



# A Family of Partial Cubes with Minimal Fibonacci Dimension

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## Abstract

A partial cube  $G$  is a graph that admits an isometric embedding into some hypercube  $Q_k$ . This implies that vertices of  $G$  can be labeled with binary words of length  $k$  in a way that the distance between two vertices in the graph corresponds to the Hamming distance between their labels. The minimum  $k$  for which this embedding is possible is called the isometric dimension of  $G$ , denoted  $idim(G)$ . A Fibonacci cube  $\Gamma_k$  is the partial cube obtained by deleting all the vertices in  $Q_k$  whose labels contain word 11 as factor. It turns out that any partial cube can be always isometrically embedded also in a Fibonacci cube  $\Gamma_d$ . The minimum  $d$  is called the Fibonacci dimension of  $G$ , denoted  $fdim(G)$ . In general,  $idim(G) \leq fdim(G) \leq 2 \cdot idim(G) - 1$ . Despite there is a quadratic algorithm to compute the isometric dimension of a graph, the problem of checking, for a given  $G$ , whether  $idim(G) = fdim(G)$  is in general NP-complete. An important family of graphs for which this happens are the trees. We consider a kind of generalized Fibonacci cubes that were recently defined. They are the subgraphs of the hypercube  $Q_k$  that include only vertices that avoid words in a given set  $S$  and are referred to as  $Q_k(S)$ . We prove some conditions on the words in  $S$  to obtain a family of partial cubes with minimal Fibonacci dimension equal to the isometric dimension.

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## 1 Introduction

The hypercube  $Q_k$ , also known as  $k$ -dimensional cube, extends the concept of a cube into higher dimensions. It is an undirected graph with  $2^k$  vertices corresponding to the binary strings of length  $n$  that describe the corresponding positions in the  $k$ -dimensional space. Then, the distances in this graph  $Q_k$  can be simply calculated by taking the *Hamming distance* (i.e. the number of bits in which they differ) between the corresponding strings.

A given undirected graph is a *partial cube* if it can be *isometrically embedded* into a hypercube. Or, in other words, if it admits a vertex-labeling scheme such that each vertex is labeled by a binary string in such a way that the distance between any two vertices equals the Hamming distance. Partial cubes have been shown to model a large variety of systems, from computational geometry to order theory, interconnection networks, and chemistry [23].

Partial cubes were characterized in 1973 by Djokovic [20] in terms of structural properties; in particular, he proved that partial cubes are bipartite graphs satisfying special properties on some classes of vertices. Later, in 1984, Winkler [35] gave a similar condition based on the existence of a special equivalence relation among the edges of the graph.

Not surprisingly, partial cubes, in general, admit more efficient algorithms than arbitrary graphs for several important problems including shortest paths and graph drawings. A quadratic algorithm for recognizing partial cubes was given by Eppstein [23] and is based on the computation of the Winkler relation's classes. If the graph is a partial cube, Eppstein's algorithm outputs one of the minimal length labelings for the vertices. It turns out that such minimal length is in fact the number of the Winkler equivalence classes on the set of graph edges.

In 1993, Hsu [24] introduced the *Fibonacci cube*  $\Gamma_k$  as the subgraph of the hypercube  $Q_k$  that includes only the vertices corresponding to Fibonacci strings (i.e. strings with no factor 11). As a consequence, the number of vertices of  $\Gamma_k$  is the  $(k + 2)$ -th Fibonacci number, and the graph itself has a recursive definition. Besides the remarkable combinatorial properties,  $\Gamma_k$  is also a partial cube, and therefore the distance between any pair of vertices coincides with the Hamming distance of their labels.

The *Fibonacci dimension* of a graph  $G$  was introduced in 2011 by Cabello, Eppstein and Klavzar [14] as the smallest integer  $d$  (denoted  $fdim(G)$ ) such that the graph admits an isometric embedding into the  $d$ -dimensional Fibonacci cube  $\Gamma_d$ . The authors take inspiration from the *isometric dimension* of a graph  $G$  defined as the smallest integer  $k$  (denoted  $idim(G)$ ) such that  $G$  can be isometrically embedded in the hypercube  $Q_k$ . The isometric dimension is finite if and only if  $G$  is a partial cube and, in this case, it effectively corresponds to the number of equivalence classes of the Winkler relation. In the same paper, they show that a graph  $G$  can be embedded in a Fibonacci Cube if and only if  $G$  is a partial cube and proved the relation

$$idim(G) \leq fdim(G) \leq 2 idim(G) - 1.$$

Besides those bounds, they study which partial cubes have extremal Fibonacci dimension. Partial cubes whose Fibonacci dimension is as large as possible are fully characterized.

The opposite case is difficult to deal with; more precisely it is proved that it is NP-complete to decide whether the isometric and Fibonacci dimensions of a given graph are the same. In addition, they show, however, that if  $T$  is a tree with  $n$  vertices, then  $T$  is a partial cube with  $idim(T) = fdim(T) = n - 1$ .

In this paper, we present a class of graphs (partial cubes) with minimal Fibonacci dimension, namely graphs for which  $idim$  and  $fdim$  coincide.

To introduce such graphs we do a step back to Fibonacci cubes. Inspired by those cubes in fact, in 2012, generalized Fibonacci cubes  $Q_k(f)$  were introduced as the subgraphs of  $Q_k$  that include only the vertices associated to binary words that do not contain the word  $f$  as a factor. It was proved that  $Q_k(f)$  is an isometric subgraph of the hypercube  $Q_k$  if and only if  $f$  satisfies some combinatorial conditions on the overlaps; in this case  $f$  is referred to as a *good* word, later called also an isometric word [26, 28, 29, 32, 33]. The notion of isometric word combines the distance notion with the property that a word does not appear as a factor in other words. Note that this property is important in combinatorics as well as in the investigation on similarities, or distances, on DNA sequences, where the avoided factor is referred to as an absent or forbidden word [13, 16, 17, 18, 22]. The research on isometric words is still very active [9, 12, 21, 30, 31]. Furthermore, some generalizations have been proposed by enlarging the alphabet, by considering different edit distances, and by moving to two-dimensional words as in [1, 2, 5, 6, 7, 8, 10, 11, 4, 15]. Moreover, very recently in [3] the intersection of several graphs  $Q_k(f_i)$  has been investigated in order to establish conditions for being isometric subgraphs of the hypercube  $Q_k$ . Note that this corresponds to take a set of words  $S = \{f_1, \dots, f_m\}$  and consider the graph  $Q_k(S)$  obtained from  $Q_k$  by deleting all the vertices whose labels contain some  $f_i \in S$  as factors.

In this paper we prove first that the isometric dimension of the partial cubes of the form  $Q_k(f)$  is always  $k$ , for any isometric word  $f$  of length at least 2. Moving on to sets, by definition, isometric sets  $S$  generate isometric subgraphs (and, consequently, partial cubes)  $Q_k(S)$  of  $Q_k$ , but non-isometric sets act differently from non-isometric words. We present a set  $S$  that is not isometric by showing that the graph  $Q_4(S)$  is not an isometric subgraph of  $Q_4$ , but it is a partial cube since it can be embedded in  $Q_5$  and, more specifically,  $idim(Q_4(f)) = fdim(Q_4(S)) = 5$ .

