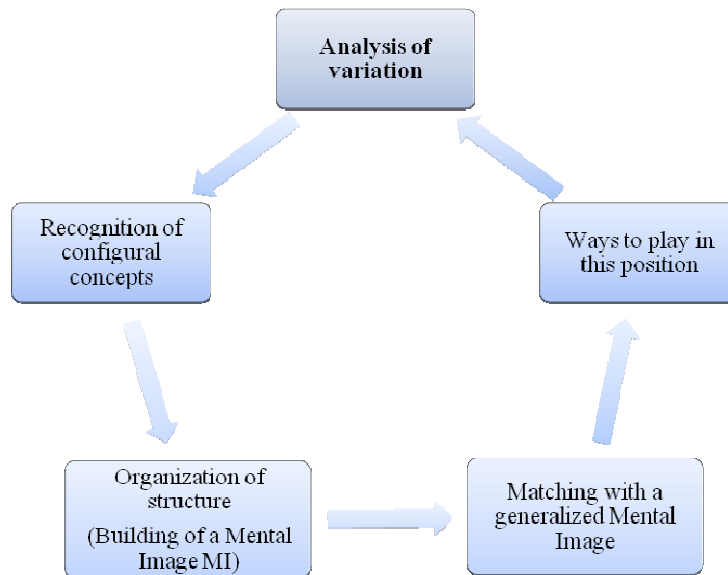*Figure 1.* Specific mental activities in intuitive thinking

Figure 2. Cycle of mental activities in "Insight" thinking

2   MENTAL ACTIVITIES, OBSTACLES AND CONFLICTUAL SITUATIONS IN CHESS

Previously we have showed the chess players' actions when facing a position: perceive, recognize, organize, match, remember, verify, re-recognize, re-organize, generalize, build a mental image, judge and choose. So we want to show that, by studying and playing chess, the player constantly solves and poses problems, and generalizes them.

Now we are going to deal with these chess activities, with emphasis on some conflictual situations.

2.1 PROBLEM POSING

A chess configural concept has conceptual and figural properties, and figural constraints could be seen as the first generators of conflictual situations.

For example, if we ask a chess player – that has just a little experience in the game – to observe the two positions in Figure 3, it is almost certain that he or she will intuitively prefer the position on the right. The term "intuitively" – as we suggested before – means that the chess player, looking at the position, can perceive some properties inherently contained in it.
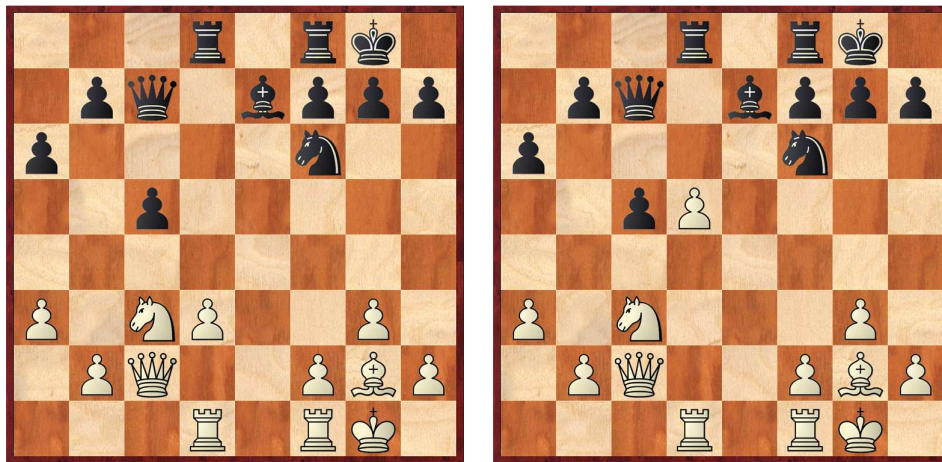

Figure 3. The instinctive property of space advantage could obscure
the changed meaning of d pawn (and consequentially of all pieces)
passing from d3 to d5

The problem that could occur is that some properties could overshadow the others, influencing the recognition of configural concepts. In fact, it is possible that in this way the player loses some important information, thus creating significant difficulties in problem posing.

