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The Clifford, Grace, and de Longchamps chains: historical developments and inversive properties

Author:
Giovanna Rinchiusa

Coordinator:
Prof. Maria Carmela Lombardo

Supervisor:
Prof. Cinzia Cerroni

Co-Supervisor:
Prof. Maria Alessandra Vaccaro

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in

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Declaration of Authorship

I, Giovanna RINCHIUSA, declare that this thesis titled, “The Clifford, Grace, and de Longchamps chains:
historical developments and inversive properties” and the work presented in it are my own. I confirm that:

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Signed:

Date:

Giovanna
Rinchiusa
14.11.2025
20:06:07
GMT+01:00

"Mathematics knows no races or geographic boundaries; for mathematics, the cultural world is one country."

David Hilbert

UNIVERSITY OF PALERMO

Abstract

Department of Mathematics and Computer Sciences

Doctor of Philosophy

The Clifford, Grace, and de Longchamps chains: historical developments and inversive properties

by Giovanna RINCHIUSA

This thesis investigates the historical origins and developments of specific configurations of points, lines, and circles within the framework of elementary geometry. The demonstrations originally presented by the authors examined in this thesis have been carefully reconstructed and analysed in detail, with particular attention to their methodological aspects. The aim of this historical reconstruction is to clarify how different approaches to geometric reasoning evolved over time and to highlight the methodological choices that shaped these developments.

The primary focus is on the chain of theorems introduced by W. K. Clifford in 1871, along with its generalisations to higher-dimensional spaces. A significant advancement in this field was made by J. H. Grace in 1898, who not only extended Clifford's chain but also applied circular inversion to it, giving the entire construction a more symmetric form.

The thesis also explores the foundational ideas of recursive geometry introduced in 1877 by G. de Longchamps, who constructed a number of chains of theorems that depended on each other in an iterative manner, starting from objects related to geometry of the triangle.

These series of theorems extend beyond the traditional boundaries of elementary geometry, as many have found significant applications in mathematical research during the second half of the 20th century. For instance, in 1972, M. S. Longuet-Higgins showed a correspondence between Clifford's and Grace's chains of configurations and n-dimensional polytopes. In 1988, H. L. Dorwart highlighted a strong connection between configurations of points and circles and block designs. Once again, in 1976, Longuet-Higgins further analysed the inversive properties of both Clifford's chain and de Longchamps' chain related to the circumcentre, highlighting their analogies and structural connections.

Throughout the topic discussed, the thesis sought to underscore the surprising and unexpected connections between various theories, including elementary geometry, synthetic geometry, the theory of polytopes, and block-design theory. In the concluding reflections, the educational potential of the addressed geometric themes is also considered.

Acknowledgements

Writing this thesis has been a long and rewarding journey. It would not have been possible without the support, guidance, and encouragement of many people, to whom I am deeply grateful.

First and foremost, I would like to thank my supervisors, Professor Cinzia Cerroni and Professor Maria Alessandra Vaccaro, who believed in me from the very beginning and supported me throughout every stage of this journey. In particular, I am grateful to Professor Maria Alessandra Vaccaro, whose close supervision and continuous encouragement have been fundamental to the development of this thesis. Working with her on several research articles has been a valuable learning experience.

Another significant figure to whom I would like to express my gratitude is Professor Aldo Brigaglia. I am honored to have had the opportunity to know him and engage in enriching discussions with him.

I would also like to express my sincere gratitude to my colleagues and dear friends, Ivana and Mario, with whom I have shared not only my academic path but also a deep and lasting friendship. We met in our first year of undergraduate studies, and over the years, what began as a shared experience in the classroom turned into an inseparable trio. Today, our bond goes far beyond the university walls. In particular, I shared the PhD years with Mario. Although I started a year later, he always encouraged me and offered valuable advice.

A heartfelt thank you goes to my friend Giovanni, who, despite not being a mathematician, has always listened to me and supported me throughout these years. His presence and understanding have meant a great deal to me.

I would also like to thank all the people I had the pleasure of meeting throughout this journey—from PhD colleagues to those I encountered during research trips—who have enriched this experience with their ideas, support, and shared moments.

Over the course of these years, my personal life has also seen significant milestones: I got married and became a mother. Experiencing these important moments alongside my academic journey has been both challenging and profoundly rewarding, shaping me in many meaningful ways. Balancing my personal life as a new mother with the demands of research was a considerable challenge. If I have succeeded in reaching this goal, it is thanks to the unwavering support of my family.
A special thanks goes to my husband, who has always stood by me, especially during the most difficult moments. He has listened patiently and encouraged me never to give up.

To my mother and father, I owe everything. This milestone belongs to them as well.
To my brother, who is also pursuing a PhD, I want to say that sharing this journey—though through different paths—is a source of great pride for me.
To my sister, I extend my heartfelt gratitude for always being there with her sensitivity and unconditional affection.
Without their constant encouragement, this achievement would not have been possible.

I want to thank my daughter, who has been my greatest source of strength. Her smile, her presence, and the love she brings into my life have given deeper meaning to every step of this journey. I am profoundly grateful to her.

Last but certainly not least, I want to remember a very dear person who, unfortunately, passed away during this journey—my grandmother, who was like a second mother to me. In her own way, she was always present. Thank you, Grandma.

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To my daughter
Eva

Chapter 1

Introduction

During the 19th century, a marked interest in certain developments within *elementary geometry* emerged across Europe. This interest arose from various motivations, including educational ones. Elementary geometry can be defined as a branch of geometry focusing on concepts that, although not requiring advanced prerequisites, should by no means be considered trivial. It deals with points, lines, triangles, quadrilaterals, circles, and conic sections in general, investigating their properties and interrelations, many of which are unexpected, elegant, and surprising. Following a period of widespread interest, elementary geometry began to decline at the beginning of the 20th century. This decline was not attributable to a paucity of creative potential; rather, as Isaak Moiseevich Yaglom (1921–1988) hypothesised, it occurred because it was during this period that *synthetic geometry*, based on deductive inference from axioms, gained prominence. It is important to note that projective geometry, which is closely connected to elementary geometry, played a significant role in the development of synthetic geometry.[1]

Within the broader framework of elementary geometry, the *new geometry of the triangle* assumed a central position during the latter half of the 19th century. It represents a renewed approach that systematically constructs remarkable points, lines, and circles from a reference triangle, and studies their interrelations through geometric transformations. This field witnessed a substantial increase in publications, including (Alasia, 1900), (Casey, 1888, 1890), (Neuberg, 1891a,b), (Poulain, 1892), (Vigarié, 1887, 1889, 1895). These works offer a comprehensive and systematic framework for understanding the properties of geometric objects associated with a triangle. In more recent times, the topic has attracted the interest of several historians, among them Pauline Romera-Lebret, whose paper (Romera-Lebret, 2014) provides an in-depth historical analysis of the subject.

Against this background, the development of chains of theorems concerning the configurations of points, lines, and circles in the plane has attracted considerable interest among mathematicians from the latter half of the 19th century onward. A prominent example is the chain devised in 1871 by William Kingdon Clifford (1845–1879). The starting point for his study was the first of ten questions posed by Jakob Steiner (1796–1863) between 1827 and 1828. Around the same time as Clifford, and using the same principle, though completely independently, the French mathematician Gaston Albert Gohierre de

[1] For more details, see (Yaglom, 1981).

Longchamps (1842–1906) developed in 1877 a number of chains of theorems that depended on each other in an iterative manner, starting from objects related to geometry of the triangle. This iterative character led him to designate this type of geometry as *Géométrie récurrente* (recursive geometry).

These chains of theorems constitute the starting point for the research conducted by the author during her Ph.D. program. This thesis is based on the author's own investigations and results. Its structure is as follows.

Chapter 2 presents Steiner's first two questions. The former had already been proposed in 1804 by William Wallace (1768–1843) (it is also known as the *Wallace–Steiner question*), and was demonstrated by various scholars. In this thesis, particular attention is given to Auguste Miquel (1816–1851), who in 1838 not only provided a proof, but also showed several properties related to the configuration in question, most notably the *Triangle Theorem*, which is closely connected to Steiner's first question. The second question was proved in 1832 by Thomas Stephen Davies (1795–1851).

Chapter 3 analyses Clifford's 1871 article, in which, using synthetic methods, he applied the theory of higher-order curves to derive the theorems of his two chains as simple corollaries. Furthermore, the chapter examines the 1898 paper by John Hilton Grace (1873–1958), in which he transformed the theorems of Clifford's first chain via circular inversion, thereby rendering the configurations more symmetric. In 1988, Harold Laird Dorwart (1901–1998) adopted the typical notation employed in the field of Combinatorics for symmetric configurations $(2^{n-1})_n$ of Clifford's first chain and observed that the configuration 4_3 is a $2-(4,3,2)$ symmetric design, wherein two blocks, that is, two circles, pass through each pair of points. In the same 1898 contribution, Grace also defined a series of theorems in the plane, of which Clifford's first chain was a particular case. He further demonstrated that there is no analogue of the Wallace–Steiner question in the three-dimensional space. Nonetheless, he constructed a new chain of configurations that generalised the planar series he had previously defined. In the four-dimensional space, however, Grace demonstrated the validity of the Wallace–Steiner question, which could potentially serve as the basis for an analogous Clifford chain in that setting. The demonstrations originally presented by Clifford and Grace in their papers have been carefully reconstructed and examined in detail by the author, with particular attention to their methodological aspects.
Approximately fifty years later, Leslie Maurice Brown (1904–1999) identified a configuration that might correspond to the next step in the four-dimensional chain started by Grace. However, in 1973, Michael Selwyn Longuet-Higgins (1925–2016) provided a definitive answer to the problem, proving that no such chain exists. The final section of the chapter provides an overview of some historical references concerning the extension of the Wallace–Steiner question and the Miquel Triangle Theorem into n-dimensional spaces. This chapter is based on the work presented in (Rinchiusa and Vaccaro, 2024).

Chapter 4 explores the fundamental ideas underlying recursive geometry of de Longchamps, tracing its origins and presenting key examples of chains of theorems. Particular attention is devoted to chains related to the circumcentre, the centroid, the *Simson–Wallace line*, and the *nine-point circle* associated with a triangle. In the final part of the chapter, the author introduces a new chain related to the *nine-point conic* associated with a complete quadrangle.

Chapter 5 focuses on three mathematical contributions by Longuet-Higgins. In 1971, he provided a synthetic proof of de Longchamps' chain related to the circumcentre, which resembles the demonstration proposed by Clifford for his first series. Longuet-Higgins also established several properties of this chain. In 1972, the British mathematician analysed the correspondence between Clifford's and Grace's chains of configurations, on the one hand, and n-dimensional polytopes, on the other. This connection, originally introduced by Longuet-Higgins, has been reconstructed by the author. Furthermore, research in this area has led the author to investigate certain combinatorial properties of specific geometric configurations. In particular, the configuration $(n+2)_{n+1}$, consisting of $n+2$ points and $n+2$ hyperspheres, with $n+1$ points on each hypersphere and $n+1$ hyperspheres passing through each point, is also a geometric representation of the $2-(n+2, n+1, n)$ symmetric design. In this design, the blocks are the hyperspheres and exactly n blocks pass through any two distinct points. Moreover, the configuration $(n+2)_{n+1}$ is also a $t-(n+2, n+1, n+2-t)$ symmetric design for all t such that $t \leq n+1$. The correspondence between Clifford's and Grace's chains, and n-dimensional polytopes, as well as the connection with block-design theory, was discussed in (Rinchiusa and Vaccaro, 2024). Finally, in the third paper by Longuet-Higgins examined in this thesis, published in 1976, he studied the inversive properties of both Clifford's chain and de Longchamps' chain related to the circumcentre, determining their analogies and connections. These topics are part of a separate work (Rinchiusa and Vaccaro, 2025b), currently under review.

Chapter 6 offers concluding reflections. In particular, it highlights the didactic potential of several topics that have been addressed throughout the thesis, with specific reference to two publications written by the author: (Rinchiusa and Vaccaro, 2023, 2025a).

Lastly, in Appendix A, some notions concerning polytopes are presented in order to support the reading of Chapter 5. In particular, the author provides an explanation of how the polytopes $h\gamma_n$ are composed, with the aim of facilitating understanding of the connection between these polytopes and Clifford's and Grace's configurations.

Chapter 2

The first two Steiner questions

Between 1827 and 1828, Steiner proposed ten questions[1] concerning a complete quadrilateral, *i.e.*, the configuration determined by four lines in a plane intersecting pairwise (Steiner, 1827-28). The first two of these questions are presented here, forming the starting point for Clifford's first chain and de Longchamps' chain related to the circumcentre, which will be explored in Chapter 3 and Chapter 4.

1. Four lines in a plane, taken three at a time, form four triangles such that the circles circumscribed about them all pass through a single point P (see Figure 2.1).

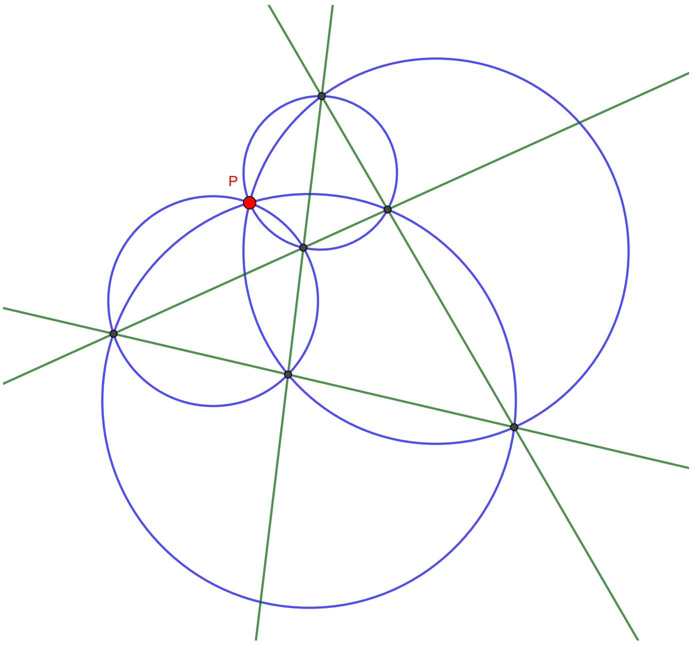


FIGURE 2.1: Steiner's first question.

2. The centres of the four circles, together with point P, all lie on the same circle (see Figure 2.2).

[1]As mentioned in Chapter 1 and as will be further discussed in Section 2.1, the first of these questions had already been solved in 1806. However, Steiner appears to have been unaware of this earlier result when he formulated his list of problems.

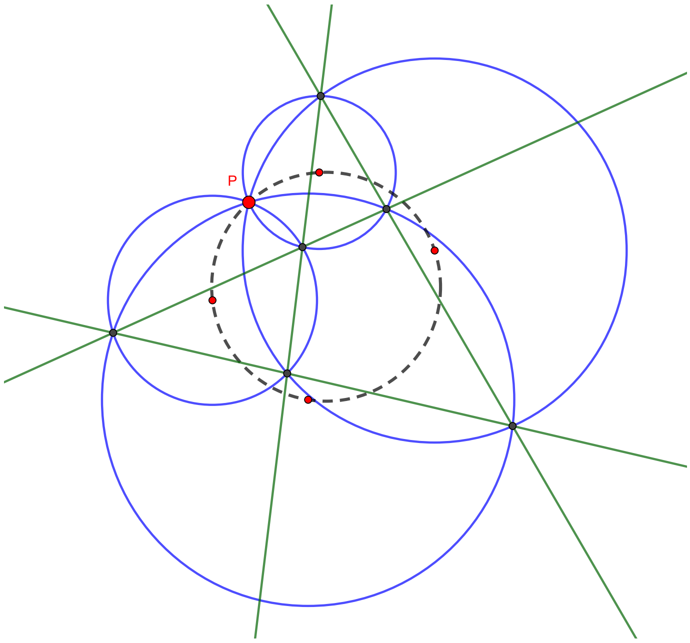


FIGURE 2.2: Steiner's second question.

2.1 The Wallace–Steiner question and Miquel's contribution

As John Sturgeon Mackay (1843–1914) has stated in (Mackay, 1890, p. 87), Steiner's first question had previously been proposed in 1804 by Wallace[2] under the pseudonym Scoticus,[3] and it had been solved in 1806 by both George Pickering and John Cavill working independently. These two proofs, which were very similar, were both published in the first volume of Leybourn's *Mathematical Repository, New Series* (Leybourn, 1806, pp. 169-170), in response to Scoticus' *Question 86*, which appears on p. 22 of the same volume. Here, the proof of Pickering is presented. He considered four lines that intersect pairwise at the points A, B, C, D, E, and F, as well as the corresponding four triangles ABC, AEF, BDE, and CDF (see Figure 2.3). Focusing on the circles circumscribing the triangles ABC and BDE, in addition to sharing the point B, they also intersect at another point, denoted by P. After drawing the segments PC and PD, Pickering proved that the opposite angles of the quadrilateral $PDFC$ are supplementary. In fact, since $\widehat{PDE} = \widehat{PBA}$ as supplementary angles of the same angle $\widehat{EBP}$, and $\widehat{PBA} = \widehat{PCA}$ as angles inscribed on the same arc $\overset{\frown}{PA}$, it follows that $\widehat{PDE} = \widehat{PCA}$. However, the latter is supplementary to $\widehat{FCP}$. Therefore, the quadrilateral $PDFC$ can be inscribed in a circle. Furthermore, by drawing the segments PA and PE, Pickering showed that the quadrilateral $PEFA$ can also be inscribed in a circle, employing analogous reasoning. Consequently, the circles circumscribing respectively the

[2]For this reason, it is also referred to as the *Wallace–Steiner question*.

[3]Referring to an article by Thomas Turner Wilkinson (1815–1875) in *Mechanics' Magazine*, Vol. LV, p. 447, 1851, Mackay argued that Scoticus was a pseudonym used by Wallace.

triangles CDF and AEF also pass through point P, which is known as the *Wallace point*.

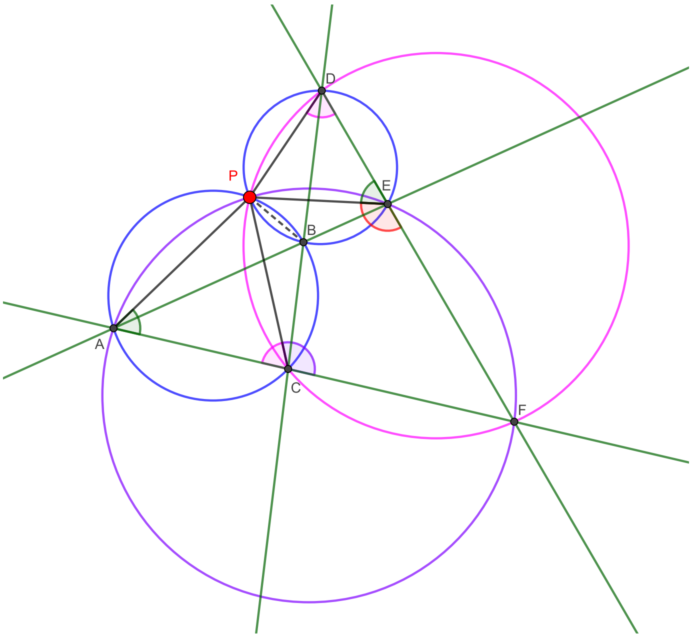


FIGURE 2.3: Pickering's proof of the Wallace–Steiner question.

Another solution to the Wallace–Steiner question was provided by the French mathematician Miquel. He was born on 6 February 1816 in Albi, France. He graduated from a school in Toulouse with a degree in literature on 5 August 1834 and a degree in science on 6 August 1835. During this period, Miquel submitted applications to the Minister of Public Instruction for a regent position or a mathematics chair at a municipal college. In response, the Minister stated that, based on the positive testimonies received regarding the candidate, he had been registered as an aspirant for these duties, and that his qualifications would be carefully reviewed as soon as a suitable opportunity arose. With regard to the request for the mathematics chair, Miquel was encouraged by Mr. Laroque, a deputy inspector who had attended his oral exam for admission to the École Normale Supérieure in Paris, science section, although he did not succeed in gaining entry. He subsequently moved to Paris, where he attended Institution Barbet[4] in 1835–36, auditing courses in physics and *mathématiques spéciales*[5] in the second division of the Collège Royal de Saint-Louis. Here, he won the first prize in algebra. Since 1836, he assumed the role of regent in various locations across France, including Nantua, Saint-Dié, Castres, Bagnols, and Vigan. His desire to return to Paris led him to repeatedly ask for permission to travel to the capital in order to

[4]The Institution Barbet was a private boys' secondary school founded in 1803 by André Marie Ruinet. Ruinet was succeeded in 1820 by his son-in-law, Jean Baptiste Brissaud, and in 1827 the school was taken over by Jean François Barbet, from which the institution takes its name.

[5]*Mathématiques spéciales* was an advanced mathematics preparatory course in the French system for admission to prestigious *Grandes Écoles* (selective higher education institutions in France).

publish some of his articles, but he continued to live in very precarious conditions[6] and he died in Vigan on 1 March 1851 at the age of 35. In addition to mathematics, Miquel had intellectual interests in politics, as evidenced by two pamphlets he published in 1844 and 1845.[7] He was aware that his political involvement posed an obstacle to his career.[8]

A reconstruction of Miquel's contributions is presented below based on the available sources, aiming to provide a clearer understanding of his role in the Wallace–Steiner question.

In 1836, Miquel proposed his solution to the Wallace–Steiner question in the French journal *Le Géomètre* (Miquel, 1836), which was founded by Antoine Philippe Guillard in 1836, the same year as the *Journal de mathématiques pures et appliquées*,[9] but it only lasted a few months. It consisted mainly of short papers, particularly questions and answers. Miquel was one of its contributors. A few years later, in 1838, Miquel's solution to the Wallace–Steiner question was published in the *Journal de mathématiques pures et appliquées* (Miquel, 1838a) along with the other properties already presented in 1836. In this paper Miquel initially showed the following theorem. Given three circles with centres O, S, and T[10] intersecting at a point P, let A be a generic point on circle O, and let D and F be the further points of intersection of circle O with circles S and T, respectively. Let us denote by B and C the points where the lines AD and AF intersect the circles S and T, respectively. The points B and C are collinear with the additional point of intersection E of circles S and T (see Figure 2.4).

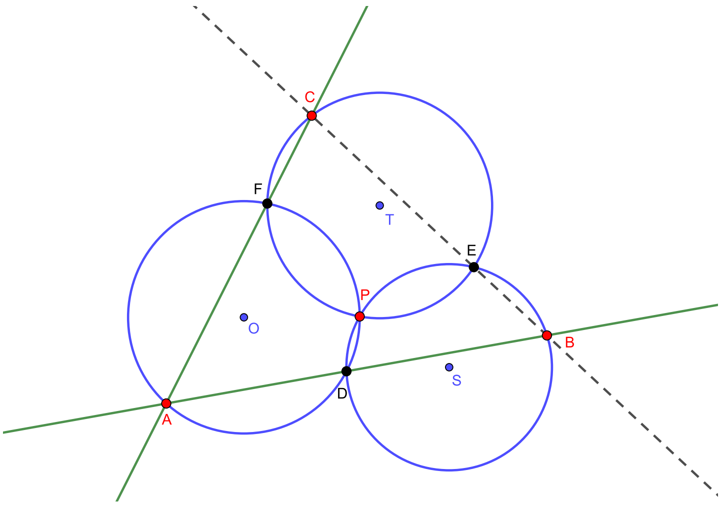


FIGURE 2.4: Theorem I of Miquel.

[6]Regents occupied the lowest rung on the social ladder of teachers. They were obliged to renew their contracts on an annual basis, their remuneration was often inadequate, and they were entrusted with classes that were often challenging.

[7]*De l'organisation du corps social*, 1844 and *Développements d'organisation sociale*, 1845.

[8]For more details on Miquel's biography, see (Gerini and Verdier, 2011, pp. 64-67) and (Huard, 2017, p. 39).

[9]This journal was founded by Joseph Liouville, from whom it takes its name, particularly in France, as the *Journal de Liouville*.

[10]The circles are labelled with the same letters as their respective centres.

In fact, if lines are drawn from P to D, E, and F, the angles $\widehat{DAF}$, $\widehat{EBD}$, and $\widehat{FCE}$ are supplementary to the angles $\widehat{FPD}$, $\widehat{DPE}$, and $\widehat{EPF}$, respectively, as they form opposite angle pairs of quadrilaterals inscribed in a circle. Thus, it can be deduced that:

$$\begin{aligned}\widehat{DAF}+\widehat{EBD}+\widehat{FCE} &= (\pi-\widehat{FPD})+(\pi-\widehat{DPE})+(\pi-\widehat{EPF})\\ &= 3\pi-(\widehat{FPD}+\widehat{DPE}+\widehat{EPF})\\ &= 3\pi-2\pi=\pi.\end{aligned}$$

Consequently, it can be concluded that $ABEC$ is a triangle, and therefore the points B, E, and C are collinear.
From this latter theorem, Miquel derived the following reciprocals:

R1. Given three circles intersecting at a point P, denote by D, E, and F the three additional points of intersection between them. If any line is drawn through E, intersecting the two circles passing through E at points B and C, then the lines DB and CF intersect at a point A, which lies on the additional circle.

R2. Given a triangle and a point on each of its sides (or their extension), the three circles that pass through one vertex of the triangle and through the two points on the sides that emerge from that vertex all intersect at a single point (see Figure 2.5).[11] This result will subsequently be referred to as the *Triangle Theorem*.

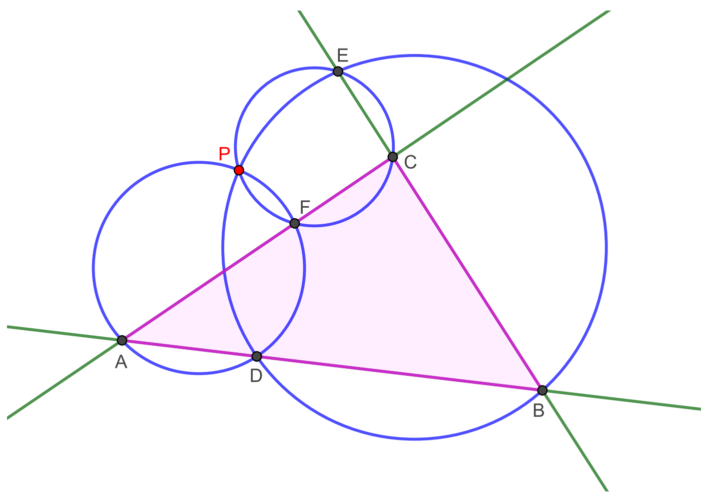


FIGURE 2.5: Miquel's Triangle Theorem.

Miquel employed this latter result to provide a solution to the Wallace–Steiner question. In fact, given four lines in a plane that intersect pairwise at the six points A, B, C, D, E, and F, applying the Triangle Theorem to triangle ABC, he proved that the circles circumscribed about the triangles ADF, BED and CEF all intersect at a point P. Subsequent application of the

[11]Concerning this, see also (Coxeter and Greitzer, 1967, pp. 61-62).

Triangle Theorem to triangle BED yielded the result that the circles circumscribed about the triangles ABC, CEF, and ADF also intersect at the same point P. Therefore, the circumcircles of the four triangles ABC, CEF, ADF, and BED all meet at the same point P (see Figure 2.6).

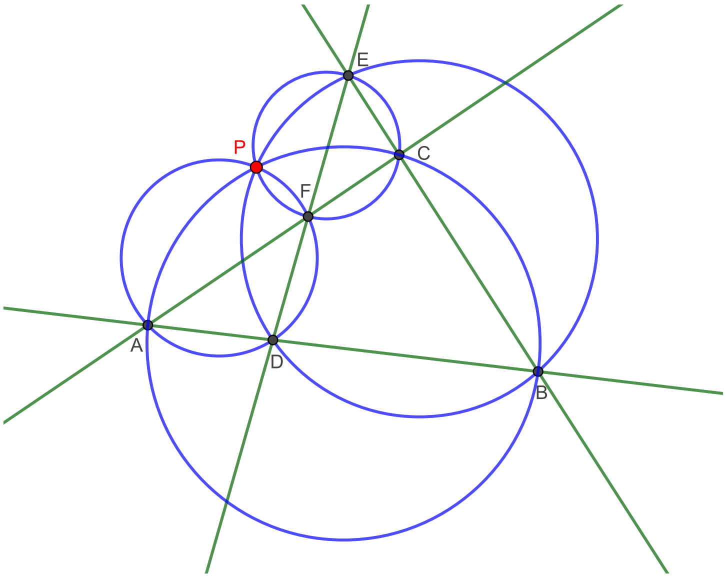


FIGURE 2.6: Theorem II of Miquel: the Wallace–Steiner question.

It should be noted that the Wallace–Steiner question can be derived from Triangle Theorem if the points D, E, and F on the sides of triangle ABC are collinear. Conversely, if point P, which is common to the three circumcircles of triangles ADF, BDE, and CFE, also lies on the circle circumscribed about triangle ABC, then the points D, E, and F must be collinear. Thus, the circle circumscribed about triangle ABC is simply the locus of points P when D, E, and F lie on a line.[12]

The Wallace–Steiner question has been a subject of interest for many mathematicians over the years. One such figure was the Italian mathematician Giuseppe Pesci, who published a solution to the Wallace–Steiner question in 1891 in his paper (Pesci, 1891) using basic Euclidean properties.[13] Another significant contribution came from Grace, who proposed a solution in 1898 in (Grace, 1898, p. 162), employing a property of cubic curves, specifically the well-known theorem according to which every cubic curve passing through eight fixed points also passes through a uniquely determined ninth.[14] In fact, as can be seen in Figure 2.6, the four circles and the four lines, taken together,

[12]See (Baker, 1930, p. 71).

[13]Pesci stated that others had already provided a proof of the question; among them, he mentioned Richard Baltzer (1818–1887) in (Baltzer, 1867).

[14]This result follows from the fact that a cubic curve, in projective geometry, is uniquely determined by nine points in general position. Fixing eight points imposes eight linear conditions, leaving a pencil of cubics passing through them. The ninth point arises as the common intersection point of all these cubics, due to Bézout's theorem, which states that two cubics intersect in nine points. Therefore, the ninth point is uniquely determined by the first eight.

can be regarded as forming four degenerate circular cubics,[15] each consisting of a circumcircle of a triangle and the corresponding omitted line. These curves, which share eight common points (the six points A, B, C, D, E, and F, along with the two circular points at infinity), intersect at an additional point.

Miquel concluded his paper with a theorem whose configuration is known as *Miquel's Pentagram*: given a convex pentagon $ABCDE$ and the circles circumscribed about the triangles ABF, BCG, CDH, DEI, and AEJ, which triangles are formed by the sides of the pentagon and the extensions of adjacent sides, the other points where these circles intersect, *i.e.*, L, M, N, P, and Q, all lie on a single circle (see Figure 2.7).[16]

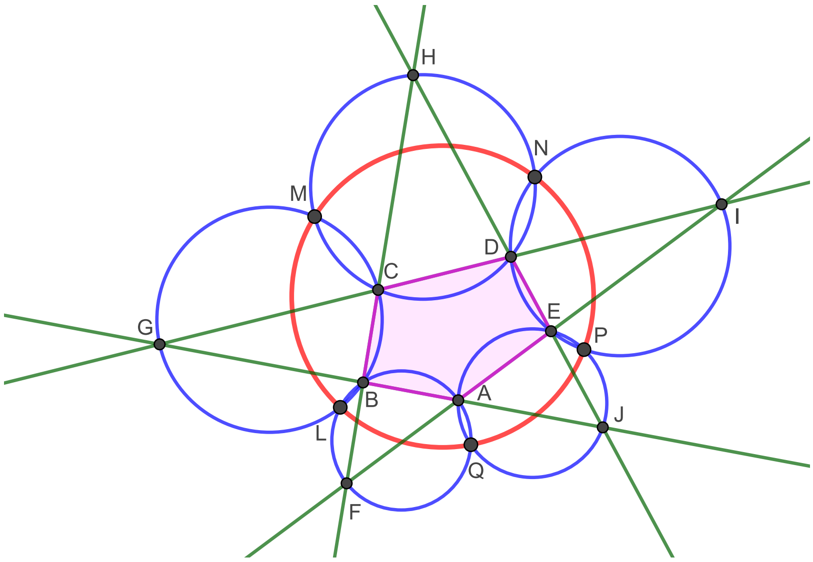


FIGURE 2.7: Theorem III: Miquel's Pentagram.

Note that the points L, M, N, P, and Q are the Wallace points of the five configurations obtained by excluding one line at a time. Moreover, the theorem remains valid even if the pentagon is not convex. In light of this, it can also be stated as follows: given five lines in a plane intersecting pairwise, the five Wallace points determined by the five complete quadrilaterals formed by excluding one line at a time all lie on a single circle. This result represents a step in the chain of theorems devised by Clifford in 1871, which will be examined in Chapter 3.

To prove Miquel's Pentagram, the French mathematician considered the circle passing through three of the five points, L, N, and Q, and demonstrated that it also passes through the other two points, M and P (see Figure 2.8). He examined the complete quadrilaterals $ABGCIF$ and $CDIEFH$, whose Wallace points are L and N, respectively. Since triangle CFI is common to the two complete quadrilaterals, the circle CFI[17] passes through both L and N.

[15]Circular cubics are cubic curves that pass through the two circular points at infinity.

[16]This theorem had already been presented in (Miquel, 1836, pp. 168-170).

[17]By a slight abuse of notation, both the triangle with vertices X, Y, and Z, and the circle circumscribed about it, will be denoted by XYZ.

He then applied property R1. to the circles CFI, ABF, and LNQ, which intersect at point L. The line FAI[18] passes through point F, additional point of intersection of circles ABF and CFI, and intersects these circles at points A and I, respectively. Since N and Q are the further intersection points of circles CFI and ABF with circle LNQ, respectively, he proved that lines AQ and IN intersect at a point R, that must lie on the circle LNQ. Next, he applied the Triangle Theorem to triangle AIR and showed that the circles IEN, RQN, and AQE intersect at a single point, which is point P. Therefore, the circle LNQ passes through point P. By similar reasoning, it can be proved that LNQ also passes through point M.[19]

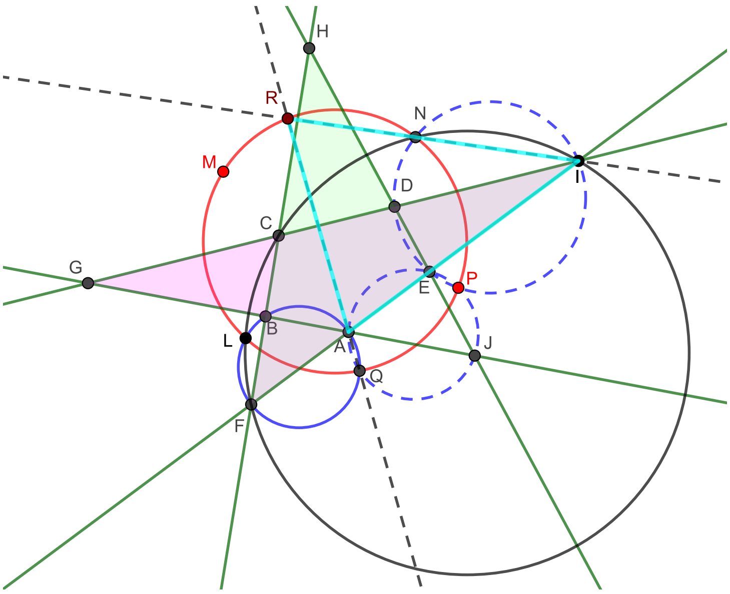


FIGURE 2.8: Proof of Miquel's Pentagram.

In the same volume of the *Journal de mathématiques pures et appliquées*, Miquel generalised the theorems already proved to the curvilinear case. To do so, he replaced the lines with circles intersecting at a single point (see Figure 2.9).[20] Furthermore, he stated that these theorems continue to hold if the circles are drawn on the surface of a sphere (Miquel, 1838b, pp. 518-519).

[18]The points F, A, and I are collinear by construction.
[19]The same proof can be found in (Catalan, 1858, pp. 92-93).
[20]It could be said that his reasoning suggests an implicit use of circular inversion.