As the main result of the paper, we present a class of graphs of the form  $Q_k(S)$  with  $S = \{11, f\}$  and give sufficient conditions on  $f$  to get partial cubes  $G$  such that  $idim(G) = fdim(G) = k$ . Finally, this class is compared with the other known family of graphs with minimal Fibonacci dimension, namely the family of trees. It turns out that the two families are incomparable under inclusion and have a non-empty intersection.

We conclude the paper with a glimpse into the future problems we plan to investigate.

## 2 Basic notation and results

In this section, we collect all the notation together with all the related results needed to present our main theorems, which combine properties on graphs and strings.

### Isometric subgraphs and partial cubes

Let  $G = (V(G), E(G))$  be a graph. The distance of two nodes  $u, v \in V(G)$ , denoted by  $\text{dist}_G(u, v)$ , is the length of the shortest path connecting  $u$  and  $v$  in  $G$ . A subgraph  $G'$  of a (connected) graph  $G$  is an *isometric subgraph* of  $G$  if for any  $u, v \in V(G')$ ,  $\text{dist}_{G'}(u, v) = \text{dist}_G(u, v)$ .

Let  $G, G'$  two graphs. An *isometric embedding*  $\beta$  of  $G'$  in  $G$  is a map  $\beta : V(G') \rightarrow V(G)$  which preserves distances, i.e., for any  $u, v \in V(G')$ ,  $\text{dist}_{G'}(u, v) = \text{dist}_G(\beta(u), \beta(v))$ . If an isometric embedding  $\beta$  of  $G'$  in  $G$  does exist, then we say that  $G'$  can be *isometrically embedded into*  $G$  and write  $G' \trianglelefteq_\beta G$ . The symbol  $\trianglelefteq$  is used to indicate that an isometric embedding does exist.

Recall that, given two binary strings  $x$  and  $y$ , their *Hamming distance*,  $\text{dist}_H(x, y)$  is the number of positions at which  $x$  and  $y$  differ.

The  $k$ -hypercube, or binary  $k$ -cube,  $Q_k$ , is a graph with  $2^k$  vertices, each associated to a binary word of length  $k$ , called its *label*. The label map, denoted by  $\lambda_k$ , of the  $k$ -hypercube  $Q_k$  is based on the Hamming distance, i.e., it is such that two vertices  $u$  and  $v$  are adjacent if and only if  $\text{dist}_H(\lambda_k(u), \lambda_k(v)) = 1$ . This implies that the distance between any two vertices  $u$  and  $v$  is simply given by the Hamming distances between the corresponding labels, i.e.  $\text{dist}_{Q_k}(u, v) = \text{dist}_H(\lambda_k(u), \lambda_k(v))$ .

A graph  $G$  is a *partial cube* if  $G$  can be isometrically embedded into a hypercube  $Q_k$ , for some  $k$ . The relevant fact is that such isometric embedding makes  $G$  inherit from  $Q_k$  a labeling of its vertices by strings in  $\Sigma^k$  that makes the distance of the vertices in  $G$  equal to the Hamming distance between their corresponding labels. Note that all the isometric subgraphs of the hypercubes are partial cubes by definition.

The *Fibonacci cube*  $\Gamma_k$  of order  $k$  is the subgraph of  $Q_k$  whose vertices are all binary words that do not contain the factor 11. It is well known that  $\Gamma_k$  is an isometric subgraph of  $Q_k$  (cf. [27]). Therefore, the Fibonacci cubes are partial cubes.

### Basic notation on words

Let  $\Sigma$  be a finite alphabet. In this paper, the alphabet is  $\Sigma = \{0, 1\}$ , although many of the definitions and results can be stated and hold for a general alphabet. If  $a \in \Sigma$ ,  $\bar{a}$  denotes the opposite of  $a$ , i.e.  $\bar{a} = 1$  if  $a = 0$  and vice versa. A word (or string)  $w$  of length  $|w| = n$  is  $w = a_1 a_2 \dots a_n$ , where  $a_1, a_2, \dots, a_n$  are symbols in  $\Sigma$ . Then,  $\bar{w} = \bar{a}_1 \bar{a}_2 \dots \bar{a}_n$  is the complement of  $w$ , and  $w^R = a_n a_{n-1} \dots a_1$  is the mirror of  $w$ . The set of all finite words over  $\Sigma$  is denoted  $\Sigma^*$  and the set of all words over  $\Sigma$  of length  $n$  is denoted  $\Sigma^n$ .

Let  $w[i]$  denote the symbol of  $w$  in position  $i$ , i.e.,  $w[i] = a_i$ . Then,  $w[i..j] = a_i \dots a_j$ , for  $1 \leq i \leq j \leq n$ , is a *factor* of  $w$  that occurs in the interval  $[i..j]$ . The *prefix* (resp. *suffix*) of  $w$  of length  $\ell$ , with  $1 \leq \ell \leq n - 1$  is  $\text{pre}_\ell(w) = w[1..\ell]$  (resp.  $\text{suf}_\ell(w) = w[n - \ell + 1..n]$ ). When  $\text{pre}_\ell(w) = \text{suf}_\ell(w) = u$  then  $u$  is an *overlap* of  $w$  of length  $\ell$ .

In the sequel, the notion of overlap with errors is also needed. A word  $w$  has a *2-error overlap* (2-*eo*, for short) of length  $\ell$  if  $\text{pre}_\ell(w)$  and  $\text{suf}_\ell(w)$  differ in two positions, namely, they have a Hamming distance equal to 2; the two positions are called *error positions*. We say that  $w$  has a 2-error overlap if  $w$  has a 2-*eo* of length  $\ell$  for some  $1 \leq \ell \leq n - 1$ .

Given two words  $w, f \in \Sigma^*$ ,  $w$  is *f-free* if  $w$  does not contain  $f$  as a factor; we also say that  $w$  *avoids*  $f$ .

### Isometric words and sets

We now shortly report some recent results on isometric words (originally called good words) that are used to define special isometric subgraphs of the hypercubes (cf. [26, 28, 29, 32]).

Let  $u, v \in \Sigma^*$  be words of equal length and  $\text{dist}_H(u, v) = d$ . A *transformation*  $\tau$  from  $u$  to  $v$  is a sequence of  $d + 1$  words  $(w_0, w_1, \dots, w_d)$  such that  $w_0 = u$ ,  $w_d = v$ , and for any  $k = 0, 1, \dots, d - 1$ ,  $\text{dist}_H(w_k, w_{k+1}) = 1$ . Moreover,  $\tau$  is *f-free* if for any  $i = 0, 1, \dots, d$ , the word  $w_i$  is *f-free*. This is the base for the following definition.