In this case, the *instinctive property* is the space advantage[3] concealing the fact that, in the position on the right, the d pawn is a past pawn, it cannot be blocked in d4 and the outposts c6 and e6 can be more dangerous than e4 and c4.

The meaning of the configural concepts "isolated pawn" is more interesting, because its judgment depends on its position, by the pieces and by the game timing.

It is clear that the problem posing actions are influenced by the intuitive recognition, and it is possible that the player is satisfied with a *figural sense* so

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[3] The space advantage is a delicate element in chess, because while it is easily perceived and really gives a significant advantage for many reasons, sometimes it obscures other (more important) elements not allowing the correctly posing of the problem.

that he/she does not need to analyze it more accurately and he/she does not enter on cycle of insight thinking.

Therefore, we can say that the intuitive recognition could be an obstacle to the posing of the problem, and by gaining experience, the chess player learns to never use just the intuitive recognition, but to repose the problem, to match it with another generalized mental image or to build a new generalized mental image altogether.

In chess thinking, the operation of re-posing the problem and generalizing new mental images is very important but, if these buildings are too rigid, it could hinder the process, and the chess player cannot return to the main problem.

This phenomenon could cause an "oversight", that is a situation in which the chess player doesn't start from the main position during the analysis, but he/she starts just from the re–posed problem. (Saariluoma, 1992). We are going to show some examples later, when we'll talk about problem solving.

2.2   FACTUAL AND CONTEXTUAL GENERALIZATION

In this section, we are going to analyze the role of generalization in chess thinking and, according to Radford (2003), we are going to use the terms "factual" and "contextual"[4]. In this work these terms will refer to chess context.

In chess reasoning we can identify two kinds of generalizations; in the first one the player builds a mental image trough which he/she can affirm "in this position I can do these ideas/moves/plans", and it is associated to a figural sense and, therefore, to the intuitive thinking (visual memory and imagination).

When a chess player generalizes factually, sometimes he/she knows what he/she has to do in front of that specific position, but maybe he/she doesn't know why the sequence continues that way.

What we want to emphasize is the fact that in this level of generalization he/she deals with *chess objects* and so with what he/she can see on the chessboard. In his/her argumentations about the reached pattern he/she uses mainly gestures (usually pointing) and his/her perception of the *chess elements* depends on his/her position in relation to the chessboard (point of view).

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[4] "A factual generalization is a generalization of actions in the form of an operational scheme (in a neo-Piagetian sense), that remains bound to the concrete level. In addition, this scheme enables the students to tackle virtually any particular case successfully. In contrast to factual generalizations, contextual generalizations generalize not only the numerical actions but also the objects of the actions. They go beyond the realm of specific figures and deal with generic objects (like the figure) that cannot be perceived by our senses. They have to be objectified and produced within the realm of reasoned discourse, that which the Greeks called *logos*", Radford (2003, p.65).

Also, their sentences in this level of generalization are never about the chess elements, much less the correct motives, hence generalizable to other similar positions.

The contextual generalization is not restricted to a particular instance in a game. A contextual generalization offers a kind of sensuous schema that can be repeated in, or applied to, other games. It usually occurs when the player studies chess, that he/she analyzes the game and try to explain the meaning of the chosen sequence (variation) to him/herself, to a friend or to a trainer.

In this level of generalization, a chess player uses less gestures and he/she refers to some chess elements and talks about "these positions". In fact, the position on the chessboard becomes a representation of those chess elements and, in order to improve his/her argumentation the chess player sometimes removes some "foggy pieces" from the chessboard or changes some pieces' placements.

In order to illustrate these concepts, we are going to focus our attention on chess endgames, trying to observe different ways to solve and generalize them and the relative obstacles that may be encountered.

The position in Figure 4 is called the *Réti endgame* and it shows the concept of diagonal in the endgames of King and pawns. In this endgame white seems to have no chance to draw the match because its King is too "far away" to help promoting its pawn and to stop the opponent pawn. On the contrary, by passing the squares g7, f6, e5, the King could draw the game promoting its pawn or taking the opponent pawn.

