

BUILDING AN ENTANGLEMENT MEASURE ON PHYSICAL GROUND

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We introduce on physical grounds a new measure of multipartite entanglement for pure states. The function we define is discriminant and monotone under LOCC; moreover, it can be expressed in terms of observables of the system.

Keywords: Entanglement measures; multipartite entanglement.

1. Introduction

Since the beginning of quantum mechanics, entanglement revealed to be a key concept for the understanding of the nature. In the last decade, in particular, the fundamental question concerning how to *quantify* entanglement has received a lot of attention [1, 2]. For this purpose, different measures of entanglement have been proposed with respect, in particular, to bipartite systems. On the contrary, entanglement in multipartite systems remains an open and debated problem. In view of the complexity of such systems it cannot indeed be understood simply by extending the tools adopted when bipartite entangled states are studied.

Consider a multipartite system consisting of N not necessarily identical subsystems, each one living in a finite-dimensional Hilbert space. In this letter we introduce a new measure of entanglement for such systems in pure states called *General Entanglement* (GE). This quantity provides a measure of the entanglement present in the system independently of how it is distributed among the finitely many possible subsystems. The GE proves to be easily computable and reduces to Meyer and Wallach's Global Entanglement [3] when qubit systems are considered. A very important aspect is that the quantity we introduce has an immediate interpretation.

Making indeed physical considerations of clear meaning we construct our new measure function by starting directly from the concept of separability. The quantity we define is moreover characterized by many appealing properties, making it very attractive both from a conceptual and an experimental point of view.

2. Construction of the Measure

As is well known, a pure state of a multipartite system is said to be completely separable if it can be written as tensor product of states of each subsystem. At the same time a state is separable with respect to an assigned subsystem if and only if, no physical quantity of the subsystem under scrutiny can be changed by acting on the rest of the system. Let us thus consider a multipartite system in a pure state $|\psi\rangle$, and let us focus on the single j th subsystem. A projective measurement^a [4] on the rest of the system is defined as

$$\mathcal{M}^j = \{\mathcal{P}_i = |\chi_i\rangle\langle\chi_i|\}, \quad (2.1)$$

with $\sum_i \mathcal{P}_i = \mathbb{I}$ and $\mathcal{P}_i = \mathbb{I}^{(j)} \otimes \mathcal{P}_i^{(r)}$. Here the projection operators \mathcal{P}_i act on the Hilbert space of the total system whereas $\mathcal{P}_i^{(r)}$ act on the Hilbert space relative to the system obtained by excluding the j th subsystem from the total one.

As a result of the measurement, the system initially in the state $\rho \equiv |\psi\rangle\langle\psi|$ is projected, with probability p_i , onto the pure state ρ_i corresponding to the obtained outcome: $\rho \xrightarrow{\mathcal{M}^j} \{p_i, \rho_i\}$. Thus, whatever the observable $\mathcal{O} = \mathcal{O}^{(j)} \otimes \mathbb{I}^{(r)}$ is, the quantity

$$R_{\mathcal{O}}^{(j), \mathcal{P}_i}(\rho) = (\text{Tr } \rho \mathcal{O} - \text{Tr } \rho_i \mathcal{O})^2, \quad (2.2)$$

