



Digital Convexity and Combinatorics on Words

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Abstract. An upward (resp. downward) digitally convex word is a binary word that best approximates from below (resp. from above) an upward (resp. downward) convex curve in the plane. We study these words from the combinatorial point of view, formalizing their geometric properties and highlighting connections with Christoffel words and finite Sturmian words. In particular, we study from the combinatorial perspective the operations of inflation and deflation on digitally convex words.

1 Introduction

Combinatorics on words and digital geometry have a long history of interactions. In particular, digital approximations of lines in the plane have a natural encoding as binary words. In this paper, we fix the binary alphabet $\{0, 1\}$, where 0 (resp. 1) represents a horizontal (resp. vertical) unary step in the grid $\mathbb{N} \times \mathbb{N}$.

For example, there is a clear and well-understood correspondence between finite words and approximations of digital segments, given by Christoffel words. For every pair of positive integers (a, b) there is one lower Christoffel word with a occurrences of 0 and b occurrences of 1, which is the digital approximation from below of the Euclidean segment from $(0, 0)$ to (a, b) . Christoffel words are detailed in many classical references [2, 12, 16]. Relationships between Christoffel words and convex digital shapes have been investigated in several papers [5, 8, 9, 18].

In this paper, we focus on digital approximations of convex curves in the plane. A binary word is called (upward) digitally convex if it is the digital approximation from below of a convex curve joining $(0, 0)$ to (a, b) and contained in the rectangle whose opposite vertices are $(0, 0)$ and (a, b) . Every such word has a occurrences of 0 and b occurrences of 1. In particular, all the digitally convex words with a occurrences of 0 and b occurrences of 1 lie between the lower

This version has been shortened due to page limit constraints. The proofs and additional results are contained in the full version of the paper.

Christoffel word with a occurrences of 0 and b occurrences of 1 and the word $1^b 0^a$, the upper left contour of the rectangle.

In [5], the authors gave a purely combinatorial characterization of digitally convex words: a word w is digitally convex if and only if all the Lyndon words in the Lyndon factorization of w are balanced (hence primitive lower Christoffel words). A primitive word (i.e., a word that cannot be written as a concatenation of copies of a shorter word) over $\{0, 1\}$ is a Lyndon word if it is lexicographically smaller (for the order $0 < 1$) than all its proper suffixes. Every nonempty word w has a unique factorization in non-increasing Lyndon words, which is called the Lyndon factorization of w .

A binary word w is balanced (or Sturmian) if for every length i , the difference between the number of occurrences of 0's and 1's in any two factors of length i of w is at most 1. Since a word over $\{0, 1\}$ is a primitive lower Christoffel word if and only if it is balanced and Lyndon, the previous characterization can be seen as a natural decomposition of digitally convex words in straight segments.

In this paper, we give combinatorial results on digitally convex words that shed light on their combinatorial properties and are related to the geometry of convex curves that these words approximate. Our main result is that every digitally convex word with Parikh vector (a, b) can be obtained with a sequence of inflation (resp. deflation) operations from the lower Christoffel word with the same Parikh vector (resp. from the word $1^b 0^a$).

The paper is organized as follows: In Sect. 2 we fix the notation and give preliminary results on words, and in particular on Christoffel words, that we will need in the sequel; in Sect. 3 we recall the notion of digitally convex words and provide some new results and characterizations. Finally, in Sect. 4 we study from the combinatorial perspective the operations of inflation and deflation on digitally convex words.

2 Preliminaries

A *word* over a finite alphabet Σ is a concatenation of letters from Σ . The *length* of a word w is denoted by $|w|$. The empty word ε has length 0. The set of all words over Σ is denoted Σ^* and is a free monoid with respect to concatenation. For a letter $x \in \Sigma$, $|w|_x$ denotes the number of occurrences of x in w . If $\Sigma = \{x_1, \dots, x_\sigma\}$ is ordered, the vector $(|w|_{x_1}, \dots, |w|_{x_\sigma})$ is the *Parikh vector* of w .