► **Definition 1.** *A word  $f$  is isometric if for any pair of  $f$ -free words  $u$  and  $v$ , with  $|u| = |v|$ , there exists an  $f$ -free transformation from  $u$  to  $v$ . It is non-isometric otherwise.*

A pair of  $f$ -free words  $(u, v)$  such that any transformation from  $u$  to  $v$  is not  $f$ -free proves that  $f$  is non-isometric and is called a pair of *witnesses* for  $f$ ; the positions where  $u$  and  $v$  differ are called *error positions*. The minimal length of a word  $u$  in a pair of witnesses for  $f$  is called the *index* of  $f$ , and denoted by  $i(f)$ .

Coming back to graphs and hypercubes we recall the following definition of *Generalized Fibonacci cubes*, or *hypercubes avoiding a word*, first introduced in [25].

► **Definition 2.** *The  $k$ -hypercube avoiding a word  $f$ , denoted  $Q_k(f)$ , is the subgraph of  $Q_k$  obtained by removing those vertices whose labels contain  $f$  as a factor.*

Note that, using this definition, the Fibonacci cube  $\Gamma_k$  coincides with  $Q_k(11)$ . Moreover, we assume implicitly that  $Q_k(f)$  is still a labeled graph keeping the original labels of the hypercube  $Q_k$  from which it is derived.

Then, one immediately realizes the strict connection between isometric subgraphs of  $Q_k$  of type  $Q_k(f)$  and isometric words. It holds that  $Q_k(f)$  is an isometric subgraph of  $Q_k$  if and only if  $f$  is isometric. Moreover, in [34], it is proved the non-obvious result that if  $f$  is not isometric then for all  $k \geq i(f)$ ,  $Q_k(f)$  is not even a partial cube, i.e.,  $Q_k(f)$  does not isometrically embed into any  $Q_{k'}$  with  $k' \geq i(f)$ . These considerations lead to the following proposition.

► **Proposition 3.** *The following statements are equivalent:*

1.  $f$  is isometric
2. for any  $k \geq |f|$ ,  $Q_k(f)$  is an isometric subgraph of  $Q_k$
3. for any  $k \geq |f|$ ,  $Q_k(f)$  isometrically embeds into  $Q_k$ .

Furthermore, note that if  $f$  is a non-isometric word and  $i(f)$  is its index, then for any  $k < i(f)$ ,  $Q_k(f) \sqsubseteq Q_k$ , while for any  $k \geq i(f)$ ,  $Q_k(f) \not\sqsubseteq Q_k$ . Finally, remark that if  $f$  is isometric, also its complement  $\bar{f}$  and its mirror  $f^R$  are isometric. Consequently, also the graphs  $Q_k(\bar{f})$  and  $Q_k(f^R)$  (that can be also obtained from  $Q_k(f)$  by, respectively, complementing or reversing all the labels) are isometric subgraphs of  $Q_k$ .

A further step in the generalization of Fibonacci cubes is done in [3] by introducing the notion of *isometric set* of words. In what follows,  $S$  denotes a finite subset of  $\Sigma^*$ . Then a word  $w$  is  *$S$ -free*, if  $w$  does not contain any word in  $S$  as a factor.

► **Definition 4.** *Let  $S \subseteq \Sigma^*$  be a finite set of words and  $\max$  be the length of a longest word in  $S$ . Then,  $S$  is isometric if for any pair  $(u, v)$  of  $S$ -free words, with  $|u| = |v| \geq \max$ , there exists a transformation from  $u$  to  $v$  such that all its intermediate words are  $S$ -free.*

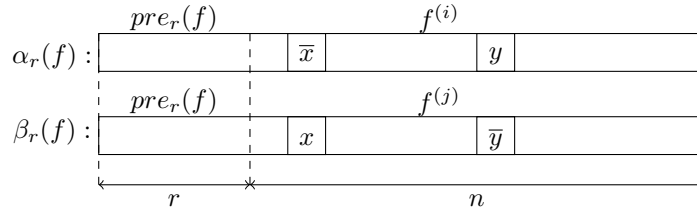
A set  $S$  is *non-isometric* if it is not isometric. If  $S$  is non-isometric, a *pair of witnesses* for  $S$  is a pair  $(u, v)$  with  $|u| = |v| \geq \max$ , such that each transformation from  $u$  to  $v$  contains a word that is not  $S$ -free.

Examples of isometric sets (cf. [3], [19]) are the sets  $D_h = \{11, 101, \dots, 10^h 1\}$ , for any  $h \geq 0$ . Note that  $D_0 = \{11\}$ . As in the case of a single word (cf. [29] and [28]), it can be shown that the non-isometricity of a set  $S$  is again connected to some properties on overlaps defined on a pair of (possible) different words in  $S$ .

► **Definition 5.** *Let  $f, g \in \Sigma^*$  with  $|g| = m$ ,  $|f| = n$ . Then,  $g$  has a 2-error overlap (or 2-eo) on  $f$  of shift  $r$  and length  $\ell$ , with  $0 \leq r \leq n - 2$ ,  $\ell = \min(m, n - r)$ , if distance  $\text{dist}_H(g[1..\ell], f[r + 1..r + \ell]) = 2$ .*

Note that the existence of a 2-eo of  $f$  on  $g$  does not imply the existence of a 2-eo of  $g$  on  $f$ . In particular, if  $g = f$  Definition 5 coincides with the classical definition of 2-eo of a word (cf. [29]).

► **Example 6.** Let  $f = 1000$  and  $g = 11$ . Then  $g$  has 2-eo on  $f$  of shift 1 and length 2 and a 2-eo of shift 2 and length 2. Instead,  $f$  has no 2-eo on  $g$ .



■ **Figure 1** The words  $\alpha_r(f)$  and  $\beta_r(f)$  corresponding to the 2-error overlap of  $f$  of shift  $r$ .

The following proposition, given in [29], characterizes isometric words.

► **Proposition 7.** *A word  $f$  is isometric if and only if  $f$  has no 2-error overlap.*

The proof of Proposition 7 is constructive. In fact, if  $f$  has a 2-error overlap of shift  $r$ , then the proof explicitly provides a “standard” pair of witnesses for  $f$ . The pair is defined as  $(\alpha_r(f), \beta_r(f))$  or  $(\eta_r(f), \gamma_r(f))$ , in the case that  $\beta_r(f)$  is not  $f$ -free. More exactly, it turns out that  $\beta_r(f)$  is not  $f$ -free if and only if the 2-error overlap of  $f$  satisfies the so-called *Condition\** (cf. [29]). Moreover, in [9], it is shown that this construction yields the pairs of witnesses of minimal distance and length. Let us recall the construction (see Fig. 1 for  $(\alpha_r(f), \beta_r(f))$ ).

► **Definition 8.** *Let  $f \in \Sigma^n$  have a 2-error overlap of shift  $r$  and error positions  $i, j$ , with  $1 \leq i < j \leq n - r$ . Then,*

- $f^{(i)}$  is the word obtained from  $f$  replacing  $f[i]$  by  $f[r + i]$ .
- if  $r$  is even and  $t = j + r/2$ , then  $f^{(j,t)}$  is the word obtained from  $f$  replacing  $f[j]$  with  $f[r + j]$ , and  $f[t]$  by  $f[i]$ .