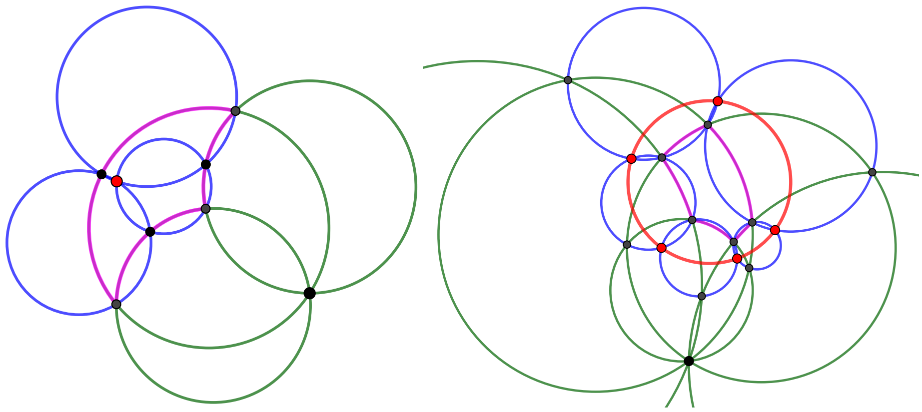


FIGURE 2.9: Miquel's curvilinear Triangle Theorem and Pentagram.

At the conclusion of the paper, Miquel established the following theorem. Let A, B, C, and D be the four vertices of a curvilinear tetrahedron, formed by the surfaces of four spheres, $ABCP$, $ABDP$, $ACDP$, and $BCDP$,[21] which all intersect at a single point P in the three-dimensional space. Let E, F, G, H, I, and K be six points lying on the edges AB, AC, AD, BC, CD, and BD, respectively. Then, four spheres can be constructed, each passing through one vertex of the tetrahedron and through the three points on the edges emerging from said vertex. These four spheres, $AEFG$, $BEHK$, $CFHI$, and $DGIK$, when considered in groups of three, share a common point on each face of the tetrahedron. Furthermore, they intersect at a single point (see Figure 2.10).

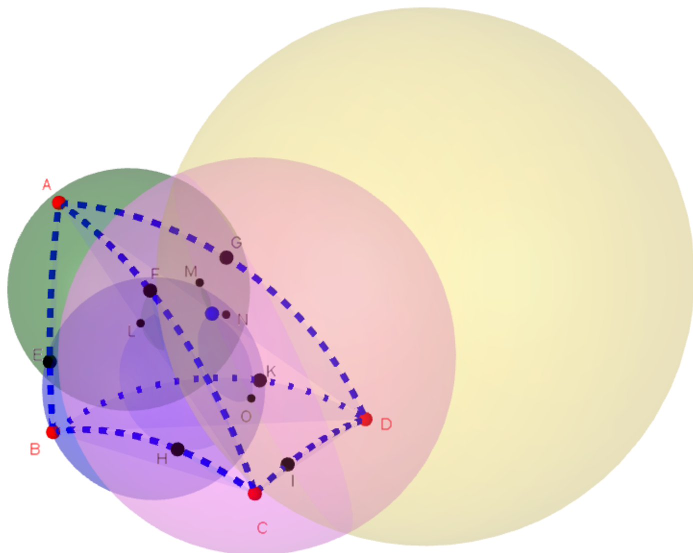


FIGURE 2.10: Miquel's curvilinear tetrahedron.

To prove this result, Miquel began by observing that ABC is a curvilinear triangle on the sphere $ABCP$ formed by the circles ABP, BCP, and ACP, which

[21] The notation $XYZT$ refers to the unique sphere passing through the four points X, Y, Z, and T, provided that they are not coplanar.

all pass through the point P, and that the points E, F, and H lie on the arcs $\overset{\frown}{AB}$, $\overset{\frown}{AC}$, and $\overset{\frown}{BC}$, respectively. By applying the curvilinear Triangle Theorem to ABC, it follows that the circles AEF, BEH, and CFH intersect at a single point L on the sphere $ABCP$. Moreover, since these circles arise as the intersections of the three spheres $AEFG$, $BEHK$, and $CFHI$ with the sphere $ABCP$, it follows that these three spheres also intersect at point L on $ABCP$. In a similar manner, it can be shown that the other triplets of spheres among $AEFG$, $BEHK$, $CFHI$, and $DGIK$ intersect at points M, N, and O, which lie respectively on the spheres $ACDP$, $ABDP$, and $BCDP$.
Then, he considered the intersections of the three spheres $ABCP$, $ABDP$, and $ACDP$ with the sphere $AEFG$, yielding the circles AEF, AFG, and AEG, which intersect at the point A and pairwise at the points E, F, and G. These three points form the vertices of a curvilinear triangle on the sphere $AEFG$, and the points L, M, and N lie on its sides EF, FG, and EG, respectively. Thus, by applying the curvilinear Triangle Theorem to EFG, the three circles ELN, FLM, and GMN are shown to intersect at the same point on the sphere $AEFG$. Since these circles arise as the intersections of the spheres $BEHK$, $CFHI$, and $DGIK$, respectively, with $AEFG$, it follows that all four spheres, $AEFG$, $BEHK$, $CFHI$, and $DGIK$, intersect at a single point (Miquel, 1838b, pp. 521-522).

After proving this theorem, Miquel observed that if the point P, where the four spheres generating the curvilinear tetrahedron intersect, lies at infinity, the curvilinear tetrahedron reduces to an ordinary tetrahedron. This led him to formulate the *Tetrahedron Theorem*, which will be referred to in Chapter 3. He returned to this topic later, in 1844 and 1845, dealing with similar issues in (Miquel, 1844) and (Miquel, 1845). Specifically, in the first article he demonstrated some properties related to curvilinear angles, while in the second he proved the validity of the Triangle Theorem, the Wallace–Steiner question, and the Pentagram Theorem on a spherical surface. Furthermore, he presented another result concerning the intersections of six spheres, three at a time, which extends the analogous theorem about four circles intersecting pairwise, proved in (Miquel, 1838b, pp. 517-518).

2.2 Steiner's second question: Davies' proof

As mentioned in Section 2.1, Mackay in (Mackay, 1890, p. 87) stated that Cavill and Pickering proved the Wallace–Steiner question in 1806 in Leybourn's *Mathematical Repository, New Series*. However, neither of these proofs presents any additional properties related to the complete quadrilateral. The earliest writings in the *Repository* concerning the additional properties of the complete quadrilateral date back to 1832, when Davies proposed new such properties and solved them in the sixth volume of the journal (Leybourn, 1835, pp. 124-127, 229). One of these, *Question 555*, corresponds to Steiner's second question. This section presents Davies' reasoning to prove the question. He did not limit himself to solving his proposition (Question 555); he also presented other properties he had known for years. He further stated that the

solutions he provides are not the original ones, as the manuscript containing them has been lost. Consequently, he had to re-investigate the problems, and although he could not recall the methods he had previously employed, he was certain they differed from the current ones and, in his view, were likely superior in several respects.
Davies divided the proof into two parts. Given four lines intersecting pairwise at the points A, B, C, D, E, and F, let M, N, R, and S be the centres of the circles ABC, ADF, CEF, and BED, respectively, and let P denote the Wallace point of the configuration (see Figure 2.11). In the first part of the proof, he showed that points M, N, R, and S lie on the same circle (*Prop. I*), and in the second part he demonstrated that this circle also passes through the point P (*Prop. II*).

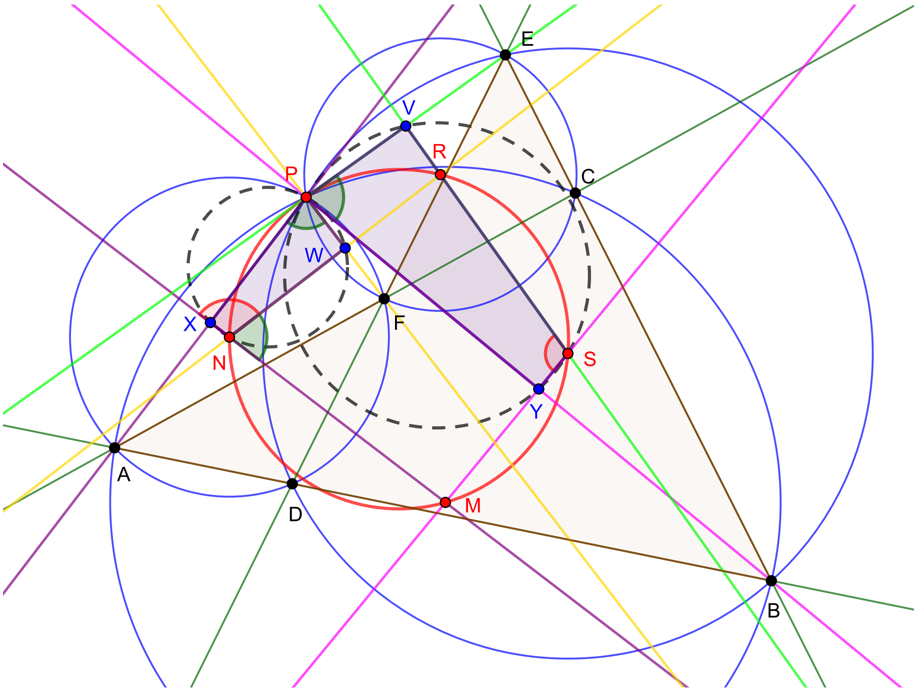


FIGURE 2.11: Davies' proof of *Prop. I*.

Let X, Y, V, and W denote the intersection points of the pairs of lines $AP \cap MN$, $BP \cap MS$, $EP \cap RS$, and $FP \cap NR$, respectively. Since the circles ABC and BED, with centres M and S respectively, intersect at points B and P, it follows that the lines MS and BP are perpendicular. Similarly, the other pairs of lines are perpendicular. Furthermore,

$$\begin{aligned} \widehat{BPE} &= \widehat{BPC} + \widehat{CPE} \\ &= \widehat{BAC} + \widehat{CFE} \\ &= \widehat{DAF} + \widehat{AFD} \\ &= \widehat{DPF} + \widehat{APD} \\ &= \widehat{APF}, \end{aligned}$$

which leads to the relation

$$\widehat{YPV} = \widehat{XPW}. \tag{2.1}$$

Since $\widehat{PVS}$ and $\widehat{SYP}$ are both right angles, the points S, V, P, and Y must lie on the same circle, and therefore $\widehat{VSY} + \widehat{YPV} = \pi$. Similarly, since $\widehat{NXP}$ and $\widehat{PWN}$ are two right angles, the points P, X, N, and W are concyclic, and $\widehat{WNX} + \widehat{XPW} = \pi$. Thus,

$$\widehat{VSY} + \widehat{YPV} = \widehat{WNX} + \widehat{XPW}. \tag{2.2}$$

By combining Equations (2.1) and (2.2), one obtains $\widehat{VSY} = \widehat{WNX}$, *i.e.*, $\widehat{RSM} = \widehat{RNX} = \pi - \widehat{MNR}$. It follows that the points M, N, S, and R lie on the same circle (Leybourn, 1835, pp. 229-230).
To prove that the point P lies on the previously determined circle (passing through M, N, R, and S), let us draw perpendiculars from the two centres S and R to the line BE (see Figure 2.12).

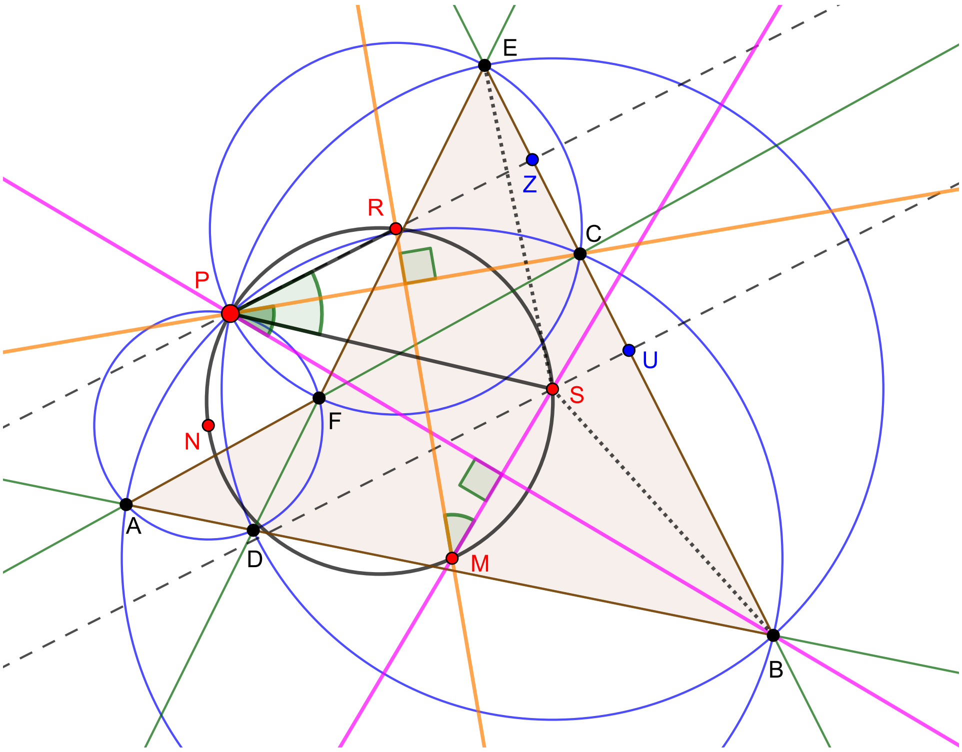


FIGURE 2.12: Davies' proof of *Prop. II*.

Let U and Z denote the feet of the two perpendiculars. Since the pairs of lines CP and MR, and BP and MS are perpendicular, one obtains that $\widehat{SMR} = \widehat{BPC}$.

Now,

$$\begin{aligned}
\widehat{BPC} &= \widehat{BPE} - \widehat{CPE} \\
&= \frac{1}{2}\widehat{BSE} - \frac{1}{2}\widehat{CRE}^{22} \\
&= \widehat{USE} - \widehat{ZRE} \\
&= (\pi - \widehat{SEU}) - (\pi - \widehat{REZ}) \\
&= \widehat{REZ} - \widehat{SEU} \\
&= \widehat{REU} - \widehat{SEU} = \widehat{RES} = \widehat{SPR}.^{23}
\end{aligned}$$

Thus, $\widehat{SMR} = \widehat{BPC} = \widehat{SPR}$; consequently, P lies on the circle through the points M, N, R, and S (Leybourn, 1835, p. 231).

[22] In fact, $\widehat{BPE}$ and $\widehat{BSE}$ constitute a pair of inscribed and central angles that subtend the same arc $\overset{\frown}{BE}$. An analogous relationship holds for $\widehat{CPE}$ and $\widehat{CRE}$.

[23] The triangles ERS and RPS are congruent by the third criterion of congruence.

Chapter 3

Clifford's and Grace's chains

The chapter begins with a brief biographical overview of two mathematicians, Clifford and Grace, whose contributions (Clifford, 1871; Grace, 1898) form the basis of the discussion. It then turns to an analysis of Clifford's 1871 paper, in which he used synthetic methods to derive the theorems of his two chains. This is followed by an examination of Grace's 1898 article, where Clifford's first chain is reformulated through circular inversion[1] to yield more symmetric configurations, and new chains of theorems are introduced in higher-dimensional spaces.
In addition to reconstructing the reasoning behind Clifford's and Grace's results, often presented in a concise or implicit manner in their works, this chapter includes detailed figures illustrating each step of the planar chains. These visual representations aim to clarify the geometric configurations and support a more intuitive understanding of the constructions involved.

3.1 Biographical sketches of William Kingdon Clifford and John Hilton Grace

William Kingdon Clifford was born on 4 May 1845 in Exeter, England. After attending several schools in his hometown, he was awarded a scholarship in 1860, at the age of fifteen, to study mathematics and classical studies at King's College London. There, he excelled not only in mathematics and classical studies, but also in English literature. During this period, he produced his first mathematical paper, titled *The Analogues of Pascal's Theorem*, which was published in the *Quarterly Journal of Pure and Applied Mathematics* in 1863. That same year, Clifford won a scholarship to study mathematics and natural philosophy at Trinity College Cambridge, where he graduated in 1867. He achieved the distinction of being Second Wrangler in the Mathematical

[1]Given a circle with centre O and radius r, a *circular inversion* is defined as the transformation that maps each point P in the plane, distinct from O, to a point P' lying on the ray OP such that the following relation holds: $\overline{OP} \cdot \overline{OP'} = r^2$. For the properties of inversion, see, for example, (Coxeter and Greitzer, 1967).

Tripos[2] of 1867 and also placed second in the prestigious Smith's Prize,[3] underscoring the originality and depth of his mathematical insights. In 1870, he joined a British expedition to Italy to observe a solar eclipse and survived a shipwreck off the coast of Sicily. The following year, he was appointed professor of applied mathematics and mechanics at University College London. In 1874, he was elected a Fellow of the Royal Society and was also an active member of the London Mathematical Society. In 1875, he married Lucy Lane. From 1876 onwards, he began to suffer from nervous exhaustion, likely caused by an excessive workload. In 1878, he travelled to Madeira for health reasons, but his condition worsened. He died on 3 March 1879.
The main mathematical contributions of Clifford concern geometry, algebra, and mechanics. He is best known for formulating *Clifford algebra*, a generalisation of complex numbers and quaternions. In 1870, he hypothesised that matter might be a manifestation of the curvature of space, a theme that would later be developed in Einstein's theory of general relativity. As well as his mathematical achievements, Clifford was a philosopher of science and a staunch proponent of Darwinism.[4]

John Hilton Grace was born on 21 May 1873 in Halewood, England. After receiving his early education at the local village school and the Liverpool Institute, he won a scholarship in 1892 to Peterhouse, University of Cambridge. In 1895, he took the Mathematical Tripos and was among the most highly regarded candidates for the title of Senior Wrangler. In 1897, he was elected a Fellow of Peterhouse, and later he became a college lecturer at both Peterhouse and Pembroke colleges. Between 1903 and 1916 he was a member of the council of the London Mathematical Society, acting as secretary from 1904 to 1915, and subsequently as vice-president. In 1908, he was elected a Fellow of the Royal Society. During 1916 and 1917 he was a visiting professor in Lahore, Pakistan. Upon returning to the United Kingdom, he continued to contribute to the academic community by temporarily replacing a professor engaged in military service in Scotland between 1917 and 1919. In 1922, due to declining health, Grace retired from active academic life. He initially settled in Norfolk and later relocated to Cambridge. He spent the final months of his life in Huntingdon, where he died on 4 March 1958.
Grace's early research (between 1898 and 1904) defined his key areas of interest. However, this period was followed by a twelve-year gap, most likely due to his intensive teaching at Cambridge. He briefly returned to publishing during his time away from Cambridge. After 1918, his productivity declined again due to health issues. Nevertheless, he published consistently between 1926 and 1930 and produced two final papers later in life. Grace was primarily a geometer, approaching invariant theory from a distinctly geometric perspective. His most notable achievement was the book *The Algebra*

[2] The Mathematical Tripos was Cambridge's most demanding mathematics examination; students were ranked by score, with the top achiever titled Senior Wrangler, followed by Second, Third, and so on.

[3] The Smith's Prize consisted of two annual awards, each conferring upon two research students in mathematics and theoretical physics at the University of Cambridge since 1769.

[4] For further details on the biography of Clifford, refer to (Chisholm, 2002).

of Invariants, published in 1903, which he co-authored with Alfred Young (1873–1940). His writings in this field are renowned for their ingenuity and elegance.[5]

3.2 Clifford's paper

Clifford began his paper (Clifford, 1871, p. 124) by referring to a footnote given by George Salmon (1819–1904) in (Salmon, 1863, p. 232):

> [...] It follows in the same way from Cor. 4, p. 193, that the circles circumscribing the four triangles pass through the same point, viz. the focus of the same parabola. If we are given five lines M. Auguste Miquel has proved (see Catalan's *Théorèmes et Problèmes de Géométrie Elémentaire*, p. 93) that the foci of the five parabolas which have four of the given lines for tangents lie on a circle.

Salmon had mentioned Miquel's well-known Pentagram Theorem, making reference to the collection of theorems and problems by Eugène Charles Catalan (1814–1894) (Catalan, 1858, pp. 92-93), but he had stated it in a different form involving parabolas. He further stated that the result derives from *Corollary 4* in (Salmon, 1863, p. 193),[6] according to which the circumcircle of the triangle formed by any three tangents to a parabola passes through its focus (see Figure 3.1).[7]

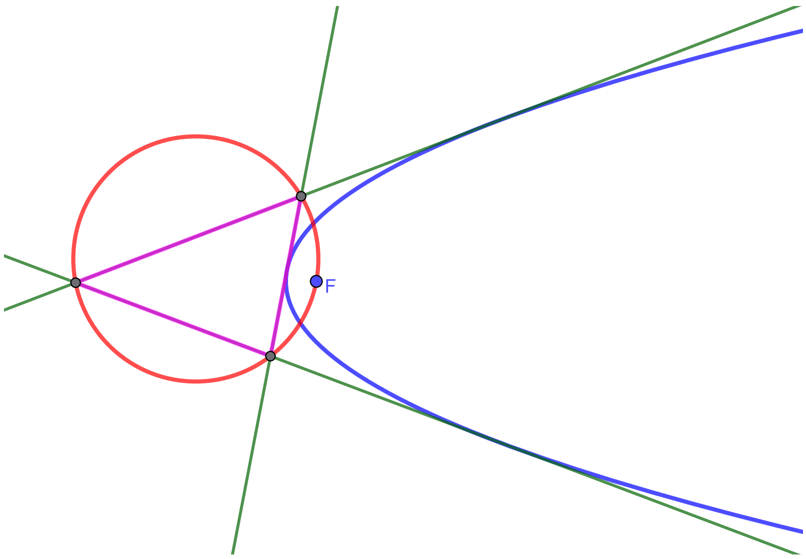


FIGURE 3.1: Salmon's Corollary 4.

[5]For further details on the biography of Grace, refer to (Todd, 1958).

[6]This corollary was already present in the second edition of *A Treatise on Conic Sections* (1850) by Salmon.

[7]As Mackay noted in (Mackay, 1890, p. 84), this property had been demonstrated in 1801 by Wallace in the second volume of *The Mathematical Repository*, pp. 54-55. In his proof, Wallace also showed that the feet of the perpendiculars dropped from the focus of a parabola to three of its tangents lie on the tangent line to the parabola at the vertex. This line, known as the Simson–Wallace line, will be mentioned again later.

Therefore, given four lines in a plane that intersect pairwise, since there exists a unique parabola tangent to them, it follows from Corollary 4 that the circumcircles of the four triangles determined by these lines all pass through the same point, *i.e.*, the focus of the parabola. Thus, the Wallace point of this configuration is none other than the focus of the parabola tangent to the four given lines. Salmon then pointed out that, given five lines in a plane intersecting pairwise, one can consider five sets of four lines each, and for each such set there exists a unique parabola tangent to the four lines; the foci of the five resulting parabolas all lie on a single circle. This result, later proved by Clifford, expressed in a different form what Miquel had already demonstrated in (Miquel, 1838a, pp. 486-487) and for the curvilinear case in (Miquel, 1838b, p. 519) and (Miquel, 1845, p. 349). Clifford cited the theorem proved by Miquel, referring precisely to Miquel's 1845 paper.

The main steps that led Clifford to the construction of his first chain are now presented in broad outline. Clifford's paper was almost entirely devoted to the classification of third-class curves;[8] in fact, he derived his proof of Miquel's theorem as a simple corollary (Clifford, 1871, p. 137) of a more general result. In his discussion, Clifford employed what is commonly referred to as synthetic geometry, although he expressed his disagreement with this terminology, preferring instead the term *organic geometry*.[9] He analysed third-class curves having double tangents[10] and reduced them to two projective equivalence classes, represented by a cardioid (named hyperbolic) and a tricuspid hypocycloid[11] (named elliptic) respectively, depending on whether the contact points with the double tangents are real or imaginary (see Figure 3.2).

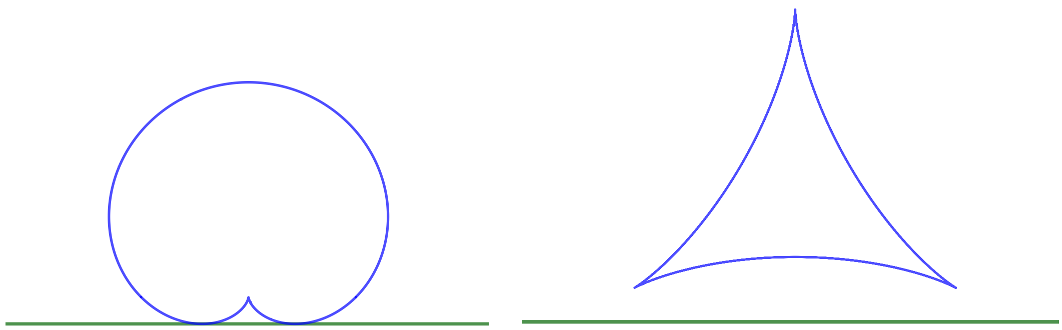


FIGURE 3.2: Hyperbolic and elliptic third-class curves with double tangents.

[8]The class of a curve is the number of tangents to the curve that can be drawn from a generic point in the plane.

[9]This terminology probably reflects that Clifford was influenced by the Newtonian tradition of organic geometry, which was popular in England at the time.

[10]Nowadays the term *bitangents* is more commonly used.

[11]Concerning the tricuspid hypocycloid, Clifford quoted the article *Sur l'hypocycloïde à trois rebroussements* by Luigi Cremona (1830–1903), which was published in *Crelle's Journal* in 1865.

He then introduced the concept of a *double parabola*, a third-class curve that has the line at infinity as a double tangent and any two points at infinity as points of contact (Clifford, 1871, p. 133) (see Figure 3.3).

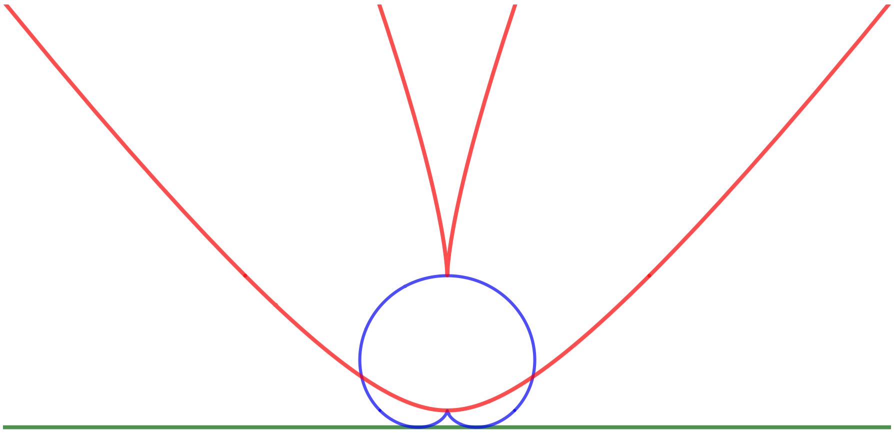


FIGURE 3.3: Double parabola.

In general, a third-class curve is determined by nine tangents. However, a double parabola is characterised by a double tangent and is therefore completely determined by six tangents. Therefore, there exists a simple infinity of double parabolas tangent to any five given lines. Furthermore, since the focus[12] of a curve is the point of intersection of the tangents drawn through the two circular points at infinity, in a double parabola there is only one focus. Denoting the circular points at infinity as I and J, and given an arbitrary line drawn through I, among the double parabolas that are tangent to five given lines there exists only one that has the line through I as a tangent. Once this line is fixed, there exists only one line through J that is tangent to the resulting double parabola. This follows from the fact that the double parabola is a third-class curve and has the line at infinity as a double tangent. Naturally, the converse also holds: given a tangent through J, the tangent through I is uniquely determined. These two tangents intersect at the focus of the double parabola, and the locus of the foci of the double parabolas tangent to five lines is a conic section passing through the two circular points at infinity, *i.e.*, a circle. Clifford thus proved that the locus of the foci of all double parabolas tangent to five given lines is a circle. Among the double parabolas tangent to five given lines, there are five degenerate cases, each consisting of the point at infinity on one of the five lines and the parabola tangent to the other four lines. Since the focus of a degenerate form is the focus of the ordinary parabola, Clifford proved as a corollary to the above that the foci of the five parabolas that have four of the five given lines as tangents all lie on a single circle.[13] Whereas there is only one double parabola Γ tangent to six lines, he deduced a second corollary: given six lines and omitting one of them at a

[12]For the definition of the focus of a curve, and more generally for notions closely related to the geometry of plane curves, see (Salmon, 1873).

[13]Note that Clifford obtained Miquel's Pentagram Theorem using Salmon's proof.

time, one obtains six circles that all intersect at the focus of Γ. Summarising the results presented so far, a chain of theorems begins to emerge.

1. Given three lines in a plane intersecting pairwise, there exists a circle that passes through the three intersection points (see Figure 3.4).[14]

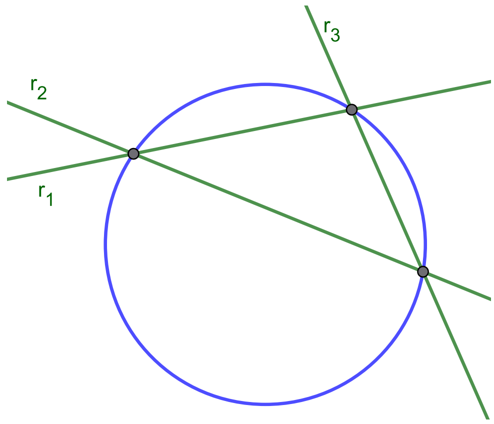


FIGURE 3.4: Clifford's first chain – three lines.

2. Given four lines in a plane intersecting pairwise, the four circles determined in the preceding step intersect at a single point (see Figure 3.5).[15]

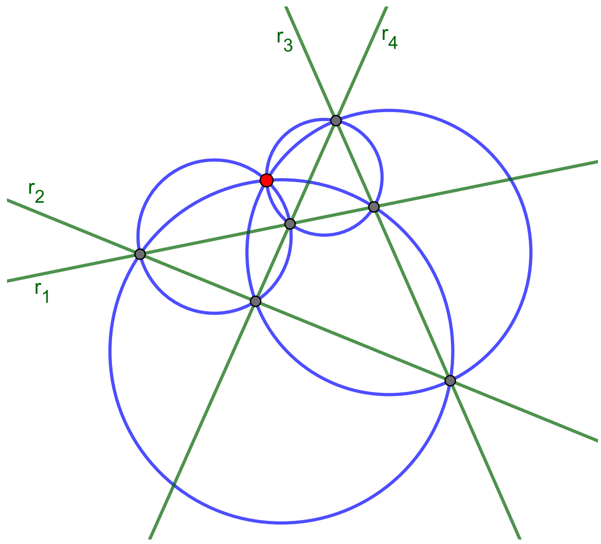


FIGURE 3.5: Clifford's first chain – four lines.

3. Given five lines in a plane intersecting pairwise, the five points determined in the preceding step lie on a single circle (see Figure 3.6).[16]

[14] This is the Proposition 5 from Book IV of Euclid's *Elements*.
[15] This is the Wallace–Steiner question.
[16] This is Clifford's Corollary 1.

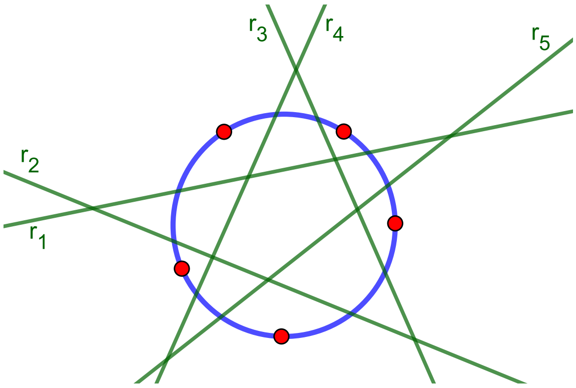


FIGURE 3.6: Clifford's first chain – five lines.

4. Given six lines in a plane intersecting pairwise, the six circles determined in the preceding step all intersect at a single point (see Figure 3.7).[17]

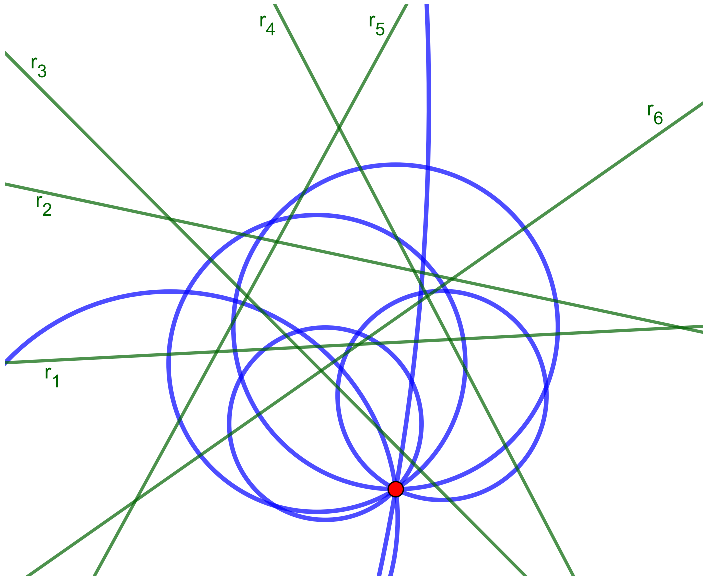


FIGURE 3.7: Clifford's first chain – six lines.

Continuing to use purely synthetic methods, Clifford proved that this chain of theorems can be extended indefinitely, namely:

- $2n+1$ lines in a plane that intersect pairwise determine $2n+1$ points lying on a single circle,
- $2n+2$ lines in a plane, intersecting pairwise, determine $2n+2$ circles that all intersect at a single point.

In his proof, Clifford retraced the reasoning discussed above, starting from a curve of class $n+1$ tangent to the line at infinity n times, which he called an *n-fold parabola*. Since such a curve is completely determined by $2n+2$

[17]This is Clifford's Corollary 2.

tangents, there exists a simple infinity of n-fold parabolas tangent to $2n+1$ given lines, and the locus of the foci of these curves is a conic section passing through the two circular points at infinity, *i.e.*, a circle. Similarly, among the n-fold parabolas tangent to $2n+1$ given lines, there exist $2n+1$ degenerate cases, each consisting of the point at infinity on one of the $2n+1$ lines and the $(n-1)$-fold parabola[18] determined by the remaining $2n$ lines. The focus of a degenerate form coincides with the focus of the corresponding $(n-1)$-fold parabola. Therefore, given $2n+1$ lines in a plane intersecting pairwise, the foci of the $2n+1$ $(n-1)$-fold parabolas tangent to $2n$ given lines all lie on a single circle. Furthermore, given $2n+2$ lines in a plane intersecting pairwise, omitting one line at a time results in $2n+2$ circles that all intersect at a single point, *i.e.*, the focus of the n-fold parabola tangent to the given lines. In this way, Clifford showed that the series of theorems described above extends indefinitely.[19]

In 1920, Walter Buckingham Carver (1879–1961) in (Carver, 1920) posed the question of whether exceptional cases might exist in which the theorems of Clifford's chain fail to hold. As early as 1878, Seligmann Kantor[20] (1857–1903) had already observed in (Kantor, 1878) that if five given lines are tangent to a tricuspid hypocycloid, the five associated points lie on a single straight line rather than on a single circle (see Figure 3.8). However, if six lines are tangent to a tricuspid hypocycloid, they determine six new lines which do not intersect at a single point, but which still possess the property of being tangent to another hypocycloid.

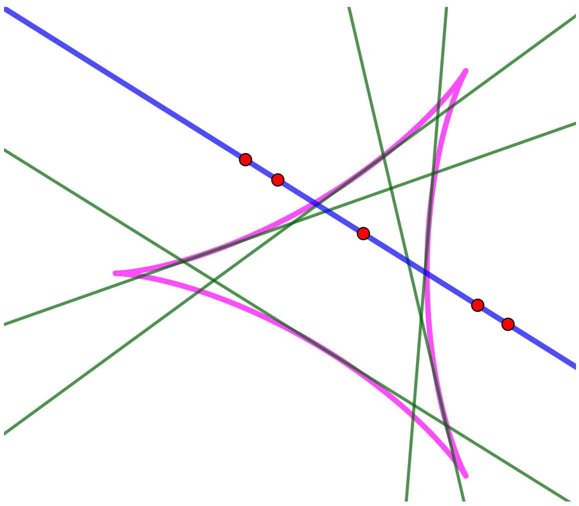


FIGURE 3.8: An exceptional case proved by Kantor in 1878.