Let $w = uv$, with $u, v \in \Sigma^*$. We say that u is a *prefix* of w and that v is a *suffix* of w . A *factor* of w is a prefix of a suffix (or, equivalently, a suffix of a prefix) of w . If $w = w_1 w_2 \cdots w_n$, with $w_k \in \Sigma$ for all k , we let $w[i..j]$ denote the nonempty factor $w_i w_{i+1} \cdots w_j$ of w , whenever $1 \leq i \leq j \leq n$. A factor u of a word $w \neq u$ is a *border* of w if u is both a prefix and a suffix of w ; in this case, w has *period* $|w| - |u|$. A word w is *unbordered* if its longest border is ε ; i.e., if its smallest period is $|w|$. For a word w , the *n -th power* of w is the word w^n obtained by concatenating n copies of w .

Two words w and w' are *conjugates* if $w = uv$ and $w' = vu$ for some words u and v . The conjugacy class of a word w contains $|w|$ elements if and only if w is *primitive*, i.e., $w = v^n$ for some v implies $n = 1$.

A nonempty word $w = w_1w_2 \cdots w_n$, $w_k \in \Sigma$ for all k , is a *palindrome* if it coincides with its reversal $w^R = w_nw_{n-1} \cdots w_1$. The empty word is also assumed to be a palindrome.

The following proposition, whose proof is straightforward, is well known (cf. [4,6]).

Proposition 1. *A word is a conjugate of its reversal if and only if it is a concatenation of two palindromes. Moreover, these two palindromes are uniquely determined if and only if the word is primitive.*

2.1 Lyndon Words

A primitive word over an ordered alphabet is a *Lyndon word* if it is lexicographically smaller than all its conjugates (or, equivalently, lexicographically smaller than all its proper suffixes).

In the binary case, we have the class of Lyndon words for the order $0 < 1$ and the class of Lyndon words for the reversed order $1 < 0$. Whenever not otherwise specified, we assume the order $0 < 1$.

The following theorem, originally due to Chen, Fox, and Lyndon (1958), is a classical result in combinatorics on words (see [11]):

Theorem 1. *Every word w has a unique non-increasing factorization in Lyndon words.*

The factorization from the previous theorem is called the *Lyndon factorization* (or sometimes the Chen–Fox–Lyndon factorization) of w .

Example 1. Let $w = 0101001001$. The Lyndon factorization of w is

$$01 \cdot 01 \cdot 001 \cdot 001.$$

Example 2. Let $w = 1100$. The Lyndon factorization of w is

$$1 \cdot 1 \cdot 0 \cdot 0.$$

A way to obtain the Lyndon factorization is by using the following:

Property 1. If u and v are Lyndon words, then $u < v$ if and only if uv is a Lyndon word.

Starting from the trivial factorization $w = w_1 \cdots w_n$, with $|w_i| = 1$ for every i , one repeatedly applies Property 1 until the factors occur in non-increasing order.

Every Lyndon word w of length $|w| \geq 2$ has a *standard factorization* $w = uv$, where v is the lexicographically least proper suffix of w (or, equivalently, the longest proper suffix of w that is a Lyndon word), see [11].

2.2 Balanced and Christoffel Words

Definition 1. A word over $\{0, 1\}$ is balanced if the number of 0's (or, equivalently, 1's) in every two factors of the same length differs by at most 1.

We let Bal denote the set of balanced words over the alphabet $\{0, 1\}$. Balanced words are precisely the finite factors of infinite Sturmian words [12]. Balance is a *factorial property*, i.e., every factor of a balanced word is itself balanced.

Christoffel words are powers of balanced Lyndon words. But let us introduce them from the point of view of digital geometry.

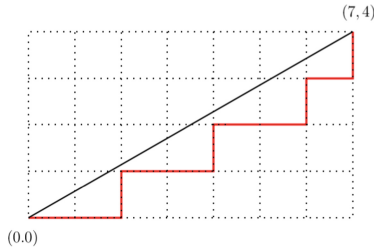
In what follows, we fix the alphabet $\{0, 1\}$ and represent a word over $\{0, 1\}$ as a path in $\mathbb{N} \times \mathbb{N}$ starting at $(0, 0)$, where 0 (resp., 1) encodes a horizontal (resp., vertical) unit segment. For every pair of nonnegative integers (a, b) (not both 0), every word w that encodes a path from $(0, 0)$ to (a, b) must have exactly a zeroes and b ones, i.e., it has Parikh vector $(a, b) = (|w|_0, |w|_1)$. We define the *slope* of a word w with Parikh vector (a, b) as the rational number b/a if $a \neq 0$, or ∞ otherwise.

