



Review article

Noisy sine-Gordon breather dynamics: A short review

Duilio De Santis^a, Davide Valenti^{a,*}, Bernardo Spagnolo^{a,b}, Giovanni Di Fresco^{a,c},
Angelo Carollo^a, Claudio Guarcello^{d,e,*}

^a Dipartimento di Fisica e Chimica “E. Segrè”, Group of Interdisciplinary Theoretical Physics, Università degli Studi di Palermo, I-90128 Palermo, Italy

^b Stochastic Multistable Systems Laboratory, Lobachevsky University, 603950 Nizhny Novgorod, Russia

^c International School for Advanced Studies (SISSA), via Bonomea 265, 34136 Trieste, Italy

^d Dipartimento di Fisica “E.R. Caianiello”, Università degli Studi di Salerno, I-84084 Fisciano, Salerno, Italy

^e INFN, Sezione di Napoli, Gruppo Collegato di Salerno - Complesso Universitario di Monte S. Angelo, I-80126 Napoli, Italy

ARTICLE INFO

Keywords:

Noisy sine-Gordon model
Solitons
Breathers
Stochastic processes
Noise-induced effects
Josephson junctions

ABSTRACT

The aim of this review is to address recent developments in the emergence and dynamics of sine-Gordon (SG) breathers under dissipation, external drives, and environmental noise. We then discuss various strategies for breather detection in a specific physical platform: the long Josephson junction (LJJ). We note that our findings rely on a rather general SG framework, whose importance extends well beyond the scope of Josephson electronics. While the results hold significance in contexts involving nonequilibrium and collective phenomena, we also emphasize their connection to the Josephson world, as they provide direct insights relevant to that domain.

To set the stage for our discussions in the Josephson context, we begin by deriving the celebrated Josephson relations. From the basic Josephson junction building block, we obtain the SG model for the electrodynamics of the LJJ and provide an overview of the model's properties. We then discuss recent research findings on: (i) supratransmission-induced traveling breathers and their detection; (ii) the noise-enhanced stability effect on breather dynamics; (iii) the AC-locking of noise-induced breathers and their statistical (resistive switching) signature; (iv) the thermal transport fingerprint of Josephson breathers.

1. Introduction

The sine-Gordon (SG) equation is a prototypical nonlinear field theory hosting solitons and rich dynamical phenomena [1–7]. Its mathematical elegance and physical ubiquity—spanning Josephson junctions, ultracold atoms, optical waveguides, as well as magnetic, mechanical, and even biological systems—have made it a cornerstone of nonlinear science [8–15]. The SG model is especially renowned for its exact solvability and ability to support topological solitons (kinks) and their bound-states (breathers), offering insights into universal aspects of non-equilibrium dynamics—both in classical and quantum domains [16–30].

In this work, we focus on the emergence and dynamics of SG breathers. These are soliton excitations exponentially localized in space and ‘breathing’, i.e., oscillating, in time [2,4]. In particular, we concisely review recent efforts aimed at solving a problem open for decades in the field of superconducting electronics: the experimental detection of breathers in long Josephson junctions (LJJs), which are devices made by weakly coupling two superconductor electrodes. Mainly due to their fast oscillations at THz frequencies and their sensitivity to dissipation,

breathers are indeed experimentally elusive, with standard techniques in the field, such as the current–voltage characteristics, being unable to provide clear evidence of their existence [31,32]. This open challenge highlights the need for an in-depth exploration of the physics of SG breathers, seeking ways to experimentally track them using the tools currently available in superconducting electronics. Here, we review recent advancements in this topic, drawing from Refs. [33–41].

In particular, Refs. [33–35] illustrate how magnetic pulses, applied at the edge of an LJJ, can be employed to generate single traveling breathers. In the presence of dissipation, external currents, and thermal fluctuations, the analysis quantifies the efficiency of this generation method by exploring the (ω, A) parameter space, where ω and A represent the frequency and amplitude of the external field, respectively. Specifically, large regions in the (ω, A) plane are found where exclusively breathers emerge in the system, starting from a breather-free initial state. One of the most intriguing results is the thermal-fluctuation-induced broadening of these regions. More precisely, there is a non-zero probability of generating breathers-only even

* Corresponding authors.

E-mail addresses: davide.valenti@unipa.it (D. Valenti), cguarcello@unisa.it (C. Guarcello).

<https://doi.org/10.1016/j.chaos.2025.116641>

Received 15 April 2025; Received in revised form 22 May 2025; Accepted 23 May 2025

Available online 13 June 2025

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outside the (ω, A) areas identified in the absence of noise. This technique for the controlled generation of individual traveling breathers is also employed, in Ref. [33], to develop a first breather detection protocol in LJJs.

Refs. [36,37] demonstrate that a noise source, such as a noisy electrical current in the case of an LJJ, can enhance the stability of breathers. In other words, stochastic forcing can sustain the soliton excitation well beyond its deterministic lifetime, which is typically proportional to the inverse of the dissipation coefficient. This seemingly counter-intuitive phenomenon leads to a non-monotonic behavior of the breather's characteristic lifetime as a function of the noise amplitude. The implication of these results is quite clear: noise allows for the control—even the enhancement—of the stability of soliton excitations, instead of degradation.

In Refs. [38–40], an LJJ driven by a spatially uniform AC current, in the presence of dissipation and thermal noise, is examined. Notably, for sufficiently high noise amplitudes, arrays of extremely stable breathers, resonant with the AC driving, are found to emerge out of a noisy (breather-free) background. The noise amplitude becomes an important resource, enabling the generation of multiple breathers without compromising their stability. As a result, the probability of breather generation exhibits a non-monotonic behavior as a function of noise amplitude. Within this generation framework, a statistical approach for breather detection named ‘resistive switching statistics’ is introduced in Ref. [38]. Through a proper statistical analysis of voltage measurements, this work shows that one can determine whether breather excitations are present in an LJJ.

In Ref. [41], a third approach—significantly different from the above two—is proposed for the experimental detection of breathers. In this case, thermally unbalanced LJJs are considered, i.e., junctions where the two electrodes reside at different temperatures. In such a scenario, by combining the SG model with an appropriate diffusive model for temperature distribution in the system, the presence of a breather is demonstrated to increase heat transfer between the junction electrodes. This phenomenon results in spatially localized heating, which can be measured within present-day Josephson experimental setups. Interestingly, the precise shape of the temperature peak caused by the breather is closely tied to its oscillation frequency.

This paper is structured as follows. In Section 2, we begin with the derivation of the Josephson relations (Section 2.1) and the SG model in the LJJ context (Section 2.2 and Section 2.3). In Section 3, we discuss recent developments in the study of SG breathers—briefly mentioned above—establishing a clear connection with Josephson physics: (i) supratransmission-induced traveling breathers and their detection (Section 3.1); (ii) the noise-enhanced stability effect on breather dynamics (Section 3.2); (iii) the AC locking of noise-induced breathers and their statistical (resistive switching) signature (Section 3.3); (iv) the thermal transport fingerprint of Josephson breathers (Section 3.4). Finally, conclusions and potential research directions are discussed in Section 4.

2. From Josephson junctions to the sine-Gordon model

2.1. Josephson effect

The Josephson effect is a remarkable demonstration of the macroscopic quantum coherence inherent in the superconducting state, emerging when two superconductors are weakly linked via an electrical junction. Josephson first predicted this phenomenon theoretically in 1962 while investigating the properties of a superconductor-insulator-superconductor (SIS) junction [42], as illustrated in Fig. 1. Shortly thereafter, Anderson and Rowell provided experimental confirmation [43]. Since its discovery, extensive theoretical advancements and potential applications have been explored [44–50]. The Josephson effect has become a cornerstone in superconductivity research, particularly following the identification of both DC and AC Josephson effects, which

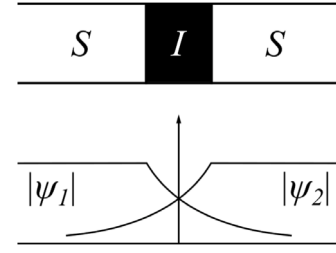


Fig. 1. Josephson junction. Schematic representation of an SIS Josephson junction. ψ_1 and ψ_2 denote the pair wave functions corresponding to the two superconductors. Source: Adapted from Ref. [44].

have paved the way for precision measurements. In particular, the AC Josephson effect has been instrumental in developing the Josephson voltage standard, now widely used in metrology to define the volt [51]. On the technological front, Josephson junctions play a crucial role in various applications, including superconducting quantum interference devices (SQUIDs), which are extensively utilized in magnetic field sensing and fundamental physics experiments [44–49]. Furthermore, they have significantly contributed to the progress of quantum computing, where they are integral to the design of qubits for quantum processors [47,48,50,52–54]. Josephson junctions have also been proposed as candidates for dark matter axion detection [55–60].

An intuitive derivation of the governing equations for a Josephson junction can be achieved within the macroscopic quantum model (MQM) of superconductivity, as discussed by Feynman [61].¹ This phenomenological approach postulates a macroscopic quantum wave function to characterize the collective behavior of Cooper pairs in the superconductor. While it does not explain superconductivity's microscopic origins, the MQM effectively describes superconductivity as a coherent phenomenon involving all Cooper pairs. In this framework, the dynamics of an individual carrier is not the focus; rather, a single wave function governs the collective state. This complex-valued wave function can be expressed as

$$\psi = \sqrt{\rho} e^{i\theta}, \quad (1)$$

where θ represents a global phase factor for all Cooper pairs, and ρ denotes their density, such that $|\psi|^2 = \rho$. Utilizing this formalism in conjunction with a two-level system model leads directly to the fundamental equations of the Josephson effect.

We begin by considering two superconductors, labeled 1 and 2, which are initially isolated and not coupled. Under these conditions, their respective pair wave functions evolve independently over time, following the Schrödinger equations

$$\begin{aligned} i\hbar\psi_1 &= E_1\psi_1, \\ i\hbar\psi_2 &= E_2\psi_2, \end{aligned} \quad (2)$$

where Newton's notation is used for time derivatives, and E_1 and E_2 denote the ground-state energies of the superconductors.

When weak coupling is introduced, as in the SIS junction shown in Fig. 1, Cooper pairs can tunnel between the superconductors. This interaction is modeled by adding a coupling term K

$$\begin{aligned} i\hbar\psi_1 &= E_1\psi_1 - K\psi_2, \\ i\hbar\psi_2 &= E_2\psi_2 - K\psi_1. \end{aligned} \quad (3)$$

The parameter K , which characterizes the junction, depends on the overlap of the pair wave functions. In the absence of a vector potential A , K is taken as real.

¹ This derivation is detailed in Ref. [44], while an alternative derivation of the Josephson equations was provided by Landau and Lifshitz [62] and was recently reviewed in Ref. [50].

Assuming identical superconductors, we have that $E_1 = E_2$. With a voltage difference V across the junction, then $E_1 - E_2 = -2eV$ holds, leading to the time evolution equations

$$\begin{aligned} i\hbar\dot{\psi}_1 &= -eV\psi_1 - K\psi_2, \\ i\hbar\dot{\psi}_2 &= +eV\psi_2 - K\psi_1. \end{aligned} \quad (4)$$

Expressing the wave functions as

$$\begin{aligned} \psi_1 &= \sqrt{\rho_1}e^{i\varphi_1}, \\ \psi_2 &= \sqrt{\rho_2}e^{i\varphi_2}, \end{aligned} \quad (5)$$

and substituting in Eq. (4), we derive the pair current density

$$J_s = \frac{2K\rho_0}{\hbar} \sin \varphi \equiv J_c \sin \varphi, \quad (6)$$

where $\varphi \equiv \varphi_1 - \varphi_2$, $\rho_0 = \rho_1 = \rho_2$ is assumed, and J_c is a characteristic current of the junction. We also get

$$\frac{\partial \varphi}{\partial t} = \frac{2e}{\hbar} V. \quad (7)$$

Eqs. (6) and (7) are the so-called Josephson equations.

According to Eq. (7), a zero voltage drop across the junction results in a constant phase difference—not necessarily zero. In Eq. (6), a constant φ indicates that a finite (super)current density, reaching a maximum of J_c , can pass through the barrier even without a voltage drop. This describes the DC Josephson effect.

For a constant applied voltage V , integrating Eq. (7) yields

$$\varphi = \varphi_0 + \frac{2e}{\hbar} Vt, \quad (8)$$

and inserting this into the current expression gives an oscillating supercurrent with frequency $\omega = (2e/\hbar)V$,

$$J_s = J_c \sin \left(\varphi_0 + \frac{2e}{\hbar} Vt \right). \quad (9)$$

This is the AC Josephson effect, which at 1 mV corresponds to a frequency of approximately 483.6 GHz.

When a vector potential $\mathbf{A}$ is present, the coupling amplitude must be modified by introducing a phase shift, leading to the gauge-invariant phase difference

$$\varphi = \varphi_1 - \varphi_2 - \frac{2e}{\hbar} \int_1^2 \mathbf{A} \cdot d\mathbf{l}. \quad (10)$$

2.2. Long josephson junctions

An LJJ is a spatially extended device where one dimension, typically along the x -axis, is significantly larger than the typical length scale of the system, i.e., the Josephson penetration depth, λ_J . At the same time, the perpendicular dimension, along the y -axis, remains small in comparison to λ_J . A precise definition of λ_J will be provided later. Here, we introduce the spatial dependence of the phase difference and demonstrate that the dynamics of these quasi-one-dimensional junctions can be effectively described by the SG equation for $\varphi(x, t)$. The junction's geometry plays a crucial role in determining its properties. Several configurations have been explored over time, including the overlap, inline, and annular geometries. This discussion will primarily focus on the overlap geometry; for further details, see Refs. [63,64].

A schematic representation of an overlap-geometry Josephson tunnel junction is shown in Fig. 2(i). It consists of two superconducting layers separated by a thin, uniform insulating oxide layer of thickness t_{ox} . Given that the width of the junction satisfies $W \ll \lambda_J$, the electromagnetic fields within the insulating barrier can be approximated as uniform along the y -direction. Consequently, the applied bias current can be treated as uniformly distributed along the x -direction.

The supercurrent tunneling through the barrier is governed by the relations

$$\begin{aligned} J_s &= J_c \sin \varphi, \\ \frac{\partial \varphi}{\partial t} &= \frac{2e}{\hbar} V = \frac{2\pi}{\Phi_0} V, \end{aligned} \quad (11)$$

where J_s is the Josephson current per unit length along x , with a maximum amplitude J_c . Here, $\varphi = \varphi(x, t)$ represents the phase difference between the superconducting wave functions, and $V = V(x, t)$ is the voltage drop across the insulating layer. The magnetic flux quantum is given by $\Phi_0 = h/(2e) \simeq 2 \times 10^{-15}$ Wb.

By combining Eq. (11) with Maxwell's equations, the LJJ can be modeled using the lumped transmission-line circuit shown in Fig. 2(ii).² In this equivalent model, I_1 and I_2 denote the superconducting and normal components of the current flowing along x . The inductance per unit length,

$$L_p = \frac{2\lambda_L + t_{ox}}{W} \mu_0, \quad (12)$$

accounts for the magnetic energy stored in the superconducting layers and the oxide barrier. Here, $\mu_0 = 4\pi \times 10^{-7}$ H/m. The series resistance per unit length, R_p , models dissipative current flow due to thermally excited quasiparticles. The capacitance per unit length,

$$C = \frac{W}{t_{ox}} \epsilon_r \epsilon_0, \quad (13)$$

accounts for the stored electric energy in the barrier, with ϵ_r being the relative dielectric constant of the oxide layer and $\epsilon_0 = (1/36\pi) \times 10^{-9}$ F/m. The tunneling current from quasiparticles is described by GV , where $1/G$ represents an effective normal resistance. Finally, J_b denotes the externally applied bias current per unit length, and J_T a thermal current term to be introduced later.

Using Kirchhoff's laws for the circuit in Fig. 2(ii), we obtain

$$\frac{\partial V}{\partial x} = -R_p I_2 = -L_p \frac{\partial I_1}{\partial t}. \quad (14)$$

Applying Kirchhoff's current law, we get

$$\frac{\partial I_1}{\partial x} + \frac{\partial I_2}{\partial x} = -C \frac{\partial V}{\partial t} - GV - J_c \sin \varphi + J_b. \quad (15)$$

Eqs. (11), (14), and (15) can be rewritten together as (adopting a concise subscript notation for partial derivatives)

$$\frac{L_p}{R_p} \varphi_{xx} + \varphi_{xx} - L_p C \varphi_{tt} - G L_p \varphi_t = \frac{2\pi L_p J_c}{\Phi_0} \left(\sin \varphi - \frac{J_b}{J_c} \right). \quad (16)$$

Introducing the normalized space and time coordinates, x' and t' , as follows

$$x' = \frac{x}{\lambda_J}, \quad t' = t \omega_p, \quad (17)$$

where the Josephson penetration depth is defined as

$$\lambda_J = \sqrt{\frac{\Phi_0}{2\pi J_c L_p}}, \quad (18)$$

and the Josephson plasma frequency is given by

$$\omega_p = \sqrt{\frac{2\pi J_c}{\Phi_0 C}}, \quad (19)$$

Eq. (16) transforms into

$$\beta \varphi_{x't't'} + \varphi_{x't't'} - \varphi_{t't't'} - \alpha \varphi_{t't'} = \sin \varphi - \gamma. \quad (20)$$

Here, the parameters α , β , and γ are defined as follows

$$\alpha = \frac{G}{\omega_p C}, \quad \beta = \frac{\omega_p L_p}{R_p}, \quad \gamma = \frac{J_b}{J_c}. \quad (21)$$

It is important to note that λ_J and ω_p are characteristic quantities that describe the system's dynamics. The Josephson penetration depth represents the characteristic distance over which an externally applied magnetic field penetrates the LJJ. The Josephson plasma frequency

² This model represents the LJJ as a series of current-biased Josephson junctions coupled via surface current terms. Detailed discussions on the fundamental circuit elements can be found in Refs. [44–47].