[18]Analogously to the n-fold parabola, the $(n-1)$-fold parabola is defined as a curve of class n that is tangent to the line at infinity $n-1$ times.

[19]As Grace noted in (Grace, 1900, p. 193), Clifford's first chain was also obtained in 1895 by Paul Serret (1827–1898) in (Serret, 1895b), although he arrived at the result in a very different way and without reference to Clifford's construction. Serret considered a generalisation of the concept of a rectangular hyperbola, *i.e.*, a curve of order n whose asymptotes intersect at a point and are parallel to the sides of a regular polygon. Hence, the locus of the foci considered by Clifford corresponds to the locus of the intersection points of the asymptotes.

[20]Kantor had presented in (Kantor, 1877) a series of theorems corresponding to those of Clifford's chain.

In 1900, Frank Morley (1860–1937) provided in (Morley, 1900) an analytical proof of the theorems of Clifford's chain and demonstrated that an odd number of lines satisfying certain conditions will determine a straight line rather than a circle. In the concluding section of the same article, Morley laid the groundwork for the developments he would later formalise in (Morley, 1929), where he obtained Clifford's chain as a property of a sequence of curves C^n, each with $n-2$ cusps, and their osculants.

In the same 1871 paper, Clifford, starting from the theorems concerning an odd number of lines in the chain that he had developed, introduced a new series of theorems:

1. If, from any point on the circle determined by three lines, perpendiculars are dropped to these lines, then the feet of these perpendiculars lie on a single line. This line is known as the Simson–Wallace line[21] (see Figure 3.9).

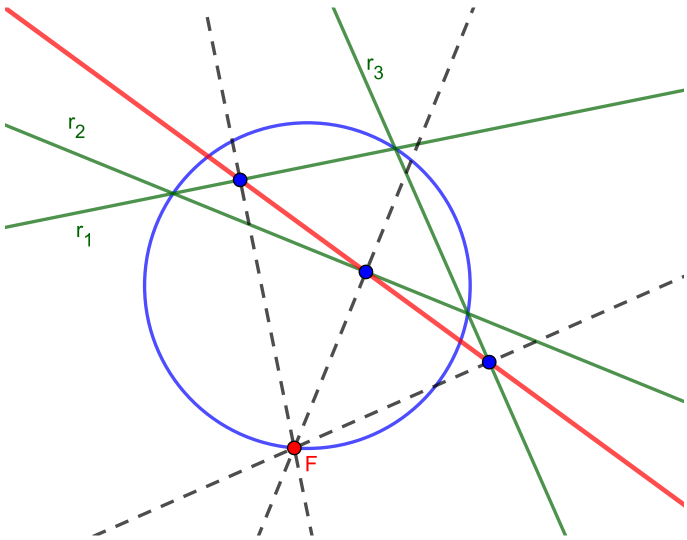


FIGURE 3.9: The Simson–Wallace line.

2. If, from any point F on the circle determined by five lines, perpendiculars are dropped to these lines, then the feet of these perpendiculars all lie on a conic passing through F (see Figure 3.10).

[21] For more details on the historical origins of the Simson–Wallace line, see (Palladino and Vaccaro, 2016) and (Palladino and Vaccaro, 2018).

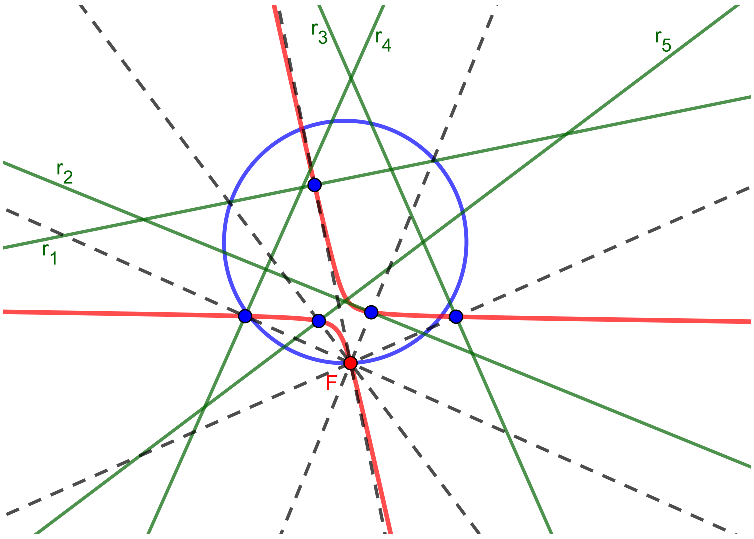


FIGURE 3.10: The conic associated with the five-line configuration.

In general, if, from any point F on the circle determined by $2n + 1$ lines, perpendiculars are dropped to these lines, then the feet of these perpendiculars all lie on a curve of order n passing through F with multiplicity $n - 1$. To prove this, Clifford used a *quadric inversion*, *i.e.*, a quadratic transformation previously studied by Thomas Archer Hirst (1830–1892) in (Hirst, 1865).[22] Hirst defined the quadric inversion starting from a fundamental conic and a fixed point A, taken as the origin. The inverse P' of a point P is the intersection of the polar of P with respect to the conic and the line joining P and A (see Figure 3.11).[23]

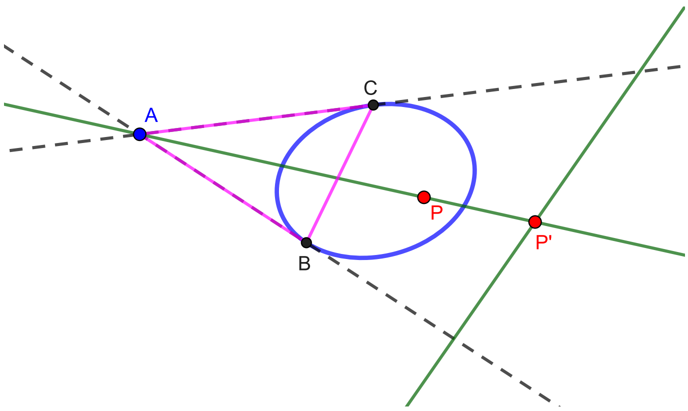


FIGURE 3.11: Hirst's quadric inversion.

[22]Hirst's paper, translated into Italian by Cremona and published in the same year in the *Annali di Matematica Pura ed Applicata*, was situated within the broader framework of quadratic transformations, a topic about which a great deal was being written at the time. An analysis of the scientific collaboration between Cremona and Hirst, based on their correspondence, can be found in (Vaccaro, 2016).

[23]Note that a circular inversion is a special case of a quadric inversion, which is also known as *triangular inversion*.

Key points of Hirst's article that clarify how Clifford employed quadric inversion in his proof are now highlighted. Adopting the notation introduced by Ludwig Immanuel Magnus (1790–1861) in (Magnus, 1832), Hirst used the term *principal points* to refer to the origin A and the two points of contact B and C of the tangents drawn from A to the fundamental conic, the term *principal triangle* to refer to the triangle having these three points as its vertices, and the term *principal lines* to refer to the polars of the principal points, *i.e.*, to the lines BC, AB, and AC. With the exception of the principal points, each of which is the inverse of every point on the corresponding polar, every point P has a unique inverse. In particular, every point on the conic is the inverse of itself. Hirst proceeded to analyse the relationship between the order of a curve and the order of its inverse. He stated that the complete quadric inverse of a curve of order n is a curve of order $2n$, with a point of multiplicity n at each of its three principal points. Hirst used the term *complete* because the inverse curve decomposes into one or more principal lines, each counted one or more times, and into an actual inverse curve of lower order that passes through the principal points. This occurs precisely when the starting curve passes through one or more principal points. Denoting by a, b, and c the multiplicities of the principal points A, B, and C, respectively, in the starting curve, if the said curve has branches passing through any principal point, then the order of its inverse is given by

$$n' = 2n - a - b - c.$$

Furthermore, denoting by a', b', and c' the multiplicities of the principal points A, B, and C, respectively, in the transformed curve, the following relationships hold:

$$\begin{aligned} a' &= n - b - c, \\ b' &= n - a - c, \\ c' &= n - a - b. \end{aligned}$$

Clifford proved that the locus of the feet of the perpendiculars dropped from the focus F of the n-fold parabola to the $2n+1$ tangents is the inverse of the reciprocal polar of the curve with respect to a circle centred at F. Since an n-fold parabola is tangent to the lines joining F and the circular points, and also tangent n times to the line at infinity, its reciprocal polar is a curve of order $n+1$ passing through F with multiplicity n and through each circular point exactly once.[24] Applying a quadric inversion where the fundamental conic is a circle centred at the focus F, and the three principal points are the two simple circular points and F with multiplicity n, one finds that the inverse of the reciprocal polar of an n-fold parabola is a curve of order n that passes through F with multiplicity $n-1$ but does not pass through the circular points.

[24] Reciprocal polarity with respect to a conic interchanges the order and the class of a curve. Since an n-fold parabola is a curve of class $n+1$, its reciprocal polar is a curve of order $n+1$.

3.3 Clifford's first chain under circular inversion

Clifford's first chain attracted the interest of several mathematicians between the late 19th and early 20th centuries. Among them was Grace, who proved the theorems of the series in 1898 by applying a circular inversion with respect to a point O[25] in the plane containing the starting lines (Grace, 1898, pp. 169-171).

The first step of the chain involves three lines, r_1, r_2, and r_3, intersecting pairwise at points Q_{12}, Q_{13}, and Q_{23}.[26] Under a circular inversion with centre O, these lines are transformed into three circles Γ_1, Γ_2, and Γ_3, all passing through O and sharing a second point pairwise. These three additional points of intersection, P_{12}, P_{13}, and P_{23},[27] determine another circle, Γ_{123}, which is the image under inversion of the circle defined by the three starting lines. This yields the symmetric configuration consisting of the four points P_{12}, P_{13}, P_{23}, and O, and the four circles Γ_1, Γ_2, Γ_3, and Γ_{123}, with the property that each circle passes through three of the points, and each point lies on three of the circles (see Figure 3.12). This symmetric configuration is referred to as $\mathcal{CL}_3$.

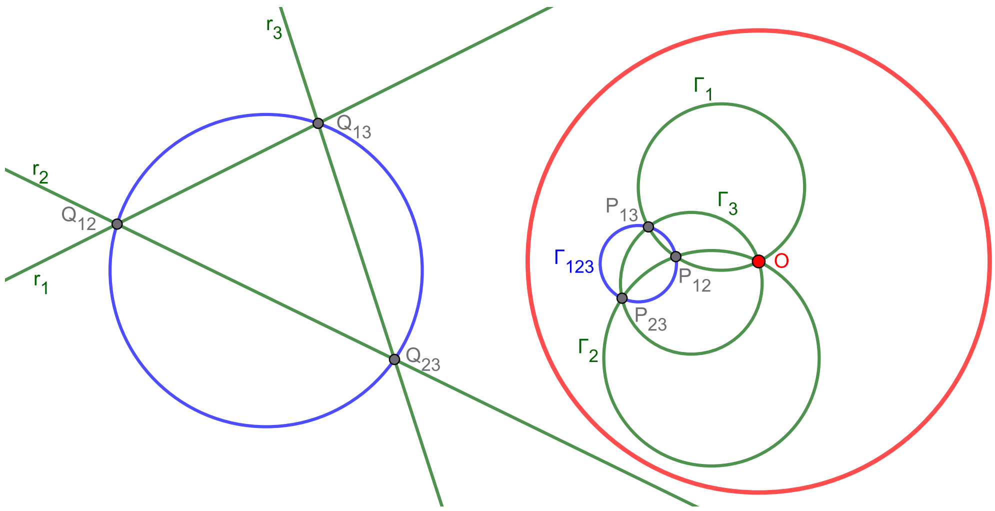


FIGURE 3.12: Clifford's configuration $\mathcal{CL}_3$.

These observations reveal a structural connection with Combinatorics, especially in the context of symmetric designs.
A $t-(v,k,\lambda)$ design $(v \geq k \geq t \geq 1 \quad \text{and} \quad \lambda \geq 1)$ is a pair $(\mathcal{X}, \mathcal{A})$ that satisfies the following properties:

- $\mathcal{X}$ is a set of v elements, called *points*;
- $\mathcal{A}$ is a family of subsets of $\mathcal{X}$, each of cardinality k, called *blocks*;

[25]By circular inversion with respect to a point is meant circular inversion with respect to any circle that has its centre at the point in question (the radius is not relevant to the present discussion).
[26]Q_{ij} denotes the point of intersection of the lines r_i and r_j.
[27]It should be noted that P_{12}, P_{13}, and P_{23} are the inverses of the points Q_{12}, Q_{13}, and Q_{23}.

- every t-subset of distinct points of $\mathcal{X}$ occurs in exactly λ blocks.

A t-design is *symmetric* if the number of blocks equals the number of points, that is, if $|\mathcal{A}| = v$.[28] Thus, the configuration $\mathcal{CL}_3$ can be regarded as a geometric representation of the *biplane*, *i.e.*, a $2-(4,3,2)$ symmetric design, whose blocks are the circles Γ_1, Γ_2, Γ_3, and Γ_{123}, and exactly two blocks pass through any two distinct points.[29]

Considering the second step of Clifford's chain, four lines in the plane, r_1, r_2, r_3, and r_4, intersecting pairwise, are transformed under a circular inversion with respect to the point O into four circles Γ_1, Γ_2, Γ_3, and Γ_4, all passing through O and intersecting pairwise at the six additional points P_{12}, P_{13}, P_{14}, P_{23}, P_{24}, and P_{34}. According to the previous step, these six points, taken three at a time, define four further circles: Γ_{123}, Γ_{124}, Γ_{134}, and Γ_{234}, which all intersect at a common point P_{1234}.[30] The symmetric configuration $\mathcal{CL}_4$ is thus obtained, consisting of the eight points P_{12}, P_{13}, P_{14}, P_{23}, P_{24}, P_{34}, P_{1234}, and O, and the eight circles Γ_1, Γ_2, Γ_3, Γ_4, Γ_{123}, Γ_{124}, Γ_{134}, and Γ_{234}, with the property that each circle passes through exactly four points, and each point lies on exactly four circles (see Figure 3.13).

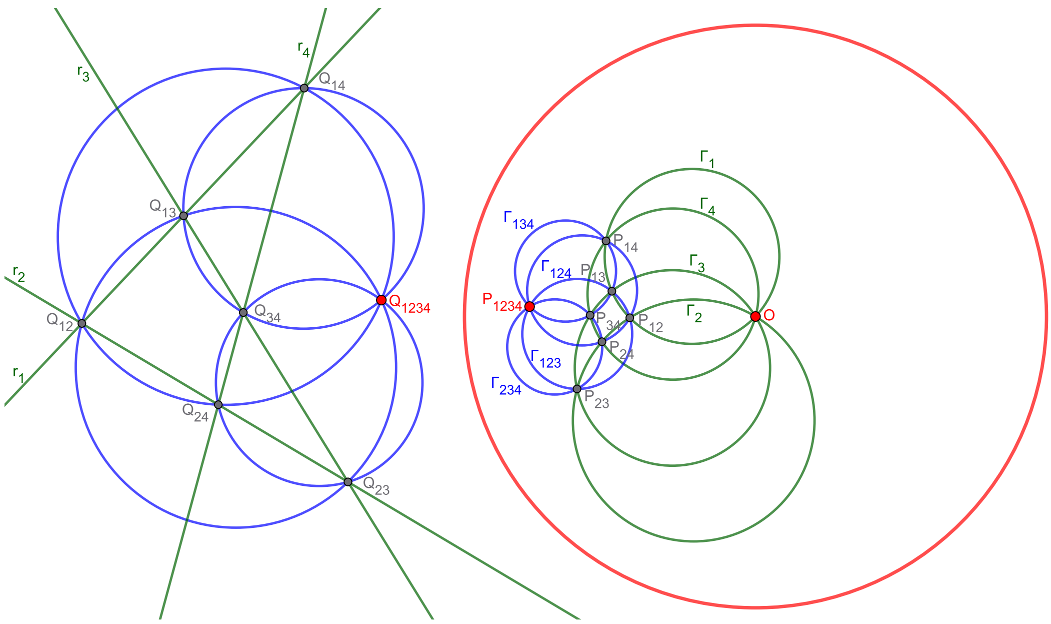


FIGURE 3.13: Clifford's configuration $\mathcal{CL}_4$.

Motivated by the analogy with configuration $\mathcal{CL}_3$, the author has investigated the possibility of interpreting $\mathcal{CL}_4$ as a 2-design. However, this interpretation cannot be sustained, as certain pairs of points, such as O and P_{1234}, do not lie simultaneously on any circle of the configuration. Such pairs will be referred to as *opposite points* in Chapter 5.

In general, the theorems of Clifford's chain are invariant under circular inversion, resulting in more harmonious configurations. Given $m \geq 3$ circles

[28]For details on the theory of t-designs, see (Rosen, 2000).
[29]This interpretation of the configuration $\mathcal{CL}_3$ as a biplane was first observed by Dorwart in (Dorwart, 1988).
[30]P_{1234} is the image of the point Q_{1234} determined by the four starting lines.

passing through the same point (the inverses of m starting lines), one obtains the complete configuration $\mathcal{CL}_m$, consisting of 2^{m-1} points and 2^{m-1} circles, with m points on each circle and m circles passing through each point.[31] This configuration, also denoted by $(2^{m-1})_m$,[32] is completely symmetric in that all its elements (points and circles) play the same role: each point lies on an equal number of circles, and each circle passes through an equal number of points.[33] Despite this symmetry, it should be noted that, except for the case $m = 3$, the configurations $\mathcal{CL}_m$ do not satisfy the conditions to be interpreted as 2-designs.

3.4 Extensions of Clifford's first chain in space of higher dimensions: Grace's contribution

This section will discuss attempts to determine analogous chains of theorems to Clifford's in spaces of higher dimensions. In particular, the focus will be on the contributions of Grace in three dimensions and those of Grace and Brown in four dimensions.

Before defining a chain similar to Clifford's in the three-dimensional space, Grace firstly wanted to construct a series of plane configurations that generalise such a chain. His starting point was the Triangle Theorem, *i.e.*, the property already demonstrated by Miquel in (Miquel, 1838a, p. 486).[34] Grace proved this theorem without making any reference to the French mathematician, by using the theorem according to which every cubic curve passing through eight fixed points also passes through a ninth. In fact, as can be seen in Figure 2.5, the three circles and the three lines, taken together, can be regarded as forming three degenerate circular cubics that, having eight points in common (the six points of the configuration along with the two circular points at infinity), intersect at an additional point (Grace, 1898, p. 167). Hence the following:

1. Given three lines intersecting pairwise and a point on each line, the three circles passing through a vertex of the triangle formed by these lines, and through the two points on the sides emerging from that vertex, all intersect at a single point (see Figure 2.5).[35]

[31] Dennis William Babbage (1909–1991) presented in (Babbage, 1969) the special case in which the circles of the obtained configuration are all congruent so long as the starting circles are. The case of three circles had already been proved in 1916 by Roger Arthur Johnson (1890–1954) in (Johnson, 1916).

[32] Using this notation, $\mathcal{CL}_3 = 4_3$ and $\mathcal{CL}_4 = 8_4$.

[33] For more details on configurations of points and circles, see (Dorwart, 1988). In his article, Dorwart pointed out that Theodor Reye (1838–1919) had mentioned the configurations 8_4 and 16_5 in a paper published in 1882. Furthermore, Dorwart noted that Reye was the first to use the term "configuration" in the context it is currently understood in geometry, in 1876.

[34] See also Section 2.1.

[35] This is the Triangle Theorem.

2. Given four lines intersecting pairwise and four concyclic points, one on each line, there are four sets of three lines each. The four points determined by the preceding step all lie on a single circle (see Figure 3.14).

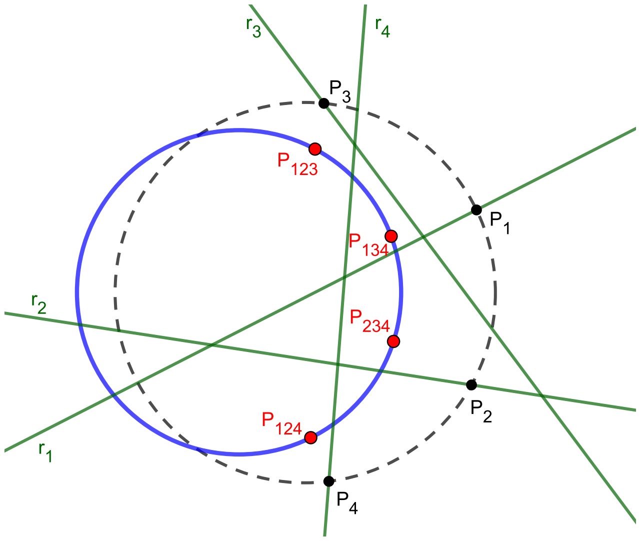


FIGURE 3.14: The second step in Grace's chain in the plane.

3. Given five lines intersecting pairwise and five concyclic points, one on each line, there are five sets of four lines each. The five circles determined by the preceding step all intersect at a single point (see Figure 3.15).

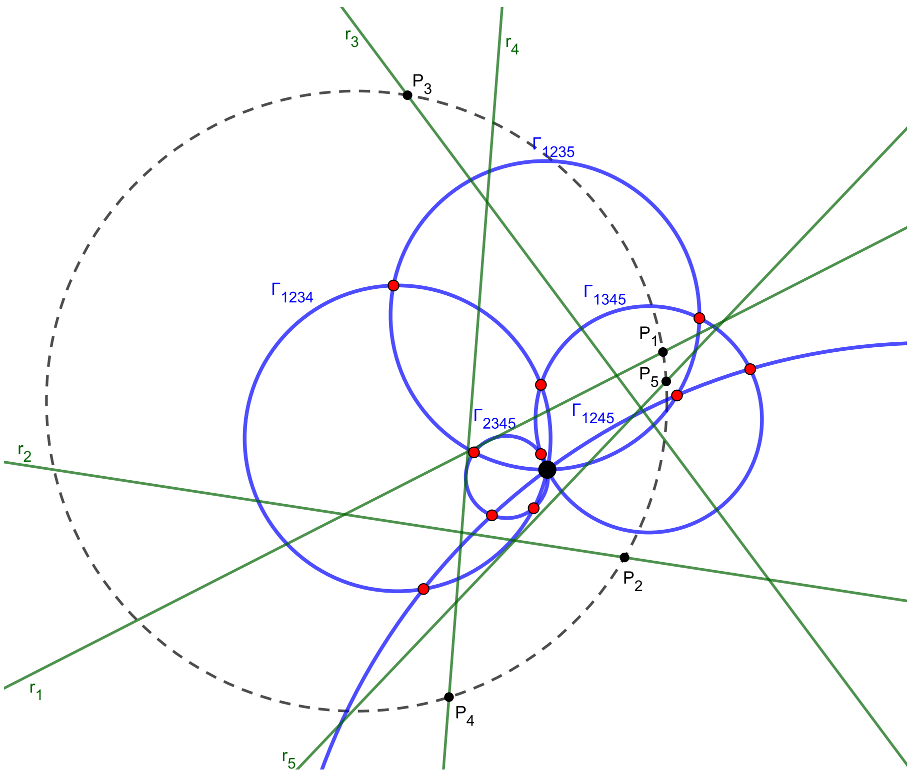


FIGURE 3.15: The third step in Grace's chain in the plane.

To prove the theorems of this new series, Grace employed the Triangle Theorem and the configurations of Clifford's chain under circular inversion. The series, which extends indefinitely via recursion, reduces to Clifford's chain when the circle on which the starting points lie degenerates into a straight line (Grace, 1898, pp. 171-173).

In the three-dimensional space, it is known that four planes, intersecting three at a time at a point, define a tetrahedron whose vertices, A, B, C, and D, lie on a sphere. Applying a spherical inversion, *i.e.*, an inversion with respect to a sphere centred at a point O in said space, these planes are transformed into four spheres all passing through O and sharing, three at a time, four additional points, A', B', C', and D'. These points are, respectively, the images of the original tetrahedron's vertices A, B, C, and D under the spherical inversion centred at O (see Figure 3.16). Furthermore, these four additional points determine another sphere, which is the image of the sphere circumscribed about the original tetrahedron. Thus, one obtains the symmetric configuration 5_4, consisting of five points and five spheres, with four points on each sphere and four spheres passing through each point.

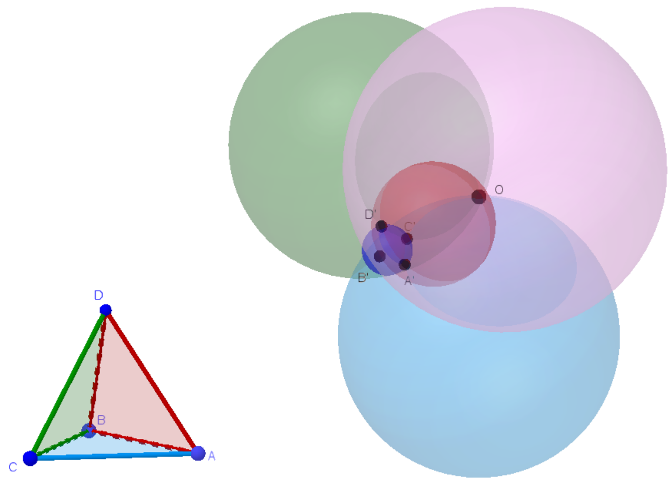


FIGURE 3.16: Grace's configuration 5_4.

It should be noted that this configuration is a geometric representation of the $2-(5,4,3)$ symmetric design, whose blocks are the spheres, and exactly three spheres pass through any two distinct points. This observation is made here to emphasise its combinatorial significance.

In attempting to define the analogue of the Clifford chain in the three-dimensional space, Grace posed the problem of extending the Wallace–Steiner question; that is, whether the five spheres that circumscribe the five tetrahedra determined by five planes, each triple of which intersects at a point, all pass through the same point. By means of purely theoretical considerations, he proved (Grace, 1898, pp. 162-163) that in three dimensions there is no analogue of the Wallace–Steiner question, but, as will be shown later, there is one in the four-dimensional space. Thus, Grace's objective shifted to constructing a series of configurations in space that would generalise the chain he had defined in the plane. To this end, he proved the

Tetrahedron Theorem, which is the three-dimensional equivalent of the Triangle Theorem: given a tetrahedron and a point on each of its edges (or their extensions), the four spheres passing through one vertex of the tetrahedron and through the three points on the edges emerging from that vertex intersect at a single point (see Figure 3.17).

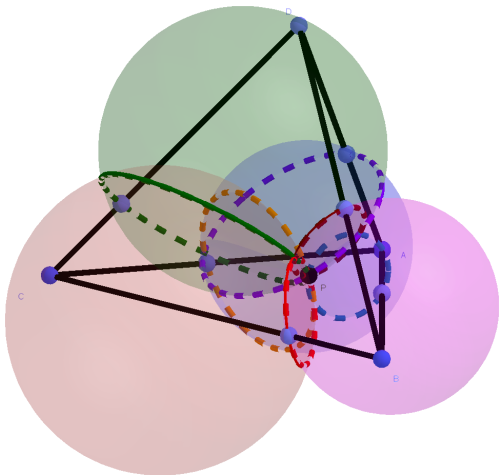


FIGURE 3.17: The Tetrahedron Theorem.

As shown earlier in Chapter 2, this theorem had already been proved by Miquel in 1838. However, Grace was almost certainly unaware of Miquel's proof, as he attributed the theorem to Samuel Roberts (1827–1913) (Roberts, 1881, pp. 117-118).[36] In his proof, Grace considered cubic cyclides, that is, cubic surfaces passing through the circle at infinity. Each sphere passing through one vertex of the tetrahedron and through the three points on the edges emerging from it (see Figure 3.17), taken together with the plane containing the face opposite that vertex, forms a degenerate cubic cyclide. Each such cubic surface passes not only through the ten specified points (the four vertices and the six points, one on each edge of the tetrahedron), but is also required to pass through five additional points. Considering the spheres three at a time and intersecting them with the plane containing the three vertices through which they pass, the three circle sections intersect at a single point in the plane, in accordance with the Triangle Theorem. Since each cubic cyclide not only passes through the aforementioned ten points but also contains four other points, one on each face of the tetrahedron, there exists a further point common to all four spheres considered (Grace, 1898, p. 168).
Grace used this theorem under a spherical inversion centred at a point O in space. Thus, the four planes defining a tetrahedron are transformed into four spheres S_a, S_b, S_c, and S_d, all passing through O, and the edges of the tetrahedron become the circle sections of these spheres. These circles, taken three at a time, identify four intersection points, P_{abc}, P_{abd}, P_{acd}, and P_{bcd}, images of the tetrahedron's vertices. Considering six points, P_{ab}, P_{ac}, P_{ad}, P_{bc}, P_{bd}, and P_{cd}, one on each circle section, it follows that the four spheres

[36] Julian Lowell Coolidge (1873–1954), in (Coolidge, 1910), noted that this theorem proved by Roberts had already been implicitly established by Miquel in (Miquel, 1845).

determined by the quaterns of points (P_{abc}, P_{ab}, P_{ac}, P_{bc}), (P_{abd}, P_{ab}, P_{ad}, P_{bd}), (P_{acd}, P_{ac}, P_{ad}, P_{cd}), and (P_{bcd}, P_{bc}, P_{bd}, P_{cd}), drawn in grey in Figure 3.18, all pass through the same point P.

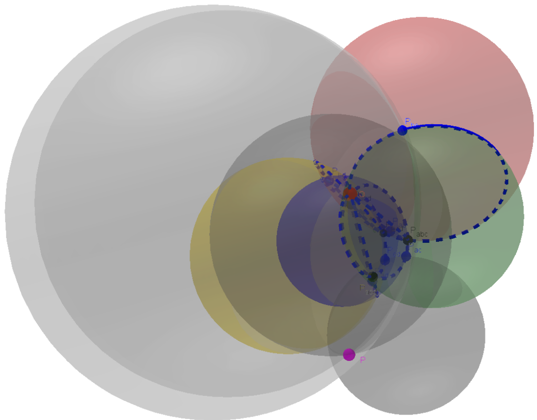


FIGURE 3.18: The Tetrahedron Theorem under a spherical inversion.

Thus, Grace established a series of configurations in the three-dimensional space that extended the chain he had defined in the plane:

1. Given three planes intersecting at a point, and a point on each line of intersection, one obtains four points that all lie on a single sphere.

2. Given four planes that intersect three at a time at a point, and a point on each intersection line, there are four sets of three planes each, and the four spheres obtained from these sets by the preceding step intersect at a single point (see Figure 3.17).[37]

3. Given five planes that intersect three at a time at a point, and a point on each intersection line, there are five sets of four planes each, and the five points obtained from these sets by the preceding step all lie on a single sphere.

4. Given six planes that intersect three at a time at a point, and a point on each intersection line, there are six sets of five planes each, and the six spheres obtained from these sets by the preceding step intersect at a single point.

After proving the first steps of this series via the Tetrahedron Theorem under spherical inversion, Grace asserted that the chain can be extended indefinitely and suggested a recursive proof (Grace, 1898, pp. 173-177).
Grace did not identify any symmetry analogous to that found in Clifford's chain under circular inversion. However, by assuming that the points on the intersection lines of the planes lie in a common plane, he constructed three configurations of points and spheres based on m planes:

- 16 points and 10 spheres for $m = 5$;

[37]This is the Tetrahedron Theorem.

- 72 points and 27 spheres for $m = 6$;
- 576 points and 126 spheres for $m = 7$.

In each case, every sphere contains 2^{m-2} points, and each point lies on m spheres. Grace did not exclude the possibility of constructing other configurations with a greater number of planes. In fact, for $m = 8$ he conjectured the existence of a configuration of points and spheres with 64 points on each sphere and 8 spheres passing through each point. However, due to the complexity arising from the large number of points, he concluded the relevant section of his paper without providing further details (Grace, 1898, pp. 181-188).

In the four-dimensional space, it is first observed that five hyperplanes, intersecting four at a time at a point, define a hypersphere circumscribed about the polytope extension of the tetrahedron, the *simplex*,[38] formed by these hyperplanes. When the five hyperplanes are inverted with respect to a point O in said space, they are transformed into five hyperspheres passing through O, any four of which intersect at an additional common point. The five points thus determined, corresponding to the vertices of the simplex, all lie on a single hypersphere (the image of the hypersphere circumscribed about the simplex). This results in the symmetric configuration 6_5, consisting of six points and six hyperspheres, with five points on each hypersphere and five hyperspheres passing through each point. As in the previously examined cases in the plane and in three-dimensional space for the configurations 4_3 and 5_4, the configuration 6_5 is a geometric representation of the $2-(6,5,4)$ symmetric design, whose blocks are the hyperspheres, and exactly four hyperspheres pass through two distinct points.
Given six hyperplanes intersecting four at a time at a point, Grace demonstrated the analogue of the Wallace–Steiner question in the four-dimensional space: the six hyperspheres obtained by excluding one hyperplane at a time all intersect at a single point. To prove this, he pointed out that each hypersphere, taken together with the omitted hyperplane, forms a degenerate cubic hypersurface passing through the sphere at infinity and through the 15 intersection points of the hyperplanes taken four at a time. After establishing the number of linear conditions that the six cubic hypersurfaces must satisfy, he deduced that, in addition to these 15 points, they must have 16 additional points in common. To determine 15 of these additional points, the Wallace–Steiner question can be applied to the 15 planes of intersection of the six hyperplanes, taken two at a time. Consequently, he concluded that the six hyperspheres intersect at one additional point. He then employed an inversion to derive a configuration of 32 points and 12 hyperspheres, with 16 points on each hypersphere and 6 hyperspheres passing through each point. This proof could constitute, in the four-dimensional space, the beginning of a chain analogous to Clifford's, but Grace did not proceed further (Grace, 1898, pp. 163-164).
In 1954, many years after Grace's article, Brown in (Brown, 1954) sought to advance research in this area by attempting to develop the subsequent steps

[38] For the nomenclature and properties of polytopes, see Appendix A.

in Grace's chain in the four-dimensional space. Specifically, he sought to determine the analogue of the third step in Clifford's chain. In so doing, Brown demonstrated the existence of a configuration of 576 points and 56 hyperspheres, with 72 points on each hypersphere and 7 hyperspheres passing through each point. To achieve this result, he used, in addition to the six-hypersphere theorem proved by Grace, a purely computational method. The configuration found by Brown, however, does not ensure the validity of the conjecture of the theorem from which it originates, *i.e.*, that the seven points obtained from seven hyperspheres passing through one point all lie on a single hypersphere. Longuet-Higgins provided an exhaustive answer to this problem in (Longuet-Higgins, 1973a, pp. 37-38), proving, by means of numerical methods that he had already used in an earlier article (Longuet-Higgins, 1971a), that there is no third step in the chain started by Grace.

3.5 Generalisation of the Wallace–Steiner question and Miquel's Triangle Theorem in higher dimensions

Chapter 3 has outlined the history of the chain of theorems developed by Clifford, which stems from the Wallace–Steiner question, and the planar chain devised by Grace, whose starting point is Miquel's Triangle Theorem. Miquel undoubtedly played a key role in the development of both chains, not only because in 1838 he proved the two theorems from which the chains originated, but especially because, in the same year, he also presented the Tetrahedron Theorem, which is an extension of the Triangle Theorem to the three-dimensional space.