Definition 2. Given a pair of nonnegative integers (a, b) (not both 0), the lower (resp., upper) Christoffel word $w_{a,b}$ (resp., $W_{a,b}$), of slope b/a , is the word encoding the path from $(0, 0)$ to (a, b) that is closest from below (resp., from above) to the Euclidean segment, without crossing it.

In other words, $w_{a,b}$ (resp. $W_{a,b}$) is the digital approximation from below (resp., from above) of the Euclidean segment joining $(0, 0)$ to (a, b) . For example, the lower Christoffel word $w_{7,4}$ is the word 00100100101 (see Fig. 1).

By construction, we have that the upper Christoffel word $W_{a,b}$ is the reversal of the lower Christoffel word $w_{a,b}$ (and the two words are conjugates, see below).

Notice that by definition we allow a or b (but not both) to be 0, that is, we have $w_{n,0} = W_{n,0} = 0^n$ and $w_{0,n} = W_{0,n} = 1^n$, so that the powers of words of length 1 are both lower and upper Christoffel words.



and are primitive words. If instead $a = n\alpha$ and $b = n\beta$ for some $n > 1$, then $w_{a,b} = (w_{\alpha,\beta})^n$ (resp., $W_{a,b} = (W_{\alpha,\beta})^n$). Hence, $w_{a,b}$ (resp., $W_{a,b}$) is primitive if and only if a and b are coprime.

Therefore, letting ϕ denote the Euler totient function, for every $n > 1$ there are $2\phi(n)$ primitive Christoffel words of length n (in particular, there are $\phi(n)$ primitive lower Christoffel words and $\phi(n)$ primitive upper Christoffel words).

Theorem 2 ([1]). *Let w be a word over $\{0, 1\}$. Then w is a primitive (lower or upper) Christoffel word if and only if it is balanced and unbordered.*

In particular, the set of primitive lower Christoffel words is precisely the set of balanced Lyndon words over the alphabet $\{0, 1\}$ for the order $0 < 1$.

Proposition 2 ([2]). *For every coprime positive integers a, b , the primitive lower Christoffel word $w_{a,b}$ is the greatest (for the lexicographic order) word among all Lyndon words having Parikh vector (a, b) .*

Proposition 3 ([7]). *For all coprime positive integers a, b , the lower Christoffel word $w_{a,b}$ is the smallest (in the lexicographic order) word among all balanced words having Parikh vector (a, b) .*

Theorem 3 ([2]). *Let (a, b) and (c, d) be pairs of nonnegative integers, both distinct from $(0, 0)$, such that $b/a \neq d/c$. Then*

$$w_{a,b} < w_{c,d} \iff \frac{b}{a} < \frac{d}{c}.$$

Since primitive lower Christoffel words are Lyndon words, every primitive lower Christoffel word of length $|w| \geq 2$ has a standard factorization.

On the other hand, a primitive lower Christoffel word is a conjugate of its reversal (the corresponding upper Christoffel word); hence, by Proposition 1, every primitive lower Christoffel word of length $|w| \geq 2$ has a unique *palindromic factorization*.

From the geometric point of view, the standard factorization divides $w_{a,b}$ in two shorter Christoffel words, and it determines the point S of the encoded path that is *closest* to the Euclidean segment from $(0, 0)$ to (a, b) ; the palindromic factorization, instead, divides $w_{a,b}$ in two palindromes and determines the point S' that is *farthest* from the Euclidean segment (see [3, 17]). An example is given in Fig. 2.

3 Digitally Convex Words

We now introduce the main object of study of this paper.