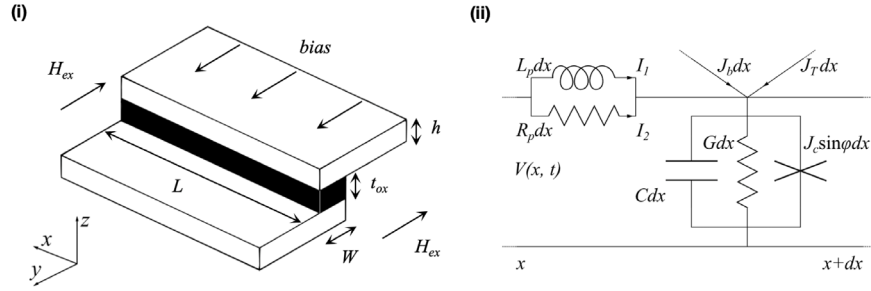


Fig. 2. LJJ and its circuit analog. Panel (i): Schematic of an overlap-geometry Josephson tunnel junction. The arrows indicate the bias current J_b and the external magnetic field H_{ex} . Assumptions: $\lambda_L \ll h \ll W \ll \lambda_J \ll L$, where λ_L is the London penetration depth, which characterizes the decay length of a magnetic field inside a superconductor. Panel (ii): Equivalent circuit model of the LJJ, incorporating a thermal noise source. Source: Adapted from Ref. [65].

corresponds to the fundamental frequency of small-amplitude oscillations in an unbiased junction. The velocity of electromagnetic wave propagation within the junction (the so-called Swihart velocity) is [66]

$$\bar{c} = \lambda_J \omega_p = \frac{1}{\sqrt{L_p C}}. \quad (22)$$

Eq. (20) governs the temporal and spatial evolution of voltages and currents, representing a perturbed version of the SG equation. In the absence of dissipation and bias, it simplifies to the SG equation, which we discuss in the next section.

In terms of the normalized time variable, the second Josephson equation can be rewritten as

$$V = \frac{\Phi_0 \omega_p}{2\pi} \varphi_{t'} \equiv V_N \varphi_{t'}. \quad (23)$$

By appropriately combining this expression with the time integral of Eq. (14), we obtain

$$I_1 = -(J_c \lambda_J) \varphi_{x'} \equiv -I_N \varphi_{x'}, \quad (24)$$

which serves as a basis for defining the boundary conditions of the junction. The superconducting surface current I_1 generates a magnetic field perpendicular to the junction's length. Consequently, setting an external magnetic field in the y direction, denoted as H_{ex} , corresponds to prescribing $\varphi_{x'}$ at the junction's boundaries. For a junction of length $l = L/\lambda_J$, the boundary conditions are given by

$$\varphi_{x'}(0, t') = \varphi_{x'}(l, t') = -\frac{H_{ex} W}{I_N} \equiv q. \quad (25)$$

Here, q is a dimensionless parameter representing the external magnetic field.

Finally, thermal fluctuations can be incorporated into this model by introducing a white Gaussian noise current source associated with the conductance of the fundamental junction element, as illustrated in Fig. 2(ii). Assuming that the noise contributions from individual junctions are uncorrelated [67–69], the autocorrelation function is given by

$$\langle J_T(x_1, t_1) J_T(x_2, t_2) \rangle = 2k_B T G \delta(x_1 - x_2) \delta(t_1 - t_2), \quad (26)$$

where k_B is Boltzmann's constant, T is the absolute temperature, and δ represents the Dirac delta function. Expressing this equation in

normalized units of J_c and in terms of the dimensionless coordinates x' and t' , we obtain

$$\langle \gamma_T(x'_1, t'_1) \gamma_T(x'_2, t'_2) \rangle = \frac{2\alpha I_T}{J_c \lambda_J} \delta(x'_1 - x'_2) \delta(t'_1 - t'_2), \quad (27)$$

where $I_T = (2ek_B T)/\hbar$ represents the thermal current. Consequently, the stochastic partial differential equation describing the LJJ dynamics takes the form

$$\beta \varphi_{x'x't'} + \varphi_{x't'} - \varphi_{t't'} - \alpha \varphi_{t'} = \sin \varphi - \gamma - \gamma_T(x', t'). \quad (28)$$

In the following sections, we will disregard higher-order dissipative terms by setting $\beta = 0$. For simplicity, since the normalized coordinates will be predominantly used, x' and t' will hereafter be referred to as x and t , unless stated otherwise.

2.3. The sine-Gordon equation

The SG equation for the field $\varphi(x, t)$ is given by

$$\varphi_{xx} - \varphi_{tt} = \sin \varphi. \quad (29)$$

This nonlinear hyperbolic partial differential equation can be interpreted physically as Newton's second law applied to a chain of pendula connected by harmonic coupling, where φ corresponds to the rotation angle. This analogy, depicted in Fig. 3 [44,70–72], provides an intuitive way to understand the three primary excitation types in the SG system: plane waves, kinks, and breathers.

Plane waves, also referred to as plasma oscillations, phonons, or radiation depending on context, arise from the small oscillations of individual pendula. In the small-angle approximation ($\varphi \ll 2\pi$), the sine function in Eq. (29) can be linearized, reducing the equation to

$$\varphi_{xx} - \varphi_{tt} = \varphi, \quad (30)$$

which is the Klein–Gordon equation [70]. This equation admits solutions of the form

$$\varphi(x, t) \sim e^{i(kx - \omega t)}. \quad (31)$$

The fundamental mode, corresponding to a wave number $k = 0$, involves a uniform oscillation of all pendula in the chain without spatial variation. In contrast, higher-order modes with $k > 0$ exhibit a standing

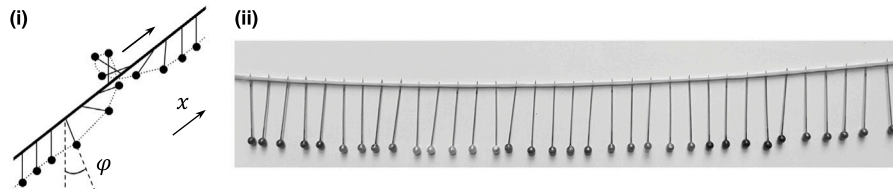


Fig. 3. SG model based on coupled pendula. Panel (i): A moving kink represented as a traveling disturbance in a chain of elastically coupled pendula, with φ being the rotational angle. Adapted from Ref. [2]. Panel (ii): A simplified mechanical model of the SG system using dressmaker pins embedded in an elastic band [2,44].

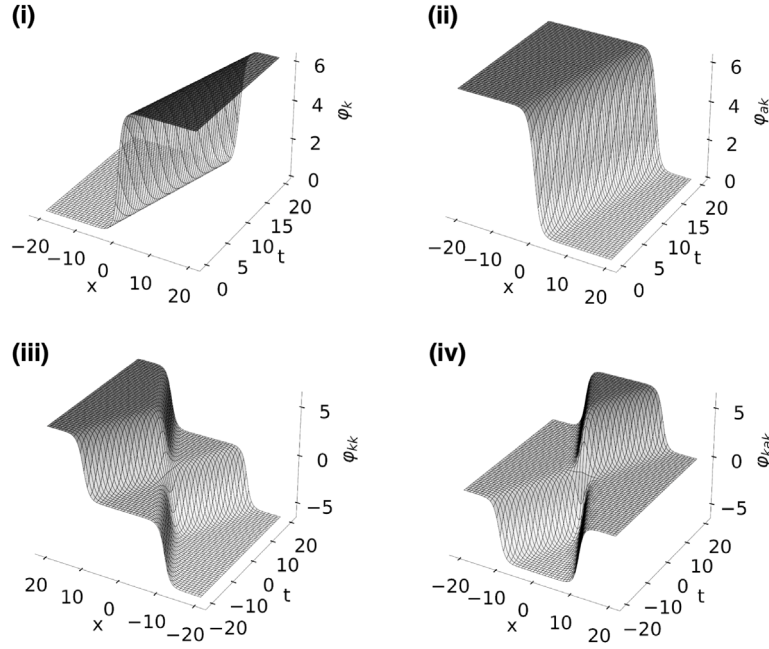


Fig. 4. Kink, antikink, and scattering solutions. Panel (i): Surface plot of the kink solution [Eq. (32)] with $v = 0.5$. Panel (ii): Surface plot of the antikink solution [Eq. (32)] with $v = 0.5$. Panel (iii): Kink-kink scattering solution [Eq. (34)] with $v = 0.5$. Panel (iv): Kink-antikink scattering solution [Eq. (35)] with $v = 0.5$.

wave pattern, featuring one or more spatial nodes along the chain. These modes follow the dispersion relation $\omega^2 = 1 + k^2$, indicating that for $\omega = 1$, the wavelength is infinitely long—akin to the behavior observed in metals at the plasma frequency. Moreover, for $\omega < 1$, the wave vector becomes imaginary, preventing wave propagation and giving rise to a forbidden band gap in the system.

Kink (ϕ_k) and antikink (ϕ_{ak}) solitons, which are exact traveling wave solutions of the SG equation, are given by

$$\phi_{k,ak}(x, t) = 4 \arctan \exp \left(\pm \frac{x - vt}{\sqrt{1 - v^2}} \right), \quad (32)$$

where $v < 1$ represents the velocity. These solutions, shown in Fig. 4(i) and (ii), correspond to localized 2π twists in the mechanical analog, highlighting their topological character. Additionally, kinks exhibit Lorentz contraction, narrowing as $|v| \rightarrow 1$, a direct consequence of the Lorentz invariance of the SG equation under transformations of the form

$$X = \frac{x - vt}{\sqrt{1 - v^2}}, \quad T = \frac{t - vx}{\sqrt{1 - v^2}}. \quad (33)$$

More complex solutions include soliton interactions, such as the kink-kink collision

$$\phi_{kk}(x, t) = 4 \arctan \left[v \sinh \left(\frac{x}{\sqrt{1 - v^2}} \right) \operatorname{sech} \left(\frac{vt}{\sqrt{1 - v^2}} \right) \right] \quad (34)$$

and the symmetric kink-antikink collision

$$\phi_{kak}(x, t) = 4 \arctan \left[\frac{1}{v} \sinh \left(\frac{vt}{\sqrt{1 - v^2}} \right) \operatorname{sech} \left(\frac{x}{\sqrt{1 - v^2}} \right) \right]. \quad (35)$$

The previous expression, if v is taken as an imaginary parameter,

$$v = i \frac{\omega}{\sqrt{1 - \omega^2}}, \quad \omega < 1, \quad (36)$$

leads to a bound state solution known as the stationary breather

$$\phi_{sb} = 4 \arctan \left[\frac{\sqrt{1 - \omega^2}}{\omega} \sin(\omega t) \operatorname{sech} \sqrt{1 - \omega^2} \right]. \quad (37)$$

This mode, shown in Fig. 5, is spatially localized and oscillates in time with frequency ω . Similar to plasma oscillations, breather modes arise

from the oscillatory motion of the individual pendulum. The amplitude of oscillation is position-dependent. However, in this case, no spatial nodes are present along the chain. Notably, breathers populate the model's plasma gap, $0 < \omega < 1$.

Through Lorentz transformation, Eq. (37) generalizes to a moving breather with envelope velocity v_e ,

$$\phi_b = 4 \arctan \left\{ \frac{\sqrt{1 - \omega^2}}{\omega} \sin \left[\frac{\omega(t - v_e x)}{\sqrt{1 - v_e^2}} \right] \operatorname{sech} \left[\frac{\sqrt{1 - \omega^2}(x - v_e t)}{\sqrt{1 - v_e^2}} \right] \right\}. \quad (38)$$

A variety of analytical methods can be utilized to obtain the soliton solutions discussed here, as well as to construct the complete hierarchy of SG solutions. For instance, the Bäcklund transformation allows one to generate new solutions from a known one, such as the trivial solution $\phi = 0$. Due to Bianchi's commutativity theorem, this process can be iterated with reasonable efforts. Detailed descriptions of this approach are available in Refs. [2,73]. Additionally, Hirota's method has been successfully applied to derive N -soliton solutions for the SG equation [74]. Around the same period, Ablowitz et al. achieved analogous results using the inverse scattering transform formalism [75].

The energies linked to the fundamental soliton solutions—specifically, kinks (E_k) and breathers (E_b)—are of particular physical significance. In this context, it is worth noting that the SG equation can

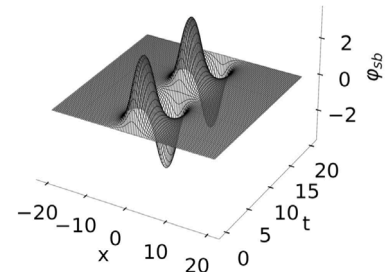


Fig. 5. Breather solution. Surface plot of the stationary SG breather mode [Eq. (37)] with $\omega = \pi/5$.

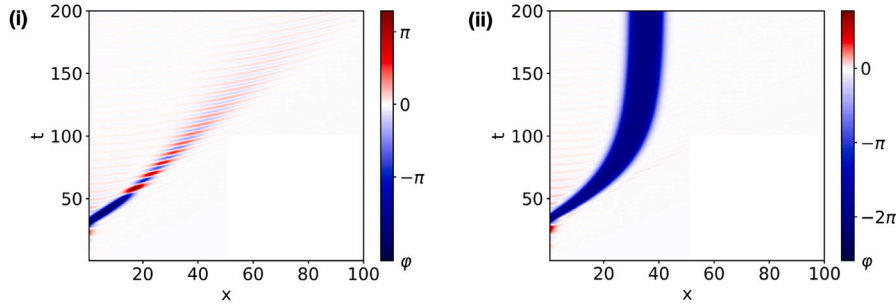


Fig. 6. Characteristic NST-induced breather and kink-type excitations. Panel (i): Field profile of a breather. Panel (ii): Field profile of a kink-antikink pair. The two panels display numerical simulations of NST-induced traveling modes in a noise-free scenario. The parameter values used are $t_f = 200$, $\Omega = 0.5$, and $A = 2.3$ for panel (i), while panel (ii) corresponds to $\Omega = 0.45$ and $A = 2.25$.

Source: Adapted from Ref. [35].

be obtained from the Hamiltonian

$$H = \frac{1}{2} \int_{-\infty}^{\infty} dx [\varphi_t^2 + \varphi_x^2 + 2(1 - \cos \varphi)], \quad (39)$$

which leads to the expression

$$E_k = \frac{8}{\sqrt{1 - v^2}} \quad (40)$$

for the energy of a kink traveling at velocity v . Similarly, the energy of a breather with envelope velocity v_e is given by

$$E_b = 16 \sqrt{\frac{1 - \omega^2}{1 - v_e^2}}, \quad (41)$$

as detailed in Appendix A. Both Eqs. (40) and (41) highlight the relativistic and massive characteristics of the elementary excitations in the SG model.

3. Noisy sine-Gordon breathers: Emergence, dynamics, and detection

Within the context of LJJ, an SG kink (or fluxon) corresponds to a quantum of magnetic flux, denoted as Φ_0 . Due to its capacity for stable storage and precise manipulation, this nonlinear wave plays a significant role in numerous applications [9,76–81]. Notably, the quantum dynamics of a single fluxon has been experimentally verified [82].

In contrast, breathers are considerably more elusive than kinks. This is primarily due to their tendency to decay radiatively in the presence of dissipation, as discussed in Appendix B, unless specific forms of external driving are applied [83,84]. Additionally, their oscillatory nature results in a very low average voltage, which typically falls below the sensitivity threshold of high-frequency oscilloscopes [31]. Consequently, despite extensive numerical and theoretical investigations dedicated to breathers [16,76,83–93], experimental confirmation of breather modes in LJJ remains elusive [31,32].

The excitation and detection of breather-like states have also been extensively explored in discrete systems [94,95]. For instance, in parallel arrays of Josephson junctions, which are modeled by the discrete SG equation (also referred to as the Frenkel-Kontorova model [21,96,97]), the existence of oscillobreathers has been theoretically predicted. However, due to their rapid oscillations, experimental verification is challenging [6,98]. On the other hand, rotobreather states, which can be tracked by measuring local DC voltages across the Josephson junction array, have been successfully observed in Josephson ladders [99,100].

Recent progress, reviewed here and detailed further in Refs. [33–41], has significantly advanced the understanding of the emergence and dynamics of SG breathers in scenarios involving dissipation, external drives, and, notably, environmental noise. These insights have been leveraged to propose and explore various strategies for detecting breathers in LJJ.