The words  $\alpha_r(f)$ ,  $\beta_r(f)$ ,  $\eta_r(f)$ , and  $\gamma_r(f)$  corresponding to the 2-error overlap of  $f$  of shift  $r$  are

- $\alpha_r(f) = pre_r(f)f^{(i)}$  and  $\beta_r(f) = pre_r(f)f^{(j)}$ .
- if  $r$  is even,  $\eta_r(f) = pre_r(f)f^{(i)}suf_{r/2}(f)$  and  $\gamma_r(f) = pre_r(f)f^{(j,t)}suf_{r/2}(f)$ .

The following theorem generalizes to any set  $S \subseteq \Sigma^*$  an analogous result for sets of words of equal length in [3]. Its proof rephrases the other one and is here sketched for the sake of completeness.

► **Theorem 9.** *Let  $S \subseteq \Sigma^*$ . If  $S$  is non-isometric, then there exist  $s_1, s_2 \in S$  (possibly  $s_1 = s_2$ ) such that  $s_1$  has a 2-er on  $s_2$ .*

**Proof.** Let  $S$  be a non-isometric set. Consider a pair  $(u, v)$  of witnesses for  $S$  of minimal distance and its error positions, i.e. the positions where  $u$  and  $v$  differ. The minimality implies that the word obtained by changing any error position in  $u$  contains the occurrence of a word of  $S$  covering that error position and another one, since  $v$  is  $S$ -free. Therefore, there exist two error positions  $i$  and  $j$  that are covered by both the words of  $S$ , say  $s_1$  and  $s_2$ , occurring after their change in  $u$ . This gives the thesis. ◀

The following definition extends Definition 2 from a single word to a set. The graphs will be referred to as generalized Fibonacci cubes, yet.

► **Definition 10.** *The  $k$ -hypercube avoiding a set  $S$ , denoted  $Q_k(S)$ , is the subgraph of  $Q_k$  obtained by removing those vertices that contain a word in  $S$  as a factor.*

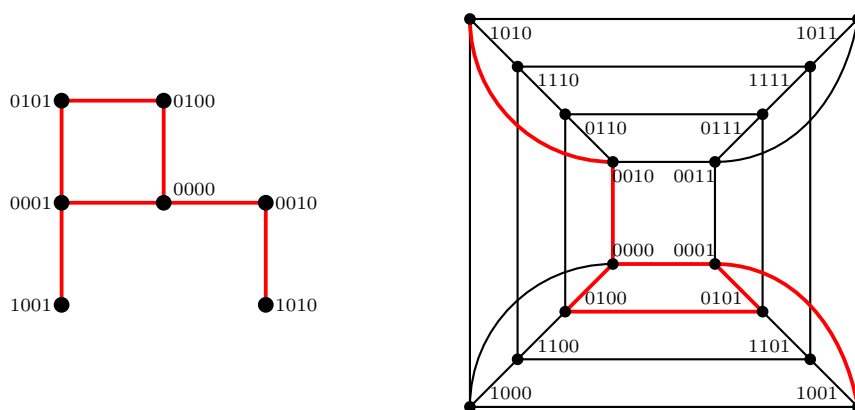


Figure 2 (a)  $Q_4(\{11, 1000\})$  and (b) hypercube  $Q_4$  with  $Q_4(\{11, 1000\})$  in red.

Note that if  $S = \{w_1, w_2, \dots, w_n\}$ , then  $Q_k(S) = Q_k(w_1) \cap Q_k(w_2) \cap \dots \cap Q_k(w_n)$ , that is the intersection of the hypercubes avoiding each word in the set, respectively. Furthermore, also in this case we assume that vertices of the subgraph  $Q_k(S)$  keep the labels they had in the whole hypercube  $Q_k$ . This immediately establishes the following correspondence between isometric sets of words and subgraphs of the hypercubes that avoid a set that are also isometric subgraphs.

► **Proposition 11.** *Let  $S \subseteq \Sigma^*$  and let  $\max$  be the length of a longest word in  $S$ . The set  $S$  is an isometric set of words iff for any  $k \geq \max$ ,  $Q_k(S)$  is an isometric subgraph of  $Q_k$ .*

We conclude with an example of a non-isometric set that will be referred to in the results of the next section.

► **Example 12.** Let  $S = \{11, 1000\}$ .  $S$  is a non-isometric set. In fact,  $(u, v)$  with  $u = 1001$  and  $v = 1010$  is a pair of witnesses for  $S$  because any transformation from  $u$  to  $v$  meets either 11 or 1000. According to Proposition 11,  $Q_4(S)$  is not an isometric subgraph of  $Q_4$ . Let  $x$  be the vertex in  $Q_4(S)$  labeled 1001 and  $y$  be the one labeled 1010. Referring to Fig. 2, one can see that  $dist_{Q_4(S)}(x, y) = 4$ , whereas  $dist_H(1001, 1010) = 2$ .

### Isometric and Fibonacci dimensions of a graph

In this last part of the section, we report some results from Cabello et al. [14] where Fibonacci dimension of a graph is first introduced and investigated. The *isometric dimension* of a graph  $G$ , denoted by  $idim(G)$ , is the smallest integer  $k$  such that  $G$  admits an isometric embedding into  $Q_k$ . If no such  $k$  exists,  $idim(G) = \infty$ . Therefore  $idim(G)$  is finite if and only if  $G$  is a partial cube.

Let  $\beta : V(G) \rightarrow V(Q_k)$  be an isometric embedding of a graph  $G$  into the hypercube  $Q_k$ . The embedding  $\beta$  is *redundant* if there exists  $i \in \{1, 2, \dots, k\}$  such that the labels of all vertices in  $\beta(V(G))$  have the same  $i$ -th coordinate. It is *irredundant* otherwise. If an embedding is redundant, another embedding can be found into a lower-dimensional hypercube by omitting the redundant coordinate.

Let us recall the following result that will be largely used in the sequel.

► **Proposition 13.** *Let  $G$  be a graph. An isometric embedding  $\beta : V(G) \rightarrow V(Q_k)$  is irredundant if and only if  $k = idim(G)$ .*

The authors then introduce the *Fibonacci dimension* of a graph  $G$ ,  $fdim(G)$ , as the smallest integer  $d$  such that  $G$  admits an isometric embedding into the Fibonacci cube  $\Gamma_d$ , and set  $fdim(G) = \infty$  if there is no such  $d$ . This is strictly related to the isometric dimension since it is shown that a graph isometrically embeds in some hypercube  $Q_k$  iff it isometrically embeds in some Fibonacci cube  $\Gamma_d$ . Note that in general  $d \geq k$ . They established upper and lower bounds as stated in the following proposition.

► **Proposition 14.** *Let  $G$  be a graph. Then  $fdim(G)$  is finite if and only if  $idim(G)$  is finite. Moreover,  $idim(G) \leq fdim(G) \leq 2idim(G) - 1$ .*

In [14], graphs  $G$  with maximal Fibonacci dimension  $2idim(G) - 1$  have been fully characterized. While the algorithm to compute the isometric dimension of a graph is quadratic (cf. [23]), it is proved, instead, that deciding for a given graph  $G$  whether or not  $idim(G) = fdim(G)$  is an NP-hard problem. In general, it is difficult to find graphs with “small” Fibonacci dimension. An important family of graphs with this property is the set of trees, for which they prove the following result.