As for the generalisation of the Wallace–Steiner question to the n-dimensional space, it is only possible when n is even. At the end of the 19th century, Grace demonstrated it in four dimensions (Grace, 1898, pp. 163-164) and Hermann Kühne (1867–1907) proved it in the n-dimensional space (Kühne, 1898). Regarding the extension of the Triangle Theorem, in addition to being proved in three dimensions by Miquel himself in (Miquel, 1838b, pp. 521-522), Roberts in (Roberts, 1881, pp. 117-118) and Grace in (Grace, 1898, p. 168), it was also generalised to n-dimensional space by Mellen Woodman Haskell (1863–1948) in (Haskell, 1903).[39]

These questions of elementary geometry, in which many mathematicians were involved from the end of the 19th century to the middle of the 20th century, continue to fascinate scholars in the new millennium. In fact, in 2009, Hiroshi Maehara and Norihide Tokushige published a paper (Maehara and Norihide, 2009) in which they extended both the Wallace–Steiner question and Miquel's theorem to the n-dimensional space. Their work not only serves as evidence of relatively recent interest in such issues, but is also noteworthy because the key idea of their proof is closely related to

[39] For more details, see also (Baker, 1924), (Baker, 1925) and (Gabbatt, 1926).

n-dimensional polytopes, which constitute the focus of Longuet-Higgins' article (Longuet-Higgins, 1972), which will be discussed later in Chapter 5.

Before concluding this chapter, it is worth emphasising that the chain of theorems developed by Clifford has been the subject of study by many mathematicians, in addition to those already mentioned. Among others are: Henri Lebesgue (1875–1941) in (Lebesgue, 1916); Francis Puryer White (1893–1969) in (White, 1925); Eric Harold Neville (1889–1961) in (Neville, 1926); H. Lob in (Lob, 1933); A. Narasinga Rao (1893–1967) in (Narasinga Rao, 1932, 1937a,b, 1938, 1945); Paul Ziegenbein (1905–1983) in (Ziegenbein, 1940); Herbert William Richmond (1863–1948) in (Richmond, 1941a,b). These studies, which explore various extensions, configurations, and interpretations of Clifford's theorems, represent a substantial corpus of geometric research. In consideration of the depth and historical significance of these developments, it is proposed that they may serve as a valuable foundation for the author's future research.

Chapter 4

The *Géométrie récurrente* of Gaston de Longchamps

At the same time as Clifford, and based on the same principle, though working entirely independently, the French mathematician de Longchamps, starting from objects related to triangle geometry, developed several series of theorems that depended on each other in an iterative way. This iterative nature led him to introduce, for the first time in 1877, the term *Géométrie récurrente* (recursive geometry) in (De Longchamps, 1877, p. 341) to describe a set of geometric constructions based on recursive processes applied to classical configurations, particularly those involving remarkable points of the triangle and associated circles. This terminology aimed to capture the systematic and repeatable character of certain geometric chains, where a property established at one step could be extended through successive analogous steps. In this regard, the recursive process underlying Clifford's first chain can be considered as an example of recursive geometry.

In this chapter, after a brief biographical overview of de Longchamps, the fundamental ideas behind his recursive geometry will be explored, tracing its origins and illustrating key examples of these chains. De Longchamps' arguments are reconstructed and accompanied by figures to clarify the geometric reasoning and enhance accessibility to the underlying processes. In particular, special attention will be given to chains related to the circumcentre, the Simson–Wallace line, and the nine-point circle associated with a triangle. The chapter concludes with a new chain related to the nine-point conic associated with a complete quadrangle. This chain is introduced by the author in the present study, in the spirit of de Longchamps' approach.

4.1 Biographical sketch of Gaston de Longchamps

Gaston Albert Gohierre de Longchamps was born on 1 March 1842 in Alençon, in the Orne region of France. Due to his family's financial difficulties, he was separated from them at the age of nine and sent to a provincial *lycée* (secondary school) in Douai, having been awarded a scholarship. In 1859, he moved to Paris to continue his education at Lycée Charlemagne and Lycée Louis-le-Grand. In 1863, he was admitted to the École Normale Supérieure, where he attended lectures by prominent mathematicians such as Victor Alexandre Puiseux (1820–1883), Charles Auguste Briot (1817–1882), Charles Hermite (1822–1901), and Jean Gaston Darboux (1842–1917), as well as by Michel Chasles (1793–1880) at the Sorbonne. He obtained a degree in both mathematical and physical sciences in 1865, and the following year began his teaching career in provincial lycées, including Mont-de-Marsan, Poitiers, and Niort. This teaching activity was temporarily interrupted by the outbreak of the Franco-Prussian War in 1870, during which he enlisted and served for several months as a second lieutenant. In 1878, he moved to Paris and became a mathematics teacher at Lycée Charlemagne. From 1890 onward, he held the position of professor of mathématiques spéciales at Lycée Saint-Louis. Teaching was clearly not de Longchamps' only professional activity. In 1882, he became editor of the *Journal de mathématiques élémentaires* and the *Journal de mathématiques spéciales*.[1] These periodicals featured studies, exercises, problems, and competition topics, along with solutions submitted by both professors and students. De Longchamps contributed numerous articles to both journals, occasionally signing with the pseudonym *Elgé*, a practice he began in 1895. In addition to numerous notes and papers published in journals devoted to elementary geometry,[2] such as *Nouvelles Annales de Mathématiques*, *Nouvelle Correspondance Mathématique* (renamed *Mathesis* in 1880), as well as the aforementioned periodicals, he also authored works of an educational nature. His scientific and pedagogical contributions earned him membership in several international scientific societies, including the Royal Society of Sciences of Liège, the Academy of Sciences of Amsterdam, the French Mathematical Society, the Society of Sciences of Prague, the Academy of Sciences of Lisbon, and the Society of Sciences of Guatemala. In 1900, he retired at his own request and, around the same time, was appointed examiner at the Saint-Cyr Military School, a position he held until his death on 9 July 1906.[3]

4.2 The chains of *Géométrie récurrente*

In 1864, de Longchamps wrote a short note on geometry (De Longchamps, 1864), which was presented to the editorial board of the *Revue des sociétés savantes* by Joseph-Alfred Serret (1819–1885), a member of the Académie des Sciences and professor at the Sorbonne. This work already anticipates the key idea underlying recursive geometry: for the first time, the chain related to the circumcentre is introduced, albeit with a slight inaccuracy.

[1] The *Journal de mathématiques élémentaires*, founded in 1877 by Justin Bourget and renamed in 1880 *Journal de mathématiques élémentaire et spéciales*, was split into the two periodicals *Journal de mathématiques élémentaires* and *Journal de mathématiques spéciales* in 1882, both under the co-directorship of de Longchamps. He was the sole editor of the *Journal de mathématiques spéciales* from 1891 to 1896, and of the *Journal de mathématiques élémentaires* from 1891 to 1897.

[2] His name is linked to several geometric objects, including the *de Longchamps point* and the *de Longchamps trisector*. The de Longchamps point of a triangle is the reflection of the orthocentre with respect to the circumcentre. The de Longchamps trisector is an unicursal cubic curve with the Cartesian equation $x(x^2 - 3y^2) = a(x^2 + y^2)$.

[3] For further details on the biography of de Longchamps, refer to (Brasseur, 2011, pp. 15-22) and (Lazzeri, 1907).

In the two subsequent articles published in 1866, (De Longchamps, 1866a)[4] and (De Longchamps, 1866b), he shifted his focus to quadratic transformations, which were a prominent research topic during the first half of the 19th century.[5] This subject provided a point of reference for the French mathematician, who, referring to Magnus' paper (Magnus, 1832), proved the following theorem: whenever, in a method of *comparative geometry*,[6] a point corresponds to another point, a line will correspond to a conic passing through three fixed points, and a curve of order m and class n will correspond to a curve of order $2m$ and class $2m+n$, having the three fixed points as points of multiplicity m. By applying a reciprocal polar transformation, that is, by interchanging points and lines through projective duality,[7] de Longchamps obtained quadratic transformations that map lines to lines, points to conics tangent to the three principal tangents, and conics to curves of degree six and class four, which possess three double tangent lines. After observing that the line-to-line correspondence can be established through an infinite number of geometric laws, and that the properties of higher-order curves derived from those of conics vary depending on the chosen law, he introduced the so-called *reciprocal transversals transformation*, which maps lines to lines and points to conics, as described below. Given a triangle ABC and a line r intersecting its sides (or their extensions) at points P, Q, and R, let P', Q', and R' be the reflections of P, Q, and R with respect to the midpoints M_{AB}, M_{BC}, and M_{AC} of the sides of triangle ABC. Then the points P', Q', and R' are collinear (see Figure 4.1). The line r' passing through them is known as the *reciprocal transversal* of r.

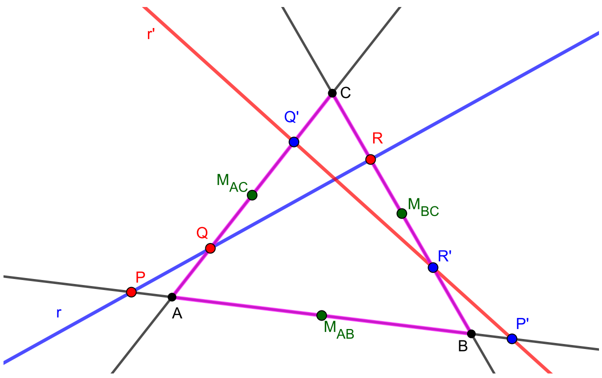


FIGURE 4.1: The reciprocal transversals r and r' associated to triangle ABC.

[4]This work attracted the attention of Chasles, who devoted almost an entire page to it at the end of his *Rapport sur les progrès de la géométrie* in 1870 (Chasles, 1870, p. 372).

[5]Transformations were employed to derive the properties of certain figures from the known properties of others.

[6]De Longchamps used the expression *Géométrie comparée*. In this thesis, it is thought that he referred to a method studying the correspondences between geometric figures of different orders and natures, analysing how properties and relationships transform from one figure to another.

[7]This approach had already been employed by Cremona in (Cremona, 1861-62). Although de Longchamps did not cite Cremona's article, the de Longchamps–Cremona correspondence, published in (Menghini, 1994), reveals an exchange of papers between the two mathematicians starting from 1873.

As shown in Figure 4.2, the line r' is parallel to the line passing through the three midpoints of the diagonals of the complete quadrilateral formed by triangle ABC and the line r. Furthermore, these midpoints and the points P', Q', and R' form two homothetic sets with centre at the centroid G of triangle ABC, and a homothety ratio of 2.

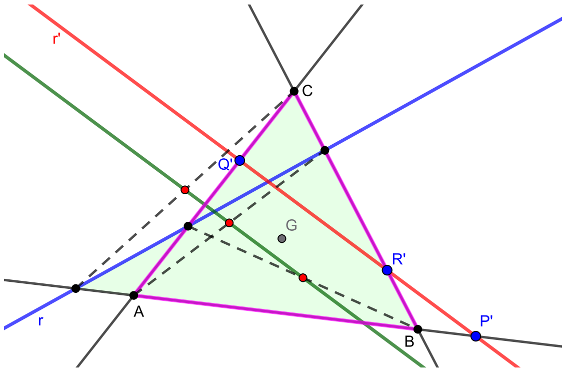


FIGURE 4.2: Property of reciprocal transversals.

In the reciprocal transversals transformation, to each point P corresponds the conic inscribed in the reference triangle ABC and tangent to the reciprocal transversals h' and k' of two lines h and k passing through P. The centre P' of this conic lies on the line joining P and the centroid G of triangle ABC, and satisfies the following relation: $\overline{GP} = 2 \cdot \overline{GP'}$ (see Figure 4.3).

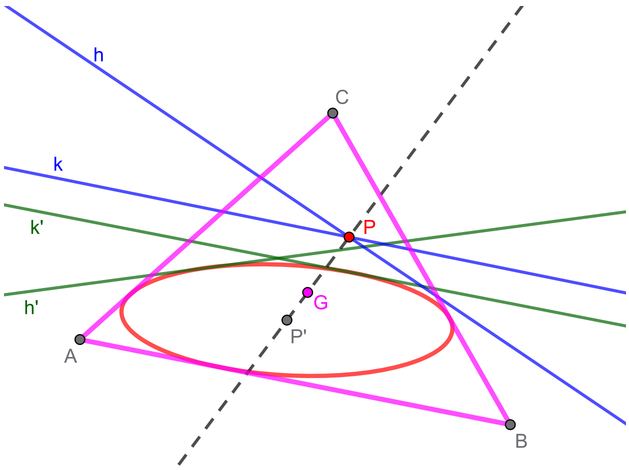


FIGURE 4.3: Image of a point.

By applying the method of transformation via reciprocal transversals, de Longchamps established a law allowing one to determine the centroid of a complete polygon, that is, a polygon whose sides are extended indefinitely, thus creating a chain of recursive geometry related to the centroid (De Longchamps, 1866a, pp. 325-330). The following is a broad outline of de Longchamps' reasoning, without delving into technical details. He began by proving the following theorem: given a triangle ABC and three lines, r, s, and t, intersecting pairwise on the sides of ABC, their reciprocal transversals, r', s', and t', also intersect pairwise on the sides of the triangle. Furthermore, let

G, g, and g' denote the centroids of triangle ABC, of the triangle determined by lines r, s, and t, and of the triangle formed by the reciprocal transversals r', s', and t', respectively. Then, the points G, g, and g' are collinear and satisfy the following relation: $\overline{Gg} = \overline{Gg'}$ (see Figure 4.4). In other words, G is the midpoint of the segment gg'.

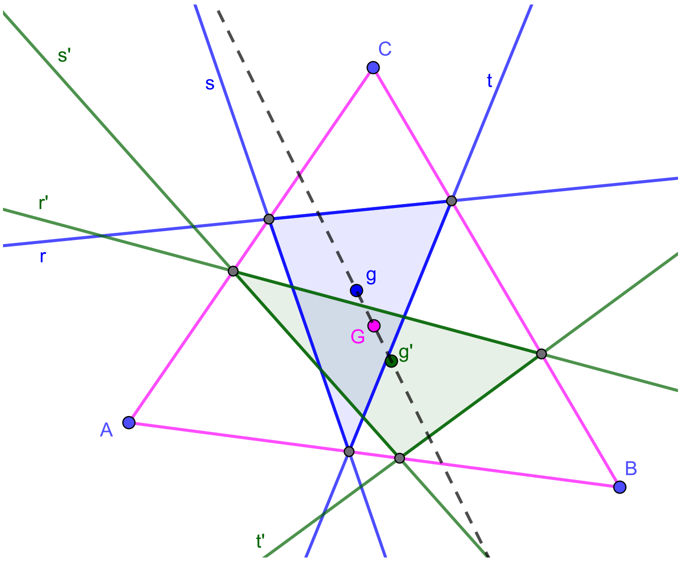


FIGURE 4.4: Theorem on three lines and their reciprocal transversals.

Considering a line r and its reciprocal transversal r', the sets of points P, Q, R, and P', Q', R', which are the intersections of r and r', respectively, with the sides of triangle ABC, can be interpreted as two infinitely flattened (degenerate) triangles. In accordance with the preceding theorem, the centroid g of the three collinear points[8] P, Q, and R, the centroid g' of the three collinear points P', Q', and R', and the centroid G of triangle ABC lie on the same line and satisfy the relation: $\overline{Gg} = \overline{Gg'}$ (see Figure 4.5).

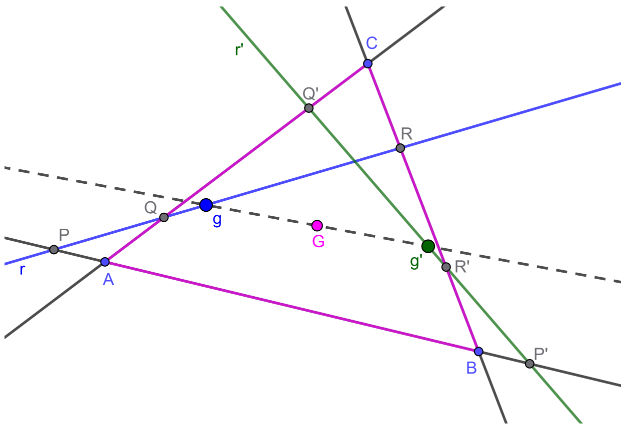


FIGURE 4.5: Corollary: the centroids g, g', and G are collinear.

[8]The centroid of three collinear points P_1, P_2, and P_3 can be found by first computing the midpoint of P_1 and P_2, and then dividing the segment joining this midpoint and P_3 in the ratio 2 : 1.

Taking these considerations into account, de Longchamps examined a complete quadrilateral determined by four straight lines, r_1, r_2, r_3, and r_4, intersecting pairwise at the points A, B, C, D, E, and F, as shown in Figure 4.6. This set of four lines can be viewed as four subsets, each consisting of a triangle formed by three of the lines, along with the one line excluded. For example, the triangle ABC, determined by lines r_1, r_2, and r_3, can be considered together with the line r_4, which intersects its sides at the points D, E, and F. Let r'_4 denote the reciprocal transversal of r_4, and let D', E', and F' be the points where r'_4 intersects the sides of triangle ABC. The following are defined:

- g_4 is the centroid of the collinear points D, E, and F on r_4,
- g'_4 is the centroid of the collinear points D', E', and F' on r'_4,
- G_4 is the centroid of triangle ABC, formed by omitting the line r_4,
- Γ is the centroid of the collinear midpoints of the diagonals AF, CD, and BE of the complete quadrilateral.

From the above, it follows that the centroids g_4, g'_4, and G_4 are collinear and satisfy the relation $\overline{g_4G_4} = \overline{G_4g'_4}$, and that the centroids g'_4, G_4, and Γ are collinear with $\overline{G_4g'_4} = 2 \cdot \overline{\Gamma G_4}$. Consequently, all four points, g_4, g'_4, G_4, and Γ lie on the same straight line, and the following relation holds: $\overline{g_4\Gamma} = \overline{\Gamma G_4}$.

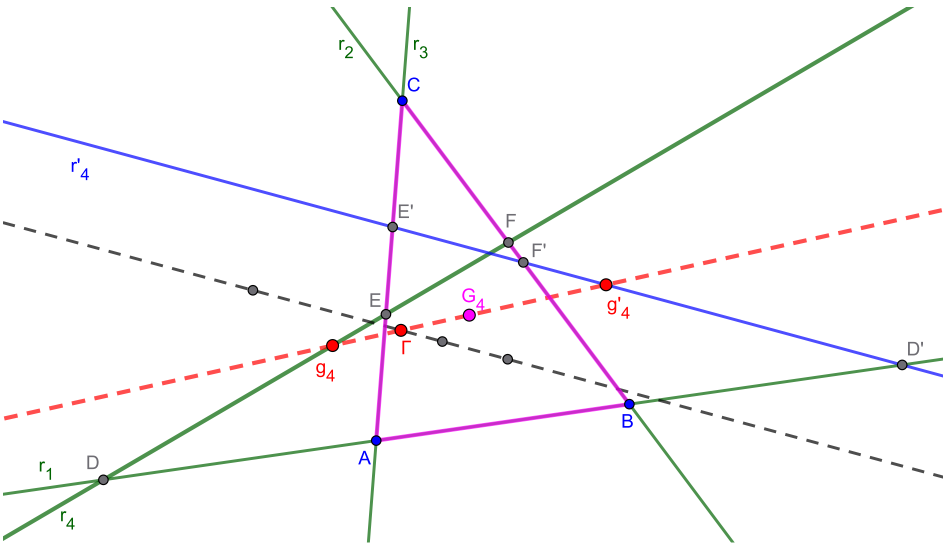


FIGURE 4.6: Collinearity of the centroids g_4, g'_4, G_4, and Γ.

The same result holds when any of the other lines is excluded; namely, $g_k\Gamma = \Gamma G_k$ for each $k = 1, 2, 3, 4$. In other words, Γ is the midpoint of each segment g_kG_k. In this way, de Longchamps defined the centroid of a complete quadrilateral and demonstrated the first step in a chain within the framework of recursive geometry.

1. Given a complete quadrilateral formed by four lines r_1, r_2, r_3, and r_4, intersecting pairwise, four triangles are obtained by excluding each line

in turn. For each k, let G_k be the centroid of the triangle formed by omitting the line r_k, and let g_k be the centroid of the three collinear points where the sides of this triangle intersect the omitted line r_k. Joining G_k and g_k for each k yields four segments. These segments, g_1G_1, g_2G_2, g_3G_3, and g_4G_4, intersect at a single point Γ, which bisects each of them (see Figure 4.7). This point Γ is referred to as the *centroid of the complete quadrilateral*.

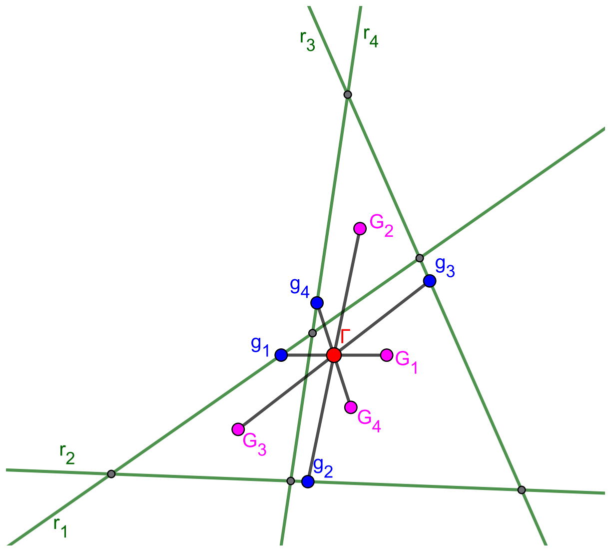


FIGURE 4.7: The centroid of a complete quadrilateral.

Before proceeding to consider complete polygons formed by more lines, de Longchamps defined the centroid of a set of n collinear points $P_1, \ldots, P_n$. This centroid can be computed recursively by first finding the centroid of the points $P_1, \ldots, P_{n-1}$, and then dividing the segment joining this centroid and the point P_n in the ratio $n-1:1$. He then proceeded to analyse a complete pentagon, defined as a configuration of five lines intersecting pairwise. Here the second step of the chain.

2. Given a complete pentagon formed by five lines, r_1, r_2, r_3, r_4, and r_5, intersecting pairwise, five complete quadrilaterals are obtained by omitting each line in turn. Let G_k now denote the centroid of the complete quadrilateral formed by excluding the line r_k, as defined in the preceding step. By joining each centroid G_k to the centroid g_k of the four collinear points of intersection between the excluded line r_k and the remaining lines, one obtains five segments, g_1G_1, g_2G_2, g_3G_3, g_4G_4, and g_5G_5, that intersect at a single point Γ. Furthermore, each of these segments is divided by Γ in the ratio $3:2$ (see Figure 4.8). The point Γ is referred to as the *centroid of the complete pentagon*.

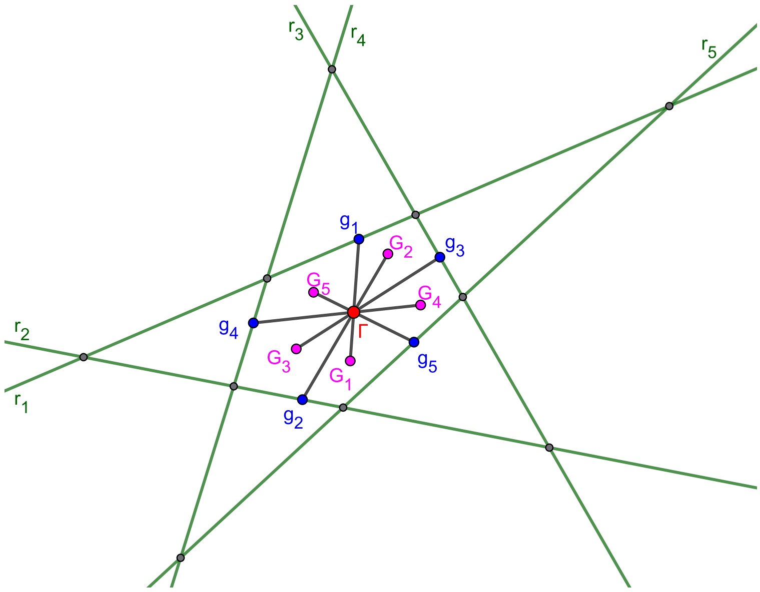


FIGURE 4.8: The centroid of a complete pentagon.

By means of a recursive process, de Longchamps generalised the theorems as follows. Given a complete polygon, defined as a configuration of n lines intersecting pairwise, by omitting one of the n lines at a time, n distinct sets of $n-1$ lines are obtained. Each of these sets determines a complete polygon of $n-1$ lines. Let G_k now denote the centroid of the complete polygon formed by excluding the line r_k, for $k = 1, \dots, n$. By joining each centroid G_k to the centroid g_k of the $n-1$ collinear points of intersection between the excluded line r_k and the remaining lines, one obtains n segments $g_k G_k$ that intersect at a single point Γ. Furthermore, each of these segments is divided by Γ in the ratio $(n-2) : 2$. The point Γ is referred to as the *centroid of the complete polygon*.

Subsequently, in later contributions (De Longchamps, 1877, 1883), the French mathematician introduced a new chain starting from the centroid of a triangle, although without providing a proof. He defined the centroid of a set of points as follows. Four points in the space,[9] P_1, P_2, P_3, and P_4, determine four triangles. Let G_k denote the centroid of the triangle obtained by excluding the point P_k. The segments P_1G_1, P_2G_2, P_3G_3, and P_4G_4 all intersect at a single point G, which divides each segment in the ratio $3 : 1$. It is worth noting that the point G coincides with the intersection point of the three bimedians[10] of the tetrahedron with vertices P_1, P_2, P_3, and P_4.
More generally, given n points in space, $P_1, \dots, P_n$, these determine n sets of $n-1$ points. Let G_k denote the centroid, defined recursively, of the set of $n-1$ points obtained by excluding the point P_k. All the segments P_kG_k intersect at a single point G, which divides each segment in the ratio $(n-1) : 1$.
De Longchamps observed that, starting from the definition of the centroid of a set of n points, and noting that a complete polygon determined by

[9]This is the only instance in which de Longchamps considers points in space rather than in the plane.
[10]A *bimedian* of a tetrahedron is a segment joining the midpoints of two opposite edges.

n lines has $\frac{n(n-1)}{2}$ vertices, the chain he had previously introduced in (De Longchamps, 1866a) to define the centroid of a complete polygon could actually be obtained through a much simpler and different method (De Longchamps, 1883, pp. 73-75).

Furthermore, following the definition of the centroid of a set of n points, he introduced an additional chain of recursive geometry related to the centroid, which provides an alternative method to determine the centroid of a system of n lines.

1. Four lines in a plane, r_1, r_2, r_3, and r_4, intersecting pairwise, determine four triangles by excluding each line in turn. Let G_k denote the centroid of the triangle formed by omitting the line r_k. In this way, four points, G_1, G_2, G_3, and G_4, are obtained. By joining each point G_k to the centroid Z_k of the triangle formed by the other three,[11] one obtains four segments, G_1Z_1, G_2Z_2, G_3Z_3, and G_4Z_4, that intersect at a single point G. This point is the centroid of the complete quadrilateral formed by the starting four lines,[12] and it divides each of the four segments in the ratio 3 : 1 (see Figure 4.9).

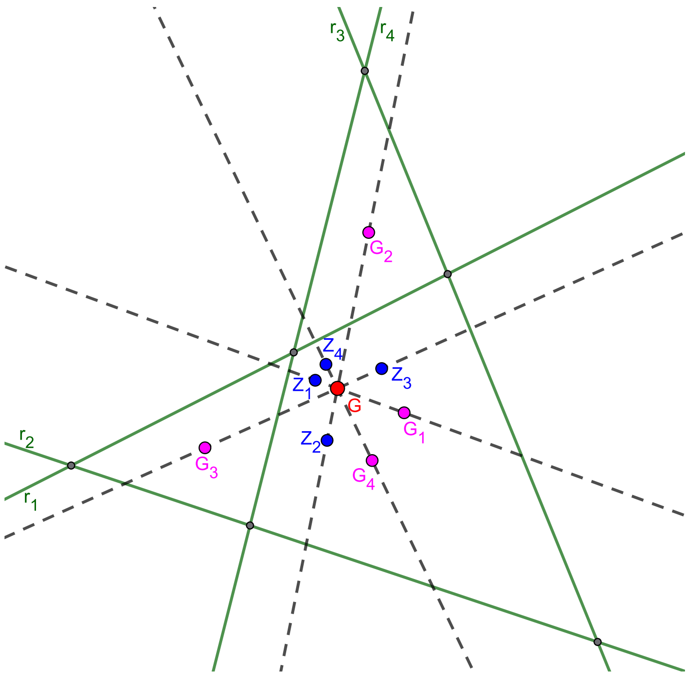


FIGURE 4.9: The centroid of a set of four lines.

2. Five lines in a plane, r_1, r_2, r_3, r_4, and r_5, intersecting pairwise, determine five complete quadrilaterals by excluding each line in turn. Let G_k denote the centroid of the complete quadrilateral obtained by omitting the line r_k, as defined in the preceding step. In this way, five points, G_1,

[11] For example, the point Z_1 is the centroid of the triangle with vertices G_2, G_3, and G_4, and so on.

[12] Note that the point G coincides with the point Γ in Figure 4.7.

G_2, G_3, G_4, and G_5, are obtained. By joining each point G_k to the centroid Z_k of the sets of points including the other four,[13] one obtains five segments, G_1Z_1, G_2Z_2, G_3Z_3, G_4Z_4, and G_5Z_5, that intersect at a single point G. This point is the centroid of the complete pentagon formed by the starting five lines,[14] and it divides each of these five segments in the ratio 4 : 1 (see Figure 4.10).

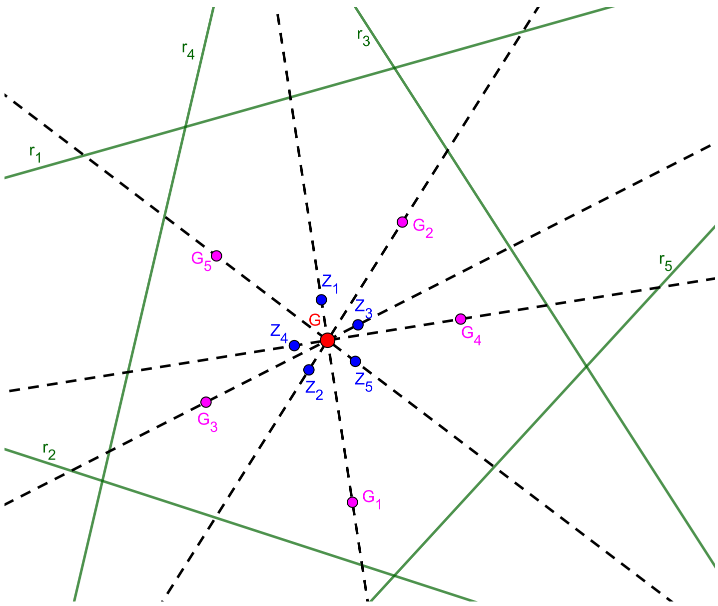


FIGURE 4.10: The centroid of a set of five lines.

De Longchamps generalised the chain as follows. A system of n lines in a plane, $r_1, \ldots, r_n$, intersecting pairwise, determines n complete polygons of $n-1$ lines each. Let G_k denote the centroid of the complete polygon formed by omitting the line r_k, for $k = 1, \ldots, n$, defined recursively. By joining each point G_k to the centroid Z_k of the set of the other $n-1$ points, one obtains n segments G_kZ_k that intersect at a single point G. This latter is the centroid of the complete polygon formed by the starting lines $r_1, \ldots, r_n$, and it divides each segment G_kZ_k in the ratio $(n-1) : 1$.

In both works (De Longchamps, 1877, 1883) the French mathematician developed several series of theorems within the framework of recursive geometry. In addition to the chains related to the centroid and the circumcentre, he introduced others focused on elements likewise associated with triangle geometry, such as the Simson–Wallace line and the nine-point circle. In the first paper he stated these results, whereas in the second he also provided their proofs. In what follows, the aforementioned chains will be presented in detail, omitting the proofs originally provided by de Longchamps, as they rely on elementary techniques of plane geometry and fall beyond the scope of the present discussion.

[13] For example, the point Z_1 is the centroid of the set of four points $\{G_2, G_3, G_4, G_5\}$, and so on.

[14] Note that the point G coincides with the point Γ in Figure 4.8.

4.2.1 The chain related to the circumcentre

A detailed description of de Longchamps' chain related to the circumcentre is presented below.

1. Given four lines in a plane, r_1, r_2, r_3, and r_4, intersecting pairwise, four triangles are formed by omitting one line at a time. The four circles circumscribed about these triangles all intersect at a single point P. Let C_{ijk} denote the circumcentre of the triangle determined by the lines r_i, r_j, and r_k, with i, j, and k all distinct, that is the centre of the circle circumscribed about the triangle. The four circumcentres C_{123}, C_{124}, C_{134}, and C_{234} all lie on a common circle, whose centre is denoted by C_{1234}.
   It is important to note that the point P also lies on the circle passing through C_{123}, C_{124}, C_{134}, and C_{234} (see Figure 4.11).

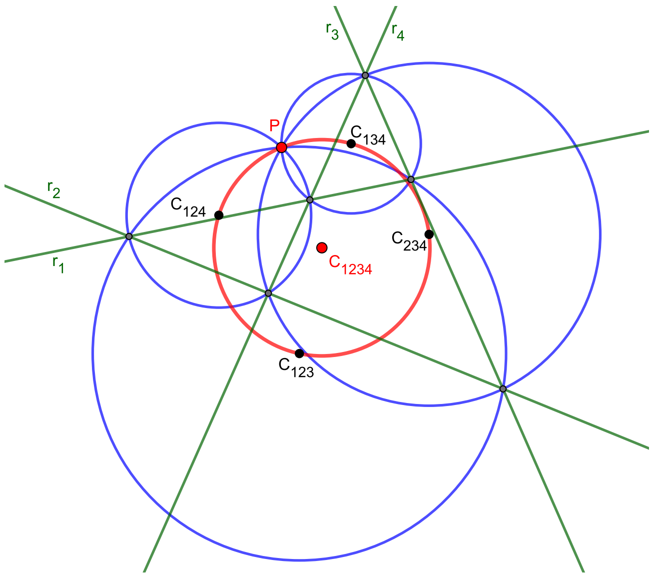


FIGURE 4.11: The chain related to the circumcentre – four lines.

2. Given five lines in a plane, r_1, r_2, r_3, r_4, and r_5, intersecting pairwise, five sets of four lines each are obtained by omitting one line at a time. According to the preceding step, each of these sets determines a circle and a point, namely its centre. The five resulting circles all intersect at a single point P, and their centres, C_{1234}, C_{1235}, C_{1245}, C_{1345}, and C_{2345}, all lie on a common circle, whose centre is denoted by C_{12345} (see Figure 4.12).

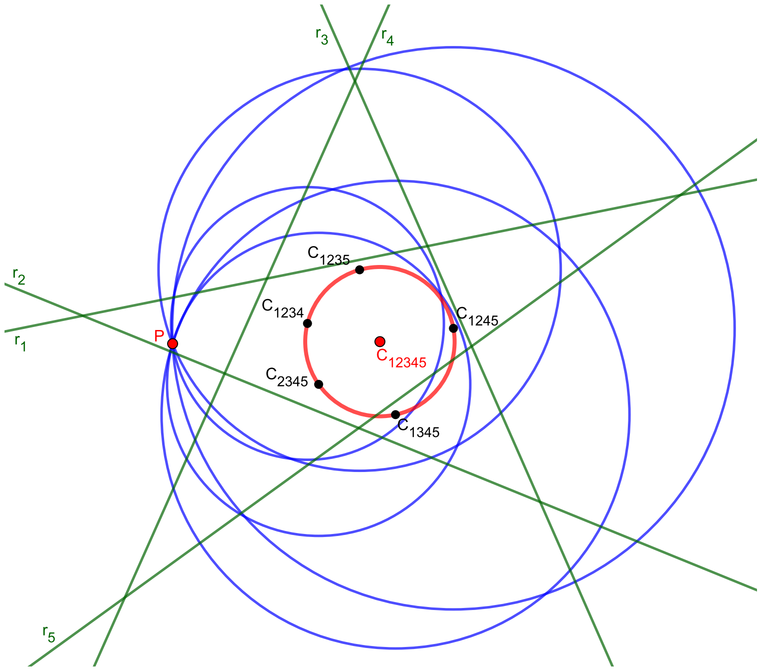


FIGURE 4.12: The chain related to the circumcentre – five lines.

