

# Time characteristics of Lévy flights in a steep potential well

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**Abstract.** Using the method previously developed for ordinary Brownian diffusion, we derive a new formula to calculate the correlation time of stationary Lévy flights in a steep potential well. For the symmetric quartic potential, we obtain the exact expression of the correlation time of steady-state Lévy flights with index  $\alpha = 1$ . The correlation time of stationary Lévy flights decreases with an increasing noise intensity and steepness of potential well.

## 1 Introduction

The analytical investigation of the time characteristics of Lévy flights in systems characterized by different potential profiles remains an open problem because it encounters some difficulties [1–5]. For instance, the theory of mean first-passage times assumes the presence of some boundary conditions which are not so evident for Markovian process having long jumps. As a consequence, a lot of results in this area were obtained by numerical simulations (see, for instance, the reviews [6–8] and references therein). At the same time, starting with the quartic potential we observe a confinement of Lévy flights, i.e., the variance of particle displacement is finite [1]. As a result, one can find, in principle, the correlation function and the power spectral density of Lévy motion in a steady state. But, unlike the stationary probability distributions being bimodal [1, 4], exact analytical results can be only obtained for some time characteristics of Lévy flights in the steep potential well profiles such as  $U(x) = \gamma x^{2m}/(2m)$ .

In this paper, we apply the method previously developed in Ref. [9], to calculate the correlation time of steady-state Lévy flights in symmetric quartic potential profiles ( $m = 2$ ).

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## 2 General method of solution

We start with the backward Kolmogorov equation for the probability density of transitions  $P(x, t|x_0, t_0)$  of arbitrary Markovian random process  $x(t)$

$$-\frac{\partial P}{\partial t_0} = \hat{L}^+(x_0, t_0) P. \quad (1)$$

For the existence of steady state, the adjoint kinetic operator  $\hat{L}^+$  should not depend on time  $t_0$ . As a result, the formal solution of Eq. (1) with the initial condition  $P(x, 0|x_0, 0) = \delta(x - x_0)$  depends only on the difference  $\tau = t - t_0 \geq 0$  and reads

$$P(x, \tau|x_0, 0) = e^{\hat{L}^+(x_0)\tau} \delta(x - x_0). \quad (2)$$

According to the definition, the correlation function  $K[\tau] = \langle x(t)x(t+\tau) \rangle$  of the stationary random process  $x(t)$  in the steady state can be calculated as

$$K[\tau] = \int_{-\infty}^{\infty} x_0 P_{\text{st}}(x_0) dx_0 \int_{-\infty}^{\infty} x P(x, \tau|x_0, 0) dx, \quad (3)$$

where  $P_{\text{st}}(x)$  is the steady-state distribution of  $x(t)$ . Substitution of Eq. (2) in Eq. (3) yields

$$K[\tau] = \left\langle x e^{\hat{L}^+(x)\tau} x \right\rangle_{\text{st}}, \quad (4)$$

where the notation  $\langle \dots \rangle_{\text{st}}$  means the average over steady-state distribution. The result (4) has been obtained in the paper [10] in a more complex way.

Since the correlation function  $K[\tau]$  of a stationary random process  $x(t)$  varies from  $K[0] = \langle x^2 \rangle$  to  $K[\infty] = \langle x \rangle^2$ , that is by the value of the variance  $\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2$ , we can define the correlation time as the width of a rectangle of the same area [9]

$$\tau_c = \frac{1}{\sigma^2} \int_0^{\infty} (K[\tau] - \langle x \rangle^2) d\tau. \quad (5)$$

It should be emphasized that the definition (5) of the correlation time is valid if the integral converges. According to the forward Kolmogorov equation for the steady state, we have  $\hat{L}(x)P_{\text{st}}(x) = 0$ , and then for an arbitrary function  $f(x)$  we find

$$\left\langle \hat{L}^+(x) f(x) \right\rangle_{\text{st}} = \int_{-\infty}^{\infty} P_{\text{st}}(x) \hat{L}^+(x) f(x) dx = \int_{-\infty}^{\infty} f(x) \hat{L}(x) P_{\text{st}}(x) dx = 0. \quad (6)$$

Substituting Eq. (4) in Eq. (5), representing  $\langle x \rangle_{\text{st}}^2$ , in accordance with Eq. (6), in the form

$$\langle x \rangle_{\text{st}}^2 = \langle x \rangle_{\text{st}} \left\langle e^{\hat{L}^+(x)\tau} x \right\rangle_{\text{st}}, \quad (7)$$

and evaluating the integral, we finally arrive at

$$\tau_c = -\frac{1}{\sigma^2} \left\langle x, \frac{1}{\hat{L}^+(x)} x \right\rangle_{\text{st}}, \quad (8)$$

where  $\langle f(x), g(x) \rangle \equiv \langle f(x)g(x) \rangle - \langle f(x) \rangle \langle g(x) \rangle$ . As seen from Eq. (8), we have to find a particular solution of the following integro-differential equation

$$\hat{L}^+(x) \psi(x) = -x, \quad (9)$$

and then calculate the correlation time as [11]

$$\tau_c = \frac{1}{\sigma^2} \langle x, \psi(x) \rangle_{\text{st}}. \quad (10)$$

