Research Article

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Existence results of variable exponent double-phase multivalued elliptic inequalities with logarithmic perturbation and convections

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Abstract: In this study, we deal with a multivalued elliptic variational inequality involving a logarithmic perturbed variable exponents double-phase operator. Additionally, it features a multivalued convection term alongside two multivalued terms, one defined within the domain and the other on its boundary. Under the noncoercive framework, we establish the existence results of weak solutions for the multivalued inequality by employing a surjective theorem for multivalued pseudomonotone operators along with the penalty technique. On the other hand, we prove the compactness of solution set by employing the S_+-property of the associated perturbed variable exponent double-phase operator. Finally, we focus on special cases to the multivalued inequality, where K is a bilateral constraint set, and the two multivalued terms are Clarke's generalized gradients with respect to two locally Lipschitz functions.

Keywords: logarithmic perturbed double-phase operator with variable exponents, multivalued convection term, multivalued variational inequality, existence and compactness result, penalty method, bilateral constraint

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1 Introduction

In this article, we concentrate on the multivalued variational inequality given below and establish the corresponding existence and compactness results: find $\pi \in K$ such that

$$\langle J(\pi) + G(\pi, \nabla\pi) + F_1(\pi) + F_2(\pi), v - \pi\rangle \geq 0, \quad \text{for all } v \in K, \tag{1.1}$$

where Ω is a bounded open set in $\mathbb{R}^N (N \geq 2)$ with a Lipschitz boundary $\partial\Omega$, and we divide $\partial\Omega$ into two relatively open subsets: Γ and its complement $\Gamma_0 = \partial\Omega\backslash\Gamma$ such that $\partial\Omega = \Gamma \cup \Gamma_0$. Furthermore, consider K, a closed convex subset of $\mathcal{V}^{\Gamma_0}$ defined as

$$\mathcal{V}^{\Gamma_0} = \{\pi \in W^{1,A}(\Omega) : \pi|_{\Gamma_0} = 0\}, \tag{1.2}$$

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where the definition of space $W^{1,A}(\Omega)$ will be given in Section 2. Moreover, G denotes a lower-order multi-valued operator depending on the gradient of unknown functions (multivalued convection term), formulated by the related multivalued function $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to 2^{\mathbb{R}}\backslash\{\varnothing\}$. However, F_1 and F_2 are two multivalued operators generated by multivalued function $f_1 : \Omega \times \mathbb{R} \to 2^{\mathbb{R}}\backslash\{\varnothing\}$ and the boundary multivalued function $f_2 : \Gamma \times \mathbb{R} \to 2^{\mathbb{R}}\backslash\{\varnothing\}$, respectively. Among (1.1), the nonlinear and nonhomogeneous operator J mapping from $W^{1,A}(\Omega)$ to $W^{1,A}(\Omega)^*$ is given by

$$J(v) \coloneqq -\operatorname{div}\left(\frac{A'(x, |\nabla v|)}{|\nabla v|}\nabla v\right), \tag{1.3}$$

for all $v \in W^{1,A}(\Omega)$ with $A : \Omega \times [0, +\infty) \to [0, +\infty)$ given by

$$A(x, t) = [t^{p(x)} + \vartheta(x)t^{q(x)}]\log(e + kt), \tag{1.4}$$

for all $x \in \Omega$ and all $t \in [0, +\infty)$, in which $p, q \in C(\overline{\Omega})$ fulfilling the assumptions $1 < p(x) < N$, $p(x) \leq q(x)$. In addition, $0 \leq \vartheta(\cdot) \in L^1(\Omega)$, e represents the Euler constant, and k is a positive constant. Note that to make sure the partial differential operator (1.3) is actually a double-phase operator, we assume that the domain where $p(x) < q(x)$ is not a subset to the domain where ϑ equals zero, i.e., $\Omega_< \coloneqq \{x \text{ in } \Omega : p(x) < q(x)\} \subsetneqq \Omega_0 \coloneqq \{x \text{ in } \Omega : \vartheta(x) = 0\}$.

The partial differential double-phase operator with k-logarithmic perturbation specified in (1.3) and (1.4) was introduced by Lu et al. [36], who studied some critical properties of operator (1.3) (including continuity, boundedness, coercivity, strict monotonicity, and S_+-property), as well as some basic results of the Musielak-Orlicz (Sobolev) spaces driven by operator (1.3). One of the main characteristics of the multivalued variational inequality (1.1) is that it involves the so-called double-phase operator (which is fully nonlinear and has unbalance growth) and multivalued convection (this leads to the invalidity of variational method, so, our main method is based on topological approaches). As far as we know, the classical double-phase operator has the following form:

$$\operatorname{div}(|\nabla\pi|^{p-2}\nabla\pi + \vartheta(x)|\nabla\pi|^{q-2}\nabla\pi),$$

with the related energy functional:

$$\pi \mapsto \int_\Omega (|\nabla\pi|^p + \vartheta(x)|\nabla\pi|^q)\mathrm{d}x. \tag{1.5}$$