In this example, the figural constraint fails to accept what the "reason" is saying, and we can see how, in a factual generalization of the diagonal chess element, an existing generalized mental sensuous image, "the square rule", disregards this process.
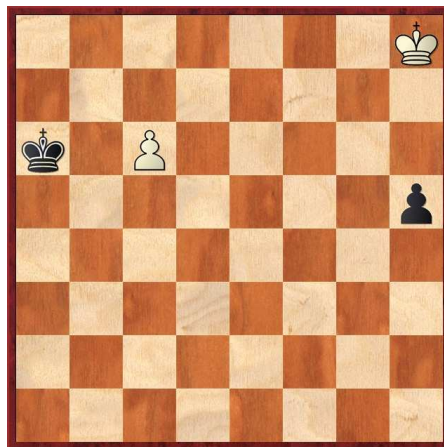

Figure 4. The Réti endgame show the diagonal idea
in Endgame of King and pawns

In fact, the square rule is the configural concept that allows the player to understand how far away is the king compared to the opponent pawn. Its use, supported by a strongly figural sense (the square shape), gives to the square rule a hierarchical position on the mental image in this kind of endgames.

In short, the player doesn't accept that he/she can reach the pawn despite staying out of the square, and so, it's possible that this endgame could be seen as an exception, special occurrence and often this is not easily accepted by the students.

So, although they have studied that this endgame is a draw, the students frequently try to find the win plan for the black.

This phenomenon could be found in every situations in which a new configural concept contrasts with the one previously accepted and this "figural reject" doesn't allow the player to factually generalize it.

In the endgame in Figure 5 we can observe an example of factually but not contextually generalization.
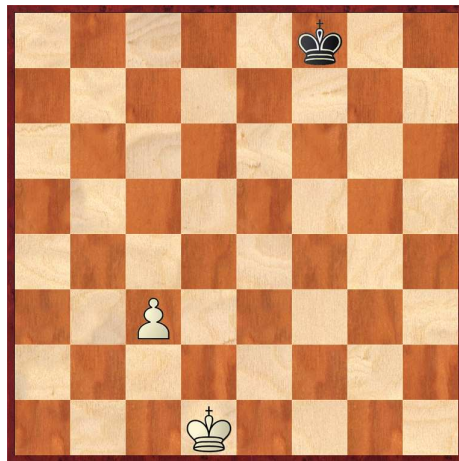
Figure 5. In this endgame white wins the game reaching with the king the critical squares b5, d5 and c5 using the opposition idea.

In this task it's required to find a plan that allow white to win the game[5].

The objective of this endgame is to reach the critical squares with the king (for the pawn in c3 the critical squares are b5, c5 and d5) and to achieve this goal using a chess configural concept, the opposition.

The correct answer is: 1. Kc2 Ke7, 2. Kb3 Kd7, 3. Kb4 Kc6, 4. Kc4 Kd6, 5. Kb5 and white reaches a critical square and so he can win "easily" the game.

In our research, we have submitted this task to players with different expertise, and we have observed the fact that most of them know the way to win,

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[5] In endgame it is more easy to use the term "plan to win the game", because in many cases there is only one plan to achieve a victory. In fact, in other moments of the game it is difficult to talk about a plan to win.

but only few players (the better ones) know why it is this the only way to win (or which is the aim to achieve).

For example an expert chess player meeting this position has answered by moving immediately the king following the path c2-b3-b4-c4 and he has ended the task saying that the game is won because he can give opposition (correct answer) and gain space advantage (in this moment of the game and without pawn structures however, it is totally wrong to talk about space advantage).

What happened? Why doesn't he know the real objective and talks about "space advantage"?

According with what we have already shown, we can say that he/she knows how to continue this position and every position belonging to this type of positions. He/she has generalized factually the idea of opposition, but he/she doesn't have achieved a higher level of generalization (contextual).