By employing a recursive approach, de Longchamps formulated a generalisation of this chain of theorems as follows. Given a system of n lines in a plane, $r_1, \ldots, r_n$, intersecting pairwise, n sets of $n-1$ lines each are formed by omitting one line at a time. From each of these sets, a circle and a point (namely its centre) are defined recursively. In total, one obtains n circles and n corresponding centres, which satisfy the following properties:

i. the n circles all intersect at a single point P;

ii. their centres lie on a common circle, whose centre is denoted by $C_{12\ldots n}$.

It should be noted that, in general, the point P does not lie on the circle whose centre is $C_{12\ldots n}$, contrary to what occurs in the configuration with four lines. However, this property holds when the starting n lines are all tangent to a common parabola, in which case the point P coincides with the focus of that parabola (see Figure 4.13).
Regarding this matter, in a note in (De Longchamps, 1877, p. 308), the French mathematician acknowledged an error in his earlier paper on the chain related to the circumcentre (De Longchamps, 1864). He further noted that the theorems in the chain constitute a generalisation of a result already proved by William Irvine Ritchie (1850–1903) in (Ritchie, 1875), concerning n lines tangent to a common parabola.[15]

[15]The same property is also stated by Joseph Jean Baptiste Neuberg (1840–1926) in (Neuberg, 1877).

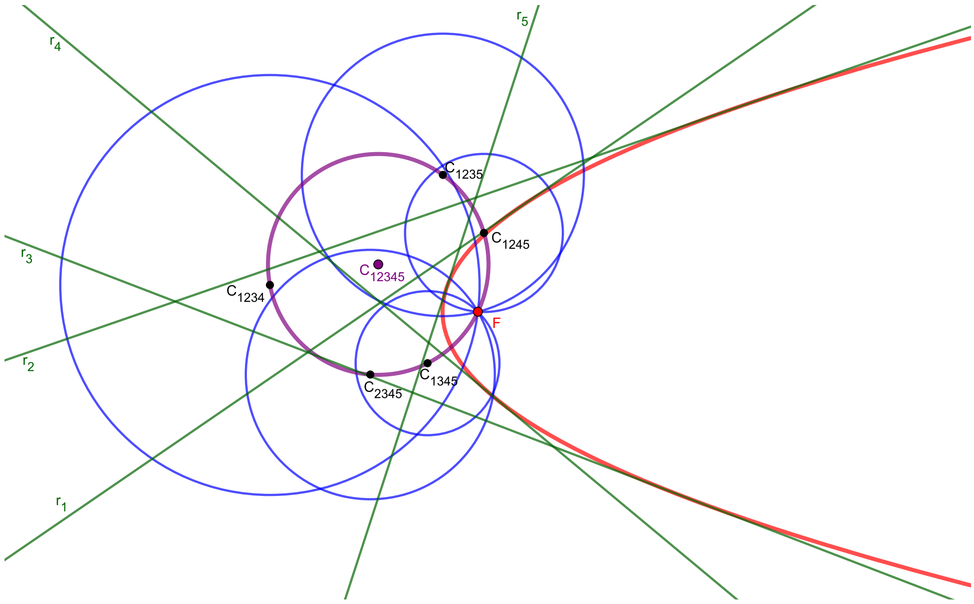


FIGURE 4.13: Special case of the second step of the chain related to the circumcentre.

Before proceeding with the presentation of the other chains in recursive geometry, it should be noted that the theorem underlying the first step in the series related to the circumcentre is essentially a direct application of Steiner's first and second questions, which were discussed in Chapter 2. In his work, de Longchamps treated as well known the theorem stating that the four circumcircles of the four triangles intersect at a single point, yet he did not attribute this result to any author. Instead, he proved that the centres of these four circumcircles are concyclic, but, once again, he made no reference to Steiner.

In light of this, it can be stated that the chain under discussion shares the same starting point as Clifford's first chain, which was presented earlier in Chapter 3. Nevertheless, despite this common origin and the fact that the method employed, namely, the recursive process used to construct both chains, is essentially the same, neither mathematician made any reference to the other's papers.

4.2.2 The chain related to the Simson–Wallace line

The following is a presentation of de Longchamps' chain related to the Simson–Wallace line. Consider a circle Γ and a point M on it, referred to as the *fundamental point*.

1. Three points P_1, P_2, and P_3 on Γ define an inscribed triangle. Dropping perpendiculars from point M to the sides of the triangle, the feet of these perpendiculars, denoted by H_{12}, H_{13}, and H_{23},[16] all lie on a

[16]The point H_{ij} is the foot of the perpendicular from point M to the side P_iP_j.

straight line r_{123}. This is the Simson–Wallace line[17] of triangle $P_1P_2P_3$ with respect to point M (see Figure 4.14).

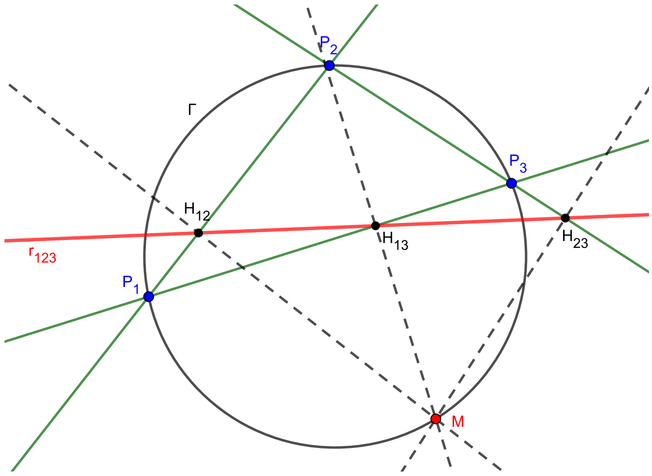


FIGURE 4.14: Chain related to the Simson–Wallace line – three points.

2. Four points P_1, P_2, P_3, and P_4 on Γ determine four distinct sets of three points, each forming a triangle. According to the preceding step, each of these sets determines a Simson–Wallace line with respect to point M. Dropping perpendiculars from point M to the four Simson–Wallace lines r_{123}, r_{124}, r_{134}, and r_{234}, the feet of these perpendiculars, H_{123}, H_{124}, H_{134}, and H_{234},[18] all lie on a new straight line, denoted by r_{1234} (see Figure 4.15).

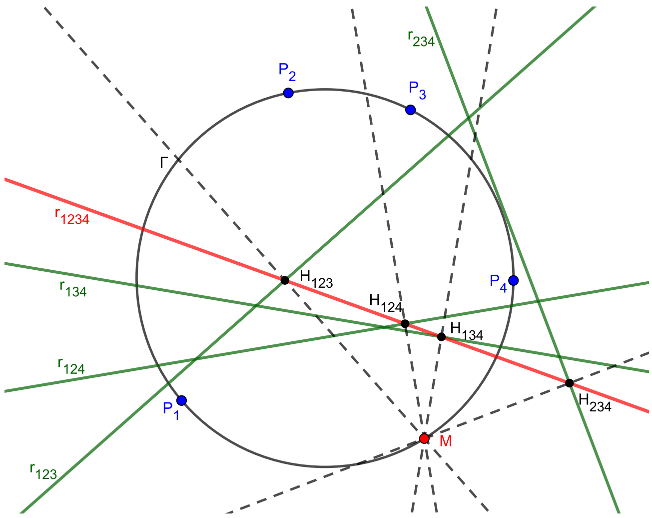


FIGURE 4.15: Chain related to the Simson–Wallace line – four points.

[17]In his papers, de Longchamps designates the Simson–Wallace line as the Simson line, while referring the reader to (Catalan, 1872, p. 30) for the proof.

[18]The point H_{ijk} is the foot of the perpendicular from point M to the line r_{ijk}.

3. Five points P_1, P_2, P_3, P_4, and P_5 on Γ determine five distinct sets of four points each. According to the preceding step, each of these sets determines a line with respect to the fundamental point M. Dropping perpendiculars from point M to these five lines, r_{1234}, r_{1235}, r_{1245}, r_{1345}, and r_{2345}, the feet of these perpendiculars, H_{1234}, H_{1235}, H_{1245}, H_{1345}, and H_{2345}, all lie on a new straight line, denoted by r_{12345} (see Figure 4.16).

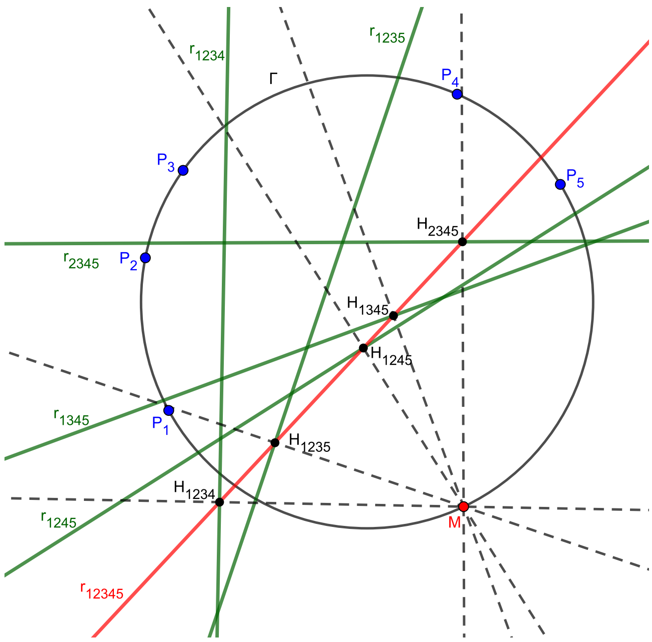


FIGURE 4.16: Chain related to the Simson–Wallace line – five points.

This process can be repeated recursively, each time generating a new set of collinear points by dropping perpendiculars from the fundamental point M to the lines obtained in the preceding step. In general, given n points $P_1, \ldots, P_n$ on Γ, n sets of $n-1$ points are formed by omitting one point at a time. From each of these sets, a line is defined recursively. Dropping perpendiculars from point M to these n lines, the feet of the perpendiculars all lie on a straight line, denoted by $r_{12\ldots n}$.[19]

Before proceeding further, it is important to observe that in the second step of the chain under consideration, namely, when four points are taken on the circle Γ, the four Simson–Wallace lines, r_{123}, r_{124}, r_{134}, and r_{234}, associated with the four triangles obtained by omitting one point at a time, form a complete quadrilateral whose Wallace point (see Chapter 2) coincides precisely with the fundamental point M. As shown in Figure 4.17, the points H_{12}, H_{13}, H_{14}, and P_1, together with the fundamental point M, all lie on the same circle, and the segment MP_1 is the diameter of this circle. The same holds for all quadruples of points of the form H_{ij}, H_{ik}, H_{il}, and P_i, where i, j, k, l are all

[19]De Longchamps also investigated further properties concerning the triangles formed by the lines obtained by excluding one point at a time from the initial set of n points, as well as the angles formed between the line determined by the n points and those obtained in the preceding step.

distinct. Indeed, by construction, the line MH_{ij} is perpendicular to the line P_iP_j; thus, all angles $P_i\widehat{H_{ij}}M$ are right angles.

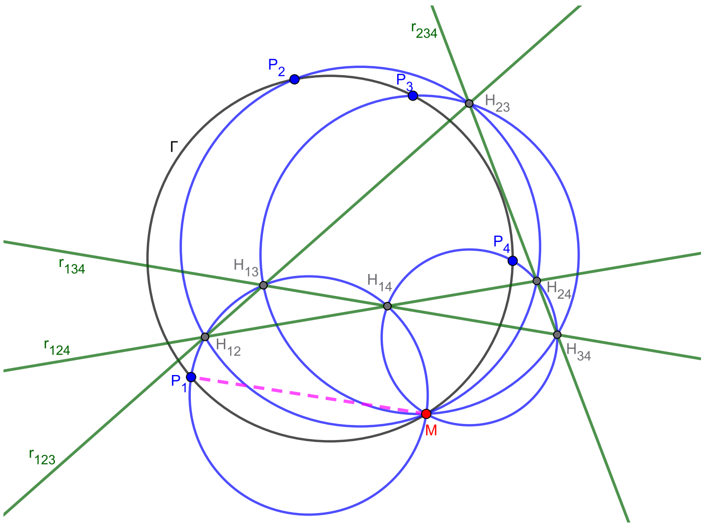


FIGURE 4.17: The Wallace point of the complete quadrilateral coincides with the fundamental point M.

4.2.3 The chain related to the nine-point circle

Another geometric object considered by de Longchamps within the framework of recursive geometry is the nine-point circle associated with a triangle ABC. This circle, also known as *Euler circle* or *Feuerbach circle*, passes through nine remarkable points of the triangle: the midpoints M_{AB}, M_{BC}, and M_{AC} of its three sides; the feet of the altitudes D, E, and F dropped from each vertex; and the midpoints G, H, and I of the segments joining each vertex to the orthocentre O (see Figure 4.18).[20]

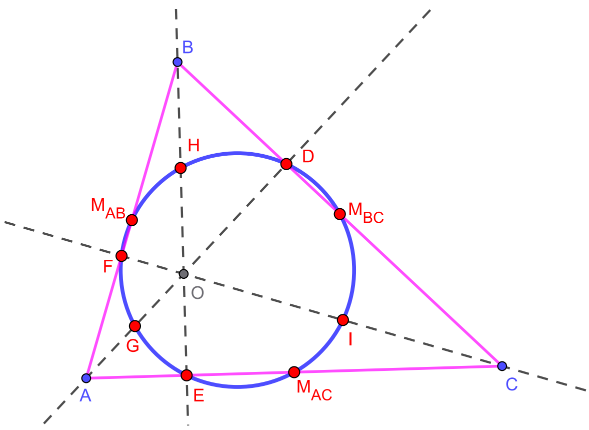


FIGURE 4.18: The nine-point circle associated with a triangle.

What follows is a description of de Longchamps' chain related to the nine-point circle. Consider a circle Γ with centre O.

[20]For more details on the historical origins of the nine-point circle associated with a triangle, see (Vaccaro, 2020).

1. Four points P_1, P_2, P_3, and P_4 on the circle Γ determine four distinct triangles. Let W_{ijk} denote the centre of the nine-point circle associated with the triangle having vertices P_i, P_j, and P_k, where i, j, k are all distinct. The four points thus obtained, W_{123}, W_{124}, W_{134}, and W_{234}, all lie on the same circle, whose centre, denoted by W_{1234}, is precisely the intersection point of the four nine-point circles.
Moreover, the lines P_1W_{234}, P_2W_{134}, P_3W_{124}, and P_4W_{123} all intersect at a point, denoted by S_{1234}. The centre O of Γ and the points W_{1234} and S_{1234} are collinear, and the following relation holds: $\overline{OS_{1234}} = 2 \cdot \overline{S_{1234}W_{1234}}$ (see Figure 4.19).[21]

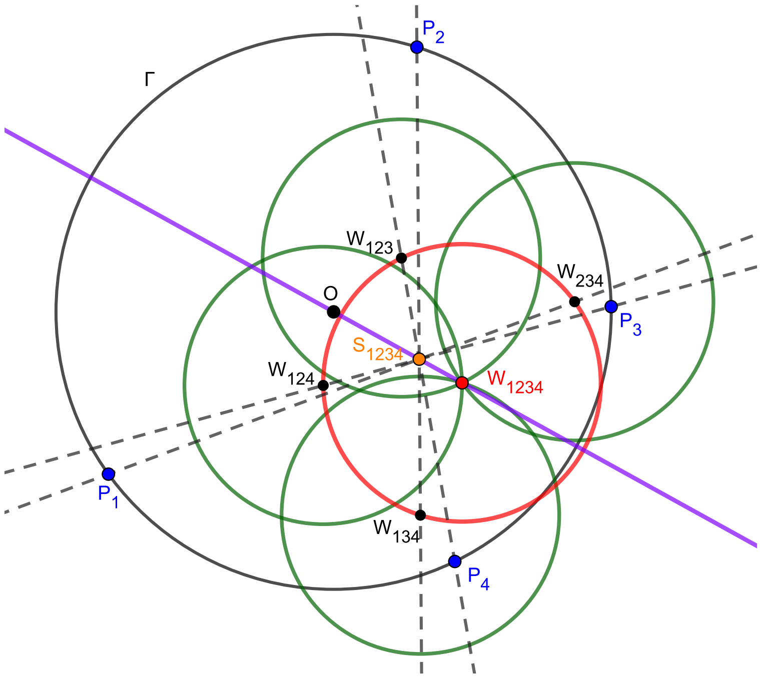


FIGURE 4.19: Chain related to the nine-point circle – four points.

2. Five points P_1, P_2, P_3, P_4, and P_5 on the circle Γ determine five distinct sets of four points each. According to the preceding step, each of these sets defines a circle and a point, namely its centre. The five resulting points, W_{1234}, W_{1235}, W_{1245}, W_{1345}, and W_{2345}, all lie on the same circle, whose centre, denoted by W_{12345}, is precisely the intersection point of the five circles determined in the preceding step.
Moreover, the lines P_1W_{2345}, P_2W_{1345}, P_3W_{1245}, P_4W_{1235}, and P_5W_{1234} all intersect at a point, denoted by S_{12345}. The centre O of Γ and the points W_{12345} and S_{12345} are collinear, and the following relation holds: $\overline{OS_{12345}} = 2 \cdot \overline{S_{12345}W_{12345}}$ (see Figure 4.20).

[21]To prove this, de Longchamps showed that the quadrilaterals $P_1P_2P_3P_4$ and $W_{123}W_{124}W_{134}W_{234}$ are similar with a ratio of $2:1$.

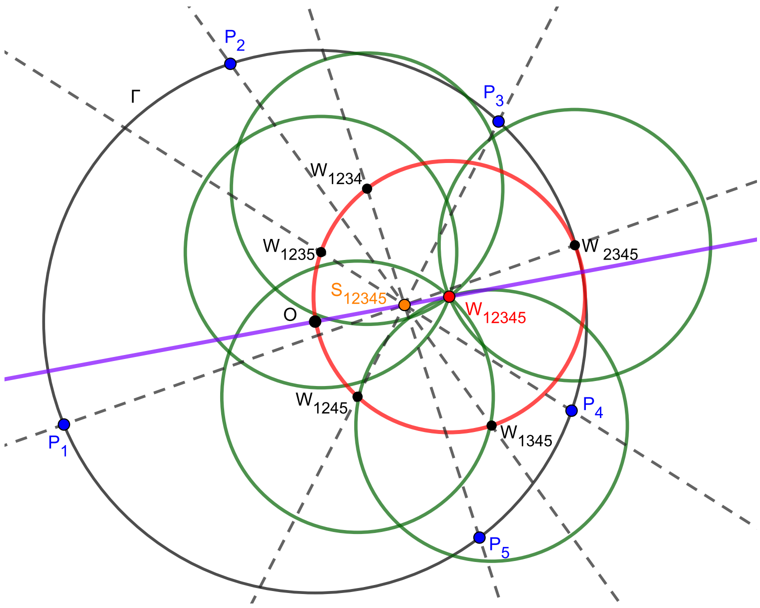


FIGURE 4.20: Chain related to the nine-point circle – five points.

In general, n points $P_1, \ldots, P_n$ on the circle Γ determine n distinct sets of $n-1$ points. Each of these sets defines a circle and a point, namely its centre. The n resulting points all lie on the same circle, whose centre, denoted by $W_{12\ldots n}$, is precisely the intersection point of the n circles recursively determined in the preceding step.
Moreover, the lines $P_i W_{1\ldots(i-1)(i+1)\ldots n}$, where $i = 1, \ldots n$, all intersect at a point, denoted by $S_{12\ldots n}$. The points O, $W_{12\ldots n}$, and $S_{12\ldots n}$ are collinear and the following relation holds: $\overline{OS_{12\ldots n}} = 2 \cdot \overline{S_{12\ldots n} W_{12\ldots n}}$. This means that the point $S_{12\ldots n}$ divides the segment $OW_{12\ldots n}$ in the ratio 2 : 1.

4.2.4 The chain related to the nine-point conic

In light of de Longchamps' analysis of the series related to the nine-point circle, an attempt has been made by the author to construct the initial steps of a chain related to the nine-point conic associated with a complete quadrangle. The latter is a configuration of four distinct points, A, B, C, and D, in the plane, no three of which are collinear, together with the six lines joining each pair of these points. It also defines three diagonal points, obtained as the intersections of the three pairs of opposite sides.[22]
The nine-point conic is a generalisation of the nine-point circle. It was introduced in 1844 by Steiner in (Steiner, 1844), although the term itself was coined later, in 1863, by the Italian mathematicians Nicola Trudi (1811–1884) and Giuseppe Battaglini (1826–1894). Given a complete quadrangle, the nine-point conic associated with it is defined as the locus of the centres of all conics in the pencil passing through the four points, A, B, C, and D, of the complete quadrangle.
This conic passes through nine remarkable points: the six midpoints of the

[22] Note that a complete quadrangle is different from a complete quadrilateral.

sides of the complete quadrangle, M_{AB}, M_{BC}, M_{CD}, M_{AD}, M_{AC}, and M_{BD}, as well as the three diagonal points, E, F, and G (see Figure 4.21).[23]

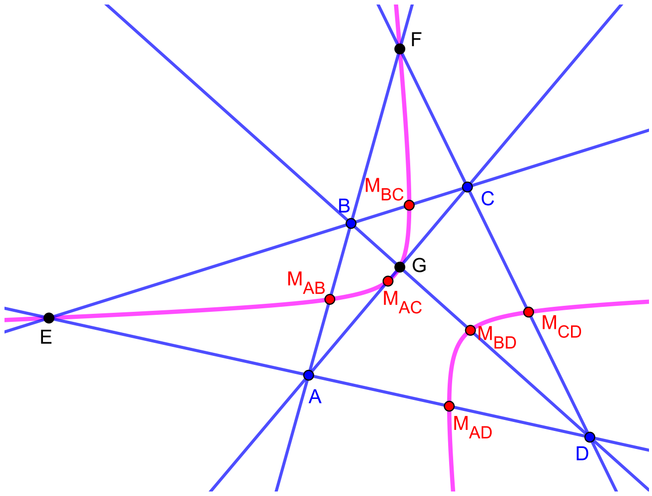


FIGURE 4.21: The nine-point conic associated with a complete quadrangle.

The type of nine-point conic associated with a complete quadrangle is determined by the relative positions of its four points within the plane. In particular, if one of the points, A, B, C, or D, lies inside the triangle formed by the other three, then the nine-point conic is an ellipse. As a special case, if this point coincides with the orthocentre of the triangle, the nine-point conic reduces to the classical nine-point circle. Conversely, if each of the four points lies outside the triangle determined by the other three, the nine-point conic is a hyperbola. Furthermore, if one of the points lies on the circle circumscribed about the triangle formed by the other three, the conic becomes a rectangular hyperbola (also known as an equilateral hyperbola). In the limiting case where one of the four points lies on a side of the triangle formed by the other three, the nine-point conic degenerates.
In all cases, the centre of symmetry of the conic coincides with the centroid of the quadrangle with vertices A, B, C, and D. This centroid is the intersection point of the lines joining the midpoints of the opposite sides; for example, the lines $M_{AB}M_{CD}$ and $M_{AD}M_{BC}$. This point bisects each of these segments.[24]

The chain of recursive geometry related to the nine-point conic associated with a complete quadrangle is presented below.[25]

1. Five points P_1, P_2, P_3, P_4, and P_5 on an ellipse $\mathcal{E}$ determine five distinct sets of four points each, obtained by excluding one point at a time. Each set defines a complete quadrangle and its associated nine-point conic. The five resulting nine-point conics all intersect at the same point, which coincides with the centre O of the ellipse $\mathcal{E}$.

[23]For more details on the nine-point conic and its historical origins, see (Vaccaro, 2020).
[24]Regarding this, see also (Scimemi, 2007).
[25]The notation used in this subsection differs from that of the other chains to avoid complicating the writing in the demonstration.

Let C_k denote the centre of symmetry of the nine-point conic associated with the complete quadrangle formed by omitting the point P_k. The five points C_1, C_2, C_3, C_4, and C_5 lie on a common ellipse, which is similar to the original ellipse $\mathcal{E}$, and whose centre is denoted by W. Moreover, the five lines P_1C_1, P_2C_2, P_3C_3, P_4C_4, and P_5C_5 all intersect at a single point, denoted by S. Finally, the points O, S, and W are collinear, and satisfy the following relation: $\overline{OS} = 4 \cdot \overline{SW}$ (see Figure 4.22).

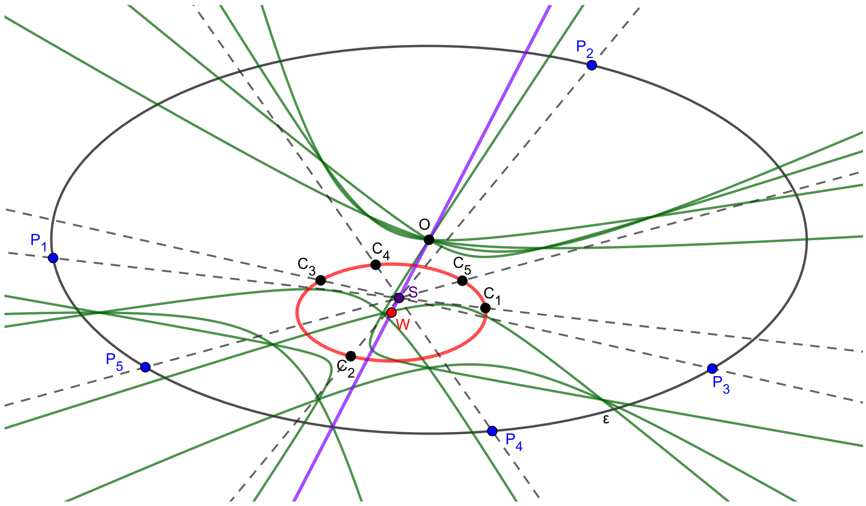


FIGURE 4.22: Chain related to the nine-point conic – five points.

The five nine-point conics obtained are all hyperbolas since the points $P_1, \ldots, P_5$ lie on an ellipse and are therefore positioned in such a way that this property holds, as explained above.
The proof of the first step of the chain is presented below.
Consider the two complete quadrangles determined by the quadruples (P_1, P_2, P_3, P_4) and (P_1, P_2, P_3, P_5) obtained by omitting the points P_4 and P_5, respectively. The corresponding nine-point conics associated with these complete quadrangles have centres of symmetry denoted by C_4 and C_5. Let M_{ij} denote the midpoint of the segment joining P_i and P_j (see Figure 4.23).
Consider the triangles $P_2P_4P_5$ and $P_2M_{24}M_{25}$. Since $\overline{P_2P_4} = 2 \cdot \overline{P_2M_{24}}$ and $\overline{P_2P_5} = 2 \cdot \overline{P_2M_{25}}$, and the angle at P_2 is common to both triangles, it follows by the second criterion of triangle similarity that the two triangles are similar with ratio $2 : 1$. In particular, it can be deduced that

$$P_4P_5 \parallel M_{24}M_{25} \quad \text{and} \quad \overline{P_4P_5} = 2 \cdot \overline{M_{24}M_{25}}.$$

It has already been established that the centre of symmetry of the nine-point conic coincides with the centroid of the corresponding complete quadrangle. It follows that the points M_{25}, M_{13}, and C_4 are collinear and that $\overline{C_4M_{25}} = \overline{C_4M_{13}}$. Similarly, the points M_{24}, M_{13}, and C_5 are collinear and satisfy the relation $\overline{C_5M_{24}} = \overline{C_5M_{13}}$.
Now consider the triangles $M_{13}M_{25}M_{24}$ and $M_{13}C_4C_5$. These triangles are

similar by the second criterion of triangle similarity with ratio 2 : 1. In particular,

$$M_{24}M_{25} \parallel C_4C_5 \quad \text{and} \quad \overline{M_{24}M_{25}} = 2 \cdot \overline{C_4C_5}.$$

It follows that

$$P_4P_5 \parallel C_4C_5 \quad \text{and} \quad \overline{P_4P_5} = 2 \cdot \overline{M_{24}M_{25}} = 4 \cdot \overline{C_4C_5}.$$

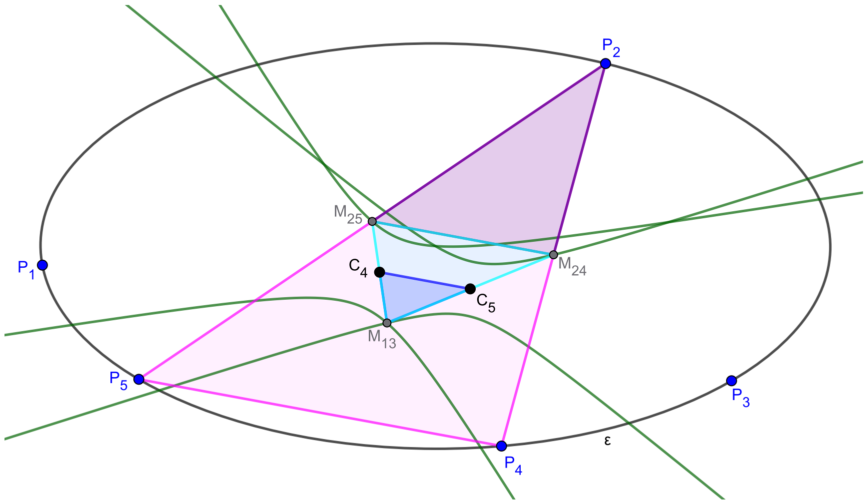


FIGURE 4.23: Proof of the first step of the chain related to the nine-point conic (I).

Similarly, it can be shown that for any indices i and j,

$$P_iP_j \parallel C_iC_j \quad \text{and} \quad \overline{P_iP_j} = 4 \cdot \overline{C_iC_j}.$$

Consequently, the polygons $P_1P_2P_3P_4P_5$ and $C_1C_2C_3C_4C_5$ are similar with ratio 4 : 1 (see Figure 4.24).

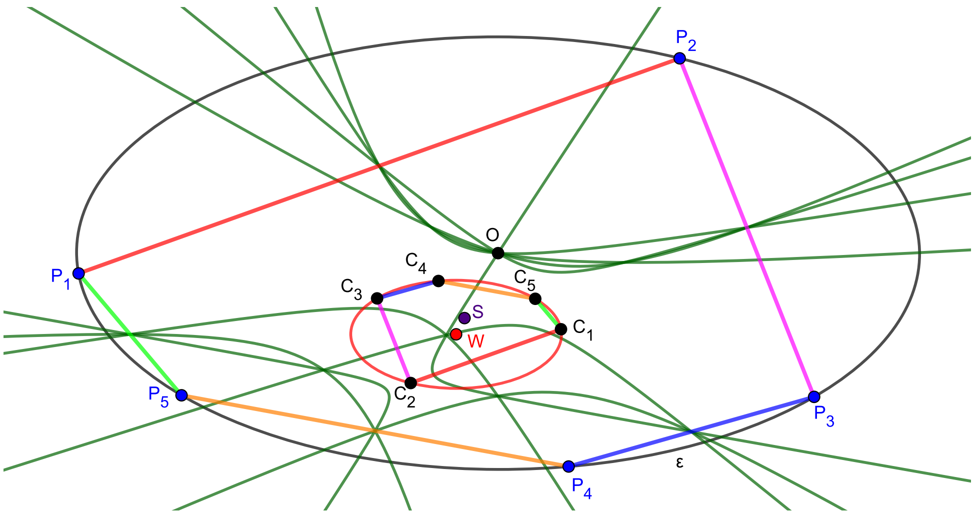


FIGURE 4.24: Proof of the first step of the chain related to the nine-point conic (II).

Moreover, since the points P_1, P_2, P_3, P_4, and P_5 lie on the ellipse $\mathcal{E}$, the points C_1, C_2, C_3, C_4, and C_5 also lie on an ellipse similar to the original one.[26] The centre of this similarity is the point S, *i.e.*, the intersection point of the lines P_kC_k. Therefore, the centre O of the ellipse $\mathcal{E}$, the point S, and the centre W of the new ellipse are collinear and satisfy the relation: $\overline{OS} = 4 \cdot \overline{SW}$.

By similar reasoning, the second step of the chain related to the nine-point conic can be established, along with its subsequent generalisation.

2. Six points $P_1, \ldots, P_6$ on an ellipse $\mathcal{E}$ determine six distinct sets of five points each, obtained by excluding one point at a time. According to the preceding step, each such set defines an ellipse similar to $\mathcal{E}$ and a point, namely its centre. The six resulting ellipses all intersect at the same point W. Let C_k denote the centre of the ellipse formed by omitting the point P_k. The six points C_1, C_2, C_3, C_4, C_5, and C_6 lie on a common ellipse, whose centre is precisely W. Moreover, the six lines P_1C_1, P_2C_2, P_3C_3, P_4C_4, P_5C_5, and P_6C_6 all intersect at a single point S. Finally, the centre O of the ellipse $\mathcal{E}$ and the points S and W are collinear and satisfy the relation: $\overline{OS} = 4 \cdot \overline{SW}$ (see Figure 4.25).

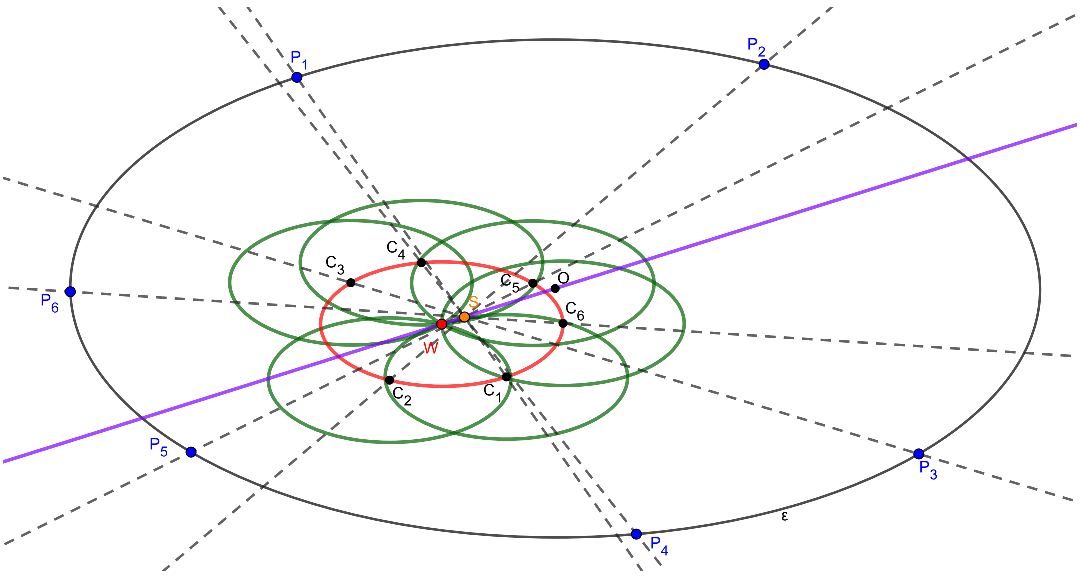


FIGURE 4.25: Chain related to the nine-point conic – six points.

In general, n points $P_1, \ldots, P_n$ on the ellipse $\mathcal{E}$ determine n distinct sets of $n-1$ points. Each of these sets defines an ellipse and a point, namely its centre. The n resulting ellipses all intersect at a single point W. Let C_k denote the centre of the ellipse formed by omitting the point P_k. The n points $C_1, \ldots, C_n$ lie on the same ellipse, whose centre is precisely W. Moreover, the n lines P_iC_i all intersect at a point S. Finally, the centre O of $\mathcal{E}$, and the points S and W are collinear, and the following relation holds: $\overline{OS} = 4 \cdot \overline{SW}$.

[26] The similarity between the two pentagons $P_1P_2P_3P_4P_5$ and $C_1C_2C_3C_4C_5$ implies the similarity between the two circumscribed ellipses.

4.2.5 Additional considerations

In order to provide a comprehensive overview of de Longchamps' contributions to the field of recursive geometry, it is necessary to address two issues before proceeding to the next chapter.

The first concerns the fact that, in his 1877 paper, he also proposed a chain related to the orthocentre (De Longchamps, 1877, pp. 343-344), which, unlike his other series, he did not revisit in (De Longchamps, 1883). In the following, this chain is presented without accompanying proofs, which are nevertheless regarded as elementary. Consider a circle Γ.