► **Theorem 15.** *For any tree  $T$ ,  $fdim(T) = idim(T) = |E(T)|$ .*

Note that trees are very special partial cubes since they have the maximal possible isometric dimension, namely equal to the number of edges. This property derives from the acyclicity and it is the one exploited to compute the Fibonacci dimension.

### 3 Isometric and Fibonacci Dimension of hypercubes avoiding words

The main result of this section is to present a new family of graphs with minimal Fibonacci dimension. We start by investigating the isometric and Fibonacci dimension of the graphs that are hypercubes avoiding a word or a set of words introduced previously. It is worth to note that any subgraph of the  $k$ -hypercube can be trivially viewed as the hypercube avoiding the set of all the  $k$ -length words that label the omitted vertices. Then, in principle, any partial cube is isometric to a hypercube that avoids a set  $S$  of words. Here we restrict to the case when  $S$  has one or two elements.

Let us introduce some notation. For a set  $S$  of words, let  $L_k(S)$  be the set of words of length  $k$  avoiding  $S$ , that is,  $L_k(S) = \Sigma^k \setminus \Sigma^* S \Sigma^*$ . In the case of  $S = \{f\}$ , we will simply write  $L_k(f)$ . The set  $L_k(S)$  is *redundant at position  $i$* , for some  $i \in \{1, 2, \dots, k\}$ , if all the strings in  $L_k(S)$  have the same symbol at position  $i$ . It is *redundant* if there exists  $i \in \{1, 2, \dots, k\}$  such that  $L_k(S)$  is redundant at position  $i$ . It is *irredundant* otherwise.

► **Remark 16.** Note that if  $Q_k(S)$  is an isometric subgraph of  $Q_k$ , then the identity function is an isometric embedding of  $Q_k(S)$  into  $Q_k$  and, moreover, it is irredundant if and only if  $L_k(S)$  is irredundant. In fact, note that  $L_k(S) = \lambda_k(V(Q_k(S)))$ .

Let us first consider hypercubes avoiding a single word  $f$ .

► **Proposition 17.** *Let  $f \in \Sigma^*$ ,  $k \geq |f|$ . If  $|f| = 1$  then  $Q_k(f)$  is a partial cube with  $idim(Q_k(f)) = fdim(Q_k(f)) = 0$ . If  $|f| = 2$  then  $Q_k(f)$  is a partial cube with  $idim(Q_k(f)) = fdim(Q_k(f)) = k$ .*

**Proof.** Let  $G_k = Q_k(f)$ . Consider the two different cases. If  $|f| = 1$  then either  $f = 1$  or  $f = 0$ . If  $f = 1$  ( $f = 0$ , resp.)  $G_k$  has a single vertex  $v$  labeled  $0^k$  ( $1^k$ , resp.). Then  $G_k$  isometrically embeds in  $Q_0 = \Gamma_0$  and then  $idim(G_k) = fdim(G_k) = 0$ . Let  $|f| = 2$ . If  $f = 11$  then  $G_k = \Gamma_k$  isometrically embeds in  $Q_k$  and in  $\Gamma_k$ , and  $idim(G_k) = fdim(G_k) = k$ . The same holds for  $f = 00 = \bar{1}\bar{1}$ . If  $f = 01$  then  $L_k(01) = \{1^i 0^{k-i} \mid 0 \leq i \leq k\}$  and the

edges connect nodes with labels  $1^i 0^{k-i}$  and  $1^{i+1} 0^{k-i-1}$ , for any  $i = 0, \dots, k-1$ , respectively. Therefore,  $G_k$  is a chain of  $k+1$  nodes, i.e. a special case of a tree, and by Theorem 15  $idim(G_k) = fdim(G_k) = k$ . The same result holds for  $f = 10 = \overline{01}$ . ◀

Proposition 17 shows the isometric and Fibonacci dimensions of  $Q_k(f)$  for  $|f| = 1, 2$ . In what follows, the other cases are considered, i.e. words  $f$  of length at least 3.

The hypercube avoiding an isometric word in  $Q_k$  isometrically embeds in  $Q_k$ , but, in principle, its isometric dimension could be smaller than  $k$ . Nevertheless, the following proposition shows that this is never the case.

► **Proposition 18.** *Let  $f$  be a word with  $|f| \geq 3$ . If  $f$  is an isometric word, then, for any  $k \geq 0$ ,  $Q_k(f)$  is a partial cube with  $idim(Q_k(f)) = k$ .*

**Proof.** Let  $G_k = Q_k(f)$ . Suppose  $f$  isometric and let  $|f| = n \geq 3$ . If  $k < n$ , then  $G_k = Q_k$  and therefore  $idim(G_k) = idim(Q_k) = k$ .

For  $k \geq n$ , since  $f$  is isometric, the identity function  $\beta$  is an isometric embedding of  $G_k$  into  $Q_k$ . Therefore, from Proposition 13, it suffices to show that  $\beta$  is irredundant or, equivalently, in view of Remark 16, that  $L_k(f)$  is irredundant. Then we proceed by induction on  $k$ .

For the basis, if  $k = n$ , then  $L_k(f) = \Sigma^k \setminus \{f\}$  and, therefore,  $|L_k(f)| = 2^k - 1$ . Since  $k = n \geq 3$ , it holds  $2^k - 1 > 2^{k-1}$ ; hence, there is no  $i \in \{1, 2, \dots, k\}$  such that all the strings in  $L_k(f)$  have the same symbol at position  $i$ , i.e.  $L_k(f)$  is irredundant.

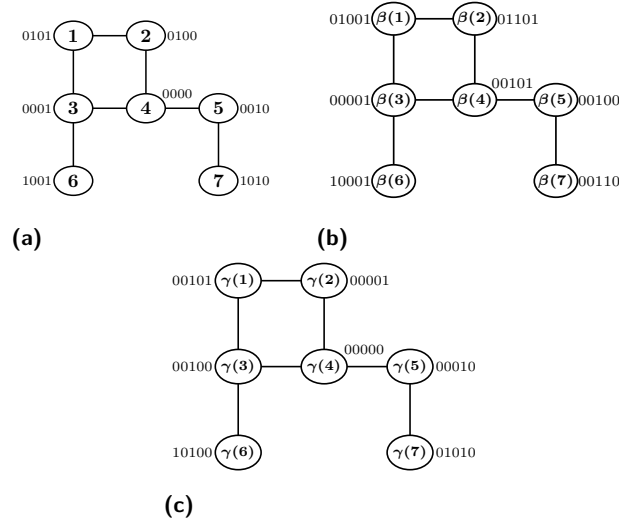
Suppose now that  $L_k(f)$  is irredundant and let us prove that  $L_{k+1}(f)$  is irredundant. Let  $f = awb$ , with  $a, b \in \Sigma$  and  $w \in \Sigma^*$ . Now, for any  $i = 1, \dots, k$ , consider two words, say  $u_i, v_i \in L_k(f)$ , such that  $u_i[i] \neq v_i[i]$ . Then, the words  $x_i = u_i \bar{b}$  and  $y_i = v_i \bar{b}$  are in  $L_{k+1}(f)$  and  $x_i[i] \neq y_i[i]$ , for any  $i = 1, \dots, k$ . Moreover, the words  $x = \bar{a}u_k$  and  $y = \bar{a}v_k$  are in  $L_{k+1}(f)$  and  $x[k+1] \neq y[k+1]$ . Hence  $L_{k+1}(f)$  is irredundant. ◀

► **Remark 19.** Note that, if  $f$  is a non-isometric word, with  $|f| \geq 3$  and index  $i(f)$ , then the equality  $idim(Q_k(f)) = k$ , proved in the previous proposition, is still valid for any  $k < i(f)$ . Indeed, for any  $k < i(f)$ , the identity function is an isometric embedding of  $Q_k(f)$  into  $Q_k$  and a reasoning, similar to the one used in Proposition 18, shows that  $Q_k(f)$  is a partial cube and  $idim(Q_k(f)) = k$ .