is zero if ρ is separable with respect to the subsystem j . This statement is in addition true whatever the chosen projective measurement \mathcal{M}^j is. If, on the contrary, the quantity $R_{\mathcal{O}}^{(j), \mathcal{P}_i}(\rho)$ vanishes for any $\mathcal{O}^{(j)}$, \mathcal{M}^j and for any outcome i , we may claim with certainty that the state of the j th subsystem is not correlated with the rest of the system in any way. Under this condition ρ must be separable with respect to the subsystem j ; thus, if this property is true for every subsystem, the state must be completely separable. In light of these considerations we introduce the quantities $\mathcal{E}_{\mathcal{M}^j}^{(j)}(\rho) = \sum_i p_i \max_{\mathcal{O} \in \Omega} R_{\mathcal{O}}^{(j), \mathcal{P}_i}(\rho)$, where Ω is the set of all the observables $\mathcal{O}^{(j)} \otimes \mathbb{I}^{(r)}$, $\mathcal{O}^{(j)}$ acting on the state space of the j th subsystem. By definition this quantity gives an estimate of the *average departure* from the separability condition. $\mathcal{E}_{\mathcal{M}^j}^{(j)}(\rho)$ is indeed equal to zero with certainty only if the state is separable with respect to j . Let us however point out that it goes to infinity in the opposite case being $R_{\alpha \mathcal{O}}^{(j), \mathcal{P}_i}(\rho) = \alpha^2 R_{\mathcal{O}}^{(j), \mathcal{P}_i}(\rho)$, $\alpha \in \mathbb{R}$. Confining ourselves however to the set Ω of all normalized observables, with respect to a prefixed norm, the quantity $\mathcal{E}_{\mathcal{M}^j}^{(j)}(\rho)$ becomes finite. Let us moreover observe that, since $R_{\mathcal{O}}^{(j), \mathcal{P}_i}(\rho) = R_{\mathcal{O}}^{(j), \mathcal{P}_i}(\rho)$ with

^aActually, we are dealing with *maximum tests* [4], in the sense that degeneracies are completely removed.

$\tilde{\mathcal{O}} = \mathcal{O} - r\mathbb{I}, r \in \mathbb{R}$, $R_{\mathcal{O}}^{(j), \mathcal{P}_i}(\rho)$ does not depend on $\text{Tr } \mathcal{O}$. Thus, without loss of generality we put $\Omega = \{\mathcal{O} = \mathcal{O}^{(j)} \otimes \mathbb{I}^{(r)}, \mathcal{O}^\dagger = \mathcal{O}, \text{Tr } \mathcal{O} = 0, \|\mathcal{O}\| = 1\}$. The quantity $\mathcal{E}_{\mathcal{M}^j}^{(j)}(\rho)$ evaluated in this set Ω , gives an estimate of the degree of entanglement existing between the j th subsystem and the rest of the system. In other words, the greater $\mathcal{E}_{\mathcal{M}^j}^{(j)}(\rho)$ is, the greater is the influence on the j th subsystem stemming from the measurement on the rest of the system. Thus, when a system consisting of N subsystems is in a pure state $\rho \equiv |\psi\rangle\langle\psi|$, we are naturally led to adopt, as a measurement of entanglement, the quantity

$$E_g(\rho) = \frac{1}{N} \sum_{j=1}^N \max_{\mathcal{M}^j} \mathcal{E}_{\mathcal{M}^j}^{(j)}(\rho). \quad (2.3)$$

In what follows we will refer to this measure as *General Entanglement*. By definition $E_g(\rho)$ may be evaluated for pure states of arbitrarily large multipartite systems, whose N constituents have finite-dimensional Hilbert spaces. It is important to stress that, unlike the Global Entanglement of Meyer and Wallach [3] or its generalizations proposed by Rigolin *et al.* [5], our definition does not require that such Hilbert spaces have the same dimensions.

3. Main Properties and Applications

We now prove that GE is a *good* entanglement measure for pure states [6–9]. For this purpose, we begin by demonstrating the following

Theorem 1. *General Entanglement is discriminant, that is $E_g(\rho) = 0 \Leftrightarrow \rho$ is completely separable.*

Proof. Complete separability of ρ implies $E_g(\rho) = 0$ being $R_{\mathcal{O}}^{(j), \mathcal{P}_i}(\rho) = 0$ whatever the observable \mathcal{O} and the subsystem j are. Conversely, $E_g(\rho) = 0$ implies $\mathcal{E}_{\mathcal{M}^j}^{(j)}(\rho) = 0$ whatever j and \mathcal{M}^j are. Since in addition, in correspondence to an outcome i with $p_i \neq 0$ $\text{Tr } \rho \mathcal{O} = \text{Tr } \rho_i \mathcal{O}$ for any \mathcal{O} , then $\rho^{(j)} \equiv \text{Tr}_r \rho$ and $\rho_i^{(j)} \equiv \text{Tr}_r \rho_i$ coincide. But, ρ_i is pure and separable with respect to j ; thus $\rho_i^{(j)}$, and therefore $\rho^{(j)}$ is pure, too. Then ρ is separable with respect to j . Since such a property holds for any subsystem j , the state $\rho \equiv |\psi\rangle\langle\psi|$ is completely separable. \square

Another remarkable feature of our GE is its invariance under local unitary operations. It is indeed possible to prove the following

Theorem 2. $E_g(\rho) = E_g(U\rho U^\dagger)$, with $U^\dagger = U^{-1}$ and $U = U^{(1)} \otimes U^{(2)} \otimes \dots \otimes U^{(N)}$.