Definition 3. *A binary word with a occurrences of 0 and b occurrences of 1 is an upward (resp. downward) digitally convex word if it is the best approximation from below (resp. above) of an upward (resp. downward) convex curve that joins $(0, 0)$ and (a, b) and is contained in the rectangle whose opposed vertices are $(0, 0)$ and (a, b) .*

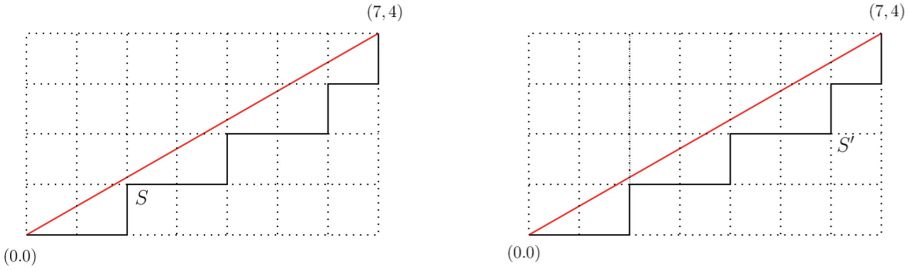


Fig. 2. The standard factorization $0Q1 \cdot 0P1 = 001 \cdot 00100101$ (left) and the palindromic factorization $0P0 \cdot 1Q1 = 00100100 \cdot 101$ (right) of the lower Christoffel word $w_{7,4}$. The point S determined by the standard factorization is the closest to the Euclidean segment, while the point S' determined by the palindromic factorization is the farthest.

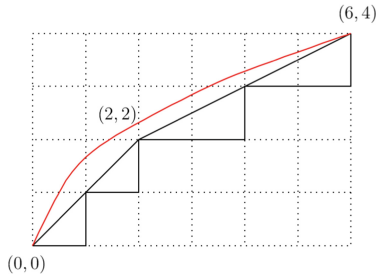


Fig. 3. The (upward) digitally convex word $w = 0101001001$ and its decomposition in two Christoffel words 0101 and 001001 . In red, one of the possible approximated upward convex curves that join $(0,0)$ and $(6,4)$. (Color figure online)

For example, the word $w = 0101001001$ of Example 1 is (upward) digitally convex, as shown in Fig. 3.

Proposition 4. *We have:*

1. *A word is upward (resp. downward) digitally convex if and only if its reverse is downward (resp. upward) digitally convex;*
2. *A word is upward (resp. downward) digitally convex if and only if its binary complement is downward (resp. upward) digitally convex.*

Geometrically, given fixed a and b positive integers, the upward digitally convex words with Parikh vector (a, b) are all above the Christoffel word $w_{a,b}$ and below the word $1^b 0^a$. Of course, the downward digitally convex words with Parikh vector (a, b) are all below the Christoffel word $W_{a,b}$ and above the word $0^a 1^b$. This will be stated formally in Theorem 9.

Every balanced word is, as we will see, both upward and downward digitally convex. On the contrary, digitally convex words are not necessarily balanced, as shown by the word $w = 0101001001$ of Example 1. We have the following characterization.

Proposition 5. *Let w be a binary word, \tilde{w} its reverse, and \bar{w} its binary complement. The following are equivalent:*

1. w is balanced;
2. w and \tilde{w} are both upward (resp. downward) digitally convex;
3. w and \bar{w} are both upward (resp. downward) digitally convex;
4. w is upward digitally convex and downward digitally convex.

The Lyndon factorization can be used to characterize digitally convex words, as it was shown by Brlek et al. in the following

Theorem 4 ([5]). *A word w is upward digitally convex if and only if all the Lyndon words in the Lyndon factorization of w are balanced (hence primitive lower Christoffel words).*

An analogous result characterizes downward digitally convex words. In fact, by a simple symmetry argument, we have that a binary word is downward digitally convex if and only if all the Lyndon words for the order $1 < 0$ in the Lyndon factorization of w are balanced (hence primitive upper Christoffel words).

Remark 1. The union of upward and downward digitally convex words is a factorial language that is also closed under reversal and binary complement, and includes the language of balanced words.

Because of the symmetries between upward and downward digitally convex words, from now on we will focus on upward digitally convex words only, which we will simply call digitally convex words. Analogous results hold, of course, for downward digitally convex words.

3.1 Minimal Forbidden Words

Minimal forbidden words are a useful tool for characterizing factorial languages.

Definition 4. *Let L be a factorial language. A word w is a minimal forbidden word for L if w does not belong to L but all proper factors of w do.*

Let $\mathcal{MF}(L)$ denote the set of minimal forbidden words of the language L . A word $w = xvy$, x, y letters, belongs to $\mathcal{MF}(L)$ if and only if

1. $xvy \notin L$;
2. $xv, vy \in L$.