After having determined the isometric dimension of a hypercube avoiding an isometric word, let us consider its Fibonacci dimension. Recall that the computation of  $fdim$  of a partial cube is a NP-hard problem in the general case. By definition,  $Q_k(\{11, f\}) = Q_k(11) \cap Q_k(f) = \Gamma_k(f)$ . Nevertheless, there is no evident relation between the  $idim(Q_k(\{11, f\}))$  and the  $fdim(Q_k(f))$  that indeed can be strictly greater than  $idim(Q_k(f))$ , as in the next example.

► **Example 20.** Let  $f = 1000$  and consider the graph  $Q_4(1000)$ . Since  $f$  is isometric, then, by Proposition 18,  $idim(Q_4(1000)) = 4$ . In order to compute  $fdim(Q_4(1000))$  we need to find the minimum  $k$  such that  $Q_4(1000)$  can be embedded into the Fibonacci cube  $\Gamma_k$ . Recall that the number of vertices of  $\Gamma_k$  is the  $(k+2)$ -th Fibonacci number. Note that  $Q_4(1000)$  has 15 vertices; then it surely cannot be embedded in  $\Gamma_4$  that has only 8 vertices. More precisely, the smallest Fibonacci cube to be considered is  $\Gamma_6$  that has 21 vertices, then  $fdim(Q_4(1000)) \geq 6$ .

We experimentally checked that the same argument based on counting vertices can be applied also to all graphs  $Q_k(1000)$  with  $k \leq 15$ . Then, we expect that  $idim(Q_k(1000)) < fdim(Q_k(1000))$  for all  $k \geq 4$ .



■ **Figure 3**  $Q_4(\{11, 1000\})$  with (a) its labeling  $\lambda_4$ , (b) an isometric embedding  $\beta$  in  $Q_5$  and (c) an isometric embedding  $\gamma$  in  $\Gamma_5$ .

In the next example we come back to Example 12 where the word  $f = 1000$  is considered together with 11.

► **Example 21.** Let  $S = \{11, 1000\}$  and consider the graph  $Q_4(S)$ . In Example 12 it is proved that  $Q_4(S)$  cannot be isometrically embedded in  $Q_4$ . Nevertheless, it is a partial cube. In fact Eppstein’s algorithm in [23] produces the isometric embedding  $\beta$  of  $Q_4(S)$  in  $Q_5$  shown in Fig. 3(b), yielding  $idim(Q_4(S)) = 5$ . Moreover, also  $fdim(Q_4(S)) = 5$  since the map  $\gamma$  shown in Fig. 3(c) is an isometric embedding of  $G_4$  in  $\Gamma_5$ . Then, for this partial cube, we have that  $idim(Q_4(S)) = fdim(Q_4(S)) = 5$ .

The previous example is emblematic. The set  $S = \{11, 1000\}$  is non-isometric; then the graph  $Q_4(S)$  does not isometrically embed in  $Q_4$ , but it isometrically embeds in  $Q_5$ . This proves that “sets” and “single words” have different properties regarding the non-isometricity. (Recall that in [32] it is shown that a word  $f$  is isometric if and only if  $Q_k(f)$  is a partial cube). Another interesting fact is that  $idim(Q_4(S)) = fdim(Q_4(S))$ . In the next section we will introduce a family of sets  $S = \{11, f\}$  for some word  $f$  for which this always holds.

### 3.1 Main results

In this section we investigate the isometric and Fibonacci dimensions of hypercubes avoiding a set of words  $S$  in the special case that  $11 \in S$ . Afterwards, the analysis is restricted to the sets of cardinality 2, namely  $S = \{11, f\}$ , for some word  $f$ .

► **Theorem 22.** *Let  $S \subseteq \Sigma^*$ , with  $11 \in S$ . Let  $\max$  be the length of a longest word in  $S$  and let  $k \geq \max$ . If  $S$  is an isometric set then  $Q_k(S)$  is a partial cube with  $idim(Q_k(S)) \leq fdim(Q_k(S)) \leq k$ ; moreover,  $idim(Q_k(S)) = k$  iff  $L_k(S)$  is irredundant.*

**Proof.** Let  $G_k = Q_k(S)$ . Since  $11 \in S$ , then  $V(G_k) \subseteq V(\Gamma_k)$  and this implies that, for any  $u, v \in G_k$ ,  $dist_{G_k}(u, v) \geq dist_{\Gamma_k}(u, v)$ .

Moreover, both  $G_k$  and  $\Gamma_k$  are isometric subgraphs of  $Q_k$ . Hence,  $\text{dist}_{G_k}(u, v) = \text{dist}_{Q_k}(u, v)$  and  $\text{dist}_{\Gamma_k}(u, v) = \text{dist}_{Q_k}(u, v)$ . So  $\text{dist}_{Q_k}(u, v) = \text{dist}_{G_k}(u, v) \geq \text{dist}_{\Gamma_k}(u, v) = \text{dist}_{Q_k}(u, v)$ . Therefore, for any  $u, v \in G_k$ ,  $\text{dist}_{G_k}(u, v) = \text{dist}_{\Gamma_k}(u, v)$ , i.e.,  $G_k$  is an isometric subgraph of  $\Gamma_k$  and  $\text{fdim}(G_k) \leq k$ .

Moreover, the claim  $\text{idim}(G_k) = k$  if and only if  $L_k(S)$  is irredundant, follows from Remark 16 and Proposition 13.  $\blacktriangleleft$

► **Corollary 23.** *Let  $S$  be a set with  $11 \in S$ . If  $S$  is isometric and  $\text{idim}(Q_k(S)) = k$ , then  $\text{fdim}(Q_k(S)) = k$ .*

Let us now show a characterization of sets  $S = \{11, f\}$  which are isometric (Theorem 24) and a characterization of sets  $S = \{11, f\}$  for which  $L_k(S)$  is irredundant (Theorem 25).