Proof. Putting $\tilde{\rho} = U\rho U^\dagger$ for every admissible measurement $\mathcal{M}^j = \{\mathcal{P}_i\}$, consider the *transformed* measurement $\widetilde{\mathcal{M}^j} \equiv \{\widetilde{\mathcal{P}_i} = U\mathcal{P}_i U^\dagger\}$ satisfying (2.1). It is immediate

to convince oneself that since

$$\tilde{p}_i = \text{Tr } \tilde{\rho} \tilde{\mathcal{P}}_i = p_i, \quad \tilde{\rho}_i = \frac{1}{\tilde{p}_i} \tilde{\mathcal{P}}_i \tilde{\rho} \tilde{\mathcal{P}}_i = U \rho_i U^\dagger, \quad (3.1)$$

then

$$R_{U\mathcal{O}U^\dagger}^{(j), \tilde{\mathcal{P}}_i}(\tilde{\rho}) = R_{\mathcal{O}}^{(j), \mathcal{P}_i}(\rho), \quad (3.2)$$

so that

$$\max_{\mathcal{O} \in \Omega} R_{\mathcal{O}}^{(j), \tilde{\mathcal{P}}_i}(\tilde{\rho}) = \max_{\mathcal{O} \in \Omega} R_{\mathcal{O}}^{(j), \mathcal{P}_i}(\rho). \quad (3.3)$$

Thus $\mathcal{E}_{\mathcal{M}^j}^{(j)}(\tilde{\rho}) = \mathcal{E}_{\mathcal{M}^j}^{(j)}(\rho)$ and, therefore, $E_g(\rho) = E_g(U\rho U^\dagger)$. \square

To obtain an explicit expression for the GE we normalize the observables of the set Ω with respect to the *trace scalar product*:

$$(A, B) = \text{Tr}(AB) \Rightarrow \|A\|^2 = \text{Tr } A^2. \quad (3.4)$$

This choice of the set Ω allows us to prove that GE is monotonic under LOCC. To demonstrate this remarkable property it is convenient to prove in advance the following general lemma.

Lemma 3. *Let $\{A_k\}$ be an orthonormal basis (with respect to (3.4)) in the vector space of traceless Hermitian $D \times D$ matrices. For every $D \times D$ Hermitian matrix σ with $\text{Tr } \sigma = 1$ we have*

$$\sum_k (\text{Tr } \sigma A_k)^2 \equiv \sum_k \langle A_k \rangle_\sigma^2 = \text{Tr } \sigma^2 - \frac{1}{D}. \quad (3.5)$$

Proof. Expanding σ in the basis $\{\mathbb{I}, A_k\}$: $\sigma = \frac{1}{D}\mathbb{I} + \sum_k r_k A_k$, we obtain

$$\sum_k (\text{Tr } \sigma A_k)^2 = \sum_k r_k^2, \quad \text{Tr } \sigma^2 = \frac{1}{D} + \sum_k r_k^2. \quad (3.6) \quad \square$$