Theorem 5. ([14]) *There is a bijection between factorial languages and their sets of minimal forbidden words.*

As a consequence, $\mathcal{MF}(L)$ uniquely determines L .

Let Bal be the (factorial) language of balanced words over $\{0, 1\}$. The next theorem gives a characterization of the minimal forbidden words for the language of binary balanced words.

Theorem 6 ([10]). $\mathcal{MF}(\text{Bal}) = \{yvx \mid \{x, y\} = \{0, 1\}, xvy \text{ is a non-primitive Christoffel word}\}$.

Example 3. The word 000101 is not balanced, but all its proper factors are. Indeed, 100100 is the square of the primitive upper Christoffel word 100.

Corollary 1 ([10]). *For every $n > 0$, there are exactly $n - \phi(n) - 1$ words of length n in $\mathcal{MF}(\text{Bal})$ that start with 0, and they are all Lyndon words. Here ϕ is the Euler totient function.*

In 2011, Provençal [15] studied the minimal forbidden words of the set \mathcal{DC} of digitally convex words. Indeed, the set of digitally convex words is a factorial language. In his paper, Provençal attributes this result to a private communication of C. Reutenauer .

As a consequence, a word is digitally convex if and only if all its Lyndon factors are balanced.

Theorem 7 ([15]). $\mathcal{MF}(\mathcal{DC}) = \{u(uv)^k v \mid k \geq 1, uv \text{ is the standard factorization of a primitive lower Christoffel word}\}$.

We now give an alternative description:

Theorem 8. $\mathcal{MF}(\mathcal{DC})$ is the set of words in $\mathcal{MF}(\text{Bal})$ that start with 0. Hence, $\mathcal{MF}(\mathcal{DC}) = \{0w1 \mid 1w0 \text{ is a non-primitive Christoffel word}\}$. In particular, all words in $\mathcal{MF}(\mathcal{DC})$ are Lyndon words.

By Corollary 1, we have:

Corollary 2. $\mathcal{MF}(\mathcal{DC})(n) = n - 1 - \phi(n)$.

3.2 Dominance Order

Definition 5. Over $\{0, 1\}^n$, we consider the dominance order defined by $u \sqsubseteq v$ if for every $1 \leq i \leq n$, $|u[1\dots i]|_1 \leq |v[1\dots i]|_1$ (or, equivalently, $|u[1\dots i]|_0 \geq |v[1\dots i]|_0$).

Notice that the dominance order is a partial order, and that the lexicographic order is a linear extension of it.

Theorem 9. Let $a, b > 0$ and $n = a + b$. For every digitally convex word u with Parikh vector (a, b) , and for every $1 \leq k \leq n$, we have that $w_{a,b}[1\dots k]$ is lexicographically smaller than or equal to $u[1\dots k]$, hence in particular $w_{a,b}[1\dots k] \sqsubseteq u[1\dots k]$.

Corollary 3. For every pair (a, b) , the lower Christoffel word $w_{a,b}$ is the smallest (in the lexicographic order) digitally convex word having Parikh vector (a, b) .

The previous theorem essentially says that all the paths encoded by upward digitally convex words with Parikh vector (a, b) stay above the path encoded by the lower Christoffel word $w_{a,b}$ (and below the path encoded by $1^b 0^a$).

By symmetry, we also have that all the paths encoded by downward digitally convex words with Parikh vector (a, b) stay above the path encoded by $0^a 1^b$ and below the upper Christoffel word $W_{a,b}$.

Definition 6. *To each binary word w it is associated an integer sequence s_w such that $s_w[i] = |w[1\dots i]|_1$. The meet (resp. join) of two binary words u and v is defined as the binary word $w = u \wedge v$ (resp. $w = u \vee v$) whose associated sequence is $s_w[i] = \min\{s_u[i], s_v[i]\}$ (resp. $s_w[i] = \max\{s_u(i), s_v(i)\}$).*

In other words, $u \wedge v$ (resp. $u \vee v$) is precisely the maximum lower bound (resp. minimum upper bound) of the set $\{u, v\}$ with respect to the dominance order. From a geometrical perspective, the meet (resp. join) of u and v turns out to be the word delimiting the intersection (resp. union) of the areas below them.