1. Four points P_1, P_2, P_3, and P_4 on Γ determine four distinct sets of three points, each forming a triangle. Let O_{ijk} denote the orthocentre of the triangle formed by the points P_i, P_j, and P_k, where i, j, and k are all distinct. The four orthocentres O_{123}, O_{124}, O_{134}, and O_{234} all lie on a circle congruent to Γ, whose centre is denoted by O_{1234} (see Figure 4.26).[27]

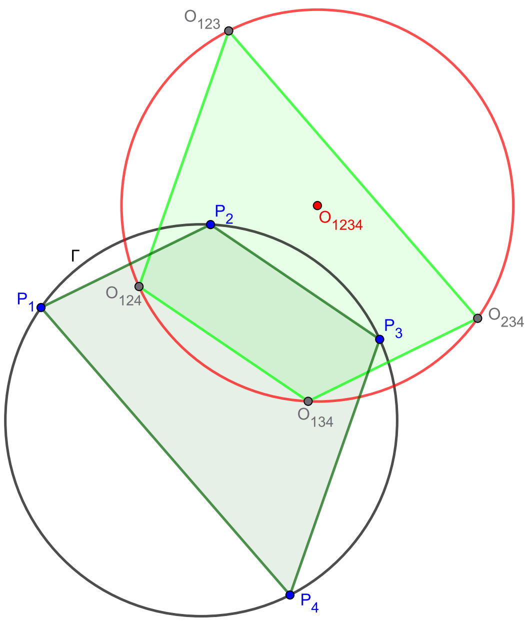


FIGURE 4.26: Chain related to the orthocentre – four points.

2. Five points P_1, P_2, P_3, P_4, and P_5 on Γ determine five distinct sets of four points. The five points resulting from the preceding step, O_{1234}, O_{1235}, O_{1245}, O_{1345}, and O_{2345}, all lie on a circle congruent to Γ, whose centre is denoted by O_{12345} (see Figure 4.27).

[27] De Longchamps deduced this property from a theorem established by Catalan in (Catalan, 1872, p. 29), according to which the quadrilateral with vertices O_{123}, O_{124}, O_{134}, and O_{234} is congruent to the quadrilateral $P_1P_2P_3P_4$.

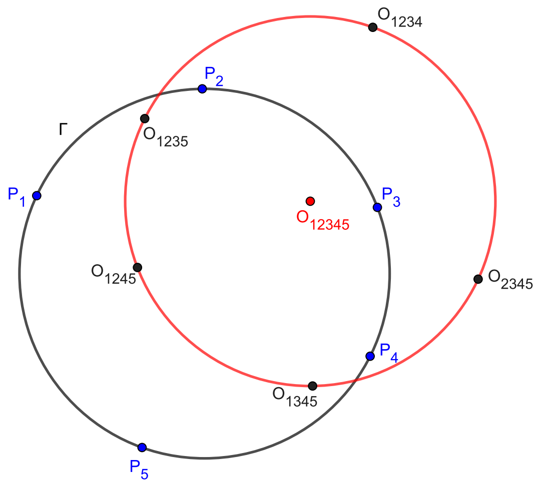


FIGURE 4.27: Chain related to the orthocentre – five points.

More generally, n points $P_1, \ldots, P_n$ on Γ determine n distinct sets of $n-1$ points. The n points obtained recursively by the preceding step all lie on a circle congruent to Γ, whose centre is denoted by $O_{12\ldots n}$.[28]

A second observation is that de Longchamps concluded his 1883 paper (De Longchamps, 1883, pp. 125-126) by presenting a new method for constructing chains of recursive geometry. Let A_1, B_1, C_1, and D_1 be four distinct points in the plane. Taken three at a time, they define four triangles: $A_1B_1C_1$, $A_1B_1D_1$, $A_1C_1D_1$, and $B_1C_1D_1$. Let A_2 be a remarkable point of triangle $B_1C_1D_1$; similarly, let B_2, C_2, and D_2 be defined in the same way for the corresponding triangles. Therefore, from the initial quadrilateral $A_1B_1C_1D_1$, a new quadrilateral $A_2B_2C_2D_2$ is obtained. Proceeding in the same manner, let A_3 be the same type of remarkable point of triangle $B_2C_2D_2$, and define B_3, C_3, and D_3 analogously. In this way, a third quadrilateral $A_3B_3C_3D_3$ is constructed. The same construction is applied iteratively at each step, using the points of the previous quadrilateral, thus generating an infinite sequence. After $n-1$ steps, one obtains the quadrilateral $A_nB_nC_nD_n$; therefore, in total, there are n quadrilaterals and $4n$ triangles. De Longchamps concluded his study by establishing that if the chosen remarkable point in the construction is the orthocentre, then the nine-point circles of the $4n$ triangles formed after $n-1$ steps all intersect at a single point (see Figures 4.28 and 4.29).

[28]This chain of theorems appears to highlight certain geometric properties of potential interest. It may therefore be worth considering for future research in this area.

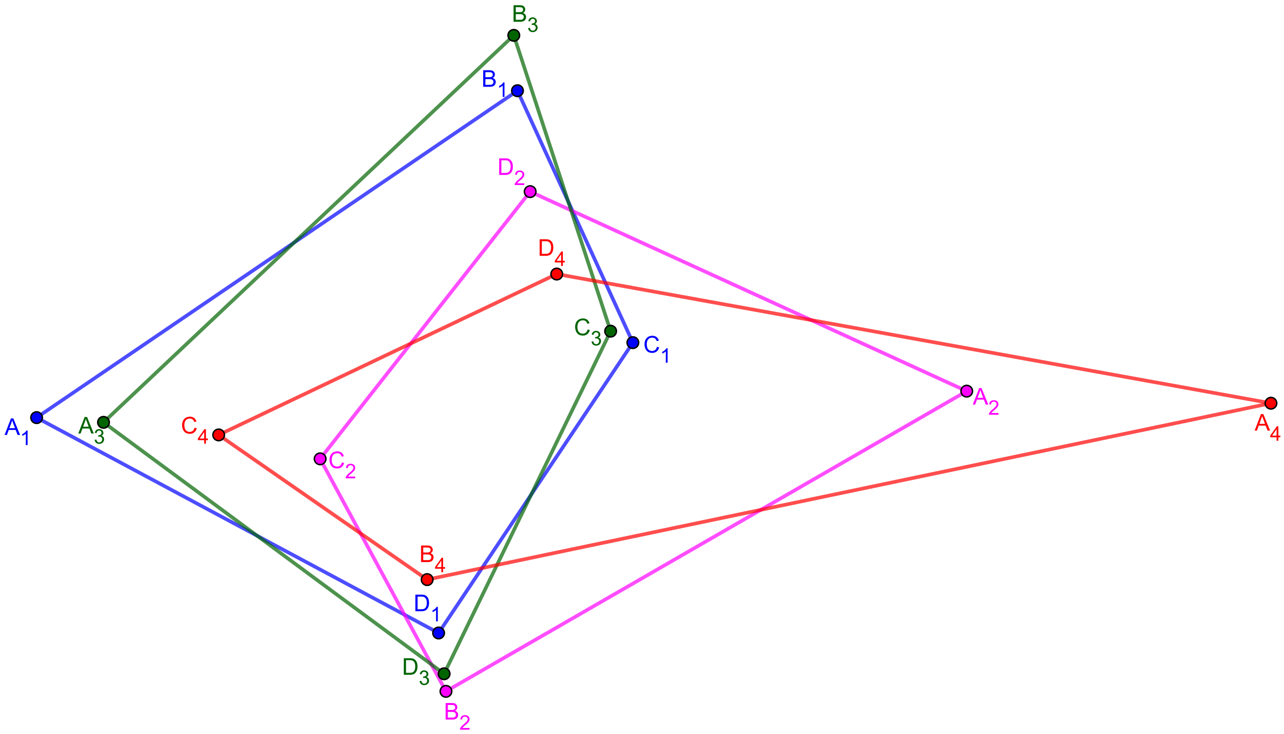


FIGURE 4.28: The four quadrilaterals $A_1B_1C_1D_1$, $A_2B_2C_2D_2$, $A_3B_3C_3D_3$, and $A_4B_4C_4D_4$.

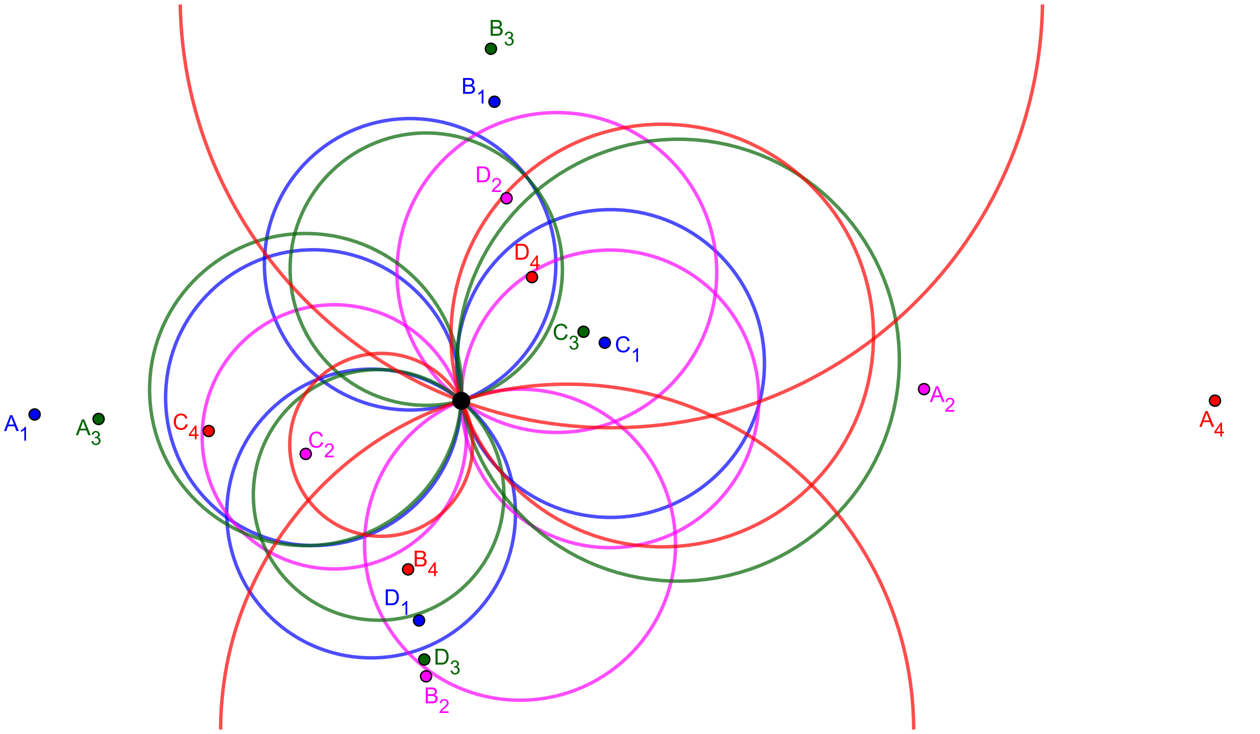


FIGURE 4.29: All sixteen nine-point circles intersect at a single point.

Chapter 5

The contributions of Longuet-Higgins

This chapter focuses on Longuet-Higgins' contributions to the study of Clifford's chain and de Longchamps' chain related to the circumcentre. In particular, it refers to three key papers. In the first presented here, (Longuet-Higgins, 1972), he analysed the link between Clifford's and Grace's chains and their relation to n-dimensional polytopes. In the second article, (Longuet-Higgins, 1971b), he provided a synthetic proof of de Longchamps' chain. Finally, in the third paper, (Longuet-Higgins, 1976), he studied the inversive properties of both Clifford's and de Longchamps' chains, highlighting their analogies and connections.
Before delving into the analysis of his contributions, a brief biography of Longuet-Higgins will be presented.

5.1 Biographical sketch of Michael Selwyn Longuet-Higgins

Born on 8 December 1925 in Lenham, Kent, Michael Selwyn Longuet-Higgins was the youngest of three children in a family with a background in the ecclesiastical, legal, and educational professions. In 1929, his family moved to Deal, Kent, and he attended the Pilgrims' School at Winchester. In 1939, he won a major scholarship to Winchester College, where he studied for the following four years. In 1943, he continued his mathematics studies at Trinity College, Cambridge, also on a scholarship. During the Second World War, his course was accelerated, allowing him to complete his degree in two years. He was then recruited by the Admiralty Research Laboratory in Teddington, where he joined Group W, a team focused on the propagation of long-distance ocean waves in anticipation of military operations in the Pacific. There, he applied his mathematical skills to wave propagation and microseisms, developing innovative tools. In 1948, after completing his war service, Longuet-Higgins returned to Cambridge for doctoral research. In 1951, he was awarded a Prize Fellowship at Trinity College and a Harkness Fellowship, which enabled him to undertake research at the Scripps Institution of Oceanography. In the early 1950s, he collaborated with Harold Scott MacDonald Coxeter (1907–2003) and Jeffrey Charles Percy Miller (1906–1981) on a study of uniform polyhedra, culminating in their influential 1954 paper

(Coxeter, Longuet-Higgins, and Miller, 1954). This collaboration with Coxeter reflected his lifelong interest in geometry and structural configurations, which dated back to his school years, when he built and studied polyhedral wire models. From 1954, he held positions at the National Institute of Oceanography, worked as a Royal Society Research Professor at the University of Cambridge from 1969, and was later affiliated with the Institute for Nonlinear Science at the University of California, San Diego. His career was marked by a characteristic blend of theoretical insight and experimental ingenuity. Beyond his scientific work, Longuet-Higgins expressed his enthusiasm for geometry through education and invention, creating the magnetic construction toy *Rhombo*, inspired by quasi-crystals and Penrose tilings. He died on 26 February 2016 in Cambridge, remembered for his unique combination of scientific rigour, geometric insight, and inventive spirit.[1]

5.2 The link between the configurations of Clifford's chain (and its extensions) and n-dimensional polytopes

This section analyses the correspondence established by Longuet-Higgins in (Longuet-Higgins, 1972) between certain configurations discussed in Chapter 3 and the n-dimensional polytopes studied by Coxeter. The notation introduced by Coxeter and adopted herein is explained in Appendix A. Further details can be found in (Coxeter, 1930) and (Coxeter, 1973).
In his 1956 review of Brown's article (Brown, 1954), Coxeter intuited that the 576 points and 56 hyperspheres of Brown's configuration[2] can be interpreted as the vertices and cells of a 7-dimensional polytope that he denoted 1_{32} (Coxeter, 1956). Intrigued by Coxeter's observation, Longuet-Higgins investigated whether the points and circles configurations of Clifford's first chain under a circular inversion exhibit a similar correspondence with polytopes. In the same paper, Longuet-Higgins extended this relationship to various configurations in higher-dimensional spaces. One reason why the correspondence is important is that the presence of a polytope ensures the existence of the relevant configuration, assuming the configuration in the preceding step exists.

Longuet-Higgins began by demonstrating that Clifford's configuration $\mathcal{CL}_m = (2^{m-1})_m$ corresponds to the polytope $1_{1\,m-3}$, namely the hemicube $h\gamma_m$ (Longuet-Higgins, 1972, pp. 445-449).
It is well known that three circles in a plane, all passing through a single point, determine the symmetric configuration 4_3, consisting of four points and four circles, with three points on each circle and three circles passing through each point. By identifying the points and circles with the vertices

[1]For further details on the biography of Longuet-Higgins, refer to (Sajjadi and Hunt, 2018).

[2]Brown's configuration is presented at the end of Section 3.4.

and triangular faces of a tetrahedron (see Figure 5.1), this configuration can be related to the simplex α_3, that is, the polytope 1_{10}.

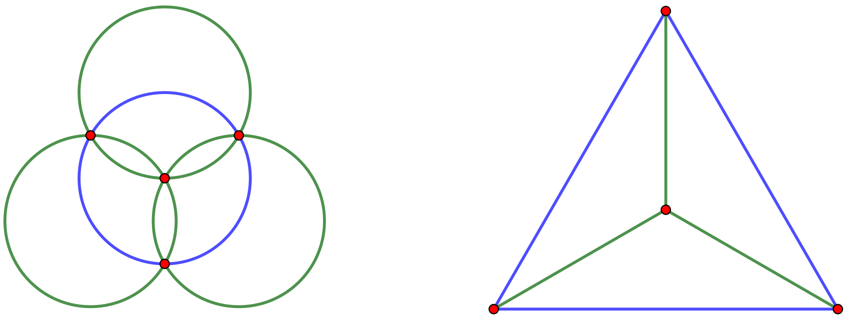


FIGURE 5.1: Correspondence between the configuration 4_3 and the simplex α_3.

It has already been established that four circles passing through a common point define the symmetric configuration 8_4, consisting of eight points and eight circles, with four points on each circle and four circles passing through each point. If, from among the four starting circles, one circle at a time is excluded, then four sets of three circles each are formed. Consequently the configuration 8_4 must contain each of the four configurations, which in turn derive from just three circles. Considering the associated polytope, it can be deduced that at each of its vertices there are four simplexes α_3. Any two of these simplexes share two points, any three share only one point, and any four share no points at all. As for the vertex figure of the polytope associated with the configuration, the triangles α_2 play the role of the tetrahedra α_3, so the configuration must contain four triangles α_2. Moreover, at each of its vertices, any two α_2s share exactly one point, while any three α_2s have no point in common. As for the second vertex figure, the segments α_1 take the place of the triangles α_2, so it contains four α_1s pairwise disjoint. Longuet-Higgins deduced that for the second vertex figure, the simplest solution is a square, *i.e.*, the polytope $(-1)_{11}$. From this, it follows that the vertex figure is 0_{11}, namely the octahedron β_3, and the polytope sought is 1_{11}, that is, the cross polytope β_4 (see Figure 5.2). The latter can be regarded as the four-dimensional equivalent of the octahedron.

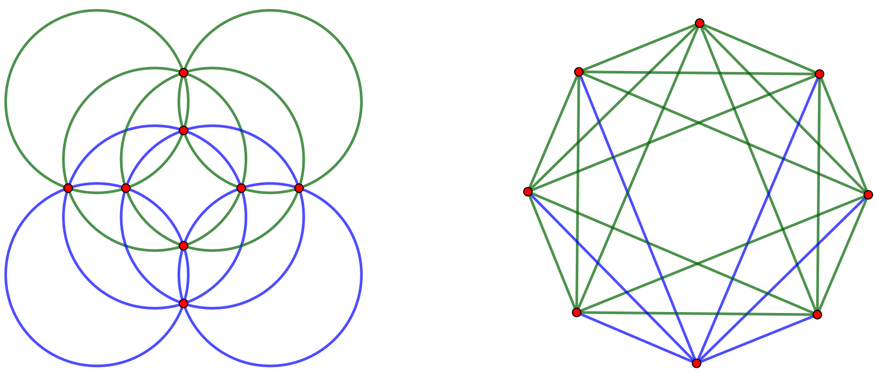


FIGURE 5.2: Correspondence between the configuration 8_4 and the cross polytope β_4.

In 2009, Wolfgang Karl Schief and Boris Georgievich Konopelchenko in (Schief and Konopelchenko, 2009, pp. 1294-1295) observed that Clifford's configuration 8_4 can also be interpreted as the configuration $6_4 8_3$, consisting of six points and eight circles, with three points on each circle and four circles passing through each point. They referred to this configuration as *octahedral*, identifying the points and circles with the vertices and faces of an octahedron, respectively (see Figure 5.3).

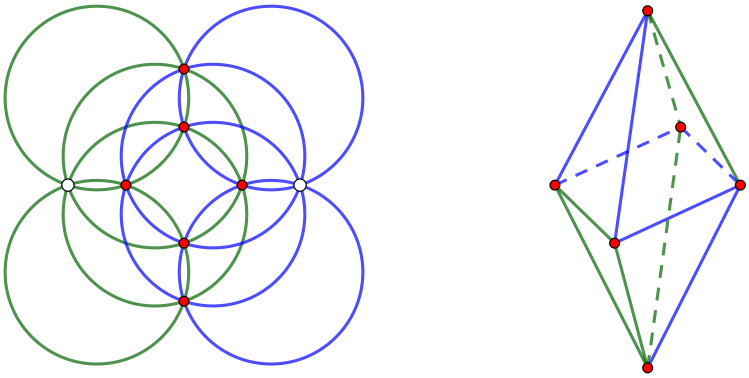


FIGURE 5.3: Correspondence between the configuration $6_4 8_3$ and the octahedron β_3.

After determining the polytope associated with the configuration 8_4, Longuet-Higgins applied a similar procedure to prove that the polytope 1_{12}, *i.e.*, the hemicube $h\gamma_5$, corresponds to the configuration 16_5. By generalisation, he demonstrated that the polytope associated with the configuration $(2^{m-1})_m$ is $1_{1\,m-3}$, namely the hemicube $h\gamma_m$, which contains 2^{m-1} vertices and 2^{m-1} simplexes α_{m-1}, corresponding respectively to the points and circles of the configuration. As shown in Appendix A, the necessary condition for the existence of the polytope $1_{1\,m-3}$ is satisfied for all m, confirming that Clifford's chain of configurations can be extended indefinitely.

As noted in Section 3.4, in the three-dimensional space, four spheres intersecting at a single point result in the symmetric configuration 5_4, consisting of five points and five spheres, with four points on each sphere and four spheres passing through each point. This configuration corresponds to the simplex α_4, *i.e.*, to the polytope 1_{20}, by associating the points and spheres of the configuration with its vertices and its cells α_3.

It has been previously established that, starting from five spheres intersecting at a single point, Grace derived the configuration $16_5 10_8$, consisting of 16 points and 10 spheres, with 8 points on each sphere and 5 spheres passing through each point. If, from among the five initial spheres, one sphere at a time is excluded, then five sets of four spheres are formed, so this configuration must contain each of the five configurations, which in turn derive from just four spheres. Considering the associated polytope, it can be deduced that at each of its vertices there are five simplexes α_4. Any two of these simplexes α_4 share two points, any three share only one point, and any four share no points at all. As for the vertex figure of the polytope associated with the configuration in question, the tetrahedra α_3 play the role of the simplexes

α_4, so the configuration must contain five tetrahedra α_3. Moreover, at each of its vertices, any two α_3s share exactly one point, while any three α_3s have no point in common. Regarding the second vertex figure, the triangles α_2 take the place of the tetrahedra α_3, so it contains five α_2s pairwise disjoint. Longuet-Higgins deduced that the simplest possible solution for the second vertex figure is a triangular prism, *i.e.*, the polytope $(-1)_{21}$. Consequently, the vertex figure is 0_{21} and the corresponding polytope is 1_{21}, namely the hemicube $h\gamma_5$.

After identifying the polytope associated with Grace's configuration $16_5 10_8$, Longuet-Higgins applied similar reasoning to demonstrate that the polytope corresponding to the configuration of six spheres intersecting at a single point is 1_{22}. In general, he proved that the polytope $1_{2\,m-4}$ is associated with Grace's configuration involving m spheres. As shown in Appendix A, the necessary condition for the existence of the polytope $1_{2\,m-4}$ is satisfied only for $m \leq 9$, so the chain of configurations developed by Grace comes to an end, contrary to what he had assumed. Regarding the configuration defined by eight spheres passing through a single point, Grace did not determine the number of points and spheres, but conjectured that each sphere would contain 64 points and that 8 spheres would pass through each point. Longuet-Higgins did not disprove this hypothesis; on the contrary, he associated the polytope 1_{24} with a configuration consisting of $17,280$ points and $2,160$ spheres, with 64 points on each sphere and 8 spheres passing through each point (Longuet-Higgins, 1972, pp. 449-451).

As noted in Section 3.4, in the four-dimensional space, five hyperspheres intersecting at a single point result in the symmetric configuration 6_5, consisting of six points and six hyperspheres, with five points on each hypersphere and five hyperspheres passing through each point. This configuration corresponds to the simplex α_5, *i.e.*, to the polytope 1_{30}, by associating the points and hyperspheres of the configuration with its vertices and its cells α_4, respectively. As already mentioned, from six hyperspheres through a single point Grace derived the configuration $32_6 12_{16}$, consisting of 32 points and 12 hyperspheres, with 16 points on each hypersphere and 6 hyperspheres passing through each point. If, from among the six initial hyperspheres, one hypersphere at a time is excluded, then six sets of five hyperspheres are formed, so this configuration must contain each of the six configurations, which in turn derive from just five hyperspheres. Considering the associated polytope, it can be deduced that at each of its vertices there are six simplexes α_5. Any two of these simplexes α_5 share two points, any three share only one point, and any four share no points at all. As for the vertex figure of the polytope associated with the configuration in question, the simplexes α_4 play the role of the simplexes α_5, so the configuration must contain six simplexes α_4. Moreover, at each of its vertices, any two α_4s have only one point in common, while any three α_4s have none. Regarding the second vertex figure, the tetrahedra α_3 take the place of the simplexes α_4, so it contains six α_3s pairwise disjoint. Longuet-Higgins deduced that the simplest possible solution for the second vertex figure is the polytope $(-1)_{31}$. Consequently, the vertex figure is 0_{31} and the corresponding polytope is 1_{31}, that is, the hemicube $h\gamma_6$.

Utilising analogous arguments, he associated the polytope 1_{32} with Brown's configuration of 576 points and 56 hyperspheres, with 72 points on each hypersphere and 7 hyperspheres passing through each point. He therefore generalised this process and demonstrated that the polytope $1_{3\,m-5}$ corresponds to the configuration defined by m hyperspheres. As shown in Appendix A, the necessary condition for the existence of the polytope $1_{3\,m-5}$ is satisfied for $m \leq 8$. Therefore, analogously to the case in the three-dimensional space, this present series of configurations does not continue any further (Longuet-Higgins, 1972, pp. 451-452). Finally, as already mentioned in Section 3.4, Longuet-Higgins in (Longuet-Higgins, 1973a, pp. 37-38) showed that the chain of theorems that corresponds to the configurations of points and hyperspheres stops at the step with six hyperspheres proved by Grace.

Longuet-Higgins continued his investigation of polytopes associated with configurations of points and hyperspheres, extending his analysis to n-dimensional spaces for $n \geq 5$. In this context, a set of $n+1$ hyperplanes, intersecting n at a time at a point, defines a hypersphere circumscribed about the n-dimensional simplex formed by these hyperplanes. When the $n+1$ hyperplanes are inverted with respect to a point in the space, they are transformed into $n+1$ hyperspheres passing through the centre of inversion. These hyperspheres, taken n at a time, intersect at one additional point. The $n+1$ points defined in this way (the inverses of the vertices of the simplex) all lie on a common hypersphere. He thus derived the symmetric configuration $(n+2)_{n+1}$, consisting of $n+2$ points and $n+2$ hyperspheres, with $n+1$ points on each hypersphere and $n+1$ hyperspheres passing through each point. This configuration is associated with the simplex α_{n+1}, that is, the polytope $1_{n-1\,0}$, by identifying the points and hyperspheres of the configuration with the vertices and the cells α_n of the polytope, respectively.
Given $n+2$ hyperspheres passing through a single point, by excluding each hypersphere at a time, one obtains $n+2$ distinct sets of $n+1$ hyperspheres. Consequently, the configuration $(n+2)_{n+1}$ must contain each of the $n+2$ configurations, which in turn derive from a different set of $n+1$ hyperspheres. Considering the associated polytope, it can be deduced that at each of its vertices there are $n+2$ simplexes α_{n+1}. Any two of these simplexes α_{n+1} share two points, any three share only one point, and any four share no points at all. Analysing the first and second vertex figures and following a reasoning analogous to that previously employed, Longuet-Higgins expected the polytope $1_{n-1\,m-n-1}$ to correspond to the configuration of m spheres passing through a point, with $m \geq n+1$ (Longuet-Higgins, 1972, pp. 452-453). Furthermore, since the necessary condition for the existence of the polytope $1_{n-1\,m-n-1}$ holds for $m \leq \frac{n^2}{n-2}$ (see Appendix A), it follows that:

- if $n = 5, 6$, there are three polytopes, $1_{n-1\,0}$, $1_{n-1\,1}$ and $1_{n-1\,2}$, associated with three configurations,

- if $n \geq 7$ there are only two polytopes, $1_{n-1\,0}$ and $1_{n-1\,1}$, corresponding to two configurations.

In analogy to the symmetric configurations 4_3, 5_4 and 6_5, it can be noted that the configuration $(n+2)_{n+1}$ is also a geometric representation of the

$2-(n+2, n+1, n)$ symmetric design, whose blocks are the hyperspheres and exactly n blocks pass through any two distinct points. As previously explained in relation to the construction of this configuration, there necessarily exists a hypersphere containing every set of $n+1$ points, hence the configuration $(n+2)_{n+1}$ is also an $(n+1)-(n+2, n+1, 1)$ symmetric design, i.e., a *Steiner system*. According to a well-known theorem[3] from design theory, such a configuration represents a $t-(n+2, n+1, n+2-t)$ symmetric design for every $t \leq n+1$.[4]

5.3 Longuet-Higgins' proof of de Longchamps' chain related to the circumcentre

In the late 19th and early 20th centuries, the chain of theorems related to the circumcentre, which was originally developed by de Longchamps (see Section 4.2.1), was rediscovered by several mathematicians. Among them, the Italian mathematician Pesci, in (Pesci, 1891), provided a proof using Euclidean properties, such as angle equalities, to establish the concyclicity of the points. Although his methods are similar to those of de Longchamps, he made no explicit reference to the French mathematician. Grace provided two proofs of the chain related to the circumcentre. In the first, presented in (Grace, 1900), he followed the approach introduced by Paul Serret in (Serret, 1895a,b) to prove the theorems of the Clifford chain, and he established an infinite series of theorems involving lines and circles in the plane. Nonetheless, as Grace himself noted, this series had already been obtained by Morley in (Morley, 1900) through more concise and purely analytical methods. In his second proof, published in (Grace, 1928), Grace adopted a method based on a projective construction in a higher-dimensional space, similar to the one used by White in (White, 1925). It is interesting to note that, while Grace explicitly referred to both Morley and Pesci, he made no mention of de Longchamps. Similarly, Morley did not mention the contributions of the French mathematician.
An explicit mention of de Longchamps appears instead in 1971, in (Longuet-Higgins, 1971b). The contributions of Longuet-Higgins to this topic are outlined below. His paper opens with a presentation of de Longchamps' chain related to the circumcentre. Although the chain has already been introduced in Section 4.2.1, it is recalled here using the notation adopted by Longuet-Higgins, in preparation for the distinct approach he employs in his proof.

1. Given four lines in a plane, the four circumcentres O_3 of the triangles formed by omitting each line in turn all lie on the same circle C_4, whose centre is denoted by O_4.

[3] If $s < t$, then a $t-(v, k, \lambda)$ design is also an $s-(v, k, \mu)$ design, where $\mu = \lambda \cdot \frac{\binom{v-s}{t-s}}{\binom{k-s}{t-s}}$.

[4] See (Rosen, 2000).

2. Given five lines in a plane, the centres O_4 of the five circles C_4, obtained by omitting each line in turn, all lie on the same circle C_5, whose centre is denoted by O_5.

3. In general, given $n+1$ lines in a plane, the $n+1$ centres O_n of the circles C_n, formed by omitting each line in turn, all lie on the same circle C_{n+1}, whose centre is denoted by O_{n+1}.

Recognising the similarity with Clifford's chain, Longuet-Higgins provided a synthetic proof of this chain, presented below, following a method analogous to that used by Clifford in his own series.

Consider four lines in the plane, α, β, γ, and δ, intersecting pairwise at six distinct points. Let I and J be the two circular points at infinity. Denote by $\mathcal{F}_3$ the family of all cubic curves passing through the six intersection points of α, β, γ, and δ, as well as through I and J. Since a general cubic curve has nine degrees of freedom,[5] specifying eight points on it leaves only one condition to be fixed in order to determine the cubic curve uniquely. Given a tangent at I, the cubic curve is uniquely determined and, consequently, the tangent at J is also uniquely determined. Therefore there is a one-to-one correspondence between tangents at I and tangents at J. The locus of the intersection points P of the tangent lines to the cubic curves in $\mathcal{F}_3$ at the points I and J is a conic section passing through the two circular points at infinity, I and J. Therefore this locus, denoted by C_4, is necessarily a circle.

The cubic curves in $\mathcal{F}_3$ also intersect the line IJ at a ninth point, denoted by K. If the point K lies on one of the starting lines, for instance on α, then the cubic curve intersects the line α in at least four distinct points: the three points of intersection of α with the other three lines, β, γ, and δ, together with the point K. Since a cubic curve cannot intersect a line in more than three points unless it contains that line as a component, it follows that the cubic curve degenerates into the union of the line α and a conic passing through I, J, and the intersection points of β, γ, and δ. This latter conic coincides with the circle circumscribed about the triangle formed by the lines β, γ, and δ. The point P corresponding to this cubic curve is the circumcentre of that triangle, and, as shown above, it lies on the circle C_4. By repeating the same reasoning with K lying on each of the other three lines, β, γ, and δ, it can be shown that the circumcentres of the four triangles determined by the four lines all lie on the same circle C_4. Hence, the first theorem in the chain presented above is established.

The centre O_4 of the circle C_4 is the point of intersection of the tangents to C_4 at the points I and J. Moreover, the tangent at I is the limit of the line IP as P tends to I, and consequently, the corresponding tangent at J, namely JP corresponds to the line IJ itself. Therefore, the centre O_4 of the circle C_4 is the intersection point between the tangent at I to the cubic curve F_3^*, which is tangent to IJ at J, and the tangent at J to the cubic curve $F_3^{*'}$, which is tangent to IJ at I.

Now consider $n+1$ lines in the plane, $\alpha, \beta, \ldots, \eta$, intersecting pairwise, with $n+1 \geq 5$. Let K be an arbitrary point on the line IJ, and denote by F_n and F_n'

[5]A curve of order n has $\frac{1}{2}n(n+3)$ degrees of freedom.

two curves of order n passing through the $\frac{1}{2}n(n+1)$ points of intersection of the $n+1$ lines $\alpha, \beta, \ldots, \eta$, as well as through the points I, J, and K. The curve F_n has a point of multiplicity $n-2$ at J along the line IJ, whereas F'_n has a point of multiplicity $n-2$ at I along the line IJ. Therefore, F_n and F'_n intersect the line IJ in n points (counted with multiplicities), and both satisfy $\frac{1}{2}n(n+3)$ conditions. Hence, if K is fixed on the line IJ, each curve is uniquely determined, whereas if K is allowed to vary along IJ, each curve has one degree of freedom. Given a tangent to F_n at I, the curve F_n is uniquely determined, as are the point K, the curve F'_n, and the tangent to F'_n at J. Therefore there is a one-to-one correspondence between tangents to F_n at I and tangents to F'_n at J. As a result, the locus of the intersection points P of the tangents to F_n at I and the tangents to F'_n at J is a conic section passing through I and J. Therefore, this locus, denoted by C_{n+1}, is necessarily a circle.
Let us denote by F^*_n the particular curve F_n which has a point of multiplicity $n-1$ at J along IJ (that is, when K coincides with J), and by $F^{*'}_n$ the particular curve F'_n which has a point of multiplicity $n-1$ at I along IJ (that is, when K coincides with I). Then the centre O_{n+1} of the circle C_{n+1} is given by the intersection of the tangent to F^*_n at I with the tangent to $F^{*'}_n$ at J. If the point K lies on one of the starting lines, for instance on α, then the curve F_n intersects the line α in at least $n+1$ points: namely, K and the n points of intersection of α with the other n lines $\beta, \gamma, \ldots, \eta$. Therefore, F_n degenerates into the union of the line α and a curve F^*_{n-1} of order $n-1$, which passes through the intersection points of the lines $\beta, \gamma, \ldots, \eta$ and has $n-2$ points of contact with IJ at J; analogously, F'_n degenerates into the union of the line α and a curve $F^{*'}_{n-1}$ of order $n-1$, which passes through the intersection points of the lines $\beta, \gamma, \ldots, \eta$ and has $n-2$ points of contact with IJ at I. Hence, the circle C_{n+1} contains the point O_n corresponding to the lines $\beta, \gamma, \ldots, \eta$. The same reasoning applies to the other points O_n obtained by letting the point K vary along the lines $\beta, \gamma, \ldots, \eta$.