Let us now focus on a single subsystem j and indicate by $D^{(j)}$ the dimension of its Hilbert space. Consider an orthonormal set of $(D^{(j)})^2 - 1$ traceless observables $\{A_k\}$ relative to the j th subsystem. Whatever the observable $\mathcal{O} \equiv \mathcal{O}^{(j)} \otimes \mathbb{I}^r \in \Omega$ is, we can write $\mathcal{O}^{(j)} = \sum_k o_k A_k$, with $\sum_k o_k^2 = 1$. For simplicity, in what follows we write $\mathcal{O}^{(j)} = \hat{\mathbf{o}} \cdot \mathbf{A}$ with $\hat{\mathbf{o}} = (o_1, o_2, \dots)$ and $\mathbf{A} = (A_1, A_2, \dots)$, and denote by $\langle \mathbf{A} \rangle_\rho$ the vector of components $\langle A_k \rangle_\rho \equiv \text{Tr}(\rho A_k \otimes \mathbb{I}^r)$ in $\mathbb{R}^{(D^{(j)})^2 - 1}$. Exploiting this notation, Eq. (2.2) may be cast in the form

$$R_{\mathcal{O}}^{(j), \mathcal{P}_i}(\rho) = [\hat{\mathbf{o}} \cdot (\langle \mathbf{A} \rangle_{\rho_i} - \langle \mathbf{A} \rangle_\rho)]^2. \quad (3.7)$$

Observing that the set Ω can be obtained simply by varying the unit vector $\hat{\mathbf{o}}$ we may write

$$\mathcal{E}_{\mathcal{M}^j}^{(j)}(\rho) = \sum_i p_i (\langle \mathbf{A} \rangle_{\rho_i} - \langle \mathbf{A} \rangle_{\rho}) \cdot (\langle \mathbf{A} \rangle_{\rho_i} - \langle \mathbf{A} \rangle_{\rho}). \quad (3.8)$$

Taking into consideration the fact that $\rho_i^{(j)}$ is pure and using Lemma 3, we have

$$\mathcal{E}_{\mathcal{M}^j}^{(j)}(\rho) = 1 - \frac{1}{D^{(j)}} + \langle \mathbf{A} \rangle_{\rho} \cdot \langle \mathbf{A} \rangle_{\rho} - 2 \langle \mathbf{A} \rangle_{\rho} \cdot \sum_i p_i \langle \mathbf{A} \rangle_{\rho_i}. \quad (3.9)$$

Starting from Eq. (2.1) it is easy to prove that $\sum_i p_i \langle \mathbf{A} \rangle_{\rho_i} = \langle \mathbf{A} \rangle_{\rho}$, and thus

$$\mathcal{E}_{\mathcal{M}^j}^{(j)}(\rho) = 1 - \frac{1}{D^{(j)}} - \langle \mathbf{A} \rangle_{\rho} \cdot \langle \mathbf{A} \rangle_{\rho}. \quad (3.10)$$

To sum up, with the choice (3.4), the quantities $\mathcal{E}_{\mathcal{M}^j}^{(j)}(\rho)$ do not depend on the measures \mathcal{M}^j and the maximization in (2.3) becomes trivial. We wish moreover to point out that, in view of Lemma 3, $0 \leq \sum_k \langle A_k \rangle_{\rho}^2 \leq 1 - \frac{1}{D^{(j)}}$. This inequality suggests rescaling $\mathcal{E}_{\mathcal{M}^j}^{(j)}(\rho)$ as follows: $\mathcal{E}_{\mathcal{M}^j}^{(j)}(\rho) \rightarrow (1 - \frac{1}{D^{(j)}})^{-1} \mathcal{E}_{\mathcal{M}^j}^{(j)}(\rho)$ obtaining $\mathcal{E}_{\mathcal{M}^j}^{(j)}(\rho) = 1 - \frac{D^{(j)}}{D^{(j)}-1} \langle \mathbf{A} \rangle_{\rho} \cdot \langle \mathbf{A} \rangle_{\rho}$. Thus, we may define the normalized General Entanglement as

$$\begin{aligned} E_g(\rho) &= 1 - \frac{1}{N} \sum_j \frac{D^{(j)}}{D^{(j)}-1} \langle \mathbf{A}^{(j)} \rangle_{\rho} \cdot \langle \mathbf{A}^{(j)} \rangle_{\rho} \\ &= 1 + \frac{1}{N} \sum_j \frac{1}{D^{(j)}-1} - \frac{1}{N} \sum_j \frac{D^{(j)}}{D^{(j)}-1} \text{Tr}(\rho^{(j)})^2. \end{aligned} \quad (3.11)$$

Thus, if the choice (3.4) is done, GE turns out to be the *mean linear entropy* of the reduced states and, when we deal with equal dimensional subsystems, it reduces to the generalized global entanglement $E_g^{(1)}$ [5]. Moreover, at least with the choice (3.4), GE is monotonic because of the monotonicity of the linear entropy.