3.1. Traveling breathers induced by supratransmission

Refs. [33–35] numerically investigate how external pulses can generate traveling breather modes in the SG system through the nonlinear supratransmission (NST) effect. The latter phenomenon dates back to pioneering work by Geniet and Leon, who demonstrated that a nonlinear system driven sinusoidally at frequencies within its forbidden band gap can transmit energy via soliton formation when the driving amplitude exceeds a critical value. Originally discovered in discrete SG chains [101,102], the NST effect arises from a fundamental nonlinear instability [103], and has since been observed across various systems including Bragg media [104], waveguide arrays [105–107], Fermi-Pasta-Ulam lattices [108–111], Klein-Gordon networks [112], discrete LC transmission lines [113], wave collisions in electrical lattices [114], multicomponent nonlinear systems [115,116], Miura-origami metastuctures [117], and Josephson devices [118–121]. For LJJ specifically, where linear plasma waves cannot propagate below ~ 1 THz, selecting a driving pulse with comparable frequency enables access to the NST. Additional examples of THz radiation controlling superconducting systems are given, e.g., in Refs. [30,122–125].

For completeness, we mention that NST-related investigations focusing on the interplay between noise and breathers—in a discrete SG setting—were conducted by Bodo et al. and can be found in Refs. [126,127]. Moreover, in a similar discrete SG context, the connections between NST and rogue waves were recently explored [128].

Refs. [33–35] examine the following noisy SG system

$$\varphi_{xx} - \varphi_{tt} - \alpha \varphi_t = \sin \varphi - \gamma - \gamma_T(x, t). \quad (42)$$

When applying an AC pulse to the $x = 0$ boundary (equivalent to a magnetic pulse in LJJ), the boundary conditions become $\varphi_x(0, t) = \tilde{A}(t) \sin(\Omega t)$ and $\varphi_x(l, t) = 0$, where l represents the normalized system length. The driving signal has frequency $\Omega < 1$ (within the plasma gap) and amplitude $\tilde{A}(t) = A f(t)$ with Gaussian turn-on/off profiles: $f(t) = \exp\left[-\frac{(t-t_{\pm})^2}{2\sigma_{\pm}^2}\right]$ for $t \lesssim t_{\pm}$ and $f(t) = 1$ otherwise. Here $t_{+} = 3\sigma_{+}$ remains fixed, while t_{-} adjusts based on field evolution monitoring at position $\tilde{x} \ll l$ to achieve single breather generation.

In Refs. [33–35], Eq. (42) undergoes numerical integration with initial conditions $\varphi(x, 0) = \arcsin \gamma$ and $\varphi_t(x, 0) = 0$ using an implicit finite-difference method. Practical parameter choices are $\Delta x = \Delta t = 0.005$, system length $l = 100$ (to minimize $x = l$ reflections³), and observation time $t_f \geq 150$ to track the breather evolution. Unless specified otherwise, in this section parameters are $\gamma = 0$, $\sigma_{+} = 10$, $\sigma_{-} = 2.5$, and $\alpha = 0.03$ [65]. It should be mentioned that the analysis proves robust across various conditions, including different α , γ , σ_{+} , and σ_{-} values, as thoroughly examined in Refs. [33–35].

³ To this end, another viable option could be to employ boundary conditions that account for external RC loads, see Refs. [129,130].

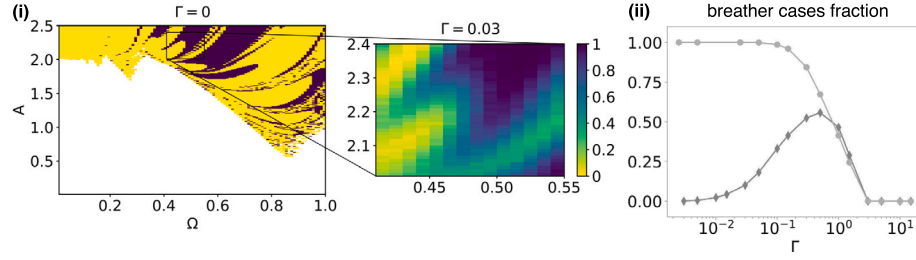


Fig. 7. Characterization of breathers generated via supratransmission. Panel (i): Phase diagram in the (Ω, A) parameter space (main panel). White regions denote parameter combinations where minimal energy transmission occurs, yellow zones indicate configurations yielding at least one persistent kink/antikink at $t_f = 200$, while purple areas correspond exclusively to breather solutions. The analysis assumes zero noise amplitude ($\Gamma = 0$). The inset displays the probability distribution of breather-only states (from $N = 520$ simulations) within a subsection of the parameter space under thermal fluctuations, with $t_f = 150$ and $\Gamma = 0.03$. Panel (ii): Breather-only occurrence probability versus noise strength Γ for $N = 10^4$ realizations. Diamond markers represent $(\Omega, A) = (0.55, 2.19)$, while circles show $(0.52, 2.45)$. Additional parameters: $t_f = 200$ and $\alpha = 0.02$. Source: Adapted from Refs. [34,35].

Refs. [33–35] identify three primary regimes when analyzing the (Ω, A) space. First, since Ω lies within the plasma gap, the linear approximation only permits evanescent waves. The NST effect predicts an Ω -dependent amplitude threshold enabling energy transmission via nonlinear entities like kinks (second regime) and breathers (third regime). Cases leaving at least one kink/antikink in the final state can be distinguished from those retaining only breathers during the simulation. With damping present, breathers vanish within a time $\sim 1/\alpha$ [Fig. 6(i)], while kinks/antikinks persist as 2π phase twists [Fig. 6(ii)]. For observation times $t_f \gg 1/\alpha$, the condition

$$\max_{x \in [0, l]} |\varphi(x, t_f)| \geq 2\pi \quad (43)$$

only holds if kinks/antikinks remain at $t = t_f$.

Fig. 7(i) presents results for the noise-free case ($\Gamma = 0$), exploring the (Ω, A) parameter space through systematic variation of the driving frequency ($\Omega \in [0, 1]$) and amplitude ($A \in [0, 2.5]$) in increments of $\Delta\Omega = \Delta A = 0.01$. The analysis reveals extensive domains supporting pure breather solutions (purple regions), alongside non-transmitting configurations (white areas) and kink-containing final states (yellow zones) at $t_f = 200$, as determined by Eq. (43).

The framework's thermal stability is evaluated in Refs. [33–35] by computing breather-only probabilities across N repeated simulations. The inset in Fig. 7(i) focuses on the subspace $[0.4, 0.55] \times [2, 2.4]$ ($\Delta\Omega = \Delta A = 0.01$) with $t_f = 150$, $\Gamma = 0.03$,⁴ and $N = 520$, demonstrating both the preservation of high-probability breather cores and noise-induced expansion into adjacent parameter regions. This aspect reveals a beneficial stochastic effect where non-breather-producing parameters gain finite excitation probabilities.

Further examination of the noise dependence selects representative points both outside and inside the deterministic breather-only zone. For $(\Omega, A) = (0.55, 2.19)$, Fig. 7(ii) (diamond markers) shows the breather-only probability peaking near $\Gamma \approx 0.6$ before suppression at higher noise levels, reflecting competing effects: optimal fluctuations enhance breather excitation, while excessive noise disrupts these solitonic bound states and generates additional kink excitations. Conversely, the high-probability case $(\Omega, A) = (0.52, 2.45)$ (circles) exhibits monotonically decreasing probability with increasing Γ .

Established results [31,86] imply that DC forcing (γ) induces breather dissociation into kink-antikink pairs. Consequently, increasing γ systematically eliminates breather-only regions, as quantitatively analyzed in Refs. [33,34].

Building on these findings, Ref. [33] proposes an experimental protocol for breather detection in LJJJs. First, scanning a portion of

(Ω, A) space under sufficient DC current bias characterizes the NST process through averaged voltage measurements. The power balance between dissipation and current bias input produces distinct voltage patterns when fluxons emerge [76] (even from breather dissociation). Conversely, minimal average voltages occur for (Ω, A) combinations below the energy transmission threshold. After confirming nonlinear excitations above threshold, focus shifts to breather-only regions thoroughly documented in Refs. [33–35]. Specifically, in unbiased junctions, breather-only (Ω, A) configurations yield negligible average voltage. If this condition holds, applying sufficient current bias ($\gamma \gtrsim 0.1$, see Ref. [31]) after time $t \gg 1/\alpha$ post-pulse should maintain low voltage if only breathers are present, since their radiative decay prevents current-driven dissociation. Conversely, fluxon-related voltage patterns indicate the presence of kink/antikink states. Thus, persistent low voltage suggests that high-frequency excitations decay within approximately $1/\alpha$, which is incompatible with linear waves due to the bandgap frequency. Combined with experimental reconstruction of the above breather-only regions, this provides compelling evidence for Josephson breathers.

3.2. Noise-enhanced stability of breather excitations

Building on the insights from the previous section, which show that noise can play a constructive role in breather generation, Refs. [36,37] explore a noisy SG equation with dissipation to determine whether stochastic fluctuations can also enhance another crucial property of these solitonic excitations: their persistence time. Specifically, in Ref. [36], the evolution of the field $\varphi(x, t)$ is governed by the equation

$$\varphi_{xx} - \varphi_{tt} - \alpha\varphi_t = \sin \varphi - \gamma_n(t), \quad (44)$$

which is numerically integrated over the domain $[-l/2, l/2] \times [0, t_f]$, where $l = 50$ and $t_f = 250$ denote the system length and observation period, respectively. Additionally, $\alpha = 0.02$ is the friction coefficient, while $\gamma_n(t)$ is a spatially uniform Gaussian noise term with zero mean and autocorrelation function $\langle \gamma_n(t_1)\gamma_n(t_2) \rangle = 2\Gamma_n\delta(t_1 - t_2)$, which can be physically interpreted as a stochastically varying electrical bias in the LJJ framework.

Unlike the setup in the previous section, Refs. [36,37] impose the boundary conditions $\varphi_x(-l/2, t) = \varphi_x(l/2, t) = 0$. To investigate breather dynamics, the initial condition is chosen as the stationary breather profile

$$\varphi_{sb}(x, t) = 4 \arctan \left[\frac{\sqrt{1 - \omega^2}}{\omega} \frac{\sin(\omega t + \vartheta)}{\cosh(\sqrt{1 - \omega^2}x)} \right], \quad (45)$$

where $\omega = 0.9 < 1$ and $0 \leq \vartheta \leq 2\pi$ correspond to the breather's oscillation frequency and phase, respectively.

Under the combined effects of dissipation and noise, as shown in Fig. 8, two primary 'decay channels' emerge: (i) stochastic energy input leads to breather dissociation into a kink-antikink pair; (ii) the breather

⁴ For LJJJs with Nb electrodes, this corresponds to $T/T_c \sim 0.1$, given $T_c = 9.2$ K, $\lambda_L = 80$ nm, $R_a = 100 \Omega \mu\text{m}^2$, $C_a = 50$ fF/ μm^2 , $t_{ox} = 1$ nm, $W = 50$ nm, BCS-like gap temperature dependence, and Ambegaokar–Baratoff critical current relation [44].

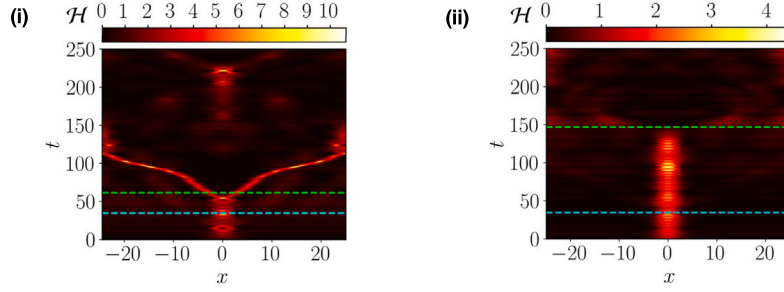


Fig. 8. Typical breather decay mechanisms in the presence of dissipation and noise. Panel (i): Energy density profile, defined as $\mathcal{H}(x, t) = [\varphi_t^2 + \varphi_x^2 + 2(1 - \cos \varphi)]/2$, for $\Gamma_n = 10^{-2}$ and $\vartheta = 0$. At $t = \tau \approx 65$, a clear kink-antikink splitting can be observed. Panel (ii): Energy density profile for $\Gamma_n = 7 \times 10^{-3}$ and $\vartheta = 0$, illustrating a radiative decay process. In this case, the breather remains stable up to $t = \tau \approx 150$. In both panels, the cyan dashed horizontal line represents the deterministic breather lifetime, given by $\ln(2)/\alpha \approx 35$. The green dashed line marks the specific decay time, i.e., $t = \tau \approx 65$ in panel (i) and $t = \tau \approx 150$ in panel (ii). Source: Adapted from Ref. [37].

dissipates into radiative modes. Since no x -dependent perturbations are present in the latest equation of motion [see Eq. (44)], the breather's evolution is trackable by monitoring its oscillation center at $x = 0$. In light of the decay mechanisms described above, for each realization, the characteristic times are defined as follows:

- $\tau_{2\pi}$: The earliest time instant at which the breather's amplitude reaches 2π . This indicates the formation of kink-type structures;
- $\tau_{1/2}$: The earliest time instant at which the breather's amplitude reaches

$$\varphi_{sb,1/2} = 2 \arctan \left(\frac{\sqrt{1 - \omega^2}}{\omega} \right),$$

which is half of the initial value [see Eq. (45)], suggesting significant radiative energy loss.

Refs. [36,37] define the breather lifetime proxy as

$$\tau = \min \{ \tau_{2\pi}, \tau_{1/2} \}, \quad (46)$$

which represents an effective measure of breather persistence in the presence of noise.⁵

Using $\tau(\Gamma_n = 0) \approx \ln(2)/\alpha$ as a reference value (see Appendix B), Fig. 9 depicts the relative variation of the average characteristic time $\langle \tau \rangle$, computed over $N = 2500$ realizations, as a function of the noise amplitude $\Gamma_n \in [10^{-5}, 10^{-2}]$. The four phase values ϑ considered are represented as follows: circles for $\vartheta = 0$, squares for $\vartheta = \pi/2$, triangles for $\vartheta = \pi$, and diamonds for $\vartheta = 3\pi/2$.

⁵ This definition applies when both $\tau_{2\pi}$ and $\tau_{1/2}$ are well-defined. If only $\tau_{2\pi}$ or $\tau_{1/2}$ is available in a given realization, then one simply sets $\tau = \tau_{2\pi}$ or $\tau = \tau_{1/2}$, respectively.

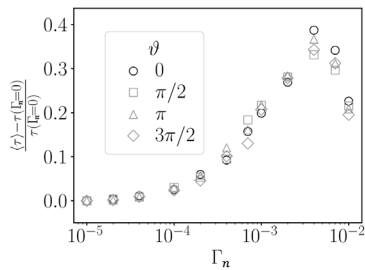


Fig. 9. Nonmonotonic behavior of the average breather characteristic time. Relative variation of $\langle \tau \rangle$, computed over $N = 2500$ realizations, as a function of the noise strength $\Gamma_n \in [10^{-5}, 10^{-2}]$. Different marker styles correspond to distinct phase values ϑ : circles for $\vartheta = 0$, squares for $\vartheta = \pi/2$, triangles for $\vartheta = \pi$, and diamonds for $\vartheta = 3\pi/2$. Source: Adapted from Ref. [37].

for $\vartheta = \pi$, and diamonds for $\vartheta = 3\pi/2$. The results confirm that the breather remains stable against noise over a broad range of noise intensities, as indicated by $\langle \tau \rangle \approx \tau(\Gamma_n = 0)$. For intermediate noise strengths, a notable increase in $\langle \tau \rangle$ is observed. However, as the noise strength increases further, breather instability and transitions to kink-type regimes emerge, leading to a subsequent decline in $\langle \tau(\Gamma_n) \rangle$. Overall, Fig. 9 demonstrates that the nonmonotonic behavior of $\langle \tau \rangle$ is consistently observed across different ϑ values, providing strong evidence of a noise-induced stabilization effect.

This phenomenon is explicitly illustrated in Refs. [36,37] (see also Fig. 8) through spatio-temporal field plots and time-resolved frequency analyses. Notably, the enhancement of τ occurs when the highly localized mode persists significantly beyond its deterministic lifetime (see Appendix B), benefiting from the stochastic energy input. The results suggest that a spatially uniform noise source can effectively prolong the lifetime of SG breathers, independently of the initial phase ϑ . Moreover, these findings remain robust against variations in breather frequency ω , random initial phase selection, and an additional space-time thermal background (see Ref. [36] for further quantitative details). Consequently, noise may serve as a tunable parameter for controlling breather dynamics in experimental and application-oriented contexts.

3.3. AC-locked breathers induced by noise

As demonstrated in earlier sections, stochastic perturbations can facilitate the excitation of breathers and enhance their stability. This naturally leads to the question: *Can a single physical mechanism achieve both generation and stabilization simultaneously under external driving and environmental effects?* It is indeed desirable to identify a method for inducing stable breathers—with near-certain probability—from a breather-free initial state in a dissipative system. Such a capability is useful across platforms for probing and applications involving SG breathers. Addressing this, Refs. [38,39] study the following noisy, AC-driven, dissipative SG system

$$\varphi_{xx} - \varphi_{tt} - \alpha \varphi_t = \sin \varphi - \eta \sin(\omega t) - \gamma_T(x, t), \quad (47)$$

with homogeneous initial conditions, $\varphi(x, 0) = \varphi_t(x, 0) = 0$, and periodic boundary conditions, $\varphi(-l/2, t) = \varphi(l/2, t)$. Unless specified otherwise, in this section the system length is fixed at $l = 50$, the observation time at $t_f = 500$, the damping coefficient at $\alpha = 0.2$, and the AC amplitude at $\eta < 1$ to maintain a perturbative SG regime. The AC frequency $\omega < 1$ is chosen to align with the sub-plasma oscillations characteristic of SG breathers.