Still using synthetic geometric methods, Longuet-Higgins also proved that the $n+1$ circles C_n all intersect at a single point Q_{n+1}. Let K and L denote two arbitrary points on the line IJ. Two curves G_n and G'_n are considered, both of order n, passing through the points I, J, K, and L, as well as through the $\frac{1}{2}n(n+1)$ points of intersection of the $n+1$ lines $\alpha, \beta, \ldots, \eta$. The curve G_n has a point of multiplicity $n-3$ at J along the line IJ, whereas G'_n has a point of multiplicity $n-3$ at I along the line IJ. Therefore, G_n and G'_n intersect the line IJ in n points (counted with multiplicities), and both satisfy $\frac{1}{2}n(n+3)$ conditions. When the points K and L are fixed on the line IJ, the corresponding curves G_n and G'_n are uniquely determined. Allowing K and L to vary along the line IJ, instead, gives rise to a net[6] of curves for each, with two degrees of freedom. These two nets are linearly correlated: each pencil of curves in the first net corresponds to a pencil in the second, and vice versa. If the point K lies on one of the initial lines, for instance on α, then the curve G_n degenerates into the union of the line α and a curve F^*_{n-1} of order $n-1$ associated with the lines $\beta, \gamma, \ldots, \eta$, denoted by $F^*_{n-1}(\beta, \gamma, \ldots, \eta)$; whereas the

[6] A net is a two-dimensional linear system of algebraic curves.

curve G'_n degenerates into the union of α and a curve $F^{*'}_{n-1}$ of order $n-1$ associated with the lines $\beta, \gamma, \ldots, \eta$, denoted by $F^{*'}_{n-1}(\beta, \gamma, \ldots, \eta)$. Therefore, the locus of the intersection points P of the tangents to G_n at I and the tangents to G'_n at J is the circle $C_n(\beta, \gamma, \ldots, \eta)$. If $K \in \alpha$ and $L \in \beta$ simultaneously, then G_n degenerates into the union of the lines α and β, and a curve $F^*_{n-2}(\gamma, \ldots, \eta)$ of order $n-2$; similarly, G'_n degenerates into the union of the lines α and β, and a curve $F^{*'}_{n-2}(\gamma, \ldots, \eta)$. Therefore, the point P coincides with the centre $O_{n-1}(\gamma, \ldots, \eta)$.

Let Q denote the common point of the circles $C_n(\beta, \gamma, \ldots, \eta)$ and $C_n(\alpha, \gamma, \ldots, \eta)$, distinct from $O_{n-1}(\gamma, \ldots, \eta)$. Since Q lies on $C_n(\beta, \gamma, \ldots, \eta)$, there exists a pair of non-degenerate curves of order $n-1$, ${}_\alpha F_{n-1}(\beta, \gamma, \ldots, \eta)$ and ${}_\alpha F'_{n-1}(\beta, \gamma, \ldots, \eta)$, tangent to the line QI at I and to the line QJ at J, respectively. By adding the line α to each, one obtains two corresponding curves of order n, ${}_\alpha G_n$ and ${}_\alpha G'_n$, tangent to QI at I and to QJ at J, respectively. Similarly, since Q also lies on $C_n(\alpha, \gamma, \ldots, \eta)$, there exists another pair of curves, ${}_\beta G_n$ and ${}_\beta G'_n$, constructed using the line β instead of α. As a result, one obtains two distinct curves in the first net, namely ${}_\alpha G_n$ and ${}_\beta G_n$, both tangent to QI at I, and two corresponding curves in the second net, namely ${}_\alpha G'_n$ and ${}_\beta G'_n$, both tangent to QJ at J. Therefore, the pencil of curves in the first net tangent to QI at I corresponds to a pencil in the second net in which two curves are tangent to QJ at J. Since the presence of two such tangents in a pencil implies that all curves in the pencil are tangent at that point, it follows that every curve in the corresponding pencil of the second net is tangent to QJ at J. Now, assume that one of the curves in the first pencil also passes through the point where a third line γ intersects the line IJ. Then it degenerates into the union of the line γ and a curve of order $n-1$, associated with the remaining lines, $\alpha, \beta, \delta, \ldots, \eta$. The corresponding curve in the second net degenerates in the same way. Since QI and QJ are the tangents to this pair of curves at I and J, respectively, it follows that Q must also lie on the circle $C_n(\alpha, \beta, \delta, \ldots, \eta)$. A similar argument shows that Q lies on all other circles C_n. This demonstrates completely de Longchamps' chain related to the circumcentre.

Longuet-Higgins did not limit himself to proving de Longchamps' chain, but also established several properties related to it.

He first raised the question of whether the points Q_{n+1} could themselves give rise to a new chain. It has already been observed in Chapter 2 that in the case $n = 3$, that is, when four lines are considered, the common point of the four circles circumscribed about the four triangles, denoted by Q_4, is the Wallace point of the configuration; and, by Steiner's second question, it also lies on the circle C_4. If $n = 4$ (*i.e.*, five lines), the point Q_5 does not, in general, lie on the circle C_5, but it lies on the circle M_5, which passes through the five points Q_4. It is easy to recognise that M_5 is the circle in the Clifford chain associated with five lines. It was Frederick Bath (1900–1982) who proved in (Bath, 1939) that $Q_5 \in M_5$, a property that will later be referred to as the *Bath Theorem*.

Grace showed in (Grace, 1928, pp. 14-15) that the six points Q_5 are not concyclic. The same can be shown for all $n \geq 6$.

On the other hand, Longuet-Higgins demonstrated that the $n+1$ points Q_n

(for $n \geq 4$) lie on a bicircular quartic B_{n+1}.[7] He also stated that B_{n+1} is unicursal,[8] and therefore corresponds to the inverse of a conic section.[9] If $n = 4$, the five points Q_4 lie on the bicircular quartic B_5, which in this case is simply the circle M_5 counted twice. Furthermore, if D_5 is the centre of the circle M_5, then the points O_5, Q_5, and D_5 are collinear (see Figure 5.4).

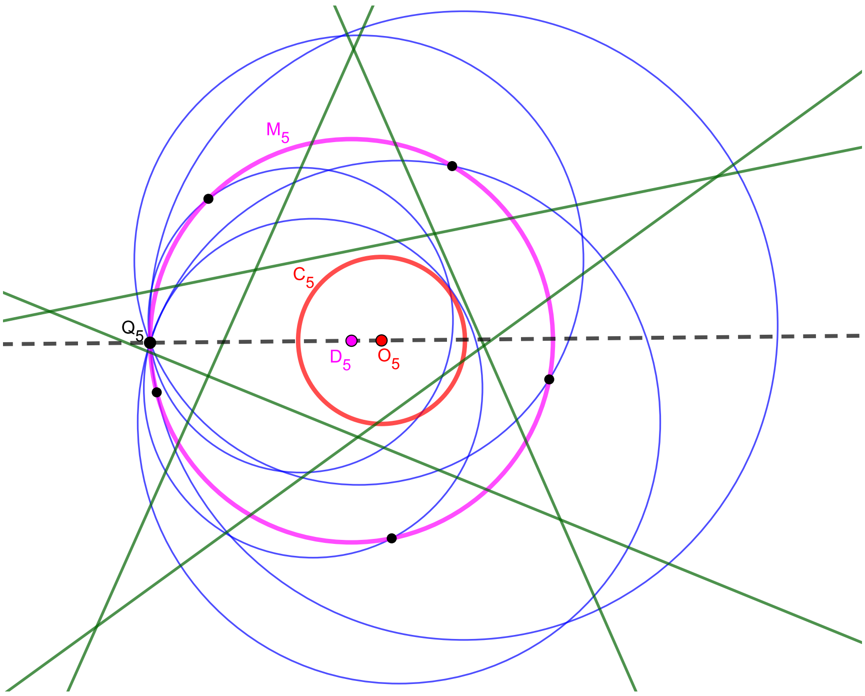


FIGURE 5.4: The points O_5, Q_5, and D_5 are collinear.

5.4 Inversive properties of Clifford's chain and de Longchamps' chain related to the circumcentre

In 1976, Longuet-Higgins investigated the inversive properties of specific geometric configurations in the plane, focusing in particular on Clifford's chain[10] and de Longchamps' chain related to the circumcentre (Longuet-Higgins, 1976).[11] As has already been noted, the starting points of the two

[7]Following the definition provided by John Casey (1820–1891) in (Casey, 1871, pp. 459-460), a *bicircular quartic* is a quartic curve having the two circular points at infinity as double points.

[8]A plane curve is said to be *unicursal* if the coordinates of its points can be expressed as rational functions of a parameter (Salmon, 1873, p. 29); equivalently, it is a curve of genus zero.

[9]Since circular inversion is a birational transformation, it preserves the genus. Therefore, the inverse of a conic (which has genus zero) is a bicircular quartic of genus zero, *i.e.*, a unicursal bicircular quartic.

[10]As already observed, chains analogous to those of Clifford under circular inversion in higher-dimensional spaces had previously been studied by Longuet-Higgins in (Longuet-Higgins, 1972).

[11]In this paper, Longuet-Higgins considered only configurations involving four and five lines.

chains coincide. Accordingly, Longuet-Higgins began by analysing Clifford's symmetric configuration $\mathcal{CL}_4$. This configuration is known to be invariant under circular inversion, as previously established in Section 3.3. Longuet-Higgins observed that the eight points P_{12}, P_{13}, P_{14}, P_{23}, P_{24}, P_{34}, P_{1234}, and O of the configuration $\mathcal{CL}_4$ are arranged into four pairs of *opposite points*:

$$(O,\, P_{1234}), \qquad (P_{12},\, P_{34}), \qquad (P_{13},\, P_{24}), \qquad (P_{14},\, P_{23}),$$

such that no circle passes through both points of any pair. Similarly, there are four pairs of *opposite circles*:

$$(\Gamma_1,\, \Gamma_{234}), \qquad (\Gamma_2,\, \Gamma_{134}), \qquad (\Gamma_3,\, \Gamma_{124}), \qquad (\Gamma_4,\, \Gamma_{123}),$$

such that no point lies on both circles of the same pair (see Figure 5.5).

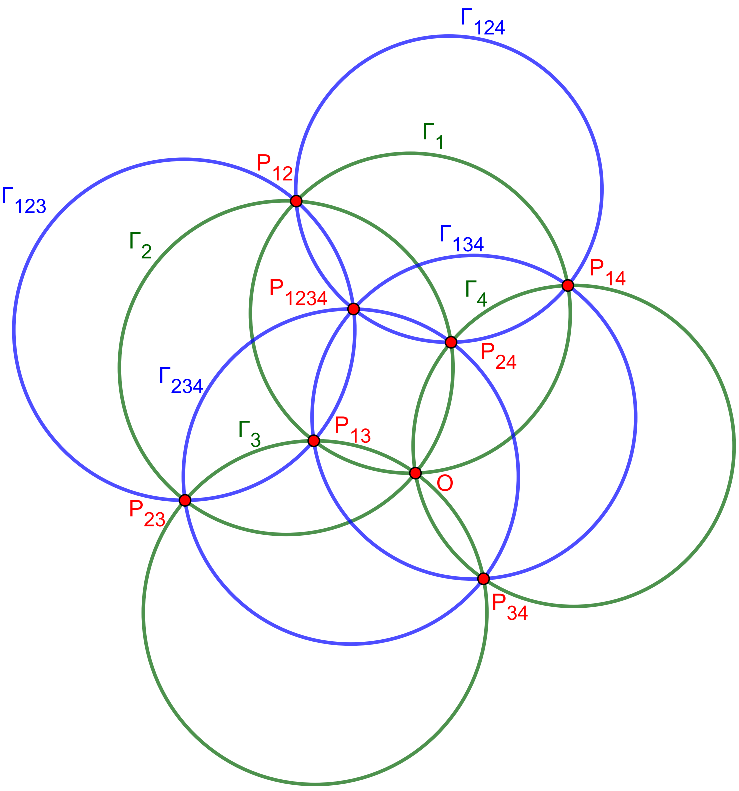


FIGURE 5.5: Configuration $\mathcal{CL}_4$: opposite points and opposite circles.

Moreover, he noted that the Wallace–Steiner question can be regarded as a special case of the configuration $\mathcal{CL}_4$ when one of the points lies at infinity.

As de Longchamps' chain related to the circumcentre involves the centres of circles, Longuet-Higgins pointed out an important property of circular inversion: the inverse of the centre of a circle does not coincide with the centre of the transformed circle. Rather, it corresponds to the image of the centre of inversion with respect to the transformed circle. More precisely, if C is the centre of a circle Σ, then the inverse point C' of C with respect to a point O

is the image of O with respect to the circle Σ', which is the inverse of Σ with respect to O (see Figure 5.6).

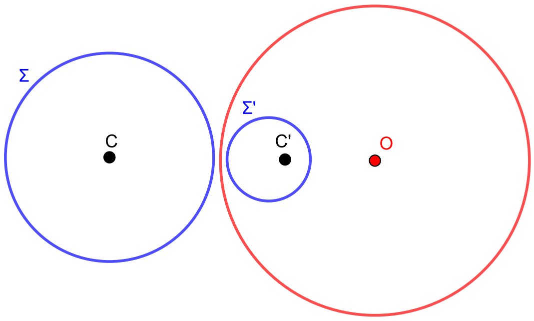


FIGURE 5.6: Circular inversion of a circle and its centre.

Building on this observation, Longuet-Higgins proceeded by transforming under circular inversion the configuration corresponding to the first step of de Longchamps' chain related to the circumcentre, according to which four lines in the plane, intersecting pairwise, determine four circles that pass through a common point, and whose centres, C_{123}, C_{124}, C_{134}, and C_{234}, all lie on the same circle (see Figure 4.11). The images of these centres under circular inversion with respect to O are the inverses of O with respect to the four transformed circles Γ_{123}, Γ_{124}, Γ_{134}, and Γ_{234}. The four points thus obtained, O_{123}, O_{124}, O_{134}, and O_{234}, all lie on the same circle, denoted by Σ_O, which is called the *polar circle* of O. It is important to note that this circle also passes through P_{1234}, the opposite point to O. This configuration is referred to as the de Longchamps configuration $\mathcal{DL}_4$ relative to the point O (see Figure 5.7).

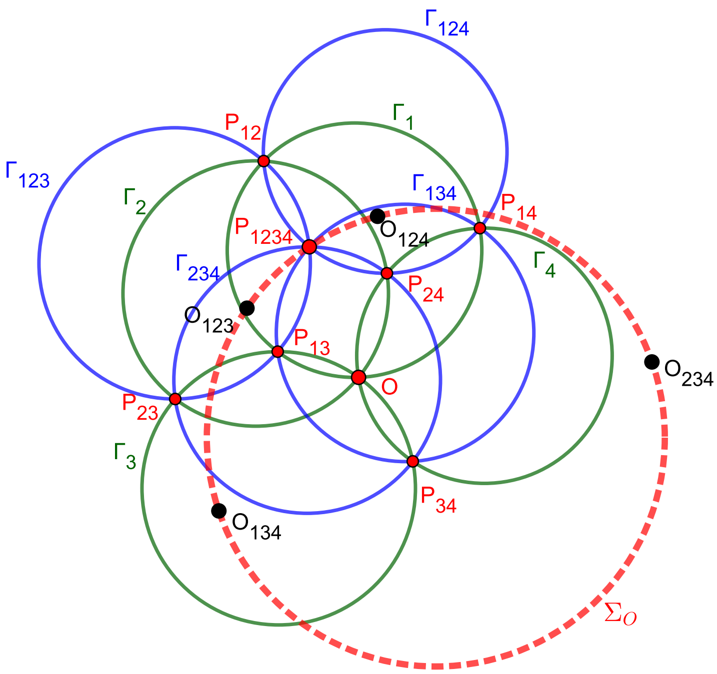


FIGURE 5.7: De Longchamps configuration $\mathcal{DL}_4$ relative to the point O.

By the symmetry of the construction, one can substitute any other point of the configuration $\mathcal{CL}_4$, namely P_{12}, P_{13}, P_{14}, P_{23}, P_{24}, P_{34}, or P_{1234}, for the point O. In this way, one obtains a configuration consisting of eight points and eight circles, with exactly one point on each circle and exactly one circle passing through each point. The eight points and their corresponding polar circles are represented in Figure 5.8, where each point and its associated polar circle are indicated using the same colour.

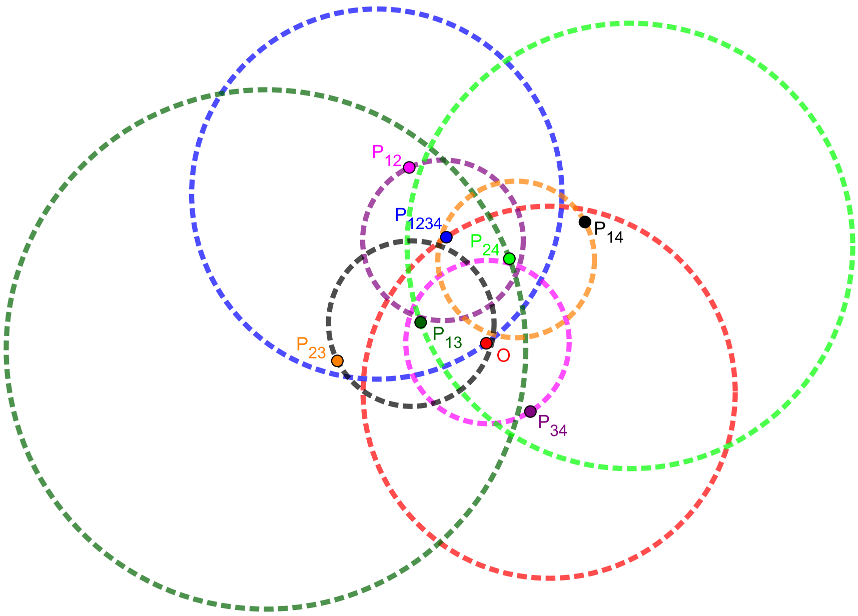


FIGURE 5.8: Points and their associated polar circles.

In general, the polar circle of a point P of the configuration $\mathcal{CL}_4$ passes through its opposite point, as well as through the images of P with respect to the four circles that do not contain it. In particular, the polar circle Σ_{1234}[12] of the point P_{1234} passes through the point O and through the images of P_{1234} with respect to the circles Γ_1, Γ_2, Γ_3, and Γ_4. Longuet-Higgins observed that if the point O lies at infinity, then the circles passing through O, namely Γ_1, Γ_2, Γ_3, Γ_4, and Σ_{1234}, become straight lines. Therefore, the images of P_{1234} under reflection[13] in the four lines Γ_1, Γ_2, Γ_3, and Γ_4 are collinear.

Longuet-Higgins applied the same reasoning to the configurations of the Clifford's and de Longchamps' chains derived from five lines in the plane intersecting pairwise. He first considered the Clifford configuration $\mathcal{CL}_5$, which is defined by five circles, Γ_1, Γ_2, Γ_3, Γ_4, and Γ_5, that all intersect at a common point O. These circles are the inverses of the five starting lines with respect to the point O. This configuration consists of sixteen points and sixteen circles. The sixteen points are:

- the centre of inversion O;

[12] For the sake of uniformity in notation, the polar circle Σ_{1234} should be more precisely denoted as $\Sigma_{P_{1234}}$, indicating that it is the polar circle associated with the point P_{1234}.

[13] Reflection in a line can be regarded as a limiting case of circular inversion, in which the centre of inversion tends to infinity and the circle becomes a straight line.

- the ten pairwise intersections of the starting lines, P_{12}, P_{13}, P_{14}, P_{15}, P_{23}, P_{24}, P_{25}, P_{34}, P_{35}, and P_{45};
- the five points, P_{1234}, P_{1235}, P_{1245}, P_{1345}, and P_{2345}, which are determined by the five configurations $\mathcal{CL}_4$ obtained from the previous step by excluding one circle Γ_i at a time.

The sixteen circles are:

- the five circles Γ_1, Γ_2, Γ_3, Γ_4, and Γ_5;
- the ten circles Γ_{123}, Γ_{124}, Γ_{125}, Γ_{134}, Γ_{135}, Γ_{145}, Γ_{234}, Γ_{235}, Γ_{245}, and Γ_{345};[14]
- the circle Γ_{12345} passing through the five points P_{1234}, P_{1235}, P_{1245}, P_{1345}, and P_{2345}.

The points and circles are arranged such that there are five points on each circle and five circles passing through each point (see Figure 5.9).

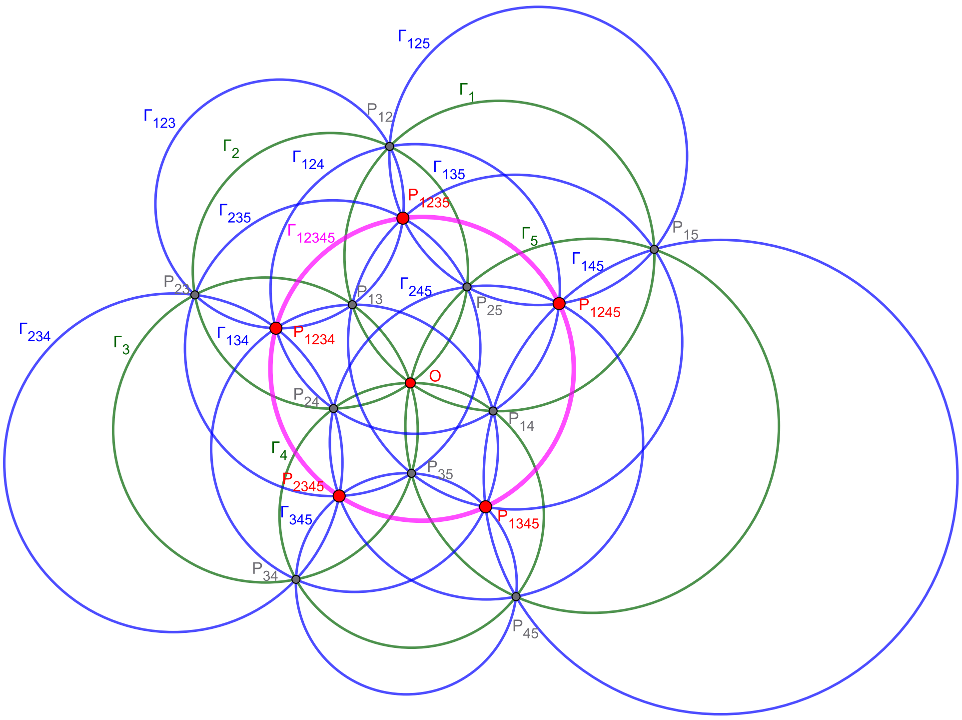


FIGURE 5.9: Configuration $\mathcal{CL}_5$.

Therefore, for each point P in the configuration $\mathcal{CL}_5$, there exist five configurations $\mathcal{CL}_4$, obtained by excluding in turn each of the five circles passing through P. In light of the above, one can associate to each point P of the configuration $\mathcal{CL}_5$ an opposite circle, *i.e.*, the circle passing through the five points opposite to P with respect to the five configurations $\mathcal{CL}_4$ containing P. For instance, the circle opposite to the point O is Γ_{12345}, which passes through the five points P_{1234}, P_{1235}, P_{1245}, P_{1345}, and P_{2345}, each of which is opposite

[14]Γ_{ijk} denotes the circle passing through the points P_{ij}, P_{ik}, and P_{jk}.

to O with respect to the five configurations $\mathcal{CL}_4$ containing O. All sixteen point–circle opposite pairs are:

$$\begin{array}{cccc} (O,\ \Gamma_{12345}), & (P_{12},\ \Gamma_{345}), & (P_{13},\ \Gamma_{245}), & (P_{14},\ \Gamma_{235}), \\ (P_{15},\ \Gamma_{234}), & (P_{23},\ \Gamma_{145}), & (P_{24},\ \Gamma_{135}), & (P_{25},\ \Gamma_{134}), \\ (P_{34},\ \Gamma_{125}), & (P_{35},\ \Gamma_{124}), & (P_{45},\ \Gamma_{123}), & (P_{1234},\ \Gamma_{5}), \\ (P_{1235},\ \Gamma_{4}), & (P_{1245},\ \Gamma_{3}), & (P_{1345},\ \Gamma_{2}), & (P_{2345},\ \Gamma_{1}).^{15} \end{array}$$

Longuet-Higgins then proceeded to investigate the second step of de Longchamps' chain, according to which five lines in the plane, intersecting pairwise, recursively define five circles that all pass through a common point.[16] The centres C_{1234}, C_{1235}, C_{1245}, C_{1345}, and C_{2345} of these five circles all lie on the same circle (see Figure 4.12). By applying a circular inversion with centre O, the five lines are transformed into the five circles Γ_1, Γ_2, Γ_3, Γ_4, and Γ_5 passing through O. From each set of four circles, obtained by excluding a circle Γ_i in turn, one obtains a polar circle of O.

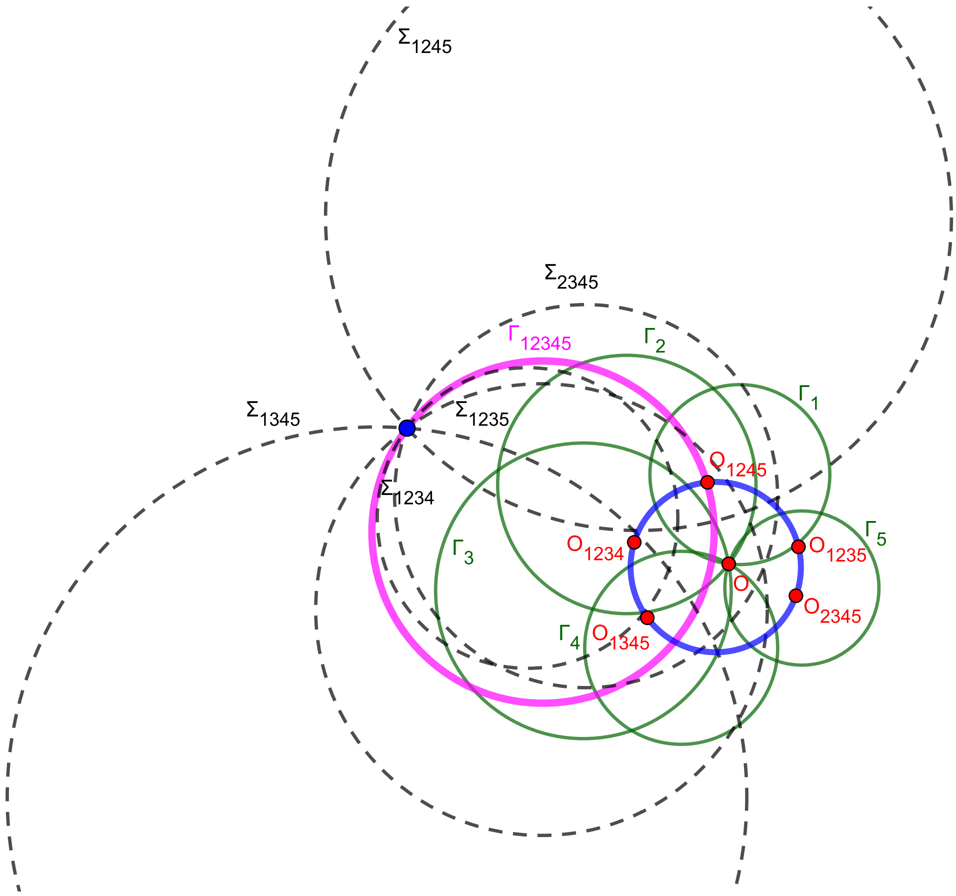


FIGURE 5.10: De Longchamps configuration $\mathcal{DL}_5$ relative to the point O.

[15] Here, the circle Γ_{ijk} is opposite to the point P_{lm}, and the circle Γ_i is opposite to the point P_{jklm}, where i, j, k, l, and m are all distinct.

[16] As observed in Section 5.3, Bath proved that this point lies on M_5, the circle in the Clifford chain associated with five lines.

Therefore, it can be deduced that there are a total of five polar circles of O: Σ_{1234}, Σ_{1235}, Σ_{1245}, Σ_{1345}, and Σ_{2345}. These circles all pass through the same point, which also lies on the circle Γ_{12345}, opposite to O.[17] Under circular inversion, the points C_{1234}, C_{1235}, C_{1245}, C_{1345}, and C_{2345} are mapped to the images of O with respect to the five polar circles of O. These inverted points, denoted by O_{1234}, O_{1235}, O_{1245}, O_{1345}, and O_{2345}, all lie on a common circle, as illustrated in Figure 5.10.

Once again, due to the symmetry of the construction, the point O can be replaced by any of the fifteen remaining points in the configuration $\mathcal{CL}_5$. In this way, each point P in the configuration is associated with a corresponding circle passing through the five images of P with respect to its five polar circles. These circles all intersect at the same point, which lies on the circle opposite to P.

By considering the point P_{12} and letting O tend to infinity, the three circles Γ_3, Γ_4, and Γ_5, which pass through O, degenerate into straight lines. Consequently, the circle Γ_{345}, which is opposite to P_{12}, becomes the circle circumscribed about the triangle determined by the lines Γ_3, Γ_4, and Γ_5. The three vertices of this triangle, namely P_{34}, P_{35}, and P_{45}, correspond to the points opposite to P_{12} with respect to the three configurations $\mathcal{CL}_4$, obtained by excluding one of the lines Γ_5, Γ_4, and Γ_3, respectively. Therefore, through P_{34}, through the reflections of P_{12} in the lines Γ_3 and Γ_4, and through the images of P_{12} under circular inversion with respect to the circles Γ_{234} and Γ_{134}, there passes a polar circle of P_{12}. The same holds for the other vertices of the triangle and for the reflections of P_{12} in the corresponding sides of the triangle. It is noteworthy that the five polar circles of P_{12} all intersect at a single point. Since the positions of the lines Γ_3, Γ_4, and Γ_5 are arbitrary in the plane, the following theorem, known by Longuet-Higgins as *Theorem T*, can be deduced: given a triangle ABC and a point P, let A', B', and C' denote the reflections of P in the sides BC, AC, and AB, respectively. Then, the four circles $AB'C'$, $A'BC'$, $A'B'C$, and ABC intersect at a single point W (see Figure 5.11).

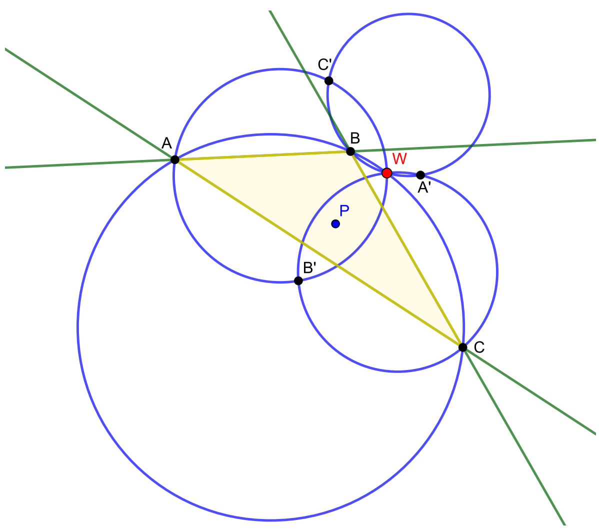


FIGURE 5.11: Theorem T.

[17] Bath Theorem under circular inversion can be recognised in this statement.

Longuet-Higgins had already proved this result in (Longuet-Higgins, 1973b) and he had observed that by applying a circular inversion with respect to the point W, the four circles $AB'C'$, $A'BC'$, $A'B'C$, and ABC are mapped to four lines that do not pass through W and intersect pairwise. According to the Wallace–Steiner question, the circles circumscribed about the four triangles determined by these four lines also intersect at a single point. By applying a circular inversion once again with respect to W, it is found that the circles $A'BC$, $AB'C$, ABC', and $A'B'C'$ intersect at a single point, denoted by W'. This result is what Longuet-Higgins referred to as *Theorem T′*. From Theorems T and T′, it follows that the eight points A, B, C, A', B', C', W, and W' form the eight points of a configuration $\mathcal{CL}_4$.
More generally, he stated that given three circles Σ_1, Σ_2, and Σ_3 passing through a common point O and intersecting pairwise again at the points A, B, and C, and letting A', B', and C' denote the images of a point P under circular inversion with respect to the three circles, then the eight circles ABC, $A'B'C'$, $AB'C'$, $A'BC'$, $A'B'C$, $A'BC$, $AB'C$, and ABC', together with the eight points A, B, C, A', B', C', W, and W' form a configuration $\mathcal{CL}_4$.

The following year, in (Longuet-Higgins, 1974), Longuet-Higgins stated the two Theorems T and T′, showing that they are inverses of each other, meaning that T can also be derived from T′ through inversion. He then presented several proofs of both theorems provided by various mathematicians.

In both his 1974 article and the 1975 paper co-authored with Cyril Frederick Parry (Longuet-Higgins and Parry, 1975), he showed a series of geometric properties that would later be revisited and further developed in his 1976 paper (Longuet-Higgins, 1976), a work he had already been preparing during those same years.
Returning to his 1976 article, Longuet-Higgins emphasised that Theorem T is closely related to Bath Theorem, thereby providing an alternative and elementary proof of Bath's result, distinct from both Bath's original demonstration (Bath, 1939) and from his own earlier proof in (Longuet-Higgins, 1971b).

Without going into detail, what follows is a broad outline of the second part of Longuet-Higgins' paper (Longuet-Higgins, 1976), serving as an example of how circular inversion is employed in his approach. He proceeded by considering a complete quadrangle $ABCD$ and the images, A', B', C', and D', of a point P under circular inversion with respect to the circles BCD, ACD, ABD, and ABC, respectively. He then applied Theorem T and Theorem T′ to triangles ABC, ABD, ACD, and BCD. In this way, he derived a symmetric configuration consisting of 16 points and 32 circles.[18] The same configuration can also be obtained by starting from A', B', C', and D' and a different point P'. Thus, he identified two opposite pentads of points

$$\begin{pmatrix} A & B & C & D & P \\ A' & B' & C' & D' & P' \end{pmatrix}$$

[18]The 16 points are the 8 points A, B, C, D, A', B', C', and D', along with 8 additional points obtained as the common intersections of four circles in each application of Theorems T and T′. The 32 circles include all circles involved in the applications of Theorems T and T′, each of which passes through exactly 4 of the 16 points.

such that each pair of points in one pentad are inverse images of each other with respect to the circle passing through the opposite triad of the other pentad. For instance, the points A and B are inverse images of each other with respect to the circle passing through C', D' and P'.
Furthermore, Longuet-Higgins showed that, by spherical inversion with respect to a point outside the plane, a similar configuration can be constructed on the surface of a sphere. In this case, the line joining any two points of a pentad passes through the pole of the plane defined by the opposite triad. Similar configurations can also be constructed on a general quadric surface.

Before concluding this chapter, and in order to provide a comprehensive overview of Longuet-Higgins' contributions on these topics, the issues addressed in 1979 together with Parry in (Longuet-Higgins and Parry, 1979) are briefly outlined here without going into detail. In that paper, the authors continued to investigate the inversive properties of the configurations $\mathcal{CL}_n$ (Clifford) and $\mathcal{DL}_n$ (de Longchamps). They showed that the elements of both configurations can be systematically arranged in a table: the elements of the $\mathcal{CL}_n$ occupy the first column, while those of the $\mathcal{DL}_n$ appear along three adjacent diagonals oriented from left to right in the lower part of the table (Longuet-Higgins and Parry, 1979, p. 542). Another interesting aspect highlighted in that study is that, while the first column of the table, corresponding to the Clifford chain, can be extended indefinitely downward by increasing the number of lines, the de Longchamps configuration admits only a limited extension. In fact, starting with six lines, as discussed in Section 5.3, the six points Q_5, obtained by omitting one line at a time, no longer lie on a circle but rather on a bicircular quartic curve. As a result, the chain structure of the $\mathcal{DL}_n$ breaks down, preventing further extension beyond six rows.[19]

[19] John Frankland Rigby (1933–2014) explored similar themes in (Rigby, 1977, 1982), building upon the foundational contributions of Longuet-Higgins and Parry.