Let us write the main result (8) in a more appropriate form, for further considerations. Using the relationship between the two kinetic operators  $\hat{L}(x)$  and  $\hat{L}^+(x)$  we obtain

$$\tau_c = \frac{1}{\sigma^2} \int_{-\infty}^{\infty} x \frac{1}{\hat{L}(x)} (\langle x \rangle_{\text{st}} - x) P_{\text{st}}(x) dx. \quad (11)$$

The operator formula (11) means that we can find the particular solution of the following integro-differential equation

$$\hat{L}(x) \phi(x) = (\langle x \rangle_{\text{st}} - x) P_{\text{st}}(x), \quad (12)$$

and then calculate the correlation time as

$$\tau_c = \frac{1}{\sigma^2} \int_{-\infty}^{\infty} x \phi(x) dx. \quad (13)$$

It should be noted that Eqs. (12) and (13) have been used in the paper [9] (see also [12]) for analytical calculations of the correlation characteristics of the ordinary Brownian motion in different potential profiles. Using the first remarkable limit we can write Eq. (13) in the form

$$\tau_c = \frac{1}{\sigma^2} \int_{-\infty}^{\infty} \phi(x) \lim_{k \rightarrow 0} \frac{\sin kx}{k} dx = \frac{1}{\sigma^2} \lim_{k \rightarrow 0} \frac{\varphi(k)}{k}, \quad (14)$$

where  $\varphi(k) = \text{Im}\{\tilde{\phi}(k)\}$  and  $\tilde{\phi}(k)$  is the Fourier transform of  $\phi(x)$ . Since, by the definition,  $\varphi(0)$  is equal to zero, Eq. (14) yields

$$\tau_c = \frac{1}{\sigma^2} \varphi'(0). \quad (15)$$

### 3 Correlation time of steady-state Lévy flights

Now we will analyze the correlation time of Lévy flights in a steep potential well  $U(x)$ . This type of anomalous diffusion can be described by the following fractional Fokker-Planck equation [6, 8]

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x} [U'(x) P] + D \frac{\partial^\alpha P}{\partial |x|^\alpha}, \quad (16)$$

where  $P(x, t)$  is the probability density function of the particle displacement and the coefficient  $D$  is the intensity of  $\alpha$ -stable Lévy noise source in the corresponding Langevin equation. Substitution of the kinetic operator from Eq. (16) into Eq. (12) yields

$$D \frac{d^\alpha \phi}{d|x|^\alpha} + \frac{d}{dx} [U'(x) \phi] = (\langle x \rangle_{\text{st}} - x) P_{\text{st}}(x). \quad (17)$$

After Fourier transform of Eq. (17) we arrive at

$$ikU' \left( -i \frac{d}{dk} \right) \tilde{\phi} + D |k|^\alpha \tilde{\phi} = - \left( \langle x \rangle_{\text{st}} + i \frac{d}{dk} \right) \vartheta_{\text{st}}(k), \quad (18)$$

where  $\vartheta_{\text{st}}(k)$  is the characteristic function of  $x(t)$  in the steady state. In the case of symmetric steep potential well ( $U(x) = \gamma x^{2m}/(2m)$ ), we have a confinement of Lévy flights at  $m \geq 2$  [1]. Taking into account that  $\langle x \rangle_{\text{st}} = 0$  in Eq. (18), we arrive at

$$\frac{d^{2m-1}\tilde{\phi}}{dk^{2m-1}} + (-1)^{m+1} \frac{D}{\gamma} |k|^{\alpha-1} \text{sgn } k \cdot \tilde{\phi} = i(-1)^m \frac{1}{k\gamma} \frac{d\vartheta_{\text{st}}}{dk}. \quad (19)$$

According to Eq. (15), we need only the imaginary part of the solution for  $k \geq 0$ . As a result, instead of solving Eq.(19), we should find a solution of the following equation

$$\frac{d^{2m-1}\varphi}{dk^{2m-1}} + (-1)^{m+1} \frac{D}{\gamma} k^{\alpha-1} \varphi = (-1)^m \frac{1}{k\gamma} \frac{d\vartheta_{\text{st}}}{dk}. \quad (20)$$