Fig. 10 presents two numerically computed energy density profiles, both revealing robust breather formation under thermal noise and AC driving. Notably, these breathers persist far beyond the expected dissipative lifetime of $\sim 1/\alpha = 5$ (see Appendix B), exhibiting positional stability—e.g., the breather at $x \approx -16.5$ in Fig. 10(i) remains localized despite the presence of noise. Such a behavior is consistent across realizations: breathers typically emerge within a few driving cycles

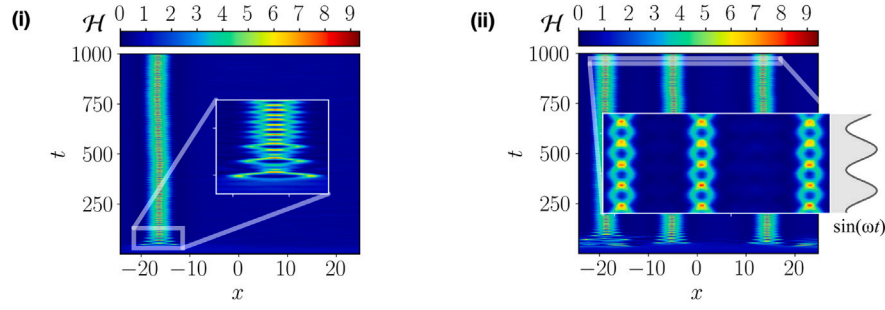


Fig. 10. Noise-induced, AC-locked breathers. Panel (i): Energy density profile $\mathcal{H}(x, t) = [\varphi_r^2 + \varphi_s^2 + 2(1 - \cos \varphi)]/2$, showing a single breather emerging from the noisy background. The region $[-21.5, -11.5] \times [30, 130]$ is enlarged to highlight the breather's formation and initial oscillations near $x \approx -16.5$. Panel (ii): Another realization yielding multiple breathers. The inset $([-22, 17] \times [950, 975])$ illustrates AC-driven mode locking. Parameters: $t_f = 1000$, $\omega = 0.6$, $\eta = 0.59$, and $\Gamma = 2.5 \times 10^{-3}$. Source: Adapted from Ref. [38].

(e.g., $t \approx 50$ in Fig. 10) and stabilize into long-lived states (here, $t_f = 1000$; no decay is observed even for larger t_f). We note that the persistence of noise-induced breathers is primarily governed by the external AC drive.

For fixed $\omega = 0.6$ and $\eta = 0.59$ (as in Fig. 10), Ref. [38] investigates the noise-strength dependence ($\Gamma \in [5 \times 10^{-5}, 2 \times 10^{-1}]$ [77]) via $N = 1000$ simulations ($t_f = 500$). Final states are categorized as follows:

- (a) *No excitations*: A flat phase profile;
- (b) *Breathers only*: Mode amplitudes satisfying

$$A_{sb} = 4 \arctan \left(\frac{\sqrt{1 - \omega^2}}{\omega} \right) < A < 2\pi;$$

- (c) *Kink-antikink pairs*: At least one 2π -step excitation is present.

We mention in passing that, with periodic boundaries, the system's topological charge (winding number in the SG mechanical analog) is conserved. Since breathers are neutral and the initial state has zero charge, kinks and antikinks must form in pairs.

Fig. 11 shows that for low Γ one gets $\mathcal{P}_a \approx 1$ (no excitations). As Γ increases, breather-dominated states ($\mathcal{P}_b \gtrsim 0.9$) prevail ($\Gamma \in [2.5 \times 10^{-3}, 5 \times 10^{-2}]$). At higher Γ , kink-antikink pairs dominate ($\mathcal{P}_c$). This nonmonotonic $\mathcal{P}_b(\Gamma)$ underscores noise as a key control parameter. Remarkably, thermal fluctuations facilitate breather formation *without* destabilizing them. Interestingly, in Nb-based annular junctions [131], the optimal $\Gamma \approx 4 \times 10^{-3}$ aligns with typical experimental conditions.

Building upon this analysis, Ref. [38] examines three further aspects: the spatial energy correlations' dependence on noise, the characteristic timescale of breather generation, and the sensitivity to variations in AC driving parameters. Additional technical discussion of these points is provided in Appendix C.

The system's geometric constraints significantly influence the breather formation process. As previously noted, periodic boundary conditions conserve the initial zero topological charge, preventing

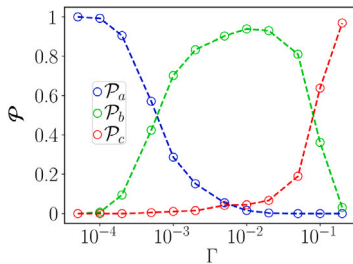


Fig. 11. Noise dependence of breather formation. Probabilities of no excitations ($\mathcal{P}_a$, blue), breathers only ($\mathcal{P}_b$, green), and kink-antikink pairs ($\mathcal{P}_c$, red) versus Γ . Source: Adapted from Ref. [38].

isolated kinks or antikinks. In contrast, Neumann boundaries permit single soliton injection at the edges, which may bind with their virtual counterparts [88,132,133] to form boundary breathers—or edge-breathers [38,39]. To compare these scenarios, a peak detection algorithm on stabilized φ configurations is implemented and location-dependent breather probabilities across multiple realizations are computed. Fig. 12's left panels display breather spatial distributions for $\omega = 0.6$ and $\eta = 0.59$, showing several Γ values within the high- $\mathcal{P}_b$ regime: panel (i) for periodic boundaries and panel (ii) for Neumann conditions. The contrast is clear—periodic boundaries yield uniform distributions while Neumann conditions produce edge-concentrated peaks. This behavior reflects fundamental physical differences: periodic systems lack preferred generation sites, whereas Neumann boundaries energetically favor edge-breathers (being single-soliton modes). Notably, Fig. 12(ii) reveals breather-free zones near the boundaries ($|x| \lesssim l/2$) due to soliton-edge interactions, though the central regions ($|x| \ll l/2$, gray shaded) resemble the behavior observed in the periodic case. These results demonstrate that boundary conditions critically affect breather spatial organization,⁶ with noise intensity primarily modulating the average excitation count.

Our boundary condition survey concludes by investigating system size dependence—particularly the minimum length l required for breather generation. Fixing parameters at the $\mathcal{P}_b$ maximum ($\omega = 0.6$, $\eta = 0.59$, $\Gamma = 0.003$; see Fig. 11), $N = 5000$ simulations across $l \in [1, 32]$ are conducted in Ref. [39] for both boundary types. Ref. [39] quantifies field fluctuations using the time-averaged spatial standard deviation

$$\bar{\sigma}_\varphi = \frac{1}{t_f - t_0} \int_{t_0}^{t_f} dt \sigma_\varphi(t), \quad (48)$$

where $t_0 = 250$ and $\sigma_\varphi(t)$ measures instantaneous spatial variation. This metric approaches zero for flat phase profiles but increases with breather-induced inhomogeneity.

Fig. 12's right panels present ensemble-averaged $\bar{\sigma}_\varphi$ versus l for periodic (iii) and Neumann (iv) boundaries. Both exhibit sharp transitions at critical lengths, with the threshold in the periodic case being approximately double that of the Neumann case. This response reflects spatial constraints: periodic systems require sufficient length ($\approx$ full breather width, blue dashed line) to accommodate complete breathers, while Neumann conditions permit edge-breathers needing only half this space (red dashed line).⁷

Prior to discuss implications for LJJ detection, we note that Ref. [39] provides extended analysis of breather properties under varying frequency-amplitude and noise conditions. A recent work [40] also demonstrates breather emergence under realistic noise with finite temporal/spatial correlations, which modify both generation probabilities

⁶ More complex boundary conditions—for example, those describing coupling to an external RC load (see Refs. [129,130])—could offer additional control over the breather statistics.

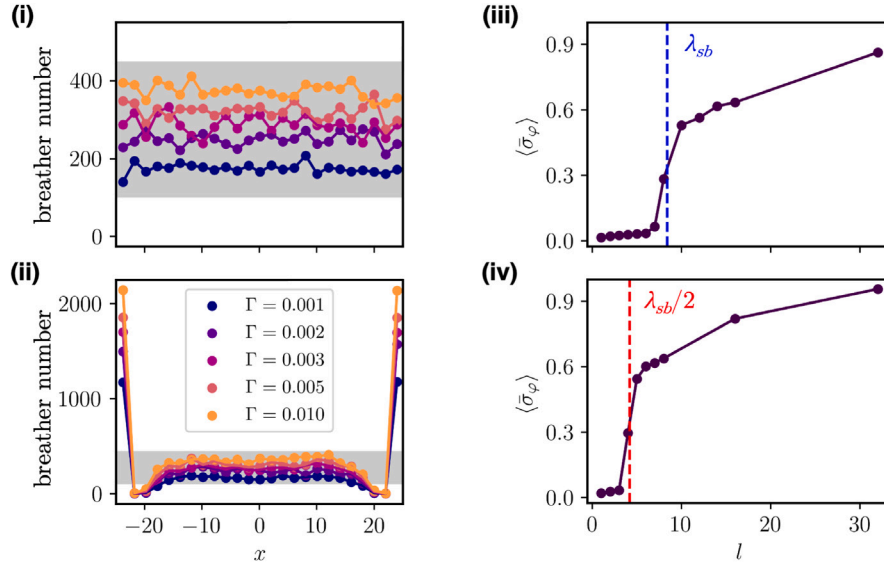


Fig. 12. Boundary condition comparison. Spatial breather distributions for various Γ values under (i) periodic and (ii) Neumann boundaries [shared legend in (ii)]. Parameters: $\Delta x = \Delta t = 0.01$, $l = 50$, $t_f = 500$, $\alpha = 0.2$, $\omega = 0.6$, $\eta = 0.59$, and $N = 5000$. Ensemble-averaged $\bar{\sigma}_\phi$ versus system size l for (iii) periodic and (iv) Neumann conditions. Dashed lines indicate breather width (blue) and half-width (red) at $\omega = 0.6$. Shared parameters: $\Delta x = \Delta t = 0.01$, $t_f = 500$, $\alpha = 0.2$, $\omega = 0.6$, $\eta = 0.59$, $\Gamma = 0.003$, $N = 5000$, and $t_0 = 250$. Source: Adapted from Ref. [39].

and timescales. Intriguingly, correlated noise can enhance breather-only probabilities compared to thermal noise, with nonmonotonic dependence on correlation parameters—see Ref. [40] for complete results.

The critical DC current required to dissociate a stationary breather into a kink-antikink pair exhibits strong phase dependence [86]. Building upon this understanding and leveraging our established framework, we examine a potential experimental protocol for breather detection in LJJ as introduced in Ref. [38], which somewhat generalizes that discussed by Gulevich et al. [31].

⁷ As shown in Ref. [39], a further difference between periodic and linear systems is that the transition from the “breather-only” to the “kink-type” regime occurs at lower noise intensities in the linear system compared to the periodic one. This difference arises because, in the linear case, single solitons can be injected from the edges when the noise is sufficiently strong, thereby driving the system into the aforementioned “kink-type” regime. In contrast, the periodic geometry lacks boundaries, so kink-antikink pair generation is the only possible mechanism, which typically requires more thermal energy than single-kink processes.

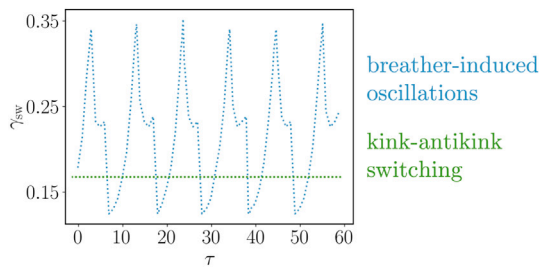


Fig. 13. Switching current statistics. Minimum current γ_{sw} producing resistive switching ($P_{sw} \geq 0.75$) versus phase offset $\tau \in [0, 59]$. The AC drive is attenuated as $\eta \exp[-0.01(t - t^* - \tau)]$ for $t > (t^* + \tau)$. The blue curve corresponds to $\Gamma = 2.5 \times 10^{-3}$ (breather-only regime), whereas the green curve corresponds to $\Gamma = 2 \times 10^{-1}$ (kink-antikink regime). Other parameters: $t_f = 500$, $\omega = 0.6$, $\eta = 0.59$, $t^* = 250$, and $N = 500$. Source: Adapted from Ref. [38].

Following the results in Figs. 11, parameters $\omega = 0.6$, $\eta = 0.59$, and $\Gamma = 2.5 \times 10^{-3}$ are selected to optimize breather formation under periodic boundary conditions. The detection methodology operates through two key stages: (i) generation of stabilized breathers, followed by (ii) characterization of their properties through controlled destruction at varying steps of their oscillations. Specifically, the AC-driven LJJ is evolved until time $t = (t^* + \tau)$, where $t^* \gg t_{\text{formation}}$ and τ represents a phase offset. Given that breathers typically emerge within $t \approx 50$ (see Fig. 10 and Appendix C), $t^* = 250$ is set to ensure field stabilization in terms of breathers. Subsequently, a smoothly ramped current bias $\gamma \{1 - \exp[-0.1(t - t^* - \tau)]\}$ ⁸ is applied while gradually reducing the AC drive,⁹ monitoring for the onset of a resistive state (indicated by kink-antikink separation and measurable voltage). This procedure is repeated to determine, for each τ , the threshold current γ_{sw} yielding switching probability $P_{sw} \geq 0.75$ across N independent realizations.

Fig. 13 reveals pronounced oscillations in $\gamma_{sw}(\tau)$ with period of order $10 \approx 2\pi/\omega$, matching the breather oscillation cycle (see the blue curve). This behavior contrasts sharply with both non-excited and kink-antikink dominated regimes, which show no τ -dependence. For low noise ($\Gamma = 5 \times 10^{-5}$) where $P_a \approx 1$ (Fig. 11), we observe $P_{sw} \approx 0$ throughout $\gamma \in [0, 0.4]$ regardless of τ . In the kink-antikink regime ($\Gamma = 2 \times 10^{-1}$, $P_c \approx 1$), the switching current remains constant at $\gamma_{sw} \approx 0.17$ for all τ values (see the green line in Fig. 13).

3.4. Heat-transfer fingerprint of the breather

In this section, we adopt the tools and insights from phase-coherent caloritronics [134,135], integrate them with our understanding of breather formation and dynamics from previous sections, and discuss a long-sought non-invasive breather detection method. This approach, discussed in Ref. [41], enables detection without disrupting the nonlinear mode.

⁸ This is adopted in Ref. [38] as a representative functional form [31]. The discussed ideas are believed to be robust to reasonable variations in the sequence details.

⁹ The amplitude decrease of the AC source should reasonably be carried out over times significantly longer than the breather period, so as not to abruptly perturb the stabilized mode and to faithfully probe its properties [38].

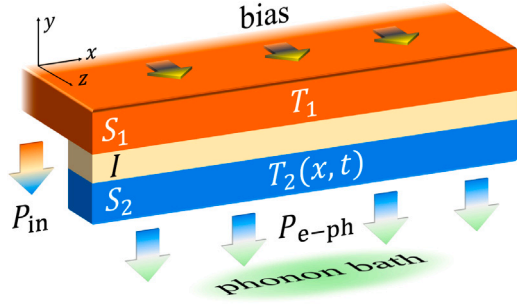


Fig. 14. Schematic of a thermally biased junction. Representation of an overlap-geometry, current-driven LJJ subjected to a thermal bias. The temperature T_1 of the first electrode (S_1) is fixed, while the second electrode (S_2) has a temperature $T_2(x, t)$ that can vary. Additionally, S_2 is thermally coupled to a phonon bath at temperature T_b . Under the condition $T_1 > T_2(x, t) \geq T_b$, the incoming thermal power $P_{in}(T_1, T_2, \varphi, V)$ directed into S_2 is illustrated, along with the outgoing heat flux $P_{e-ph}(T_2, T_b)$, which arises due to quasiparticle interactions with the phonon bath at T_b . Source: Adapted from Ref. [41].

More broadly, superconducting phase coherence has proven to be a powerful mechanism for controlling heat transport. This property has prompted the design of nonlinear caloritronic devices for various applications, including thermometry [136–139], refrigeration [140–145], memory storage [146,147], and energy conversion in engines [148, 149]. Additionally, it has been leveraged for the development of routers [150–152], switches [153], diffractors [154], and radiation detectors [155,156]. The key advantage of phase-coherent caloritronics lies in its ability to regulate temperature by tuning the Josephson phase, for instance, through externally applied magnetic fields. Within the framework of LJJs, previous works [78,79,157] have analyzed kink excitations from a caloritronic perspective, highlighting their properties and potential applications. However, the corresponding case for breathers had remained unexplored until the contributions of Ref. [41], which we now examine.