The ability of writing the General Entanglement $E_g(\rho)$ as expressed by Eq. (3.11) is remarkable not only because it shows its monotonicity but also in view of the following considerations. First of all, thanks to the physical reasoning giving rise to its definition, GE enriches the physical meaning of the linear entropy, showing that the latter is obtained by a particular choice on the class of observables used to quantify the influence on a particular subsystem of acting on the rest of the system. Moreover, exploiting the first equality of Eq. (3.11), we may express $E_g(\rho)$ in terms of mean values of local observables. This circumstance is of particular relevance from an experimental point of view, offering the possibility of testing directly in laboratory the quantity $E_g(\rho)$ here defined. In what follows we will apply the new concept of GE in order to evaluate the degree of entanglement of assigned multipartite systems. Let us begin by considering a system of N spins $\frac{1}{2}$. In correspondence to each subsystem the operators S_x , S_y , and S_z are traceless

and orthogonal to each other so that, once normalized, they provide the following useful set of operators:

$$A_1 \equiv \sqrt{2}S_z, \quad A_2 \equiv \sqrt{2}S_x, \quad A_3 \equiv \sqrt{2}S_y, \quad (3.12)$$

with $\hbar = 1$. Exploiting (3.11), it is immediate to conclude that, in the case under scrutiny, the degree of multipartite entanglement measured by GE is simply given by

$$E_g(\rho) = 1 - \frac{4}{N} \sum_j \langle \mathbf{S}^{(j)} \rangle^2. \quad (3.13)$$

This expression coincides with the Meyer–Wallach Global Entanglement [3, 10, 2] when N qubits are considered. It is of relevance to observe that, if $N = 2$, GE can be directly related to the concurrence function C [11], being in particular $E_g(\rho) = C^2(|\psi\rangle)$.

Suppose now that the system of interest consists of N spins 1. In this case, in order to construct the appropriate set of $\{A_k^{(j)}\}$ operators, let us start by considering the following linearly independent observables:

$$\begin{aligned} S_z &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad S_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \\ S_x^2 &= S_x S_x, \quad S_{xy} = S_x S_y + S_y S_x, \quad S_{xz} = S_x S_z + S_z S_x, \\ S_{yz} &= S_y S_z + S_z S_y, \quad S_y^2 = S_y S_y. \end{aligned} \quad (3.14)$$

Orthonormalizing this set by the Gram–Schmidt method [12], we obtain the following orthonormal traceless basis:

$$\begin{aligned} A_1 &= \frac{1}{\sqrt{2}}S_z, \quad A_2 = \frac{1}{\sqrt{2}}S_x, \quad A_3 = \frac{1}{\sqrt{2}}S_y, \\ A_4 &= \frac{1}{\sqrt{2}}S_{xy}, \quad A_5 = \frac{1}{\sqrt{2}}S_{xz}, \quad A_6 = \frac{1}{\sqrt{2}}S_{yz}, \\ A_7 &= \sqrt{\frac{3}{2}}S_x^2 - \sqrt{\frac{2}{3}}\mathbb{I}, \quad A_8 = \sqrt{2}S_y^2 + \frac{1}{\sqrt{2}}S_x^2 - \sqrt{2}\mathbb{I}. \end{aligned} \quad (3.15)$$

Thus (3.11) becomes

$$E_g(\rho) = 1 - \frac{3}{2N} \sum_j \langle \mathbf{A}^{(j)} \rangle_\rho^2. \quad (3.16)$$

As we have previously stressed, GE does not distinguish between “truly” N -partite entanglement [1, Sec. VII.A] and partially separable entanglement. In other words, the quantity $E_g(\rho)$ is different from zero also in correspondence to a state separable with respect to some bipartition. It indeed indicates how much *Global* Entanglement is present in the system. Anyway, if we are interested only in N -partite entanglement, a variant of GE can be introduced. Let us denote by ρ_P the

state of the system viewed as a bipartite system induced by the bipartition P . Then, the measure

$$E_g^N(\rho) = \min\{E_g(\rho), E_g(\rho_P), \forall \text{ bipartition } P\} \quad (3.17)$$

is nonvanishing if and only if, the state is N -partite truly entangled, and is less or equal than $E_g(\rho)$. If the system is not too large, the quantity $E_g^N(\rho)$ defined by (3.17) is simple to compute. In addition it is monotonic in view of the fact that a LOCC with respect to all subsystems is a LOCC with respect to a bipartition.