Proposition 6. *Let u and v be two digitally convex words. Then $u \wedge v$ is a digitally convex word.*

Hence, the set $\mathcal{DC} \cap \{0, 1\}^n$ is a *meet-semilattice* for all n , as is the set $\mathcal{DC}_{a,b}$ of all digitally convex words with Parikh vector (a, b) .

As one can expect, the join of two digitally convex words is not, in general, a digitally convex word, according to the fact that the union of convex polygons does not preserve convexity. As an example, consider $u = 010101010110001001$ and $v = 011110001001001001$. The word $u \vee v = 011110001010001001$, whose Lyndon factorization is $01111 \cdot 000101 \cdot 0001 \cdot 001$, contains a non-balanced Lyndon factor 000101 , so it is not digitally convex by Theorem 4.

4 Inflation/Deflation of Digitally Convex Words

Borrowing the terminology from [19], we call *deflation* the operation that changes a digitally convex word $w = u10v$ into a digitally convex word $w' = u01v$. The name suggests the geometrical interpretation of the operation, i.e., the removal of a point from the (discrete) digitally convex set related to w to obtain a new (discrete) digitally convex set related to w' . Similarly, the addition of a point to a digitally convex set preserving the convexity is called *inflation*, and it is realized by changing a 01 occurrence into a 10 while preserving the convexity of a digitally convex word.

Lemma 1 (see [18, 19]). *Let $w_1 = u1$ and $w_2 = 0v$ be lower Christoffel words such that $w_1 > w_2$. Then $u01v$ is digitally convex.*

Lemma 2. *Let $w = u10v$ be a lower Christoffel word. Then $u01v$ is not digitally convex.*

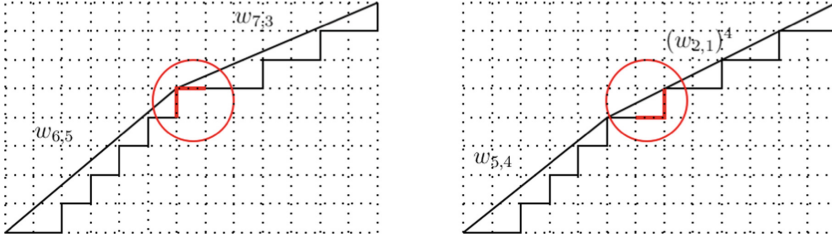


Fig. 4. On the left, the two Christoffel words with decreasing slopes $w_{6,5}$ and $w_{7,3}$. Their common point is highlighted. On the right, the deflation performed on that point produces new Christoffel words $w_{5,4}$ and $w_{8,4}$ preserving the decreasing of their slopes, and so the digital convexity.

Theorem 10 (see [18, 19]). *Let $w = u10v = w_1 \cdots w_k$ be a digitally convex word, where $w_1 > \dots > w_k$ ($k \geq 2$) are Christoffel words. Then $u01v$ is digitally convex if and only if $0v = w_i w_{i+1} \cdots w_k$ for some $1 < i \leq k$.*

Remark 2. The deflation of a digitally convex word w is a local operation, in that all elements of the Lyndon factorization of $w = u10v$ are inherited in the Lyndon factorization of $w' = u01v$, except for the two elements that overlap the given occurrence of 10. By contrast, the converse inflation operation can change an arbitrarily large number of elements in the Lyndon factorization, as shown in the next example.

Example 4. Let $f = 0100101001001 \cdots$ be the Fibonacci infinite word, fixed point of the substitution $0 \mapsto 01, 1 \mapsto 0$, and consider the Sturmian word $s = 001f$. As shown in [13], the Lyndon factorization of f is

$$\prod_{n \geq 1} \ell_{2n+1} = (01)(00101)(0010010100101) \cdots$$

where $\ell_1 = 1, \ell_2 = 0, \ell_{2n+1} = \ell_{2n} \ell_{2n-1}$, and $\ell_{2n+2} = \ell_{2n} \ell_{2n+1}$ for $n \geq 1$ give the sequence of all lower Christoffel factors of f (note that $|\ell_n| = F_n$, the n th Fibonacci number). It follows that for all $k > 1$, the Lyndon factorization of the prefix p_k of s of length $3 + \sum_{n=1}^k F_{2n+1} = F_{2k+2} + 2$ is $p_k = (00101)^2 \cdot \prod_{n=3}^k \ell_{2n+1}$.