Defining the dimensionless space and time coordinates as $x' = x/\lambda_J$ and $t' = t\omega_p$ throughout this section, we analyze the equation of motion for the LJJ

$$\frac{\partial^2 \varphi}{\partial x'^2} - \frac{\partial^2 \varphi}{\partial t'^2} - \alpha \frac{\partial \varphi}{\partial t'} = \sin \varphi - \eta \sin[\omega(t' - t'_0)] - \gamma_T(x', t'). \quad (49)$$

Below, we summarize the key physical quantities involved in Eq. (49). The characteristic length and frequency scales are given by $\lambda_J = \sqrt{\Phi_0 / (2\pi\mu_0 I_c J_c)}$ and $\omega_p = \sqrt{2\pi J_c / (\Phi_0 C)}$, where t_d represents the effective magnetic thickness, while $J_c = I_c / A$ corresponds to the Josephson critical current density, with $A = L \times W$ being the junction area, and L (W) its length (width). The junction's specific capacitance is denoted by C . Additionally, the damping coefficient is expressed as $\alpha = 1/(\omega_p R_a C)$, where R_a stands for the normal resistance per unit area. The external AC drive has a frequency ω and amplitude η , both normalized to ω_p and I_c , respectively. The parameter t'_0 represents a normalized time shift, and $\gamma_T(x', t')$ denotes a zero-mean, dimensionless noise current with correlation function

$$\langle \gamma_T(x'_1, t'_1) \gamma_T(x'_2, t'_2) \rangle = 2\alpha \Gamma \delta(x'_1 - x'_2) \delta(t'_1 - t'_2). \quad (50)$$

The noise intensity is given by $\Gamma = 2\pi k_B T / (\Phi_0 I_c)$, where $l = L/\lambda_J$ is the normalized junction length and T denotes the absolute temperature. In the case of an overlap-geometry LJJ, illustrated in Fig. 14, and in the absence of external magnetic fields, we impose the boundary conditions $(\partial \varphi / \partial x')|_{x'=0,l} = 0$.

Fig. 14 provides a visualization of the thermal biasing setup under consideration. The first electrode (S_1) maintains a fixed temperature T_1 , while the second electrode (S_2) has a temperature $T_2(x, t)$ that evolves dynamically. The latter is also thermally linked to a phonon bath at

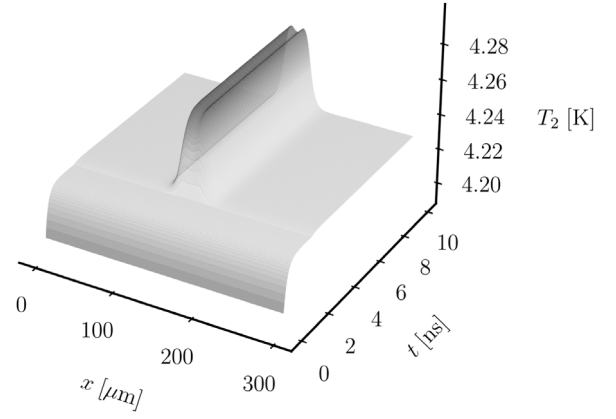


Fig. 15. Breather-enhanced thermal transport. Surface plot of the temperature $T_2(x, t)$. A stabilized breather is introduced at $x_0 = 150 \mu\text{m}$, at time $t_0 = 2 \text{ ns}$, after S_2 has reached thermal equilibrium at approximately 4.23 K. The plot highlights the localized heating effect, where $\max\{T_2\} \gtrsim 4.27 \text{ K}$, due to the breather. Parameter values: $\omega = 0.5$, $\eta = 0.33$, and $\Gamma = 0$. Source: Adapted from Ref. [41].

temperature T_b , satisfying $T_1 > T_2(x, t) \geq T_b$. The evolution of T_2 follows the heat diffusion equation [78,158,159]:

$$\frac{\partial}{\partial x} \left[\kappa(T_2) \frac{\partial T_2}{\partial x} \right] + P_{\text{tot}}(T_1, T_2, T_b, \varphi, V) = c_v(T_2) \frac{\partial T_2}{\partial t}, \quad (51)$$

where $\kappa(T_2)$ is the temperature-dependent electronic heat conductivity, and $P_{\text{tot}}(T_1, T_2, T_b, \varphi, V) = P_{\text{in}}(T_1, T_2, \varphi, V) - P_{\text{e-ph}}(T_2, T_b)$ represents the net thermal power density in S_2 . The term $P_{\text{in}}(T_1, T_2, \varphi, V)$ corresponds to incoming heat, while $P_{\text{e-ph}}(T_2, T_b)$ accounts for losses due to electron–phonon interactions. The volume-specific heat capacity is denoted by $c_v(T_2)$. The left-hand side of Eq. (51) describes the spatial diffusion of heat, whereas the right-hand side governs the internal energy evolution of S_2 .

The phase-sensitive thermal power density is given by [157]:

$$P_{\text{in}}(T_1, T_2, \varphi, V) = P_{\text{qp}}(T_1, T_2, V) - \cos \varphi P_{\text{cos}}(T_1, T_2, V). \quad (52)$$

Here, the ‘qp’ term represents an incoherent quasiparticle-mediated energy transfer from S_1 (hot) to S_2 (cold), while the ‘cos’ term describes coherent tunneling processes involving Cooper pair recombination and annihilation in both S_1 and S_2 [160,161].

For Eq. (51), we impose the initial condition $T_2|_{t=0} = T_b$ and assume thermally insulated boundaries at the device’s edges, leading to $(\partial T_2 / \partial x)|_{x=0,L} = 0$.

Detailed information regarding the above expressions, including their physical interpretation and the numerical methods used to handle them, can be found in Ref. [41]. An LJJ made of Nb/AlO_x/Nb is assumed henceforth, with $L = 300 \mu\text{m}$, $W = 0.5 \mu\text{m}$, $d_2 = 0.1 \mu\text{m}$ (thickness of S_2), $d = 1 \text{ nm}$ (thickness of the insulating layer), $R_a = 50 \Omega \mu\text{m}^2$, and $C = 100 \text{ fF} \mu\text{m}^{-2}$. Such a device is thermally biased by setting $T_1 = 7 \text{ K}$ and $T_b = 4.2 \text{ K}$ [78]. Also, by including the temperature dependence in both $t_d(T_1, T_2)$ and $I_c(T_1, T_2)$, the following values are estimated: $\lambda_J \approx 8 \mu\text{m}$ (thus $l = L/\lambda_J \approx 37$), $\omega_p \approx 0.95 \text{ THz}$, $\alpha \approx 0.2$, and $\Gamma \approx 0.0026$.

Starting with $\Gamma = 0$, a representative simulation result for $T_2(x, t)$ is illustrated in Fig. 15 for $\omega = 0.5$ and $\eta = 0.33 \approx \eta_{\text{th}}(\omega = 0.5)$, with a total simulation time of $t_f = t'_f/\omega_p = 10 \text{ ns}$. For $t < t_0 = t'_0/\omega_p = 2 \text{ ns}$, the temperature gradually relaxes towards its steady-state value in the absence of a breather, reaching approximately 4.23 K. At $t = t_0$, a stabilized breather is introduced at $x_0 = 150 \mu\text{m}$ by employing an analytical breather expression [41,83]. Interestingly, for $t > t_0$, the breather induces a localized exponential rise in temperature, resulting in a sustained double-peaked heating profile where $\max\{T_2\} \gtrsim 4.27 \text{ K}$. This pattern—reflecting the breather’s nature as a two-soliton bound

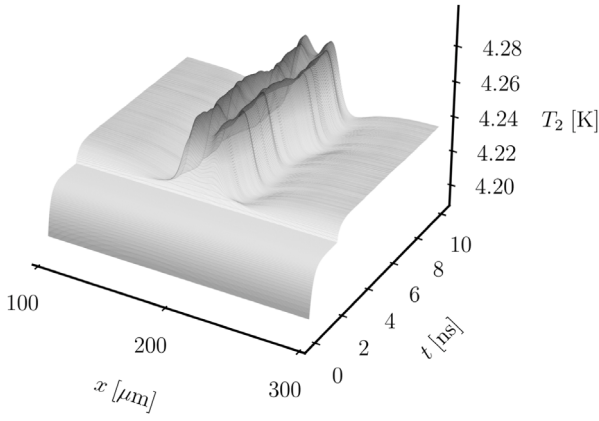


Fig. 16. AC-synchronized, noise-activated breather and its thermal signature. Temperature distribution $T_2(x, t)$ displayed as a surface plot. The plot reveals a breather state, phase-locked to the AC drive and activated by noise, which emerges near $200 \mu\text{m}$ for times $t \gtrsim t_0 = 2 \text{ ns}$, after the electrode S_2 has thermalized to its steady-state temperature of $\sim 4.23 \text{ K}$. Within the spatial interval $[100, 300] \mu\text{m}$, the breather generates a distinctive bimodal heating pattern, reaching peak temperatures exceeding 4.27 K . Simulation parameters: $\omega = 0.6$, $\eta = 0.59$, and $\Gamma = 0.0026$. Source: Adapted from Ref. [41].

state—constitutes a distinct thermal signature. Due to the inherent $\cos \varphi$ dependence in Eq. (52), the breather has a tangible thermal effect, allowing for its detection without necessitating its dissociation. As a further step in the analysis, Ref. [41] demonstrates that different breathing frequencies result in morphologically distinct local temperature peaks (not shown here).

Next, following Ref. [41], we address the integration of the thermal fingerprint with a reliable breather excitation and stabilization technique—an essential task for the actual experiment. To this end, in the previous section, we discussed how, starting from the realistic initial condition $\varphi = 0$, the combination of noise and AC forcing leads to the emergence of long-lived stable breather modes at random locations within the system.

In the simulation shown in Fig. 16, the noisy junction ($\Gamma = 0.0026$, see above) is driven by a frequency/amplitude combination of $\omega = 0.6$ and $\eta = 0.59$. After the excitation-free thermalization of electrode S_2 for $t < t_0$, a breather emerges near $200 \mu\text{m}$, motivating the focus on the region $[100, 300] \mu\text{m}$ in the plot. A key result from Ref. [41] is that the distinctive features of the breather's influence on the temperature T_2 remain intact even in this noisy scenario. As expected, in this case, the breather's center position fluctuates over time, along with variations in its amplitude. However, the overall behavior remains analogous to that shown in Fig. 15—a double-peaked local heating with $\max\{T_2\} \gtrsim 4.27 \text{ K}$.

The experimental requirements for probing this temperature profile are compatible with present-day Josephson setups [162–166]. For detailed estimations, see Ref. [41]. Overall, this approach provides a practical experimental blueprint for generating and detecting breathers in noisy, AC-driven LJJ's using current technology. Notably, this method does not require breather destruction for probing purposes.

4. Conclusion

This work provides a brief overview of recent advancements in the study of SG breathers, which are fundamental components of the SG spectrum. We discussed the impact of various perturbing terms on breather dynamics. Notably, recent findings suggest that thermal fluctuations, rather than contributing to degradation, can serve as a valuable mechanism for exciting and stabilizing breathers on demand [33–40]. Breathers are notoriously challenging to observe experimentally, and deeper insights into their dynamics could have significant implications

for various scientific communities [124,167,168]. Building on these results [33–40], we discussed recent progress toward addressing the long-standing challenge of experimentally detecting SG breathers in extended Josephson junctions [31]—devices composed of two weakly coupled superconductors that host breathers as bound states of magnetic flux quanta. We reviewed three independent and promising strategies for breather detection [33,38,41], which may also pave the way for technological applications [119,169–172].

More specifically, Refs. [33–35] present a methodology for creating single traveling breathers in the SG system through carefully designed boundary excitations. These investigations consider realistic conditions by incorporating energy dissipation, external driving, and thermal noise effects. Through systematic exploration of the driving field's frequency (Ω) and amplitude (A) parameters, the effectiveness of this excitation mechanism is quantified. The analysis reveals significant portions of the (Ω, A) parameter space that reliably produce breather states only, starting from states with no pre-existing breathers. A particularly interesting phenomenon emerges from thermal effects—the parameter regions supporting breather formation exhibit noticeable expansion. Remarkably, thermal fluctuations enable breather generation with measurable probability beyond the predicted boundaries of the noise-free case. Beyond the generation mechanism itself, Ref. [33] leverages this controlled approach to propose a first detection framework for breather states in LJJ systems.

As a further step, findings from Refs. [36,37] reveal a surprising stabilizing effect of random fluctuations on breather excitations. Contrary to conventional expectations, stochastic perturbations can significantly prolong the existence of breather modes compared to their purely deterministic evolution. This stabilization manifests through a non-monotonic relationship between the noise intensity and the breather's persistence time. The implications of this mechanism are quite interesting: rather than acting solely as a disruptive influence, noise can actively improve and regulate the lifetime of soliton-like excitations. This result suggests new possibilities for manipulating nonlinear wave phenomena through tuned stochastic forcing.

Moreover, investigations in Refs. [38–40] focus on the SG system subjected to uniform oscillating forcing, incorporating both dissipative effects and thermal fluctuations. A remarkable phenomenon emerges when noise intensity exceeds a critical threshold: highly stable breather arrays spontaneously form, synchronized with the external AC drive, even from initial conditions completely devoid of kinks or breathers. In this regime, noise amplitude transforms from a disruptive factor into a functional parameter that facilitates the creation of multiple robust breather states. The dependence of breather formation probability on noise strength displays a non-monotonic characteristic—revealing an optimal parameter range for excitation efficiency. This behavior can be characterized across various system dimensions and boundary geometries [39]. Building upon this, a novel detection methodology for LJJ's is developed based on resistive switching statistics [38]. This approach enables the identification of breather states through careful statistical processing of voltage measurement data, providing a reliable experimental signature of these stabilized nonlinear excitations.

Finally, a novel detection scheme for breather states in LJJ's is introduced in Ref. [41], exploiting thermal nonequilibrium conditions. This approach examines junctions with asymmetric electrode temperatures, where the temperature gradient between superconducting layers serves as a detection mechanism. By coupling the SG dynamics with thermal diffusion equations, the analysis reveals that breather excitations significantly modify the system's heat transport properties. The mechanism produces a characteristic localized temperature increase that falls within the resolution limits of modern Josephson junction experiments [164]. Notably, the thermal signature exhibits a direct correlation with the breather's dynamical properties—the spatial temperature profile encodes precise information about the excitation frequency. This temperature-frequency coupling provides a unique fingerprint for identifying and characterizing breather states—without destroying them—through thermal measurements.

Looking ahead, it would be intriguing to investigate the emergence and dynamics of breathers under the influence of non-Gaussian fluctuations, such as those driven by Lévy-distributed noise sources [173–175]. We note that, in this context, the barrier crossing problem, involving enhanced stability of a metastable state, has been exactly solved [176]. Lévy noise is particularly noteworthy for inducing rich dynamical phenomena in SG-like systems [177–180]. A natural question arises: could the well-known Lévy flights lead to more efficient breather generation compared to Gaussian noise?

Another fascinating direction, as highlighted in Ref. [40], involves the emergence of peculiar collective solitonic behaviors triggered by spatially correlated noise [181]. These phenomena warrant further exploration and could have broader implications within the field of statistical physics, particularly in SG-like models [68,69,182–185]. Such studies might also intersect with Kibble–Zurek-type scenarios [186], extending beyond the specific breather generation framework discussed in Ref. [40].

Before concluding this short review, we note that the study of SG kinks, breathers, and noise-related phenomena is beginning to be explored in the context of non-reciprocal systems [187–189] as well. This opens up new avenues for research that are conceptually aligned with the topics reviewed in our work.

CRediT authorship contribution statement

Duilio De Santis: Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Resources, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Davide Valenti:** Writing – review & editing, Supervision, Project administration, Methodology, Funding acquisition, Formal analysis, Conceptualization. **Bernardo Spagnolo:** Writing – review & editing, Methodology, Investigation, Formal analysis, Conceptualization. **Giovanni Di Fresco:** Writing – review & editing, Validation, Software, Resources, Investigation, Formal analysis, Data curation. **Angelo Carollo:** Writing – review & editing, Methodology, Investigation, Formal analysis. **Claudio Guarcello:** Writing – review & editing, Validation, Supervision, Project administration, Methodology, Funding acquisition, Formal analysis, Data curation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

The authors acknowledge the support of the Italian Ministry of University and Research (MUR). D.V. and A.C. acknowledge support from European Union - Next Generation EU through project THENCE - Partenariato Esteso NQSTI - PE00000023 - Spoke 2. D.V. also acknowledges support from PRIN 2020 - DAMATIRA: aDvanced Analysis and Modeling of Acoustic Responses of pLants - B77G22000050001. Lastly, A.C. acknowledges financial support from European Union-Next Generation EU through projects: Eurostart 2022 ‘Topological atom-photon interactions for quantum technologies’ and PRIN 2022–PNRR No. P202253RLY.