The exact solution of Eq. (20) can be obtained only for the Lévy index  $\alpha = 1$ . Moreover, we restrict our further considerations to the case of the quartic potential ( $m = 2$ ). Substituting these parameters in Eq. (20), we arrive at ( $k \geq 0$ )

$$\frac{d^3\varphi}{dk^3} - \beta^3 \varphi = \frac{1}{k\gamma} \frac{d\vartheta_{\text{st}}}{dk}, \quad (21)$$

where  $\beta = \sqrt[3]{D/\gamma}$ . The steady-state characteristic function  $\vartheta_{\text{st}}(k)$  has been found in the papers [4, 13] and has the form

$$\vartheta_{\text{st}}(k) = \frac{2}{\sqrt{3}} e^{-\beta|k|/2} \cos\left(\frac{\beta|k|\sqrt{3}}{2} - \frac{\pi}{6}\right). \quad (22)$$

The characteristic function (22) corresponds to the following bimodal steady-state probability density function

$$P_{\text{st}}(x) = \frac{\beta^3}{\pi(x^4 - x^2\beta^2 + \beta^4)}, \quad (23)$$

with the variance  $\sigma^2 = \beta^2$ . Substituting Eq. (22) in Eq. (21), we finally arrive at

$$\frac{d^3\varphi}{dk^3} - \beta^3 \varphi = -\frac{2\beta}{k\gamma\sqrt{3}} e^{-\beta k/2} \sin \frac{\beta k\sqrt{3}}{2}. \quad (24)$$

The general solution of Eq. (24), which tends to zero at  $k \rightarrow \infty$ , reads

$$\begin{aligned} \varphi(k) = & \frac{1}{\gamma\sqrt{3}} e^{\beta k} \int_k^\infty e^{-3\beta q/2} \left( \sin \frac{\beta q\sqrt{3}}{2} - \frac{1}{\sqrt{3}} \cos \frac{\beta q\sqrt{3}}{2} \right) \ln q \, dq \\ & + e^{-\beta k/2} \left( c_1 \sin \frac{\beta k\sqrt{3}}{2} + c_2 \cos \frac{\beta k\sqrt{3}}{2} \right) \\ & - \frac{1}{\gamma\sqrt{3}} e^{-\beta k/2} \int_0^k \left[ \sin \beta\sqrt{3} \left( \frac{k}{2} - q \right) + \frac{1}{\sqrt{3}} \cos \beta\sqrt{3} \left( \frac{k}{2} - q \right) \right] \ln q \, dq. \end{aligned} \quad (25)$$

Calculating the unknown constants  $c_1$  and  $c_2$  in Eq. (25) from the conditions  $\varphi(0) = 0$  and  $\varphi''(0) = 0$ , arising from the odd function  $\varphi(k)$ , and substituting  $\varphi'(0)$  and  $\sigma^2$  in Eq. (15) we finally arrive at

$$\tau_c = \frac{\pi}{3\sqrt{3}\gamma\beta^2} = \frac{\pi}{3\sqrt{3}\sqrt[3]{\gamma D^2}}. \quad (26)$$

The Eq. (26) is the first exact analytical result obtained in the area of Lévy flights time characteristics and confirms a power dependence of the correlation time on the noise intensity  $D$ . A power law behavior, with different scaling exponent, was previously found both analytically and numerically for mean first-passage times for free and confined Lévy flights [3, 7, 8, 14, 15] and recently in the transient dynamics of a Josephson junction in the presence of Lévy noise [16]. As might be expected, the correlation time (26) decreases with an increase of noise intensity  $D$  and the steepness  $\gamma$  of potential well.

At the same time, it should be emphasized that the correlation function  $K[\tau]$  has a non-analytical dependence of  $\tau$ , because, it cannot be expanded in power series in  $\tau$ . In fact, according to Eq. (4), the first order term in  $\tau$  of the Taylor series expansion has an infinite coefficient

$$K'[0^+] = \left\langle x \hat{L}^+(x) x \right\rangle_{\text{st}} = -\langle x U'(x) \rangle_{\text{st}} = -\gamma \langle x^{2m} \rangle_{\text{st}} = -\infty, \quad (27)$$

in contrast to the ordinary Brownian motion ( $\alpha = 2$ ), where  $K'[0^+] = -D$  for any potential profile [9]. It is quite difficult to find this non-trivial dependence analytically, thus numerical studies are welcome.

## 4 Conclusions

We have obtained the exact analytical result for the correlation time of steady-state Lévy flights in a symmetric quartic potential well. The correlation time has a power-law dependence on the noise intensity and decreases with an increase of the noise intensity and the steepness of potential well. Our analysis can be extended to the calculation of the correlation time for steeper potential wells.

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