► **Theorem 24.** *A set  $S = \{11, f\}$  is isometric if and only if*

1.  $11$  is a factor of  $f$  or
2.  $f \notin 000\Sigma^* \cup \Sigma^*000 \cup \Sigma^*0000\Sigma^*$  and  $f$  is isometric or
3.  $f \notin 000\Sigma^* \cup \Sigma^*000 \cup \Sigma^*0000\Sigma^*$ ,  $f$  is non-isometric and any pair  $(u, v)$  of witnesses of minimal distance for  $f$  is such that  $11$  is a factor of  $u$  or  $v$ .

**Proof.** Let  $S = \{11, f\}$  be an isometric set and suppose, by contradiction, that  $f$  is 11-free and, moreover,  $f \in 000\Sigma^* \cup \Sigma^*000 \cup \Sigma^*0000\Sigma^*$  or  $f$  is non-isometric and any pair of witnesses  $(u, v)$  for  $f$  of minimal distance is such that  $u$  and  $v$  are 11-free.

Now, if  $f$  is 11-free,  $f$  is non-isometric and any pair of witnesses  $(u, v)$  for  $f$  of minimal distance is such that  $u$  and  $v$  are 11-free, then  $(u, v)$  is also a pair of witnesses for  $S$  contradicting  $S$  isometric.

If  $f$  is 11-free, and  $f \in 000\Sigma^* \cup \Sigma^*000 \cup \Sigma^*0000\Sigma^*$ , three different cases must be considered:  $f = 000x$  or  $f = x000$  or  $f = x0000y$ , for some  $x, y \in \Sigma^*$ . We will show that, in any case, a pair of witnesses for  $S$  can be constructed, contradicting  $S$  isometric.

If  $f = 000x$  with  $x \in \Sigma^*$ , then the pair  $(100x, 010x)$  is a pair of witnesses for  $S$ . In fact,  $100x$  and  $010x$  are  $f$ -free, since they have the same length as  $f$ , but differ from it, and they are 11-free, since  $f$  is 11-free. Moreover, no  $S$ -free transformation exists from  $100x$  to  $010x$ . In fact, if the 1 in the first position is changed into 0, then  $f$  occurs as a factor, if the 0 in the second position is changed into 1, then  $11$  occurs as a factor. Similar arguments can be used to obtain a pair of witnesses for  $S$  in the case  $f = x000$  or  $f = x0000y$ .

For the converse, if  $11$  is a factor of  $f$ , then  $Q_k(S) = \Gamma_k(f) = \Gamma_k$ . Since  $\Gamma_k$  isometrically embeds into  $Q_k$ ,  $S$  is isometric.

Suppose, now, that condition 2. of the statement holds and that, by contradiction,  $S$  is non-isometric. Note that  $S$  non-isometric implies that  $f$  is 11-free.

Let  $(u, v)$  be a pair of witnesses of minimal distance for  $S$  and consider  $s_1, s_2 \in S$  as in the proof of Theorem 9. Since  $f$  is isometric, it cannot be  $s_1 = s_2 = f$ . Moreover, the case  $s_1 = s_2 = 11$  cannot happen. If  $s_1 \neq s_2$ , we have three cases and we will show, in any case, a contradiction with the hypothesis  $f \notin 000\Sigma^* \cup \Sigma^*000 \cup \Sigma^*0000\Sigma^*$ . Indeed, if the 2-error overlap involves the first two characters of  $f$ , then  $f = 00z$  with  $z \in \Sigma^*$ . Moreover  $z = 0y$  otherwise  $u$  would be not 11-free. Then  $f = 000y$ : a contradiction. If the overlap involves the last two positions of  $f$ , for the same reason  $f = x000$ : a contradiction. If the overlap involves two consecutive characters inside of  $f$ , then  $f = z_100z_2$  and, as before,  $z_1$  cannot end by 1 (otherwise  $u$  is not 11-free) and  $z_2$  cannot begin by 1 (otherwise  $v$  would not be 11-free). Then  $f = x0000y$  with  $x, y \in \Sigma^*$  is a contradiction again.

At last, suppose that condition 3. of the statement holds and that, by contradiction,  $S$  is non-isometric. As before, let  $(u, v)$  be a pair of witnesses of minimal distance for  $S$  and consider  $s_1, s_2 \in S$  as in the proof of Theorem 9. If  $s_1 = s_2 = f$ , let  $(u', v')$  be the pair

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of witnesses of minimal distance for  $f$  constructed on this 2-error overlap as in Definition 8. Then  $u'$  ( $v'$ , resp.) is a factor of  $u$  ( $v$ , resp.) and then  $u'$  and  $v'$  are 11-free. This contradicts the hypothesis on the witnesses of minimal distance for  $f$ . The case  $s_1 = s_2 = 11$  can not happen. Moreover, the case  $s_1 \neq s_2$  gives a contradiction with the hypothesis  $f \notin 000\Sigma^* \cup \Sigma^*000 \cup \Sigma^*0000\Sigma^*$ , as shown before.  $\blacktriangleleft$

► **Theorem 25.** *Let  $S = \{11, f\}$ , with  $|f| \geq 3$ , and  $k \geq 3$ . Then,  $L_k(S)$  is redundant if and only if  $f = 010$ .*

**Proof.** Let  $|f| \geq 3$ , and  $k \geq 3$ .

If  $f = 010$  then  $L_k(S) = \{0^k, 10^{k-1}, 0^{k-1}1, 10^{k-2}1\}$  and it is redundant at position  $i = 2$ .

Conversely, suppose that  $L_k(S)$  is redundant at position  $i$ , for some  $i = 1, \dots, k$ . Consider  $L_k(11)$  and partition it in  $L_k(11) = L_k^{i,0}(11) \cup L_k^{i,1}(11)$  where  $L_k^{i,0}(11) = \{x_1 \cdots x_k \in L_k(11) \mid x_i = 0\}$  and  $L_k^{i,1}(11) = \{x_1 \cdots x_k \in L_k(11) \mid x_i = 1\}$ . Our goal is to prove that if  $L_k(S) \subseteq L_k^{i,0}(11)$  or  $L_k(S) \subseteq L_k^{i,1}(11)$  then  $f = 010$ . Note that  $L_k(S) \subseteq L_k^{i,0}(11)$  if and only if any word in  $L_k^{i,1}(11)$  has  $f$  as a factor; analogously for the symmetric case.

Let us first prove that the case  $L_k(S) \subseteq L_k^{i,1}(11)$  cannot hold. In fact, the only factor shared by all words in  $L_k^{i,0}(11)$  is 0 against the hypothesis that  $|f| \geq 3$ . Then,  $L_k(S) \subseteq L_k^{i,0}(11)$ .

Let us now show that  $i \neq 1$  and  $i \neq k$ . Indeed, the only factor shared by all words in  $L_k^{i,1}(11)$  when  $i = 1$  ( $i = k$ , resp.) is 10 (01, resp.) against the hypothesis that  $|f| \geq 3$ .