Chapter 6

Concluding reflections

This thesis has investigated the historical origins and developments of several chains of theorems and geometric configurations, focusing particularly on Clifford's chains, Grace's extensions into higher-dimensional spaces, and de Longchamps' chains of recursive geometry. Furthermore, particular attention was given to the contributions of Longuet-Higgins. Specifically, he analysed the correspondence between Clifford's first chain and Grace's chain, under circular and spherical inversion, respectively, and the n-dimensional polytopes. He also demonstrated de Longchamps' chain related to the circumcentre using synthetic methods, with the aim of applying reasoning analogous to that employed by Clifford in his own series. Subsequently, Longuet-Higgins emphasised the connection between the two chains, Clifford's and de Longchamps', by studying their inversive properties, highlighting their analogies and affinities, and thereby enhancing the understanding of geometric configurations and their symmetries.
Although this thesis has focused on a selected group of contributions concerning chains of geometric configurations, it is important to acknowledge that the topic has also been investigated by many other mathematicians. In addition to those already mentioned throughout the chapters, more recent authors have made notable contributions. Their work has significantly advanced the development and generalisation of configuration theory, offering valuable perspectives for future research.

Throughout the topics discussed, the thesis sought to emphasise the unexpected connections between various theories. The investigation began with elementary geometry and progressed through the use of synthetic geometry methods, culminating, surprisingly, in the theory of polytopes, block-design theory, and, more broadly, Combinatorics. In this regard, adopting an expression coined by Yaglom, Combinatorics may be considered the elementary geometry of the latter half of the 20th century.

These researches have opened new and unexpected perspectives not only at the theoretical level but also at the didactic level. Indeed, the subject matters of this thesis serve as a valuable tool for teachers to design laboratory experiences, both within and beyond the curriculum, in upper secondary school. Concrete examples in this regard can be found in (Rinchiusa and Vaccaro, 2025a) and (Rinchiusa and Vaccaro, 2023): the former involves the planning and experimentation of a laboratory activity focused on de Longchamps' chain related to the circumcentre in upper secondary school, while the latter proposes a workshop centred on the series related to the

nine-point circle associated with a triangle.
Without entering into details thoroughly addressed in (Rinchiusa and Vaccaro, 2025a), the following provides a summary of the educational experience, in order to illustrate the didactic potential of recursive geometry.
It describes a learning path carried out with eleventh-grade students attending an Italian scientific upper secondary school. The aim was to stimulate the formulation of conjectures and argumentation towards demonstrative thinking, through the use of the dynamic geometry software *GeoGebra*. To this end, the topic of recursive geometry was chosen because it was believed that its nature would encourage students to identify common features in the configurations examined at each step of the chain, prompting them to formulate conjectures and construct arguments to support them. Furthermore, this theme was considered particularly appropriate due to the lack of didactic paths in upper secondary schools that focus specifically on recursion in geometry. Additionally, as mathematical induction is often perceived by students as a difficult topic, recursive processes were considered a potential key to improving their understanding of induction.
In the article, a theoretical framework is presented to support both the design and analysis of the learning experience. Two main theoretical cores are considered: Toulmin's model (Toulmin, 1958), which provides a structure for analysing students' argumentative processes; and the Vygotskian perspective on the use of cognitive artefacts in learning (Bartolini Bussi and Mariotti, 2008; Vygotsky, 1981), with particular attention to their role as mediators of mathematical knowledge. In this context, GeoGebra is considered a semiotic mediator that enables students to explore, formulate, and refine conjectures.
An important feature of GeoGebra, particularly useful in the proposed activity, is the ability to create and save macros, *i.e.*, tools that allow users to repeat a series of operations, even very complex ones, through a simple command. Within the workshop, macros proved especially valuable for analysing what occurs in the successive steps of the chain of configurations, since these depend on each other in an iterative manner.
The learning path was structured into five progressive phases:

1. **Verification of prerequisite knowledge**: revisiting basic concepts of plane geometry and discussing the definition of a circle determined by the intersection points of three lines, with an emphasis on formalising the related proof.

2. **Introduction to mathematical induction**: presenting induction through elementary arithmetic problems, then applying it to prove geometric properties such as the sum of the interior angles of a polygon.

3. **Recursive process and GeoGebra exploration**: encouraging students to analyse configurations formed by four lines and to construct circles from triples of lines using the dynamic geometry software GeoGebra, thereby fostering recursive reasoning.

4. **From argumentation to formal proof**: posing stimulus questions about the relationships between triangles, circles, and centres formed by

four lines to the class, divided into groups who collaboratively formulate conjectures and discuss their reasoning. Then, guiding students through a formal proof phase by supporting their transition from informal arguments to deductive demonstration, adapting the approach to their familiarity with formal reasoning.

5. **Generalisation**: extending the reasoning to configurations with five lines, using GeoGebra macros to manage more complex constructions and solidify understanding of recursion in geometry. Guiding students towards the generalisation of the steps in the studied chain.

Central to the approach was the use of mathematical discussion, strategically guided by the course instructor as a mediator, to support students' theoretical reasoning. In order to assess the achievement of the course objectives, recordings of classroom discussions, students' ongoing work, and interviews conducted a few weeks after the end of the activities were used, involving both students and their mathematics teacher.

By the final balance it emerged that the learning path on recursive geometry offered students the opportunity to approach plane geometry from a new perspective: from being mere observers, they became active protagonists of the activities. Not only did the students have the opportunity to address pre-existing gaps, but the repeated application of the same construction process across different configurations also helped them assimilate the fundamental concepts of Euclidean geometry that had been explored. During the classroom activity, it was observed that the vast majority of students, starting from the specific cases explored and independently reiterating the use of the defined macros, directed their attention towards configurations of the de Longchamps chain involving a greater number of lines, thereby appropriating the notion of generalisation. Considering that the class had not previously encountered the recursive process, not even in the arithmetic context, it is believed that the definition of recursion in geometry was understood. However, the final interviews revealed some confusion among students between the principle of induction and induction as an inferential process. Nevertheless, the latter was correctly employed by the students to generalise the characteristic properties of the studied chain.

Regarding the transition from the argumentative phase to the demonstration, it was apparent that the vast majority of students participated in the argumentative phase. Moreover, given that the class was not familiar with the demonstrative process, it was considered appropriate to guide students in formalising the proof of the theorem related to the configuration with four lines. Despite the majority of the class contributing effectively to the development of most of the proof, only one student was able to reach the conclusion independently. However, the learning process revealed a moderate development of students' argumentative skills. While they initially encountered difficulties in justifying even the most elementary steps, a gradual improvement was observed in their verbal reasoning, which became increasingly aware and followed appropriate logical reasoning.

Based on positive feedback from students and the classroom teacher, as well as observation evaluations, it is believed that the project yielded positive outcomes overall.

In light of this experience, it is believed that further activities can be developed focusing on Clifford chains and other chains devised by de Longchamps. Additionally, circular inversion, which is often overlooked in school curricula among geometric transformations, represents another valuable topic for learning pathways. Its power lies in providing elegant and unifying methods to address classical geometric problems and configurations, including the aforementioned chains as well as classic geometric problems, such as Apollonius' problem of tangency, revealing hidden symmetries and simplifying complex constructions.

These topics offer fertile ground for planning geometry workshops that combine theoretical exploration with practical activities supported by dynamic geometry software.

As a final consideration, it is worth emphasising that the rediscovery and reinterpretation of classical geometry problems, when integrated into well-structured mathematical workshops, not only promote deeper and more conscious engagement with mathematics among students but also encourage teachers to adopt innovative and reflective teaching practices.

Such experiences enrich everyday classroom practice and contribute to ongoing professional development, transforming teaching into a space for reflection, research, and methodological growth. They represent a valuable intersection between tradition and innovation, where the legacy of classical geometry is reimagined within contemporary educational frameworks.

To conclude this thesis, it is worth emphasising that even the simplest and seemingly most trivial issues, when explored in depth, can reveal hidden aspects and connections. As Laurent Lafforgue remarked in the introduction to (Lafforgue, 2018):

> We never stop enriching our knowledge and understanding of even the most elementary subjects, because in the two and a half millennia in which mathematics has existed, most of its progress has consisted of continually returning to the same fundamental subjects and exploring them in greater depth. The experience of mathematics teaches us that, regardless of how elementary a fundamental subject may appear at first sight, there is no limit to how deeply such a subject can be explored...[1]

[1] *On n'a jamais fini d'enrichir ses connaissances et sa compréhension même des sujets les plus élémentaires car, depuis deux millénaires et demi que les mathématiques existent, leur progrès a consisté pour une large part à reprendre encore et toujours les mêmes thèmes fondamentaux et à les approfondir toujours davantage. L'expérience des mathématiques enseigne qu'il n'y a pas de limite à l'approfondissement d'un thème fondamental, aussi élémentaire semble-t-il au premier abord...* (Lafforgue, 2018).

Appendix A

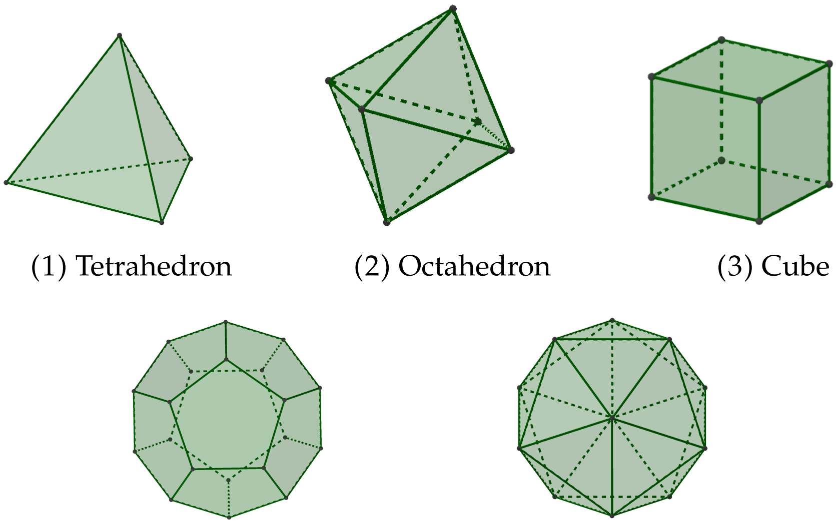
Polytopes

The term *polytope*, derived from the prefix *poly-* and the Greek word τόποσ (*tópos*, meaning "place"), appears to have been coined in 1882 by the German mathematician Ernst Reinhold Eduard Hoppe (1816–1900).[1] Approximately twenty years later, the Irish mathematician Alicia Boole Stott (1860–1940) adopted the term to refer to the n-dimensional equivalent of the concept of a polygon in two dimensions and a polyhedron in three dimensions.

According to the definition proposed by Coxeter in (Coxeter, 1973), a polytope Π_n is defined as a finite, convex region in n-dimensional space bounded by a finite number of hyperplanes. The part of a polytope that lies in a hyperplane is called a *cell*. Polygons are 2-dimensional polytopes Π_2 and have segments Π_1 as cells; polyhedra are 3-dimensional polytopes Π_3 and have polygons Π_2 as cells, and so on. Thus, each polytope Π_n consists of elements of type Π_k, where $k < n$.
If the midpoints of the edges emerging from a vertex O of the polytope Π_n all lie in a single hyperplane, then they form the vertices of an $(n-1)$-dimensional polytope, called the *vertex figure* of Π_n at O. The cells of the vertex figure at O correspond to the vertex figures at O of the cells of the polytope Π_n that surround O. The vertex figure of the vertex figure is an $(n-2)$-dimensional polytope, known as the *second vertex figure* of Π_n at O.

Giving an inductive definition, a polytope Π_n is *regular* if all of its cells and vertex figures are regular. It follows that all the cells of a regular polytope are equal, as are all of its vertex figures. What follows is a classification of regular polytopes, starting from the two-dimensional space. To this end, Schläfli notation is employed. Named after the Swiss mathematician Ludwig Schläfli (1814–1895), this notation assigns to each regular polytope a symbolic representation that reflects its fundamental structural properties.
In the plane, the symbol $\{p\}$ denotes a regular polygon with p sides and p vertices.
In the three-dimensional space, $\{p, q\}$ represents a regular polyhedron whose faces are regular polygons $\{p\}$ and which has q edges emanating from each vertex. There exist only five regular polyhedra, the so-called *Platonic solids*: the *tetrahedron* $\{3, 3\}$, the *octahedron* $\{3, 4\}$, the *cube* $\{4, 3\}$, the *dodecahedron* $\{5, 3\}$ and the *icosahedron* $\{3, 5\}$ (see Figure A.1).

[1]In (Hoppe, 1882, p. 30), Hoppe used the German term *Polytops* (as well as *Vielraum*, meaning "multi-space") to refer to a four-dimensional linearly bounded figure.



(1) Tetrahedron (2) Octahedron (3) Cube

(4) Dodecahedron (5) Icosahedron

FIGURE A.1: Platonic solids.

The Platonic solids are summarised in Table A.1, which includes their Schläfli symbols, the number of vertices N_0, the number of edges N_1, and the number of faces N_2. These quantities are linked by Euler's formula:

$$N_0 - N_1 + N_2 = 2.$$

This formula holds for all simply connected polyhedra, that is to say, all polyhedra, both regular and irregular, whose faces do not intersect each other.

TABLE A.1: Platonic solids.

Name	**Schläfli Symbol**	N_0	N_1	N_2
Tetrahedron	$\{3,3\}$	4	6	4
Octahedron	$\{3,4\}$	6	12	8
Cube	$\{4,3\}$	8	12	6
Dodecahedron	$\{5,3\}$	20	30	12
Icosahedron	$\{3,5\}$	12	30	20

In the four-dimensional space, $\{p,q,r\}$ denotes a regular polytope whose cells are regular polyhedra $\{p,q\}$ and r of them surround an edge. Its vertex figures are polyhedra $\{q,r\}$. As with polyhedra, there exist only a finite number of regular polytopes in four dimensions: the *hypertetrahedron* $\{3,3,3\}$, the *hypercube* $\{4,3,3\}$, the *hyperoctahedron* $\{3,3,4\}$, the *rhombic hyperdodecahedron* $\{3,4,3\}$, the *hyperdodecahedron* $\{5,3,3\}$, and the *hypericosahedron* $\{3,3,5\}$. Table A.2 shows, for each of these polytopes, the number of vertices N_0, edges N_1, faces N_2, and cells N_3. These quantities are linked by Euler's formula, which in the four-dimensional space becomes:

$$N_0 - N_1 + N_2 - N_3 = 0.$$

TABLE A.2: Regular polytopes in 4-dimensional space.

Name	**Schläfli Symbol**	N_0	N_1	N_2	N_3
Hypertetrahedron (5-cell)	$\{3,3,3\}$	5	10	10	5
Hypercube (8-cell)	$\{4,3,3\}$	16	32	24	8
Hyperoctahedron (16-cell)	$\{3,3,4\}$	8	24	32	16
Rhombic Hyperdodecahedron (24-cell)	$\{3,4,3\}$	24	96	96	24
Hyperdodecahedron (120-cell)	$\{5,3,3\}$	600	1200	720	120
Hypericosahedron (600-cell)	$\{3,3,5\}$	120	720	1200	600

More generally, in n-dimensional space, a regular polytope Π_n is denoted by the Schläfli symbol $\{p, q, r, \ldots, v, w\}$. It has cells of type $\{p, q, r, \ldots, v\}$, which are regular polytopes in $(n-1)$-dimensional space, and w of these cells occur in a Π_{n-3} element of Π_n. The vertex figures are the polytopes $\{q, \ldots, v, w\}$. In n-dimensional space, where $n > 4$, there exist only three types of regular polytopes:

1. the *simplex* α_n,
2. the *cross polytope* β_n,
3. the *hypercube* γ_n (also called *measure polytope*).

A simplex is defined as a finite region of n-dimensional space bounded by $n+1$ hyperplanes. It is the n-dimensional polytope that has the fewest vertices. If all edges are congruent, the simplex is regular and is denoted by α_n (see Figure A.2). Utilising Schläfli notation, the following can be deduced:

$$\alpha_2 = \{3\}, \quad \alpha_3 = \{3,3\}, \quad \alpha_4 = \{3,3,3\}, \quad \ldots, \quad \alpha_n = \{\underbrace{3,3,\ldots,3}_{n-1}\}.$$

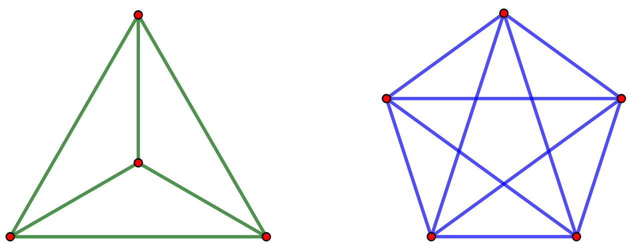


FIGURE A.2: Plane projections of the tetrahedron α_3 and the hypertetrahedron α_4.

It can be observed that the simplex α_n can be regarded as a pyramid with base α_{n-1} and apex lying outside the $(n-1)$-dimensional space of α_{n-1}. Consequently, all elements of the simplex α_n are themselves simplexes α_k, where $k < n$. Let N_k denote the number of k-dimensional elements α_k contained in α_n, then

$$N_k = \binom{n+1}{k+1}. \tag{A.1}$$

For example, the 5-dimensional simplex α_5 is composed of 6 vertices α_0, 15 edges α_1, 20 triangular faces α_2, 15 tetrahedra α_3, and 6 simplexes α_4 as cells.

In an n-dimensional Cartesian coordinate system with origin O, consider a set of n points, each located on a distinct coordinate axis and equidistant from O. These points define the vertices of a regular simplex α_{n-1}. Furthermore, the $2n$ points equidistant from O, two on each axis in both directions, form the vertices of another regular polytope, namely the cross polytope β_n, which has $2n$ simplexes α_{n-1} as its cells (see Figure A.3). Utilising Schläfli notation, the following can be deduced:

$$\beta_2 = \{4\}, \quad \beta_3 = \{3,4\}, \quad \beta_4 = \{3,3,4\}, \quad \ldots, \quad \beta_n = \{\underbrace{3,3,\ldots,3}_{n-2},4\}.$$

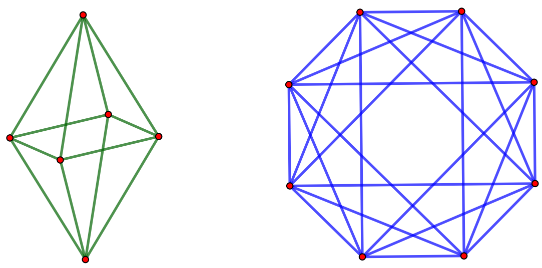


FIGURE A.3: Plane projections of the octahedron β_3 and the hyperoctahedron β_4.

It can be noted that the cross polytope β_n can be regarded as a dipyramid with base β_{n-1}. All its elements are simplexes α_k, where $k < n$. Let N_k denote the number of α_k contained in β_n, then

$$N_k = 2^{k+1}\binom{n}{k+1}. \tag{A.2}$$

For example, the 5-dimensional cross polytope β_5 is composed of 10 vertices α_0, 40 edges α_1, 80 triangular faces α_2, 80 tetrahedra α_3, and 32 simplexes α_4 as cells.

Consider a point that is moved along a line from an initial position to a final position, thereby defining a segment. This segment is then translated in a direction distinct from its own, generating a parallelogram. Subsequently, the parallelogram is translated in a direction outside the plane containing it, producing a parallelepiped. By iterating this process in n-dimensional space, the resulting figure is a polytope known as a *parallelotope*. If all translations are performed utilising vectors of equal length that are mutually perpendicular, the parallelotope will become a regular polytope referred to as a hypercube or measure polytope γ_n. It has 2^n vertices and $2^{n-1}n$ edges. Let N_k denote the number of k-dimensional elements, which are all hypercubes γ_k, where $k < n$, then

$$N_k = 2^{n-k}\binom{n}{k}. \tag{A.3}$$

For example, the 5-dimensional hypercube γ_5 is composed of 32 vertices, 80 edges, 80 square faces γ_2, 40 cubes γ_3, and 10 hypercubes γ_4 as cells (see Figure A.4). Utilising Schläfli notation, the following can be deduced:

$$\gamma_2 = \{4\}, \quad \gamma_3 = \{4,3\}, \quad \gamma_4 = \{4,3,3\}, \quad \ldots, \quad \gamma_n = \{4,\underbrace{3,\ldots,3}_{n-2}\}.$$

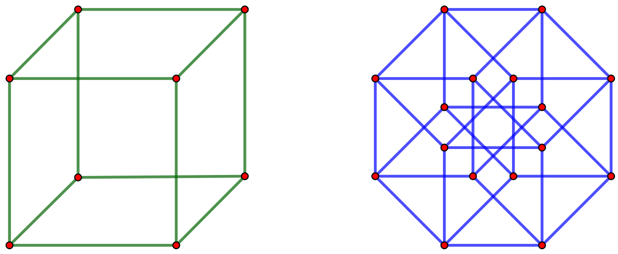


FIGURE A.4: Plane projections of the cube γ_3 and the hypercube γ_4.

In summary, it can be concluded that for $n > 4$, the simplex α_n, the cross polytope β_n, and the hypercube γ_n are the only regular polytopes whose cells are respectively α_{n-1}, α_{n-1} and γ_{n-1}, and whose vertex figures are respectively α_{n-1}, β_{n-1} and α_{n-1}.[2]

Given a regular n-dimensional polytope $\Pi_n = \{p, q, r, \ldots, v, w\}$ with even p, the vertices reachable from any vertex by traversing an even number of edges are defined as the *alternates*. The polytope formed by these alternate vertices is denoted by $h\Pi_n$. For example, in two dimensions, $h\{p\} = \{\frac{p}{2}\}$ is defined when p is even. In the three-dimensional space, there is only one case with even p, namely the hypercube $\gamma_3 = \{4,3\}$, for which $h\gamma_3 = \alpha_3$ (see Figure A.5).

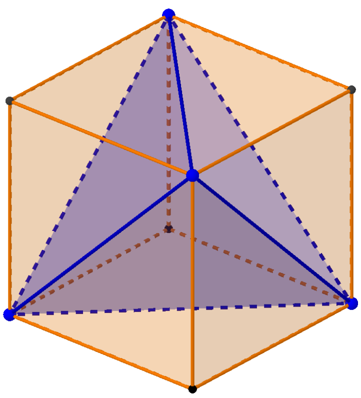


FIGURE A.5: Representation of $h\gamma_3$.

[2]For completeness of notation, it holds that $\alpha_0 = \beta_0 = \gamma_0$, all representing a single point (0-dimensional polytope), and $\alpha_1 = \beta_1 = \gamma_1$, all representing a segment (1-dimensional polytope).

For $n > 3$, the only regular n-dimensional polytope $\{p, q, \ldots, v, w\}$ with even p is the hypercube γ_n. Therefore, the only polytope for which studying makes sense is the polytope $h\gamma_n$, known as the *hemicube*. Its elements include simplexes and polytopes $h\gamma_k$, where $k < n$.
More precisely, it is composed of:

- $\frac{1}{2}N_0$ vertices,
- $\frac{1}{2}N_0$ cells of type $\{3^{n-2}\} = \alpha_{n-1}$,
- N_{n-1} cells of type $h\{4, 3^{n-3}\} = h\gamma_{n-1}$,

where N_0 and N_{n-1} denote the number of vertices and the number of cells of the polytope γ_n, respectively. From equation (A.3), one obtains

$$N_0 = 2^n \qquad \text{and} \qquad N_{n-1} = 2^{n-(n-1)}\binom{n}{n-1} = 2\binom{n}{n-1}.$$

Therefore, $h\gamma_n$ has 2^{n-1} vertices, 2^{n-1} cells α_{n-1}, and $2\binom{n}{n-1}$ cells $h\gamma_{n-1}$.

Let's now examine the number and type of $(n-2)$-dimensional elements, *i.e.*, the Π_{n-2} of $h\gamma_n$. These will correspond to the cells of the cells of $h\gamma_n$, so it is necessary to identify the cells of α_{n-1} and $h\gamma_{n-1}$. From equation (A.1), it is found that α_{n-1} has $\binom{n}{n-1}$ cells α_{n-2}.
By following the same reasoning applied to $h\gamma_n$, it is found that $h\gamma_{n-1}$ has two types of cells, namely:

$$\frac{1}{2}N_0(\gamma_{n-1}) = \frac{1}{2}2^{n-1} = 2^{n-2}$$

cells of type $\{3^{n-3}\} = \alpha_{n-2}$, and

$$N_{n-1}(\gamma_{n-1}) = N_{(n-1)-1} = N_{n-2} = 2^{n-1-(n-2)}\binom{n-1}{n-2} = 2\binom{n-1}{n-2}$$

cells of type $\{4, 3^{n-4}\} = h\gamma_{n-2}$.
Additionally, since two cells of $h\gamma_n$ share a Π_{n-2} of $h\gamma_n$, so the total number of Π_{n-2} must be divided by 2. Therefore, in $h\gamma_n$ there will be:

$$\frac{\binom{n}{n-1} \cdot 2^{n-1} + 2^{n-2} \cdot 2\binom{n}{n-1}}{2} = 2^{n-1}\binom{n}{n-1}$$

$(n-2)$-dimensional elements of type α_{n-2}, and

$$\frac{2\binom{n-1}{n-2} \cdot 2\binom{n}{n-1}}{2} = 2\binom{n-1}{n-2}\binom{n}{n-1} = 2(n-1)n = 2^2\binom{n}{n-2}$$

$(n-2)$-dimensional elements of type $h\gamma_{n-2}$.

The aim is now to determine the $(n-3)$-dimensional elements, *i.e.*, the Π_{n-3} of $h\gamma_n$. These correspond to the cells of the cells of the cells of $h\gamma_n$, that is, the cells of both α_{n-2} and $h\gamma_{n-2}$.

From equation (A.1), observe that α_{n-2} has $\binom{n-1}{n-2}$ cells α_{n-3}.
Following once again the reasoning previously applied to $h\gamma_n$ and $h\gamma_{n-1}$, it is found that $h\gamma_{n-2}$ has two types of cells, namely:

$$\frac{1}{2}N_0(\gamma_{n-2}) = \frac{1}{2}\cdot 2^{n-2} = 2^{n-3}$$

cells $\{3^{n-4}\} = \alpha_{n-3}$, and

$$N_{n-1}(\gamma_{n-2}) = N_{(n-2)-1} = N_{n-3} = 2^{n-2-(n-3)}\binom{n-2}{n-3} = 2\binom{n-2}{n-3}$$

cells $\{4, 3^{n-5}\} = h\gamma_{n-3}$.
The cells of both α_{n-2} and $h\gamma_{n-2}$ have thus been identified. As previously established, $h\gamma_n$ contains exactly $2^{n-1}\binom{n}{n-1}$ α_{n-2} and $2^2\binom{n}{n-2}$ $h\gamma_{n-2}$. Furthermore, it must be noted that each $(n-3)$-dimensional element of $h\gamma_n$ is shared by three higher-dimensional cells, hence the total number of Π_{n-3} must be divided by 3. Therefore, in $h\gamma_n$ there will be:

$$\begin{aligned}\frac{\binom{n-1}{n-2}\cdot 2^{n-1}\binom{n}{n-1} + 2^{n-3}\cdot 2^2\binom{n}{n-2}}{3} &= \frac{2^{n-1}\cdot 2\binom{n}{n-2} + 2^{n-1}\binom{n}{n-2}}{3} \\ &= \frac{2^{n-1}\cdot 3\binom{n}{n-2}}{3} = 2^{n-1}\binom{n}{n-2}\end{aligned}$$

$(n-3)$-dimensional elements of type α_{n-3}, and

$$\begin{aligned}\frac{2\binom{n-2}{n-3}\cdot 2^2\binom{n}{n-2}}{3} &= \frac{2^3\binom{n-2}{n-3}\binom{n}{n-2}}{3} = \frac{2^3(n-2)\frac{n(n-1)}{2}}{3} \\ &= \frac{2^2 n(n-1)(n-2)}{3} = \frac{2^2\binom{n}{n-3}3!}{3} = 2^3\binom{n}{n-3}\end{aligned}$$

$(n-3)$-dimensional elements of type $h\gamma_{n-3}$.
By generalising the reasoning previously applied, one may count the Π_k of $h\gamma_n$ for every $k \geq 3$. Specifically, it can be shown that $h\gamma_n$ contains exactly

$$2^{n-1}\binom{n}{k+1} \qquad \alpha_k$$

and

$$2^{n-k}\binom{n}{k} \qquad h\gamma_k.$$

In the case $k = 2$, $h\gamma_k = h\gamma_2 = \alpha_1$ and does not correspond to a face, but rather to an edge. The faces in this case are only of type α_2. Therefore, in $h\gamma_n$ there will be:

$$2^{n-1}\binom{n}{2+1} = 2^{n-1}\binom{n}{3} \qquad \alpha_2$$

and
$$2^{n-2}\binom{n}{2} \qquad h\gamma_2 = \alpha_1.$$

The goal is now to determine the number of k-dimensional elements at each vertex.
$\underline{k=1}$ (elements α_1):

$$\frac{\text{\# vertices} \cdot \text{\# edges at each vertex}}{2} = \text{\# edges},$$

therefore,

$$\text{\# edges at each vertex} = 2\frac{\text{\# edges}}{\text{\# vertices}} = 2\frac{2^{n-2}\binom{n}{2}}{2^{n-1}} = \binom{n}{2}.$$

$\underline{k=2}$ (elements α_2):

$$\frac{\text{\# vertices} \cdot \text{\# faces at each vertex}}{3} = \text{\# faces},$$

therefore,

$$\begin{aligned}\text{\# faces at each vertex} &= 3\frac{\text{\# faces}}{\text{\# vertices}} = 3\frac{2^{n-1}\binom{n}{3}}{2^{n-1}} = 3\binom{n}{3} \\ &= 3\frac{n(n-1)(n-2)}{3!} = n\frac{(n-1)(n-2)}{2} = n\binom{n-1}{2}\end{aligned}$$

$\underline{k>2}$:
As shown above, there are two types of Π_k elements, namely α_k and $h\gamma_k$. It is known that each α_k has $k+1$ vertices, thus:

$$\frac{\text{\# vertices} \cdot \#\alpha_k \text{ at each vertex}}{k+1} = \#\alpha_k,$$

and consequently,

$$\begin{aligned}\#\alpha_k \text{ at each vertex} = (k+1)\frac{\#\alpha_k}{\text{\# vertices}} &= (k+1)\frac{2^{n-1}\binom{n}{k+1}}{2^{n-1}} = (k+1)\binom{n}{k+1} \\ &= (k+1)\frac{n(n-1)\dots(n-k-1)!}{(k+1)!(n-k-1)!} \\ &= n\frac{(n-1)\dots(n-1-k)!}{k!(n-1-k)!} = n\binom{n-1}{k}.\end{aligned}$$

The number of $h\gamma_k$ at each vertex is now considered. It is observed that $h\gamma_k$ has half as many vertices as γ_k, that is, 2^{k-1}. Therefore, the following relation holds:

$$\frac{\text{\# vertices} \cdot \#h\gamma_k \text{ at each vertex}}{2^{k-1}} = \#h\gamma_k,$$

which implies:

$$\#h\gamma_k \text{ at each vertex} = 2^{k-1}\frac{\#h\gamma_k}{\#\text{vertici}} = 2^{k-1}\frac{2^{n-k}\binom{n}{k}}{2^{n-1}} = \binom{n}{k}.$$

In Table A.3, the results obtained thus far are summarised. Specifically, $h\gamma_n$ contains the following quantities and types of elements Π_k:

TABLE A.3: Summary of the number and types of elements Π_k in $h\gamma_n$.

k	Type	Total number	At each vertex
0	α_0	2^{n-1}	-
1	α_1	$2^{n-2}\binom{n}{2}$	$\binom{n}{2}$
2	α_2	$2^{n-1}\binom{n}{3}$	$n\binom{n-1}{2}$
$k \geq 3$	α_k	$2^{n-1}\binom{n}{k+1}$	$n\binom{n-1}{k}$
	$h\gamma_k$	$2^{n-k}\binom{n}{k}$	$\binom{n}{k}$

For example, when $n = 4$, the hemicube $h\gamma_4$ is composed of 8 vertices, 24 edges, 32 triangular faces α_2, 8 cells α_3, and 8 cells $h\gamma_3$; hence, $h\gamma_4 = \beta_4$. On the other hand, for $n \geq 5$, the hemicube $h\gamma_n$ does not correspond to any known regular polytope.

The *prism* of dimension $q + r$ denoted by $[\alpha_q, \alpha_r]$, is defined as the polytope whose vertices are obtained as the product of the vertices of α_q and those of α_r. For example, $[\alpha_2, \alpha_1]$ corresponds to the triangular prism in three-dimensional space. The polytope whose vertex figure is $[\alpha_q, \alpha_r]$ is denoted by 0_{qr}, and the polytope whose vertex figure is $(n-1)_{qr}$ is denoted by $n_{qr} = n_{rq}$. It follows that 1_{qr} is the polytope having vertex figure 0_{qr} and second vertex figure $(-1)_{qr} = [\alpha_q, \alpha_r]$, and is thus a polytope of dimension $q + r + 2$. The following identities hold: $1_{p0} = \alpha_{p+2}$ and $1_{p1} = h\gamma_{p+3}$. Observing that the necessary condition for the existence of the $(n + q + r + 1)$-dimensional polytope n_{qr} is

$$\frac{1}{n+1} + \frac{1}{q+1} + \frac{1}{r+1} \geq 1,$$

for the polytopes 1_{qr} this becomes

$$\frac{1}{q+1} + \frac{1}{r+1} \geq \frac{1}{2}.$$

Table A.4 lists all the possible polytopes 1_{qr}.

TABLE A.4: Polytopes 1_{qr} for various values of q and r.

$q \backslash r$	0	1	2	3	4	5	≥ 6
0	$1_{00} = \alpha_2$	$1_{01} = \alpha_3$	$1_{02} = \alpha_4$	$1_{03} = \alpha_5$	$1_{04} = \alpha_6$	$1_{05} = \alpha_7$	$1_{0r} = \alpha_{r+2}$
1	$1_{10} = \alpha_3$	$1_{11} = h\gamma_4$	$1_{12} = h\gamma_5$	$1_{13} = h\gamma_6$	$1_{14} = h\gamma_7$	$1_{15} = h\gamma_8$	$1_{1r} = h\gamma_{r+3}$
2	$1_{20} = \alpha_4$	$1_{21} = h\gamma_5$	1_{22}	1_{23}	1_{24}	1_{25}	
3	$1_{30} = \alpha_5$	$1_{31} = h\gamma_6$	1_{32}	1_{33}			
4	$1_{40} = \alpha_6$	$1_{41} = h\gamma_7$	1_{42}				
5	$1_{50} = \alpha_7$	$1_{51} = h\gamma_8$	1_{52}				
≥ 6	$1_{q0} = \alpha_{q+2}$	$1_{q1} = h\gamma_{q+3}$					

After associating a polytope with the configurations of Clifford, Grace, and Brown, the mathematician Longuet-Higgins in (Longuet-Higgins, 1972) proposed, starting from a generic polytope 1_{qr}, the construction in $(q+1)$-dimensional space of a configuration C_{qr} consisting of points and hyperspheres, with $q+r+2$ hyperspheres passing through each point. It was first observed that the polytopes 1_{0r} correspond to the configurations C_{0r} of the line, consisting of $r+3$ points and $r+3$ intervals, with $r+2$ intervals through each point. The polytopes 1_{q0} were then associated with the configurations C_{q0} of $q+3$ points and $q+3$ hyperspheres, with each point lying on $q+2$ hyperspheres, as discussed in Chapter 3. Furthermore, it was shown that polytopes of the form 1_{1r} correspond to Clifford's plane configurations C_{1r}, consisting of 2^{r+2} points and 2^{r+2} circles, with $r+3$ circles passing through each point. In the case of the polytope 1_{21}, the known configurations C_{01} and C_{11} were used to construct C_{21}, a configuration in the three-dimensional space consisting of 16 points and 10 spheres, with 8 spheres through each point. Utilising a recursive process starting from the polytopes 1_{q1}, the existence of the configurations C_{q1} was established, each consisting of 2^{q+2} points and $2(q+3)$ hyperspheres, with $q+3$ hyperspheres passing through each point. Finally, once the configurations $C_{q'r'}$ with $q' \leq q$, $r' \leq r$, and $(q', r') \neq (q, r)$ had been constructed, the recursive method was applied to associate the configuration C_{qr} with the polytope 1_{qr}, under the assumption of its existence.

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