We conclude this section saying that although the operation of generalization in chess is fundamental and it is present in every game, in most cases the chess player tries to contextualize it and achieve a higher level of generalization, but it is possible that he/she perceives only a sensuous meaning in the pieces and their relationships and not the proper theoretical chess meaning. The objectification of the relationships at the basis of the configural concepts may not have occurred.

2.3   PROBLEM SOLVING

The chess elements are controlled by conceptual constraints, but the "conceptual way" to solve a problem could be deprived of creativity because in chess analysis the flow of productive ideas could be disturbed or even inhibited by looking constantly for analytical and formal justifications.
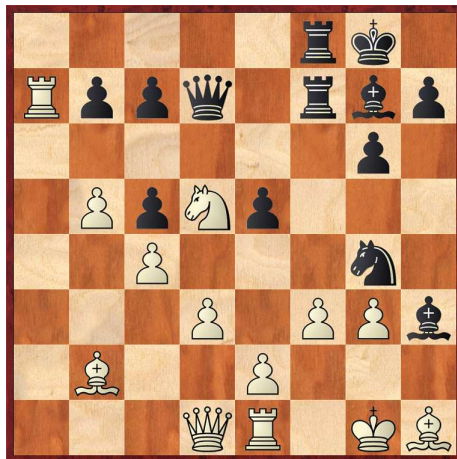

Figure 6. Choose a plan for black to attack the white king

In fact, in the position on the Figure 6 (move to black), the research of a conceptual way to find the correct move/plan could be very hard, and so the player must rely on creative thinking controlled prevalently by figural constraints.

For example, in my courses I have presented this position and asked the students to find a plan to attack the white king. Now we'll show the analysis of reflective activity of one student[6].

Initially, we noticed that the player starts by analyzing some elements and he has tried to find the plan with a logical structure. He saw the "aggressive" bishop in h3, the two rooks in the f file, the "weakness" of square e3 and the possibility to sacrifice the Knight in g4 to open the f file.

These elements do not give any information in order to find the attack plan[7], so he looked for other elements: the "strong" white rook in a7, the "weak" pawn b7 and the doubled pawns in c7 and c5, excluding the possibility to enter in an endgame[8].

During these few minutes, I stressed repeatedly the required task, and tried to put him on the right path.

It is interesting to observe that he used gestures only in some cases: pointing on square b8 to indicate the move Rb8, touching the pawn b7 to indicate its weakness and pointing on pawns b7, c7 and c5 when he considered to sacrifice them to attack the white king.

He introduced the term "candidate moves" that are the first moves of each variant.

After few minutes (5) he found (with our help) the move e4, but with the idea to attack the pawn f3 and thus freeing the f file or to move the pawn in e3. So he didn't realize to do Qd6 to attack the weak pawn in g3.

At this point, we asked him to analyze this particular plan:

1…e4, 2.BxBg7 Qd6, 3. f4 Rxf4, 4. gxRf4 RxRf4 (Figure 7).

The first thing that we noticed is that he used more gestures pointing on squares or simulating moves "in the air".

Finally, according to my predictions, the players built a mental image giving "weight" to some elements. This weight varies in function of the move that is controlled by conceptual constraints. In fact the player continued the variant with the following moves (now he always used the gestures to point the arrive-square of the moves) : 5.Ra8+ KxBg7, 6.Qa1+

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[6] We have asked him to motivate his reasoning and to talk aloud.

[7] Probably because he can't relate them to a configural concept.

[8] It is interesting to observe the nature of the terms aggressive, strong and weak (for the b7 pawn) when related to the experience of the player. Only when he refers to the square e3 with the term weak this is accepted by chess literature (e.g. a square that can't be defended by a pawn).
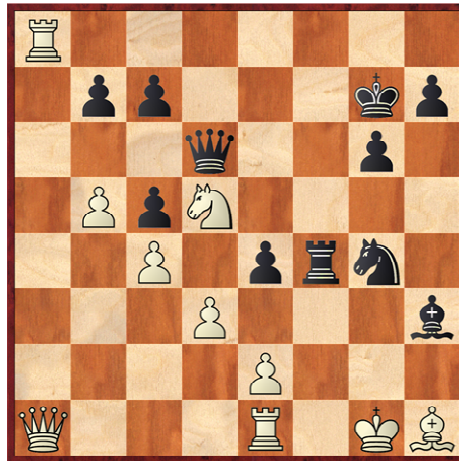
Figure 7 In this position the black pawn in e4 disappears
from the chess player mind and he proposes Rd4

…and Rd4, an illegal move! What's happened?