Appendix A. Energies of kink and breather solutions

The SG equation can be derived from the Lagrangian density

$$\mathcal{L} = \frac{1}{2} [\varphi_t^2 - \varphi_x^2 - 2(1 - \cos \varphi)], \quad (\text{A.1})$$

which corresponds to the energy density

$$\mathcal{H} = \varphi_t \mathcal{L}_{\varphi_t} - \mathcal{L} = \frac{1}{2} [\varphi_t^2 + \varphi_x^2 + 2(1 - \cos \varphi)]. \quad (\text{A.2})$$

Consequently, the total energy is given by

$$H = \int_{-\infty}^{\infty} dx \mathcal{H} = \frac{1}{2} \int_{-\infty}^{\infty} dx [\varphi_t^2 + \varphi_x^2 + 2(1 - \cos \varphi)]. \quad (\text{A.3})$$

We begin by examining the kink solution

$$\varphi_k(x, t) = 4 \arctan \left[\exp \left(\frac{x - vt}{\sqrt{1 - v^2}} \right) \right]. \quad (\text{A.4})$$

Substituting this into Eq. (A.2) yields

$$\mathcal{H} = \frac{16}{1 - v^2} \left[\exp \left(\frac{x - vt}{\sqrt{1 - v^2}} \right) \right]^2 \left\{ 1 + \left[\exp \left(\frac{x - vt}{\sqrt{1 - v^2}} \right) \right]^2 \right\}^{-2}, \quad (\text{A.5})$$

which, when integrated in Eq. (A.3), provides the kink energy

$$E_k = \frac{8}{\sqrt{1 - v^2}}. \quad (\text{A.6})$$

The energy of a stationary breather, E_{sb} , can be more easily calculated by leveraging the conservation of total energy in the SG system. Indeed, the time derivative of H is

$$\frac{dH}{dt} = \int_{-\infty}^{\infty} dx \frac{\partial \mathcal{H}}{\partial t} = \int_{-\infty}^{\infty} dx [\varphi_t \varphi_{tt} + \varphi_x \varphi_{xt} + (\sin \varphi) \varphi_t]. \quad (\text{A.7})$$

After integrating the second term in Eq. (A.7) by parts and assuming that the time and space derivatives of φ vanish as $x \rightarrow \pm\infty$, the equation simplifies to

$$\frac{dH}{dt} = \int_{-\infty}^{\infty} dx (\varphi_{tt} - \varphi_{xx} + \sin \varphi) \varphi_t = 0. \quad (\text{A.8})$$

Since the integral in Eq. (A.3) must be time-independent, it follows that $H = H(t \rightarrow 0)$. Thus, we can express the total energy as

$$H = \int_{-\infty}^{\infty} dx \lim_{t \rightarrow 0} \mathcal{H}, \quad (\text{A.9})$$

where the interchange of the limit and integration is justified by the regularity of the solutions under consideration. For a stationary breather, we obtain

$$\lim_{t \rightarrow 0} \mathcal{H} = 8(1 - \omega^2) \operatorname{sech}^2(\sqrt{1 - \omega^2} x), \quad (\text{A.10})$$

which implies that

$$E_{sb} = 16\sqrt{1 - \omega^2}. \quad (\text{A.11})$$

Applying a Lorentz boost, we arrive at

$$E_b = 16\sqrt{\frac{1 - \omega^2}{1 - v_e^2}}, \quad (\text{A.12})$$

representing the energy of a breather moving with envelope velocity v_e .

Appendix B. Breather decay due to damping

In the spatio-temporal domain defined by $-l/2 \leq x \leq l/2$ and $t \geq 0$, we analyze the dissipative SG equation

$$\varphi_{xx} - \varphi_{tt} - \alpha \varphi_t = \sin \varphi, \quad (\text{B.1})$$

subject to no-flux boundary conditions

$$\varphi_x(-l/2, t) = \varphi_x(l/2, t) = 0, \quad (\text{B.2})$$

and initial conditions

$$\varphi(x, 0) = \varphi_{sb}(x, 0) \quad \text{and} \quad \frac{\partial \varphi(x, 0)}{\partial t} = \frac{\partial \varphi_{sb}(x, 0)}{\partial t}, \quad (\text{B.3})$$

where $\varphi_{sb}(x, t)$ represents the stationary SG breather solution, given by

$$\varphi_{sb}(x, t) = 4 \arctan \left[\frac{\sqrt{1 - \omega^2}}{\omega} \frac{\sin(\omega t + \vartheta)}{\cosh(\sqrt{1 - \omega^2} x)} \right]. \quad (\text{B.4})$$

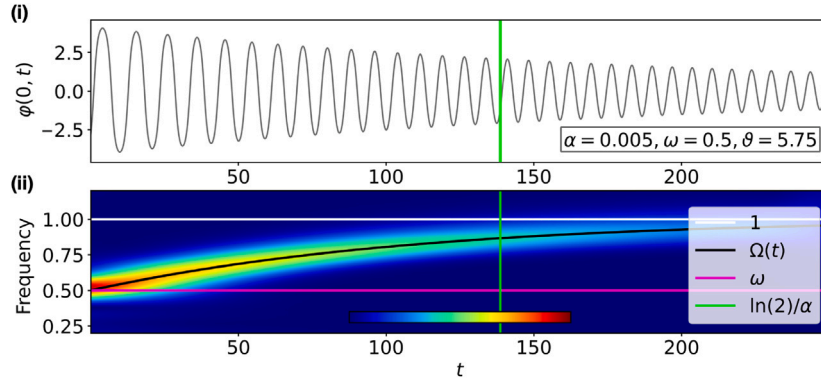


Fig. B.17. Breather dynamics under dissipation. Panel (i): Trajectory of $\phi(x=0, t)$ showing the decay of breather oscillations. Panel (ii): Scalogram of the breather oscillation signal, representing its energy content as a function of time and frequency. The color scale ranges from minimum (blue) to maximum (red) values. The white and fuchsia horizontal lines indicate the plasma frequency and the initial breather frequency, respectively. The black curve depicts the estimated frequency modulation due to dissipation, as given by Eq. (B.7), while the green vertical line marks the time $t = \ln(2)/\alpha$ at which the oscillation amplitude is halved. Simulation parameters for both panels: $t = 50$, $t_f = 250$, $\alpha = 0.005$, $\omega = 0.5$, and $\theta = 5.75$.

Source: Adapted from Ref. [36].

Here, ω (with $0 < \omega < 1$) and θ (with $0 \leq \theta \leq 2\pi$) denote the frequency and phase of the excitation, respectively. For simplicity, we set the velocity to zero to focus solely on the internal dynamics of the breather. The parameter ω determines the pulsation cycle duration $T_{\text{puls}} = 2\pi/\omega$, the energy $E_{sb} = 16\sqrt{1-\omega^2}$, and the amplitude $A_{sb} = 4 \arctan(\sqrt{1-\omega^2}/\omega)$ of the unperturbed nonlinear mode. Consequently, breathers with low frequencies exhibit higher energies and larger oscillation amplitudes, while high-frequency breathers display the opposite behavior.

As demonstrated in prior studies [76,92], dissipative effects cause a breather to decay into radiation over a timescale $\sim 1/\alpha$. This behavior is illustrated in Fig. B.17 for $\alpha = 0.005$, $\omega = 0.5$, and $\theta = 5.75$. Specifically, by using Eqs. (B.3)–(B.4), the decaying breather can be monitored by tracking the SG field at $x = 0$, i.e., $\phi(0, t)$ (Fig. B.17, upper panel). Since no perturbation explicitly depends on x , the breather remains localized at its initial position. The decaying amplitude profile, which closely follows an exponential trend, reveals a time-dependent modulation of the oscillation frequency. To analyze this, we employ the scalogram of $\phi(0, t)$,¹⁰ which quantifies the energy distribution across time and frequency. As shown in Fig. B.17 (lower panel), the breather's frequency gradually converges to the plasma frequency (indicated by the white horizontal line).

To model this behavior, we introduce a time-dependent breather amplitude of the form

$$A_1(t) = 4 \arctan \left(\frac{\sqrt{1 - \Omega(t)^2}}{\Omega(t)} \right), \quad (\text{B.5})$$

where $\Omega(t) < 1$ is a slowly varying frequency parameter. By equating Eq. (B.5) with an exponentially decaying amplitude profile that satisfies the initial condition, i.e.,

$$A_2(t) = 4 \arctan \left(\frac{\sqrt{1 - \omega^2}}{\omega} \right) \exp(-\alpha t), \quad (\text{B.6})$$

and applying standard trigonometric identities, we derive the analytical expression

$$\Omega(t) = \cos \left[(\arccos \omega) \exp(-\alpha t) \right], \quad (\text{B.7})$$

which aligns remarkably well with numerical results (Fig. B.17, lower panel, black curve). This expression, consistent with the McLaughlin–Scott perturbation theory [76], remains valid for arbitrary combinations of α and ω . In Fig. B.17, the parameters $\alpha = 0.005$ and $\omega = 0.5$ are chosen for clarity,¹¹ while $\theta = 5.75$ serves as a representative example.

Appendix C. Energy spatial correlations and timescale of breather formation

The energy density $H(x, t)$ and its spatial correlations serve as fundamental observables for characterizing localized coherent structures [191,192]. In particular, Ref. [38] considers the time average

$$\overline{H}(x) = \frac{1}{t_f} \int_0^{t_f} H(x, t) dt, \quad (\text{C.1})$$

where t_f represents the total observation time. The spatial correlation function,

$$C_{\overline{H}}(\mathcal{X}) \propto \frac{\left\langle \int \overline{H}(x) \overline{H}(x + \mathcal{X}) dx \right\rangle}{\left\langle \int \overline{H}(x) dx \right\rangle^2}, \quad (\text{C.2})$$

is then introduced. Here, $\mathcal{X}$ denotes a spatial displacement, $\langle \dots \rangle$ indicates ensemble averaging, and the integration is performed across the entire system.

As discussed in Ref. [38], Eq. (C.2) exhibits distinct behaviors in three physical regimes:

- **Spatially-uniform state:** In the absence of soliton excitations, energy distributes uniformly, yielding $C_{\overline{H}}(\mathcal{X}) = 1$ (with proper normalization);
- **Stable breather regime:** Persistent breathers create localized $\overline{H}(x)$ peaks of width $\sim \lambda_{sb} = 1/\sqrt{1-\omega^2}$ (the characteristic breather size), producing correlations that decay over this characteristic length scale;
- **Turbulent kink-antikink regime:** While transient solitons appear, their random motion and annihilation restore approximate spatial homogeneity in the long-time average.

Following Ref. [38], the system's response is quantified through $C_{\overline{H}}(\mathcal{X} = \lambda_{sb})$ versus Γ . Fig. C.18(i) demonstrates the nonmonotonic dependence of this quantity on noise strength $\Gamma \in [5 \times 10^{-5}, 2 \times 10^{-1}]$, using parameters $\Delta x = \Delta t = 0.01$, $l = 50$, $t_f = 500$, $\alpha = 0.2$, $\omega = 0.6$,

¹⁰ The scalogram represents the absolute value of the signal's continuous wavelet transform as a function of time and frequency [190]. Here, the Complex Morlet wavelet function is used.

¹¹ These values ensure that the frequency modulation is smoothly distributed over the simulation duration, making the scalogram easier to interpret.

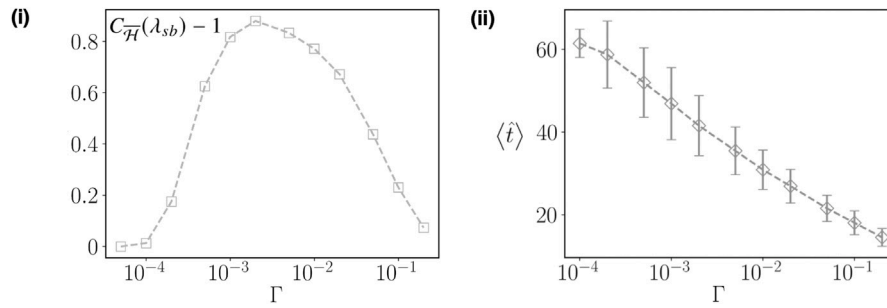


Fig. C.18. Noise-dependent energy correlations and emergence timescales. Panel (i): Spatial correlation coefficient [Eq. (C.2)] versus Γ . Panel (ii): Mean first-appearance time $\langle \hat{t} \rangle$ and its standard deviation versus Γ . Parameters as in panel (i).

Source: Adapted from Ref. [38].

$\eta = 0.59$, and $N = 1000$. The observed trend corresponds precisely to the transition between the three regimes described above.

As emphasized in Ref. [38], moderate noise levels enhance the system's responsiveness to external driving while maintaining coherent structures. The noise amplitude also governs the timescale for soliton formation. Ref. [38] tracks the first-appearance time $\hat{t}$ of excitations exceeding amplitude $A_{sb} = 4 \arctan(\sqrt{1 - \omega^2}/\omega)$ —the characteristic breather amplitude at frequency ω .

Fig. C.18(ii), reproduced from Ref. [38], reveals that stronger noise accelerates soliton emergence, with $\langle \hat{t} \rangle$ decreasing monotonically with Γ . These results confirm that environmental fluctuations play a crucial role in both the spatial organization and temporal dynamics of nonlinear excitations in the system.

Data availability

Data will be made available on request.

References

- [1] Rajaraman R. Solitons and instantons: An introduction to solitons and instantons in quantum field theory. 1982.
- [2] Scott AC. Nonlinear science: emergence and dynamics of coherent structures. Oxford; 2003.
- [3] Giamarchi T. Quantum physics in one dimension. Clarendon Press; 2003.
- [4] Dauxois T, Peyrard M. Physics of solitons. Cambridge University Press; 2006.
- [5] Mussardo G. Statistical field theory: an introduction to exactly solved models in statistical physics. USA: Oxford University Press; 2010.
- [6] Cuevas-Maraver J, Kevrekidis PG, Williams F. The Sine-Gordon model and its applications. Springer; 2014.
- [7] Torrielli A. Integrability using the Sine-Gordon and Thirring duality. IOP Publishing; 2024, p. 2053–563. <http://dx.doi.org/10.1088/978-0-7503-5899-6>.
- [8] Barone A, Esposito F, Magee CJ, Scott AC. Theory and applications of the Sine-Gordon equation. Riv Del Nuovo Cim 1971;1(2):227–67.
- [9] Ustinov AV. Solitons in Josephson junctions. Phys D 1998;123(1–4):315–29.
- [10] Asano T, Nojiri H, Inagaki Y, Boucher JP, Sakon T, Ajiro Y, et al. ESR investigation on the breather mode and the spinon-breather dynamical crossover in Cu benzoate. Phys Rev Lett 2000;84(25):5880.
- [11] Gritsev V, Polkovnikov A, Demler E. Linear response theory for a pair of coupled one-dimensional condensates of interacting atoms. Phys Rev B 2007;75(17):174511.
- [12] Ivancevic VG, Ivancevic TT. Sine-Gordon solitons, kinks and breathers as physical models of nonlinear excitations in living cellular structures. J Geom Symmetry Phys 2013;31:1–56.
- [13] Tiegels AC, Honecker A, Pruschke T, Ponomarev A, Zvyagin SA, Feyerherm R, et al. Dynamical properties of the Sine-Gordon quantum spin magnet Cu-PM at zero and finite temperature. Phys Rev B 2016;93(10):104411.
- [14] Visuri AM, Lebrat M, Häusler S, Corman L, Giamarchi T. Spin transport in a one-dimensional quantum wire. Phys Rev Res 2020;2(2):023062.
- [15] Roy A, Schuricht D, Hauschild J, Pollmann F, Saleur H. The quantum Sine-Gordon model with quantum circuits. Nuclear Phys B 2021;968:115445.
- [16] Kivshar YS, Malomed BA. Dynamics of solitons in nearly integrable systems. Rev Modern Phys 1989;61(4):763.
- [17] Greene PB, Kofman L, Starobinsky AA. Sine-Gordon parametric resonance. Nuclear Phys B 1999;543(1):423–43. [http://dx.doi.org/10.1016/S0550-3213\(99\)00018-8](http://dx.doi.org/10.1016/S0550-3213(99)00018-8).
- [18] Gritsev V, Demler E, Lukin M, Polkovnikov A. Spectroscopy of collective excitations in interacting low-dimensional many-body systems using quench dynamics. Phys Rev Lett 2007;99(20):200404.
- [19] Neuenhahn C, Polkovnikov A, Marquardt F. Localized phase structures growing out of quantum fluctuations in a quench of tunnel-coupled atomic condensates. Phys Rev Lett 2012;109(8):085304.
- [20] Dalla Torre EG, Demler E, Polkovnikov A. Universal rephasing dynamics after a quantum quench via sudden coupling of two initially independent condensates. Phys Rev Lett 2013;110(9):090404.
- [21] Citro R, Dalla Torre EG, D'Alessio L, Polkovnikov A, Babadi M, Oka T, et al. Dynamical stability of a many-body Kapitza pendulum. Ann Physics 2015;360:694–710.
- [22] Kormos M, Zaránd G. Quantum quenches in the Sine-Gordon model: A semiclassical approach. Phys Rev E 2016;93(6):062101.
- [23] Schweigler T, Kasper V, Erne S, Mazets I, Rauer B, Cataldini F, et al. Experimental characterization of a quantum many-body system via higher-order correlations. Nature 2017;545(7654):323–6.
- [24] Schweigler T, Gluzá M, Tajik M, Sotiriadis S, Cataldini F, Ji S-C, et al. Decay and recurrence of non-Gaussian correlations in a quantum many-body system. Nat Phys 2021;17(5):559–63.
- [25] Lovas I, Citro R, Demler E, Giamarchi T, Knap M, Orignac E. Many-body parametric resonances in the driven Sine-Gordon model. Phys Rev B 2022;106:075426. <http://dx.doi.org/10.1103/PhysRevB.106.075426>.
- [26] Koch R, Bastianello A. Exact thermodynamics and transport in the classical Sine-Gordon model. SciPost Phys 2023;15(4):140.
- [27] Wybo E, Bastianello A, Aidelburger M, Bloch I, Knap M. Preparing and analyzing solitons in the Sine-Gordon model with quantum gas microscopes. PRX Quantum 2023;4:030308.
- [28] Bastianello A. Sine-Gordon model from coupled condensates: A generalized hydrodynamics viewpoint. Phys Rev B 2024;109(3):035118.
- [29] Visuri A-M. Sine-Gordon dynamics in spin transport. Phys Rev A 2024;109:053310. <http://dx.doi.org/10.1103/PhysRevA.109.053310>.
- [30] Michael MH, De Santis D, Demler EA, Lee PA. Giant dynamical paramagnetism in the driven pseudogap phase of $\text{YBa}_2\text{Cu}_3\text{O}_{6+x}$. 2024, arXiv preprint arXiv:2410.12919.
- [31] Gulevich DR, Gaifullin MB, Kusmartsev FV. Controlled dynamics of Sine-Gordon breather in long Josephson junctions. Eur Phys J B 2012;85:24.
- [32] Monaco R. Traveling electromagnetic waves in annular Josephson tunnel junctions. Wave Motion 2019;88:214–25. <http://dx.doi.org/10.1016/j.wavemoti.2019.04.010>.
- [33] De Santis D, Guarcello C, Spagnolo B, Carollo A, Valenti D. Generation of travelling Sine-Gordon breathers in noisy long Josephson junctions. Chaos Solitons Fractals 2022;158:112039. <http://dx.doi.org/10.1016/j.chaos.2022.112039>.
- [34] De Santis D, Guarcello C, Spagnolo B, Carollo A, Valenti D. Supratransmission-induced travelling breathers in long Josephson junctions. Commun Nonlinear Sci Numer Simul 2022;106736. <http://dx.doi.org/10.1016/j.cnsns.2022.106736>.
- [35] De Santis D, Guarcello C, Spagnolo B, Carollo A, Valenti D. Magnetically excited breather modes in an overlap-geometry Josephson tunnel junction. Il Nuovo Cim C 2022;45(6):1–4.
- [36] De Santis D, Guarcello C, Spagnolo B, Carollo A, Valenti D. Breather dynamics in a stochastic Sine-Gordon equation: Evidence of noise-enhanced stability. Chaos Solitons Fractals 2023;168:113115. <http://dx.doi.org/10.1016/j.chaos.2023.113115>.
- [37] De Santis D, Guarcello C, Spagnolo B, Carollo A, Valenti D. Noise-enhanced stability of Sine-Gordon breathers. Il Nuovo Cim C 2023;46(4):1–4.
- [38] De Santis D, Guarcello C, Spagnolo B, Carollo A, Valenti D. Ac-locking of thermally-induced Sine-Gordon breathers. Chaos Solitons Fractals 2023;170:113382. <http://dx.doi.org/10.1016/j.chaos.2023.113382>.