4. Conclusions

In this paper we propose a new way to quantify entanglement in multipartite pure systems. In contrast to the Global Entanglement [3] and its generalizations [5], our measure does not require that all subsystems have the same dimension. Thus, GE can be applied to more general physical situations. In addition, the measure we propose turns out to be a good one being discriminant, invariant under local unitary operations and monotonic under LOCC, at least when the normalization (3.4) is adopted. Moreover, the possibility of expressing GE in terms of mean values of suitable local quantities turns out to be very attractive from the experimental point of view. By definition, the quantity we introduce does not make it possible to distinguish the many ways in which a multipartite system can be entangled. Our aim is indeed to quantify the entanglement present in a multipartite system independently of its distribution. On the other hand, the generalization of GE proposed in Eq. (3.17) allows us to distinguish genuine multipartite entangled states. The fundamental aspect of our GE is the fact that it is constructed by following a quite simple reasoning based on physical grounds. This directly provides the possibility to interpret our function in a clear way. The starting point is that, the more our physical predictions on a subsystem can be changed by acting on the rest of the system, the more the subsystem is entangled with the rest. Since it reduces to the mean linear entropy with the particular choice (3.4), GE moreover enriches the physical meaning of this entropy used as a measure of entanglement. Clearly, investigating choices other than (3.4) would be of high interest.

An important result, from the conceptual point of view, is that GE improves, with respect to the notion of monotonicity under LOCC, our capability to *physically* say that a state is more or less entangled than another. In fact, as far as monotonicity is concerned, if a state $|\psi\rangle$ can be transformed into $|\phi\rangle$ by LOCC, we physically say that $|\psi\rangle$ is more (or equal) entangled than $|\phi\rangle$. But the order imposed by LOCC is only partial; thus, let us consider two states that cannot be converted into each other. We could not physically say that a state is more entangled than the other, if we limit the concept of entanglement to a quantity that does not increase under LOCC. The physical meaning of GE provides a way to compare, on physical basis, the entanglement of such states. Thanks to the fact that GE is monotonic,

this physical meaning is not in contrast with the commonly accepted fact that entanglement is a quantity that does not increase under LOCC.

References

- [1] R. Horodecki *et al.*, *Rev. Mod. Phys.* **81** (2009) 865–942.
- [2] L. Amico, R. Fazio, A. Osterloh and V. Vedral, *Rev. Mod. Phys.* **80** (2008) 517.
- [3] D. A. Meyer and N. R. Wallach, *J. Math. Phys.* **43** (2002) 4273.
- [4] M. Le Bellac, *Quantum Physics* (Cambridge, 2006).
- [5] G. Rigolin, T. R. de Oliveira and M. C. de Oliveira, *Phys. Rev. A* **74** (2006) 022314.
- [6] V. Vedral, M. B. Plenio, M. A. Rippin and P. L. Knight, *Phys. Rev. Lett.* **78** (1997) 2275.
- [7] G. Vidal, *J. Mod. Opt.* **47** (2000) 355.
- [8] M. Horodecki, P. Horodecki and R. Horodecki, *Phys. Rev. Lett.* **84** (2000) 2014.
- [9] M. J. Donald, M. Horodecki and O. Rudolph, *J. Math. Phys.* **43** (2002) 4252.
- [10] G. K. Brennen, *Quant. Inf. Comp.* **3** (2003) 619.
- [11] W. K. Wootters, *Phys. Rev. Lett.* **80** (1998) 2245.
- [12] G. Arfken and H. Weber, *Mathematical Methods for Physicists*, 6th edn. (Academic Press, 2005).