We think that in his mental image the pawn e4[9] disappeared (Saariluoma, 1991), maybe because its function is only to permit the black Queen to attack the white king.

Therefore we gathered that conceptual constraints could be an obstacle for the chess player in creative thinking and long analysis in which the configural concepts could be disrupted by the function of its elements.

This phenomenon of disappearance could be explained by the fact that the pawn in e4 does not belong to any configural concepts. In our analysis in fact, the principal configural concepts are:

a)   Weakness of square g3, Knight in g4 and Bishop in h3;
b)   A pair of black rooks on f file;
c)   Pawns structure;
d)   Pieces activity.

And when the player moves the pawn in e4, it does not belong to any configural concepts.

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[9] In our experience, this phenomenon has occurred in the following variant too: 5.Ra8+, KxAg7, 6.Qa1+ Kh6, 7.Qc1 without seeing the defence resource 7…e3!

3   STUDYING CHESS IN CLASSROOM

In this section we consider the collaborative phases in chess studying in classroom or between two chess players who have finished a game.

During chess lessons, in the same way as mathematics ones, teaching attempts to provide knowledge and skills, computational techniques and criteria for evaluating positions. However there is a big difference between the two disciplines, because in chess, the concept of "best move" isn't always unique.

In fact, during the game, it is common the effort of finding the best move or best variant, but it often happens that the players restrict their research to find a good move or, at least, not a bad one.

Since this assessment is often biased and not impartial, how do we define the best move during the time of study involving the interaction of many individuals?

We define best move (or best variant) as the one that is socially accepted by the collective, which is understood by all individuals and that could modify the zone of proximal development (Vygotsky, 1986) of each individual of the collective.

During our chess lessons there is no asymmetry between the students and the teacher, and everyone tries to find and share ideas to solve the problems. In fact, in situation of problem posing and problem solving, the teacher submits some tasks for certain reasons, but the students, with their answers and proposals, could lead the argumentation into other directions.

This comes from the fact that every students, in front of a position, can develop a different intuitively reasoning and recognize some configural concepts rather than others. They share ideas and conflictual situations, and thus generate opinions and questions that the teacher designing the tasks has not previously imagined.

However, for a chess player, an essential moment is the comparison of ideas and analysis not only with the rest of the class or with its game opponent, but also with him/herself.

In fact, there are games in which the chess player does not understand the reason of certain moves or why he/she did some mistakes, and other ones that he/she remember perfectly.

This phenomenon depends on the emotional sphere of the game, that does not affect the player during his/her analysis process.

For us, one of the main skills for a chess player is the ability to ask to himself/herself (and to other people) questions of a metacognitive type, to accept criticism and therefore, to extend the reflection activity not only to the game, but also to its analysis, and finally, to share them with the context.

The reason of this statement is based on the fact that in chess activities the chess player analyzes at different levels: recognition of configural concepts

(with the theoretical levels with which they were generalized and so objectified), problem posing, problem solving and calculation of all variants.

It's for this reason that we propose chess like a classroom activity that could improve students' cognitive abilities (visual spatial abilities), students' meta-cognitive abilities and the capacity to interact between them with the purpose of reaching the goal of their activities.

4 CONCLUSIONS

In this article we have introduced a new concept for the analysis of chess elements and chess player thinking: configural concepts.

We have distinguished chess objects and chess elements to investigate the nature of mental activities in chess reasoning. In fact, during a game, we deal with positions in the chessboard (chess objects), and we build some mental images (chess elements) that depend on our perception of elements, by our consciousness of them and by chains of deduction that we follow to achieve some goals (Schoenfeld, 1985; English, 1998).