- [39] De Santis D, Guarcello C, Spagnolo B, Carollo A, Valenti D. Noise-induced, ac-stabilized Sine-Gordon breathers: Emergence and statistics. *Commun Nonlinear Sci Numer Simul* 2024;131:107796. <http://dx.doi.org/10.1016/j.cnsns.2023.107796>.
- [40] Di Fresco G, De Santis D, Guarcello C, Spagnolo B, Carollo A, Valenti D. Effects of correlated noise on the excitation of robust breathers in an ac-driven, lossy Sine-Gordon system. *Chaos Solitons Fractals* 2024;189:115678. <http://dx.doi.org/10.1016/j.chaos.2024.115678>.
- [41] De Santis D, Spagnolo B, Carollo A, Valenti D, Guarcello C. Heat-transfer fingerprint of Josephson breathers. *Chaos Solitons Fractals* 2024;185:115088. <http://dx.doi.org/10.1016/j.chaos.2024.115088>, <https://www.sciencedirect.com/science/article/pii/S0960077924006404>.
- [42] Josephson BD. Possible new effects in superconductive tunneling. *Phys Lett* 1962;1(7):251–3.
- [43] Anderson PW, Rowell JM. Probable observation of the Josephson superconducting tunneling effect. *Phys Rev Lett* 1963;10:230–2.
- [44] Barone A, Paternò G. Physics and applications of the Josephson effect. New York: Wiley; 1982.
- [45] Likharev KK. Dynamics of Josephson junctions and circuits. Routledge; 1986.
- [46] Orlando TP, Delin KA. foundations of applied superconductivity. Addison-Wesley; 1991.
- [47] Gross R, Marx A. Applied Superconductivity: Josephson Effect and Superconducting Electronics Available at <https://www.wmi.badw.de/teaching/LectureNotes/> (D:20250613).
- [48] Tafuri F. Fundamentals and frontiers of the Josephson effect. Springer Nature; 2019.
- [49] Braginski AI. Superconductor electronics: Status and outlook. *J Supercond Nov Magn* 2019;32:23–44.
- [50] Citro R, Guarcello C, Pagano S. Josephson junctions, superconducting circuits, and qubit for quantum technologies. In: New trends and platforms for quantum technologies. Springer Nature; 2024, p. 1–59. http://dx.doi.org/10.1007/978-3-031-55657-9_1.
- [51] Hamilton CA. Josephson voltage standards. *Rev Sci Instrum* 2000;71(10):3611–23.
- [52] Bando Y, Suta Y, Oshiyama H, Shibata N, Ohzeki M, Gómez-Ruiz FJ, et al. Probing the universality of topological defect formation in a quantum annealer: Kibble–Zurek mechanism and beyond. *Phys Rev Res* 2020;2(3):033369.
- [53] King AD, Suzuki S, Raymond J, Zucca A, Lanting T, Altomare F, et al. Coherent quantum annealing in a programmable 2000qubit ising chain. *Nat Phys* 2022;18(11):1324–8.
- [54] Andersen TI, et al. Thermalization and criticality on an analogue–digital quantum simulator. *Nature* 2024;638:79.
- [55] Kuzmin LS, Sobolev AS, Gatti C, Di Gioacchino D, Crescini N, Gordeeva A, et al. Single photon counter based on a Josephson junction at 14 GHz for searching galactic axions. *IEEE Trans Appl Supercond* 2018;28(7):1–5.
- [56] Chiarello F, Alesini D, Babusci D, Barone C, Beretta M, Buonomo B, et al. Investigation of resonant activation in a Josephson junction for axion search with microwave single photon detection. *IEEE Trans Appl Supercond* 2022;32(4):1–5.
- [57] Grimaudo R, Valenti D, Spagnolo B, Filatrella G, Guarcello C. Josephson-junction-based axion detection through resonant activation. *Phys Rev D* 2022;105:033007.
- [58] Grimaudo R, Valenti D, Filatrella G, Spagnolo B, Guarcello C. Coupled quantum pendula as a possible model for Josephson-junction-based axion detection. *Chaos Solitons Fractals* 2023;173:113745.
- [59] Grimaudo R, Valenti D, Spagnolo B, Troisi A, Filatrella G, Guarcello C. Axion field influence on Josephson junction quasipotential. *Materials* 2023;16(17):5972.
- [60] Pankratov AL, Gordeeva AV, Chiginev AV, et al. Detection of single-mode thermal microwave photons using an underdamped Josephson junction. *Nat Comm* 2025;16:3457. <http://dx.doi.org/10.1038/s41467-025-56040-4>.
- [61] Feynman RP, Leighton RB, Sands M. In: The feynman lectures on physics, vol. 1–3, Massachusetts: Addison-Wesley, Reading; 1964, available online: <https://www.feynmanlectures.caltech.edu/>.
- [62] Landau LD, Lifshitz EM. Lehrbuch der Theoretischen Physik, Bd. IX. Berlin: Akademie-Verlag; 1980.
- [63] Parmentier RD. Solitons and long Josephson junctions. In: Weinstock H, Ralston RW, editors. The new superconducting electronics. Kluwer Academic Publishers; 1993, p. 221–48.
- [64] Pedersen NF. Solitons in Josephson transmission lines. In: Trullinger SE, Zakharov VE, Pokrovsky VL, editors. Solitons. Elsevier; 1986, p. 469–501.
- [65] Lomdahl PS, Soerensen OH, Christiansen PL. Soliton excitations in Josephson tunnel junctions. *Phys Rev B* 1982;25(9):5737.
- [66] Swihart JC. Field solution for a thin-film superconducting strip transmission line. *J Appl Phys* 1961;32:461–9.
- [67] Castellano MG, Torrioli G, Cosmelli C, Costantini A, Chiarello F, Carelli P, et al. Thermally activated escape from the zero-voltage state in long Josephson junctions. *Phys Rev B* 1996;54(21):15417.
- [68] Fedorov KG, Pankratov AL. Mean time of the thermal escape in a current-biased long-overlap Josephson junction. *Phys Rev B* 2007;76:024504.
- [69] Pankratov AL. Noise self-pumping in long Josephson junctions. *Phys Rev B* 2008;78:024515.
- [70] Scott AC. A nonlinear Klein–Gordon equation. *Amer J Phys* 1969;37(1):52–61.
- [71] Fulton TA. Equivalent circuits and analogs of the Josephson effect. In: Schwartz BB, Foner S, editors. Superconductor applications: sQUIDS and machines. Plenum; 1977, p. 125–87.
- [72] Zhu C, Lei J, Wu Y, Li N, Chen D, Shi Q. Experimental observation of solitary waves in a new designed pendulum chain system. *Eur J Phys* 2015;36(4):045002.
- [73] Koers G. Collision of wave packets in the Sine-Gordon equation Available at <https://www.universiteitleiden.nl/binaries/content/assets/science/mi/scripties/koers.pdf> (D:20250613).
- [74] Hirota R. Exact solution of the Sine-Gordon equation for multiple collisions of solitons. *J Phys Soc Japan* 1972;33(5):1459–63.
- [75] Ablowitz MJ, Kaup DJ, Newell AC, Segur H. Method for solving the Sine-Gordon equation. *Phys Rev Lett* 1973;30:1262–4.
- [76] McLaughlin DW, Scott AC. Perturbation analysis of fluxon dynamics. *Phys Rev A* 1978;18:1652–80.
- [77] Guarcello C, Solinas P, Di Ventra M, Giazotto F. Solitonic Josephson-based meminductive systems. *Sci Rep* 2017;7(1):1–7.
- [78] Guarcello C, Solinas P, Braggio A, Giazotto F. Solitonic Josephson thermal transport. *Phys Rev Appl* 2018;9:034014. <http://dx.doi.org/10.1103/PhysRevApplied.9.034014>.
- [79] Guarcello C, Solinas P, Braggio A, Giazotto F. Phase-coherent solitonic Josephson heat oscillator. *Sci Rep* 2018;8(1):1–12.
- [80] Wustmann W, Osborn KD. Reversible fluxon logic: Topological particles allow ballistic gates along one-dimensional paths. *Phys Rev B* 2020;101:014516. <http://dx.doi.org/10.1103/PhysRevB.101.014516>.
- [81] Osborn KD, Wustmann W. Reversible fluxon logic with optimized CNOT gate components. *IEEE Trans Appl Supercond* 2021;31(2):1–13.
- [82] Wallraff A, Lukashenko A, Lisenfeld J, Kemp A, Fistul MV, Koval Y, et al. Quantum dynamics of a single vortex. *Nature* 2003;425(6954):155–8.
- [83] Lomdahl PS, Samuelsen MR. Persistent breather excitations in an ac-driven Sine-Gordon system with loss. *Phys Rev A* 1986;34(1):664.
- [84] Lomdahl PS, Samuelsen MR. Modulation of the breather frequency in the ac-driven Sine-Gordon system with loss. *Phys Lett A* 1988;128(8):427–32.
- [85] Scott AC. Sine-Gordon breather dynamics. *Phys Scr* 1979;20(3–4):509.
- [86] Lomdahl PS, Olsen OH, Samuelsen MR. Breather–kink-antikink-pair conversion in the driven Sine-Gordon system. *Phys Rev A* 1984;29:350–4.
- [87] Grønbech-Jensen N, Kivshar YS, Samuelsen MR. Stabilization of breathers in a parametrically driven Sine-Gordon system with loss. *Phys Rev B* 1991;43:5698–701.
- [88] Grønbech-Jensen N, Malomed BA, Samuelsen MR. Stability of breathers in the ac-driven Sine-Gordon system. *Phys Lett A* 1992;166(5):347–51. [http://dx.doi.org/10.1016/0375-9601\(92\)90720-7](http://dx.doi.org/10.1016/0375-9601(92)90720-7).
- [89] Grauer R, Kivshar YS. Chaotic and phase-locked breather dynamics in the damped and parametrically driven sine-Gordon equation. *Phys Rev E* 1993;48:4791–800.
- [90] Pascual PJ, Jiménez S, Vázquez L. Numerical simulations of a nonlinear Klein–Gordon model. Applications. In: Garrido PL, Marro J, editors. Third granada lectures in computational physics. Springer Berlin Heidelberg; 1995, p. 211–70.
- [91] Jiménez S, Pascual P, Aguirre C, Vázquez L. A panoramic view of some perturbed nonlinear wave equations. *Int J Bifurcat Chaos* 2004;14(01):1–40.
- [92] Gulevich DR, Kusmartsev FV. Perturbation theory for localized solutions of the Sine-Gordon equation: Decay of a breather and pinning by a microresistor. *Phys Rev B* 2006;74:214303.
- [93] Johnson S, Biswas A. Breather dynamics of the Sine-Gordon equation. *Commun Theor Phys (Beijing)* 2013;59(6):664–70.
- [94] Flach S, Willis CR. Discrete breathers. *Phys Rep* 1998;295:181–264.
- [95] Flach S, Gorbach AV. Discrete breathers—advances in theory and applications. *Phys Rep* 2008;467(1–3):1–116.
- [96] Braun OM, Kivshar YS. The Frenkel-Kontorova model: concepts, methods, and applications. Springer Science & Business Media; 2004.
- [97] Chelpanova O, Kelly SP, Morigi G, Schmidt-Kaler F, Marino J. Injection and nucleation of topological defects in the quench dynamics of the Frenkel-Kontorova model. *Europhys Lett* 2023;143(2):25002. <http://dx.doi.org/10.1209/0295-5075/ace27d>.
- [98] Mazo JJ, Orlando TP. Discrete breathers in Josephson arrays. *Chaos* 2003;13(2):733–43.
- [99] Trías E, Mazo JJ, Orlando TP. Discrete breathers in nonlinear lattices: Experimental detection in a Josephson array. *Phys Rev Lett* 2000;84:741–4. <http://dx.doi.org/10.1103/PhysRevLett.84.741>.
- [100] Binder P, Abramov D, Ustinov AV, Flach S, Zolotaryuk Y. Observation of breathers in Josephson ladders. *Phys Rev Lett* 2000;84:745–8.
- [101] Geniet F, Leon J. Energy transmission in the Forbidden band gap of a nonlinear chain. *Phys Rev Lett* 2002;89(13):134102.
- [102] Geniet F, Leon J. Nonlinear supratransmission. *J Phys: Condens Matter* 2003;15:2933–49.
- [103] Leon J. Nonlinear supratransmission as a fundamental instability. *Phys Lett A* 2003;319:130–6.