Now, the inflated infinite word $s' = 010f$ is still Sturmian, so that its prefixes are all digitally convex; in particular, for all $k > 1$, the Lyndon factorization of its prefix p'_k such that $|p'_k| = |p_k|$ is $p'_k = 01 \cdot \ell_{2k+2}$, thus showing that an arbitrarily large number of elements in the Lyndon factorization of a digitally convex word can give rise to a constant number (2, in this case) of such elements after inflation.

In the next lemma, we analyze how the inflation operation acts on a single Christoffel word.

Lemma 3. *Let $w = v01u$ be a primitive lower Christoffel word. Then $v10u$ is digitally convex if and only if $w = 0uv1$, i.e., $(v0, 1u)$ is the palindromic factorization of w and $(0u, v1)$ is its standard factorization.*

The previous lemma essentially states that the only possible inflation point in a Christoffel word is the point S' determined by the palindromic factorization, which is the point on the path encoding the Christoffel word that is the farthest from the Euclidean segment (see Fig. 2).

A deeper investigation reveals that the inflation operation performed as in Lemma 3 cannot be applied to any Christoffel factor of a digitally convex word while preserving the digital convexity property. The following example is from [8].

Example 5. Let w be the digitally convex word $w = w_{5,3}w_{20,11}$. The application of the inflation to $w_{5,3}$, according to Lemma 3, produces $w' = w_{3,2}w_{2,1}w_{20,11}$, which is not digitally convex any more. To gain back the digital convexity, a second inflation in $w_{20,11}$ is required, thus obtaining $w'' = w_{3,2}w_{2,1}w_{9,5}w_{11,6}$. Note that $w_{2,1}w_{9,5}$ is the standard factorization of $w_{11,6}$; so, finally, $w'' = w_{3,2}w_{11,6}w_{11,6}$.

Remark 3. Example 5 suggests that an order on the inflation operations can be established to avoid the loss of the digital convexity property. In fact, performing on w a first inflation in its Christoffel factor $w_{20,11}$ produces the digitally convex word $w_{5,3}w_{9,5}w_{11,6}$. Now, the second inflation on $w_{5,3}$ leads to w'' .

The following lemma expresses what was observed in the previous remark.

Lemma 4. *Let $w = w_1 \cdots w_k$ be a digitally convex word. where $w_1 > \dots > w_k$ ($k \geq 2$) are Christoffel words. There exists an index $1 \leq i \leq k$ such that w_i has palindromic factorization $(u0, 1v)$ and $w' = w_1 \cdots w_{i-1} u10v w_{i+1} \cdots w_k$ is digitally convex.*

Proposition 7. *Let w be a digitally convex word with Parikh vector (a, b) . There exists a sequence of applications of inflation that leads from the word $w_{a,b}$ to w .*

Proposition 8. *Let w be a digitally convex word with Parikh vector (a, b) . There exists a sequence of applications of deflation that leads from the digitally convex word 1^b0^a to w .*

We therefore arrive at the main result of this section, whose proof directly follows from Propositions 7 and 8.

Theorem 11. *Iterated inflation (resp. deflation) in the word $w_{a,b}$ (resp. 1^b0^a) produces all digitally convex words with the same Parikh vector (a, b) .*

Now, we are ready to define the order relation \leq_I on the set $\mathcal{DC}_{a,b}$ of all the digitally convex words with Parikh vector (a, b) such that, provided $u, v \in \mathcal{DC}_{a,b}$, $u \leq_I v$ if there exists a sequence of applications of the inflation operation leading from u to v . It is worthwhile proving that the structure $\mathcal{W}_{a,b} = (\mathcal{DC}_{a,b}, \leq_I)$ is a partial order, having the words 1^b0^a and $w_{a,b}$ as maximum and minimum element, respectively.

Let us indicate the partial order provided by the dominance order on the same set $\mathcal{DC}_{a,b}$ as $\mathcal{D}_{a,b} = (\mathcal{DC}_{a,b}, \sqsubseteq)$.

Theorem 12. *The partial orders $\mathcal{W}_{a,b}$ and $\mathcal{D}_{a,b}$ define the same structure on the ground set $\mathcal{DC}_{a,b}$.*

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