However, when a chess player thinks, he or she needs to see the chessboard, he/she moves his/her eyes frenetically and sometimes simulates the moves with the hands.

Therefore, when a chess player thinks about a position, what does he/she see in it? Which are the links between chess objects and chess elements?

To answer these questions we introduced the idea of configural concepts, which is built on the objectification theory.

A configural concept is made up of chess objects and their conceptual relationships. Its meaning comes from the hierarchical linkage of the conceptual relationships between the chess objects that it involves and from its position in the whole theoretical structure of the pieces in the chessboard. The organization of the parts of a configural concept depends on the goal of the active reflection of the problem; the objectification of a configural concept requires the student to notice this organization, often through a process mediated by artifacts, body, language, and signs (Radford, 2002).

We can summarize that when a chess player thinks about a chess position, he/she deals with mental entities whose organization is mediated by the chess objects and the body (perception). When he/she thinks about a position, he/she first judges it through a logical interpretation of the signs presented by sensory perceptions (Merleau-Ponty, 1945)[10], so he/she uses those knowledge that are

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[10] Merleau-Ponty shows that a judgment may be defined as a perception of a relationship between any objects of perception that it is neither a purely logical activity, nor a purely sensory activity.

objectified in the patterns recognized by him/her to produce moves, ideas or variations and finally he/she tries to validate them.

When a chess player "moves" the pieces, he/she does not change the position of them, but he/she assigns to the square involved in the moves, the positions of the pieces. Thus, in Figure 4, analyzing the variant 1. Kg7 h4, 2. Kf6 Kb6. 3. Ke5 h3, 4. Kd6 (Figure 8) he/she knows that in d6 there is the white king, in b6 there is the black king and in h3 there is the black pawn. The objects of reflection are the square and not the pieces.

However, that is not all. The chess player in his/her sequence must ensures the moves alternation and who moves in Figure 8, white or black.

To cover the time features of the moves the chess players activate an intense bodily activity that includes: rhythmic breathing, rhythmic closing of the eyelids, rhythmic moving of the fingers and saying the moves.
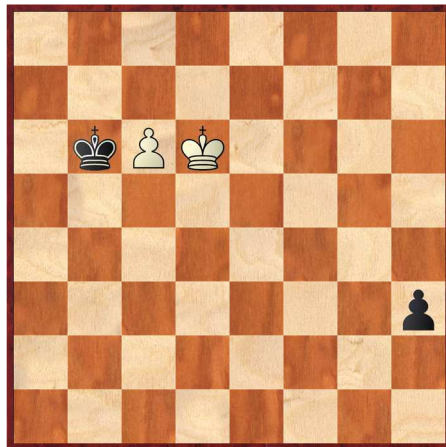

Figure 8. The position reached after 1. Kg7 h4, 2. Kf6 Kb6. 3. Ke5 h3, 4. Kd6

In this work we have sketched a scheme to show these chess activities and how conceptual and figural sense could represent obstacles in their handling.

By promoting chess as a tool to improve intellectual abilities we have noticed three fundamental phases in chess thinking: problem posing, problem solving and generalization (factual and contextual).

In the achieving of a higher level of generalization the chess player can build a networking of knowledge (or make it more stable), that allows him/her to achieve a stable form of awareness about those knowledge.

We concluded this work treating chess studying situations, in which the objectification of chess configural concepts is linked to the individuals' mediated and reflexive efforts aimed at the attainment of the goal of their activity.

In this way, we have defined the best move (or best variant) as the one that is socially accepted by the collective, which is understood by all individuals and

that could modify the zone of proximal development (Vygotsky, 1986) of each individual of the collective.

Now we want to submit two research problems:

1. Could a chess activity built *ad hoc* improve the student ability to handle a figural concept in geometry?
2. Could the ability of the chess player (student) in asking himself/herself questions of metacognitive types be extended also in a geometrical context?

An interesting development of this work is the analysis of the visual spatial abilities (Presmeg, 1986, 1995, 2006) that are involved in chess and geometrical activities, with the ideas of designing *ad hoc* activities and trying to find bilateral relationships and so some synergies between these disciplines.

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