- [104] Leon J, Spire A. Gap soliton formation by nonlinear supratransmission in Bragg media. *Phys Lett A* 2004;327(5–6):474–80.
- [105] Leon J. Nonlinear tunneling in a fiber guide array resonator. *Phys Rev E* 2004;70(5):056604.
- [106] Khomeriki R. Nonlinear band gap transmission in optical waveguide arrays. *Phys Rev Lett* 2004;92(6):063905.
- [107] Togueu Motcheyo AB, Tchinnang Tchameu JD, Siewe Siewe M, Tchawoua C. Homoclinic nonlinear band gap transmission threshold in discrete optical waveguide arrays. *Commun Nonlinear Sci Numer Simul* 2017;50:29–34. <http://dx.doi.org/10.1016/j.cnsns.2017.02.001>.
- [108] Khomeriki R, Lepri S, Ruffo S. Nonlinear supratransmission and bistability in the Fermi-Pasta-Ulam model. *Phys Rev E* 2004;70(6):066626.
- [109] Macías-Díaz JE, Bountis A. Supratransmission in β -Fermi-Pasta-Ulam chains with different ranges of interactions. *Commun Nonlinear Sci Numer Simul* 2018;63:307–21. <http://dx.doi.org/10.1016/j.cnsns.2018.04.007>.
- [110] Motcheyo ABT, Macías-Díaz JE. Energy transmission in the forbidden band-gap of a nonlinear chain with global interactions. *J Phys A Math* 2020;53(50):505701. <http://dx.doi.org/10.1088/1751-8121/abc3fd>.
- [111] Liazkhov S. Energy supply into a semi-infinite Fermi-Pasta-Ulam-Tsingou chain by periodic force loading. *Acta Mech* 2024;235:4005–27.
- [112] Bodo B, Morfu S, Marquié P, Rosse M. Klein-Gordon electronic network exhibiting supratransmission effect. *Electron Lett* 2010;46(2):123–4.
- [113] Koon KTV, Leon J, Marquié P, Tchofo-Dinda P. Cutoff solitons and bistability of the discrete inductance-capacitance electrical line: Theory and experiments. *Phys Rev E* 2007;75(6):066604.
- [114] Togueu Motcheyo AB, Tchawoua C, Tchinnang Tchameu JD. Supratransmission induced by waves collisions in a discrete electrical lattice. *Phys Rev E* 2013;88:040901. <http://dx.doi.org/10.1103/PhysRevE.88.040901>.
- [115] Anghel-Vasilescu P, Dorignac J, Geniet F, Leon J, Taki M. Nonlinear supratransmission in multicomponent systems. *Phys Rev Lett* 2010;105(7):074101.
- [116] Pechac JE, Frazier MJ. Non-reciprocal supratransmission in mechanical lattices with non-local feedback control interactions. *Crystals* 2021;11(2). <http://dx.doi.org/10.3390/cryst11020094>.
- [117] Zhang Q, Fang H, Xu J. Programmable stopbands and supratransmission effects in a stacked Miura-origami metastructure. *Phys Rev E* 2020;101:042206. <http://dx.doi.org/10.1103/PhysRevE.101.042206>.
- [118] Chevriaux D, Khomeriki R, Leon J. Theory of a Josephson junction parallel array detector sensitive to very weak signals. *Phys Rev B* 2006;73:214516.
- [119] Macías-Díaz JE, Puri A. On the transmission of binary bits in discrete Josephson-junction arrays. *Phys Lett A* 2008;372(30):5004–10.
- [120] Khomeriki R, Leon J. Fluxon dynamics in boundary driven Josephson transmission line. *J Phys Conf Ser* 2009;150(2):022035.
- [121] Macías-Díaz JE, Togueu Motcheyo AB. Energy transmission in nonlinear chains of harmonic oscillators with long-range interactions. *Results Phys* 2020;18:103210. <http://dx.doi.org/10.1016/j.rinp.2020.103210>.
- [122] Schlawin F, Dietrich ASD, Kiffner M, Cavalleri A, Jaksch D. Terahertz field control of interlayer transport modes in cuprate superconductors. *Phys Rev B* 2017;96:064526.
- [123] Schlawin F, Dietrich ASD, Jaksch D. Optical control of the current-voltage relation in stacked superconductors. *Phys Rev B* 2019;100:134510.
- [124] Dienst A, Casandru E, Fausti D, Zhang L, Eckstein M, Hoffmann M, et al. Optical excitation of Josephson plasma solitons in a cuprate superconductor. *Nat Mater* 2013;12(6):535–41.
- [125] Laplace Y, Cavalleri A. Josephson plasmonics in layered superconductors. *Adv Phys X* 2016;1(3):387–411. <http://dx.doi.org/10.1080/23746149.2016.1212671>.
- [126] Bodo B, Morfu S, Marquié P, Essimbi BZ. Noise induced breather generation in a Sine-Gordon chain. *J Stat Mech* 2009;P01026.
- [127] Bodo B, Morfu S. Noise effect in a Sine-Gordon lattice. In: 2013 22nd international conference on noise and fluctuations. 2013, p. 1–4.
- [128] Motcheyo ABT, Kimura M, Doi Y, Archilla JF. Nonlinear bandgap transmission by discrete rogue waves induced in a pendulum chain. *Phys Lett A* 2024;497:129334.
- [129] Soriano C, Costabile G, Parmentier RD. Coupling of josephson flux-flow oscillators to an external RC load. *Supercond Sci Technol* 1996;9(7):578. <http://dx.doi.org/10.1088/0953-2048/9/7/011>.
- [130] Pankratov AL. Minimizing the linewidth of the flux-flow oscillator. *Appl Phys Lett* 2008;92(8):082504. <http://dx.doi.org/10.1063/1.2839605>.
- [131] Wallraff A, Koval Y, Levitchev M, Fistul MV, Ustinov AV. Annular long josephson junctions in a magnetic field: engineering and probing the fluxon interaction potential. *J Low Temp Phys* 2000;118(5):543–53.
- [132] Costabile G, Parmentier RD, Savo B, McLaughlin DW, Scott AC. Exact solutions of the sine-Gordon equation describing oscillations in a long (but finite) Josephson junction. *Appl Phys Lett* 1978;32(9):587–9. <http://dx.doi.org/10.1063/1.90113>.
- [133] Cirillo M, Parmentier RD, Savo B. Mechanical analog studies of a perturbed Sine-Gordon equation. *Phys D* 1981;3(3):565–76.
- [134] Giazotto F, Martínez-Pérez MJ. The Josephson heat interferometer. *Nature* 2012;492(7429):401–5. <http://dx.doi.org/10.1038/nature11702>.
- [135] Martínez-Pérez MJ, Giazotto F. A quantum diffractor for thermal flux. *Nat Comm* 2014;5(1):3579.
- [136] Kuzmin L. Ultra-sensitive cryogenic thermometer based on an array of the SIN tunnel junctions. *Physica C* 2008;468(3):142–6. <http://dx.doi.org/10.1016/j.physc.2007.10.024>.
- [137] Giazotto F, Solinas P, Braggio A, Bergeret FS. Ferromagnetic-insulator-based superconducting junctions as sensitive electron thermometers. *Phys Rev Appl* 2015;4:044016. <http://dx.doi.org/10.1103/PhysRevApplied.4.044016>.
- [138] Zgirski M, Foltyn M, Savin A, Norowski K, Meschke M, Pekola J. Nanosecond thermometry with Josephson junctions. *Phys Rev Appl* 2018;10:044068. <http://dx.doi.org/10.1103/PhysRevApplied.10.044068>.
- [139] Guarcello C, Braggio A, Solinas P, Giazotto F. Nonlinear critical-current thermal response of an asymmetric Josephson tunnel junction. *Phys Rev Appl* 2019;11:024002. <http://dx.doi.org/10.1103/PhysRevApplied.11.024002>.
- [140] Hofer PP, Perarnau-Llobet M, Brask JB, Silva R, Huber M, Brunner N. Autonomous quantum refrigerator in a circuit QED architecture based on a Josephson junction. *Phys Rev B* 2016;94:235420. <http://dx.doi.org/10.1103/PhysRevB.94.235420>.
- [141] Solinas P, Bosisio R, Giazotto F. Microwave quantum refrigeration based on the Josephson effect. *Phys Rev B* 2016;93:224521. <http://dx.doi.org/10.1103/PhysRevB.93.224521>.
- [142] Gordeeva AV, Zbrozhek VO, Pankratov AL, Revin LS, Shamporov VA, Gunbina AA, et al. Observation of photon noise by cold-electron bolometers. *Appl Phys Lett* 2017;110(16):162603. <http://dx.doi.org/10.1063/1.4982031>.
- [143] Gordeeva AV, Pankratov AL, Pugach NG, Vasenko AS, Zbrozhek VO, Blagodatkin AV, et al. Record electron self-cooling in cold-electron bolometers with a hybrid superconductor-ferromagnetic nanoabsorber and traps. *Sci Rep* 2020;10(1):21961.
- [144] Pimanov DA, Frost VA, Blagodatkin AV, Gordeeva AV, Pankratov AL, Kuzmin LS. Efficiency of electron cooling in cold-electron bolometers with traps. *Beilstein J Nanotechnol* 2022;13:896–901. <http://dx.doi.org/10.3762/bjnano.13.80>.
- [145] Hwang S-Y, Sothmann B, Sánchez D. Superconductor-quantum dot hybrid coolers. *Phys Rev B* 2023;107:245412. <http://dx.doi.org/10.1103/PhysRevB.107.245412>.
- [146] Guarcello C, Solinas P, Braggio A, Di Ventra M, Giazotto F. Josephson thermal memory. *Phys Rev Appl* 2018;9:014021. <http://dx.doi.org/10.1103/PhysRevApplied.9.014021>.
- [147] Ligato N, Paolucci F, Strambini E, Giazotto F. Thermal superconducting quantum interference proximity transistor. *Nat Phys* 2022;18(6):627–32. <http://dx.doi.org/10.1038/s41567-022-01578-z>.
- [148] Scharf B, Braggio A, Strambini E, Giazotto F, Hankiewicz EM. Topological Josephson heat engine. *Commun Phys* 2020;3(1):198. <http://dx.doi.org/10.1038/s42005-020-00463-6>.
- [149] Cavaliere F, Razzoli L, Carrega M, Benenti G, Sassetti M. Hybrid quantum thermal machines with dynamical couplings. *IScience* 2023;26(3). <http://dx.doi.org/10.1016/j.isci.2023.106235>.
- [150] Timossi GF, Fornieri A, Paolucci F, Puglia C, Giazotto F. Phase-tunable Josephson thermal router. *Nano Lett* 2018;18(3):1764–9. <http://dx.doi.org/10.1021/acs.nanolett.7b04906>.
- [151] Hwang S-Y, Giazotto F, Sothmann B. Phase-coherent heat circulator based on multiterminal Josephson junctions. *Phys Rev Appl* 2018;10:044062. <http://dx.doi.org/10.1103/PhysRevApplied.10.044062>.
- [152] Acciai M, Hajiloo F, Hassler F, Splettstoesser J. Phase-coherent heat circulators with normal or superconducting contacts. *Phys Rev B* 2021;103:085409. <http://dx.doi.org/10.1103/PhysRevB.103.085409>.
- [153] Sothmann B, Giazotto F, Hankiewicz EM. High-efficiency thermal switch based on topological Josephson junctions. *New J Phys* 2017;19(2):023056. <http://dx.doi.org/10.1088/1367-2630/aa60d4>.
- [154] Guarcello C, Giazotto F, Solinas P. Coherent diffraction of thermal currents in long Josephson tunnel junctions. *Phys Rev B* 2016;94:054522. <http://dx.doi.org/10.1103/PhysRevB.94.054522>.
- [155] Guarcello C, Braggio A, Solinas P, Pepe GP, Giazotto F. Josephson-threshold calorimeter. *Phys Rev Appl* 2019;11:054074. <http://dx.doi.org/10.1103/PhysRevApplied.11.054074>.
- [156] Paolucci F. Nonlocal superconducting single-photon detector. *Phys Rev Appl* 2023;20:014003. <http://dx.doi.org/10.1103/PhysRevApplied.20.014003>.
- [157] Guarcello C, Solinas P, Giazotto F, Braggio A. Thermal flux-flow regime in long Josephson tunnel junctions. *J Stat Mech Theory Exp* 2019;2019(8):084006. <http://dx.doi.org/10.1088/1742-5468/ab3194>.
- [158] Timofeev AV, García CP, Kopnin NB, Savin AM, Meschke M, Giazotto F, et al. Recombination-limited energy relaxation in a Bardeen-Cooper-Schrieffer superconductor. *Phys Rev Lett* 2009;102:017003. <http://dx.doi.org/10.1103/PhysRevLett.102.017003>.
- [159] Fornieri A, Timossi G, Virtanen P, Solinas P, Giazotto F. $0-\pi$ phase-controllable thermal Josephson junction. *Nat Nanotechnol* 2017;12(5):425–9.
- [160] Maki K, Griffin A. Entropy transport between two superconductors by electron tunneling. *Phys Rev Lett* 1965;15:921–3. <http://dx.doi.org/10.1103/PhysRevLett.15.921>.

- [161] Golubev D, Faivre T, Pekola JP. Heat transport through a Josephson junction. *Phys Rev B* 2013;87:094522. <http://dx.doi.org/10.1103/PhysRevB.87.094522>.
- [162] Fritzsche L, Köhler H-J, Thrum F, Wende G, Meyer H-G. Preparation of Nb/Al- AlO_x /Nb Josephson junctions with critical current densities down to 1 A/cm². *Physica C* 1998;296(3–4):319–24.
- [163] Schmidt DR, Yung CS, Cleland AN. Temporal measurement of hot-electron relaxation in a phonon-cooled metal Island. *Phys Rev B* 2004;69(14):140301.
- [164] Fornieri A, Giazotto F. Towards phase-coherent caloritronics in superconducting circuits. *Nat Nanotechnol* 2017;12(10):944–52. <http://dx.doi.org/10.1038/nnano.2017.204>.
- [165] Pekola JP, Karimi B. Colloquium: Quantum heat transport in condensed matter systems. *Rev Modern Phys* 2021;93:041001. <http://dx.doi.org/10.1103/RevModPhys.93.041001>.
- [166] Gümüş E, Majidi D, Nikolić D, Raif P, Karimi B, Peltonen JT, et al. Calorimetry of a phase slip in a Josephson junction. *Nat Phys* 2023;19(2):196–200.
- [167] Žalohar J, Vičič B, Potočnik M, Soklič N, Komac M, Hölscher T, et al. Breather-type oscillations of the global tectonic shear stress fields. *J Struct Geol* 2020;140:104185. <http://dx.doi.org/10.1016/j.jsg.2020.104185>.
- [168] Liu X, Wen M, Mao X, Wang X. Optimal and efficient generation of Sine-Gordon breathers. *Phys Rev E* 2021;104:014209. <http://dx.doi.org/10.1103/PhysRevE.104.014209>.
- [169] Macías-Díaz JE, Puri A. An application of nonlinear supratransmission to the propagation of binary signals in weakly damped, mechanical systems of coupled oscillators. *Phys Lett A* 2007;366:447–50.
- [170] Fujii T, Nishida M, Hatakenaka N. Mobile qubits in quantum Josephson circuits. *Phys Rev B* 2008;77:024505.
- [171] Krasnov VM. Terahertz electromagnetic radiation from intrinsic Josephson junctions at zero magnetic field via breather-type self-oscillations. *Phys Rev B* 2011;83:174517. <http://dx.doi.org/10.1103/PhysRevB.83.174517>.
- [172] Bastrakova M, Pashin D, Pikunov P, Soloviev I, Schegolev A, Klenov N. Josephson flying qubit revival: Flux-based control optimization. *Chaos Solitons Fractals* 2025;196:116353.
- [173] Chechkin AV, Sliusarenko OY, Metzler R, Klafter J. Barrier crossing driven by Lévy noise: Universality and the role of noise intensity. *Phys Rev E* 2007;75(4):041101.
- [174] Afek G, Davidson N, Kessler DA, Barkai E. Colloquium: Anomalous statistics of laser-cooled atoms in dissipative optical lattices. *Rev Modern Phys* 2023;95(3):031003.
- [175] Baule A, Sollich P. Exponential increase of transition rates in metastable systems driven by non-Gaussian noise. *Sci Rep* 2023;13(1):3853.
- [176] Dubkov AA, Guarcello C, Spagnolo B. Enhancement of stability of metastable states in the presence of Lévy noise. *SciPost Phys* 2025;18:006.
- [177] Guarcello C, Valenti D, Augello G, Spagnolo B. The role of non-Gaussian sources in the transient dynamics of long Josephson junctions. *Acta Phys Pol B* 2013;44(5).
- [178] Guarcello C, Filatrella G, Spagnolo B, Pierro V, Valenti D. Voltage drop across Josephson junctions for Lévy noise detection. *Phys Rev Res* 2020;2(4):043332.
- [179] Guarcello C. Lévy noise effects on Josephson junctions. *Chaos Solitons Fractals* 2021;153:111531. <http://dx.doi.org/10.1016/j.chaos.2021.111531>.
- [180] Guarcello C, Filatrella G, De Santis D, Spagnolo B, Valenti D. Lévy noise-induced effects in a long Josephson junction in the presence of two different spatial noise distributions. *Chaos Solitons Fractals* 2024;187:115421. <http://dx.doi.org/10.1016/j.chaos.2024.115421>.
- [181] De Santis D, et al. Ongoing work. 2025.
- [182] Federov KG, Pankratov AL, Spagnolo B. Influence of length on the noise delayed switching of long Josephson junctions. *Int J Bifurcat Chaos* 2008;18(09):2857–62.
- [183] Augello G, Valenti D, Pankratov AL, Spagnolo B. Lifetime of the superconductive state in short and long Josephson junctions. *Eur Phys J B* 2009;70(1):145–51. <http://dx.doi.org/10.1140/epjb/e2009-00155-x>.
- [184] Pankratov AL, Gordeeva AV, Kuzmin LS. Drastic suppression of noise-induced errors in underdamped long Josephson junctions. *Phys Rev Lett* 2012;109:087003.
- [185] Guarcello C, Valenti D, Carollo A, Spagnolo B. Stabilization effects of dichotomous noise on the lifetime of the superconducting state in a long Josephson junction. *Entropy* 2015;17:2862–75.
- [186] Gordeeva AV, Pankratov AL. Defect formation in long Josephson junctions. *Phys Rev B* 2010;81:212504. <http://dx.doi.org/10.1103/PhysRevB.81.212504>.
- [187] Veenstra J, Gamayun O, Guo X, Sarvi A, Meinersen CV, Coullais C. Non-reciprocal topological solitons in active metamaterials. *Nature* 2024;627(8004):528–33. <http://dx.doi.org/10.1038/s41586-024-07097-6>.
- [188] Brandenbourger M, Gamayun O, Veenstra J, van Gorp F, Terwisscha-Dekker H, Caux J, et al. Non-reciprocal breathing solitons. 2024, arXiv preprint arXiv: 2405.10562.
- [189] Pinto-Ramos D, Alfaro-Bittner K, Rojas RG, Clerc MG. Noise-induced nonreciprocal topological dissipative solitons in directionally coupled chains and lattices. *Nonlinear Dynam* 2025.
- [190] Sauv   A. Scaleogram – tool for 1D data analysis with continuous wavelet transform built on the PyWavelets library. 2019, <https://pypi.org/project/scaleogram/>.
- [191] Brown DW, Bernstein LJ, Lindenberg K. Stochastic localization. *Phys Rev E* 1996;54:3352–60. <http://dx.doi.org/10.1103/PhysRevE.54.3352>.
- [192] Daumont I, Dauxois T, Peyrard M. Modulational instability: first step towards energy localization in nonlinear lattices. *Nonlinearity* 1997;10(3):617. <http://dx.doi.org/10.1088/0951-7715/10/3/003>.