Finally, let us show that when  $2 \leq i \leq k - 1$ , the only factor  $f$  shared by all words in  $L_k^{i,1}(11)$  is  $f = 010$ . First observe that 010 occurs in all words in  $L_k^{i,1}(11)$  at position  $i - 1$ . Further, no other factor is shared by words in  $L_k^{i,1}(11)$ . In fact,  $L_k^{i,1}(11)$  contains the word  $0^{i-1}10^{k-i}$  whose factors of length greater than or equal 3 contain two consecutive 0, whereas no such factor occurs in the word of  $L_k^{i,1}(11)$  which belongs to  $(01)^* \cup (10)^*$ .  $\blacktriangleleft$

The next example considers the case of the word 010 that is involved in the previous theorem.

► **Example 26.** Consider the word  $f = 010$  and the set  $S = \{11, 010\}$ . For any  $k \geq 3$ , the subgraph  $Q_k(S)$  has 4 vertices which form a cycle. Their labels are in  $L_k(S) = \{0^k, 10^{k-1}, 0^{k-1}1, 10^{k-2}1\}$ . Hence, for any  $k \geq 3$ ,  $Q_k(S)$  is a partial cube with  $\text{idim}(Q_k(S)) = 2$  and  $\text{fdim}(Q_k(S)) = 3$ .

The characterizations shown in Theorems 24 and 25 allow to specify Theorem 22 and provide a family of partial cubes with minimal Fibonacci dimension, i.e., equal to the isometric dimension.

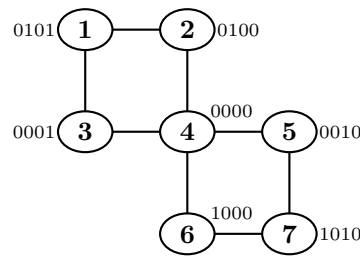
► **Theorem 27.** *Let  $f$  be a word such that  $f \neq 010$  and  $f$  satisfies conditions 1., 2., or 3. of Theorem 24. Then, for any  $k \geq |f|$ ,  $Q_k(\{11, f\})$  is a partial cube with  $\text{idim}(Q_k(\{11, f\})) = \text{fdim}(Q_k(\{11, f\})) = k$ .*

Note that the previous Example 21 showed the graph  $Q_k(\{11, 1000\})$  that does not satisfy the hypotheses of Theorem 27 and it is a partial cube with  $\text{idim}(Q_k(\{11, 1000\})) = \text{fdim}(Q_k(\{11, 1000\}))$  but  $\text{idim}(Q_k(\{11, 1000\})) \neq k$ .

Let us apply Theorem 27 and show in the next example a graph  $Q_k(\{11, f\})$  which is a partial cube with  $\text{idim}(Q_k(\{11, f\})) = \text{fdim}(Q_k(\{11, f\})) = k$ .

► **Example 28.** Let  $S = \{11, 1001\}$ ,  $k = 4$ , and consider  $G_4 = Q_4(\{11, 1001\})$ , as in Figure 4. According to Theorem 27,  $G_4$  is a partial cube with  $\text{idim}(G_4) = \text{fdim}(G_4) = 4$  because  $f \neq 010$  and  $f$  satisfies the condition 2. in Theorem 24.

More specifically, by Theorem 24,  $S$  is isometric. In fact,  $f = 1001 \notin 000\Sigma^* \cup \Sigma^*000 \cup$



■ **Figure 4**  $Q_4(\{11, 1001\})$ .

$\Sigma^*0000\Sigma^*$  and  $f$  is non-isometric. Moreover, the pairs of witnesses for  $f$  of minimal distance are  $(10001, 11011)$  and  $(100001, 101101)$ , and both pairs contain a word that is not 11-free. Furthermore,  $f \neq 010$  and then, according to Theorem 25,  $L_4(S) = \{0101, 0100, 0001, 0000, 0010, 1000, 1010\}$  is irredundant.

Theorem 27 shows a new family of subgraphs of the hypercube with minimal Fibonacci dimension, i.e., equal to the isometric dimension. Recall that also the family of trees shares the same property. Let us compare these two families in the following remark. Let  $T_k$  be the tree composed of a root with  $k$  children.

► **Remark 29.** Consider the family of partial cubes  $G_k = Q_k(\{11, f\})$  with  $idim(G_k) = fdim(G_k) = k$ , coming from Theorem 27, and compare it with the family of trees. First, neither of the two families is included in the other one. In fact,  $Q_4(\{11, 1001\})$  in Example 28 is not a tree, whereas the tree  $T_4$  cannot be described as a graph of the form  $G_4 = Q_4(\{11, f\})$ ; recall that  $idim(T_4) = fdim(T_4) = 4$ . In fact, there is no  $f$  such that  $T_4$  coincides with  $\Gamma_4(f)$ . The only vertex of degree 4 in  $\Gamma_4$  is the one labeled 0000, but the vertices at distance 2 from it do not share any factor that is not shared by some other vertex. Finally, the tree  $T_3$  is an example of graph belonging to both families, because it coincides with  $Q_3(\{11, 101\})$ .

The considerations made for  $k = 4$  can be generalized to any higher  $k$ . Therefore, one has that the trees  $T_k$ , with  $k \geq 4$ , cannot be described as  $Q_k(\{11, f\})$ . However, considering sets with more than two words, the following result holds.

► **Proposition 30.** For any  $k \geq 0$ , let  $D_k = \{11, 101, \dots, 10^k1\}$ . Then,  $Q_k(D_{k-2}) = T_k$  and  $idim(Q_k(D_{k-2})) = fdim(Q_k(D_{k-2})) = k$ .

**Proof.** It suffices to note that, by removing from  $Q_k$  the vertices whose labels contain words in  $D_{k-2}$ , the only remaining vertices are those whose labels contain, at most, one occurrence of the symbol 1, i.e.  $Q_k(D_{k-2}) = T_k$ . Therefore,  $idim(Q_k(D_{k-2})) = fdim(Q_k(D_{k-2})) = k$ . ◀

## 4 Conclusions and Future Work

In this paper, generalized Fibonacci cubes are studied in the context of the computation of their isometric and Fibonacci dimensions. It is proved that, if  $f$  is an isometric word, then the isometric dimension of the graph  $Q_k(f)$  is always equal to  $k$ . Moreover, we conjecture that for this type of graphs, the Fibonacci dimension never equals the isometric dimension.

By considering subgraphs of the hypercube of type  $Q_k(S)$ , where  $S$  is a set of words, the study has been restricted to the special case of graphs  $Q_k(\{11, f\})$  and it is shown that when they isometrically embed in  $Q_k$  then they have minimal Fibonacci dimension, namely equal to their isometric dimension, except for  $f = 010$ . The main results provide

sufficient conditions on  $f$  for  $Q_k(\{11, f\})$  to be a partial cube and, accordingly, characterize the announced family. Finally, this family is compared with the family of tree graphs with which it shares the property of minimal Fibonacci dimension.

Also an emblematic example is given, namely the graph  $Q_4(\{11, 1000\})$ , that has several peculiarities. In fact, it does not isometrically embed into  $Q_4$ ; therefore, it does not belong to the above-defined family. Nevertheless, it isometrically embeds in  $Q_5$  and  $\Gamma_5$ ; therefore, it is a partial cube with equal isometric and Fibonacci dimension, equal to 5. We believe that this kind of graphs deserves to be investigated in this setting of understanding properties of graphs with minimal Fibonacci dimension.

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