Motivated by the study associating strongly anisotropic materials, Zhikov [59] first introduced the aforementioned energy functional in the 1980s, since it can well describe the fact that the energy density changes its ellipticity and growth characteristics in response to the specific point x within the domain Ω. Energy functional given by (1.5) and variants of it are called double-phase functional because when it comes to the domain where weighted function $\vartheta(\cdot)$ equals zero, i.e., $\Omega_0 \coloneqq \{x \in \Omega : \vartheta(x) = 0\}$, it will show p-growth instead of q-growth. Nowadays, double-phase problems have gained increasing attention for its broad applications, concerning weighted anisotropic variant of the transonic flow problems, nonlinear Derrick's problem, reaction diffusion systems, nonlinear theory of composite materials, image processing, and so on (see, for instance, [2,4,9,10,27,58]). As for new class of double-phase operators, we refer to Crespo-Blanco et al. [14], who studied the double-phase operator with variable exponents:

$$\operatorname{div}(|\nabla\pi|^{p(x)-2}\nabla\pi + \vartheta(x)|\nabla\pi|^{q(x)-2}\nabla\pi),$$

and showed certain properties of this type of operator and the corresponding Musielak-Orlicz Sobolev spaces. Then, Vetro and Zeng [54] considered a double-phase energy functional exhibiting log L-perturbed and p, q-growth formulated by

$$\operatorname{div}\left(\frac{\mathcal{H}'(x, |\nabla\pi|)}{|\nabla\pi|}\nabla\pi\right),$$

with

$$\mathcal{H} = [t^p + \vartheta(x)t^q]\log(e + t)$$

(see [54] for more details involving the properties of the associated Musielak Orlicz-Sobolev space as well as existence and uniqueness results for the solution of perturbed Dirichlet double-phase problems). Moreover, we can refer to Arora et al. [1] for the detailed study of the logarithmic double-phase-type operator

$$\operatorname{div}\left(|\nabla\pi|^{p(x)-2}\nabla\pi + \vartheta(x)\left[\log(e + |\nabla\pi|) + \frac{|\nabla\pi|}{q(x)(e + |\nabla\pi|)}\right]|\nabla\pi|^{q(x)-2}\nabla\pi\right),$$

and existence and multiplicity results of the equations driven by the aforementioned logarithmic double-phase-type operator. For more impressive works dealing with multivalued double-phase problems without logarithmic perturbation, it could refer to Zeng et al. [55–57]. Particularly, double-phase problems with logarithmic perturbation can be well applied in the study of the plastic theory relating to logarithmic hardening, population dynamic problems as well as quantum mechanics (see Fuchs and Mingione [22], Fuchs and Seregin [23], Marcellini and Papi [37], Seregin and Frehse [48], and Vetro and Winkert [53]).

In particular, we can choose set K in (1.1) as the following bilateral constraint:

$$K = \{\pi \in \mathcal{V}^{\Gamma_0} : \varphi_1(x) \le \pi(x) \le \varphi_2(x), \quad \text{for a.a. } x \in \Omega\},$$

in which $\varphi_1, \varphi_2 \in W^{1,A}(\Omega)$ satisfying $\varphi_1(x) \le 0$ along with $\varphi_2(x) \ge 0$ for a.a. $x \in \Gamma_0$. In fact, the constraint set K defined earlier turns out to be a closed and convex subset of $\mathcal{V}^{\Gamma_0}$. Moreover, we introduce the multivalued operators F_1 and F_2 by the Clarke's generalized gradient corresponding to locally Lipschitz functions j_1 and j_2 respectively; then, the multivalued variational inequality (1.1) becomes equivalent to the next variational-hemivariational inequality:

$$\langle J(\pi), v - \pi\rangle + \int_\Omega \xi(v - \pi)\mathrm{d}x + \int_\Omega j_1^o(x, \pi; v - \pi)\mathrm{d}x + \int_\Gamma j_2^o(x, \pi; v - \pi)\mathrm{d}\varsigma \ge 0, \tag{1.6}$$

for all $v \in K$, where $\xi \in G(\pi, \nabla\pi)$, j_1^o, and j_2^o denote the Clarke's generalized directional derivative of j_1 and j_2, respectively.

As we know, the initial study of variational inequalities originated from calculus of variations, and developed systematically by Fichera [20] and Stampacchia [50,51], in the 1960s to address challenges in mechanics, particularly obstacle problems in elasticity. Following the seminal contributions by Lions and Stampacchia [35], the research concerning variational inequalities developed strongly, for more details with respect to variational inequalities, readers can refer to monographs [29,47,52]. It is widely recognized that variational inequalities are intimately tied to the convexity of involved energy functionals; however, if these functionals are nonconvex but locally Lipschitz, there arises a generalization of variational inequalities, i.e., hemivariational inequalities. In fact, the study for hemivariational inequalities is fundamentally grounded in the theories involving Clarke's generalized gradient of locally Lipschitz functionals, and we point out that the research for Clarke's generalized gradient has its origin in Clarke and Chang [8], Clarke [11,12]. Based on the aforementioned fundamental works, a plenty of impressive works concentrating on variational or hemivariational inequalities appeared. For instance, Carl and Le [5] provided valuable insights into existence results for multivalued variational inequalities and inclusions, where the variational-hemivariational inequalities can be seen as particular cases of multivalued variational inequalities considered. Among multivalued variational inequality (1.1), the well-posedness result related to a type of variational-hemivariational inequality that incorporates constraints and history-dependent operators was studied by Liu et al. [34]. Recently, Carl [6] studied a quasilinear elliptic hemivariational inequality with bilateral constraints and obtained the existence results for weak solutions by appropriately designed penalty technique. For the study of variational problems with bilateral constraints, we refer to Kovalevsky [31,32], and more theoretical results can be found in [13,15,25,41,46]. For the aspect of practical applications, variational inequalities and hemivariational inequalities can be widely applied in various fields, for example, calculus of variations, economics, mathematical physics, and engineering, especially in contact mechanics as well as numerical analysis (see, e.g., previous works [3,26,38–40,49]).

Another intricate aspect of problem (1.1) lies in the presence of the multivalued convection term G, along with the multivalued term F_1 and boundary-defined multivalued term F_2. In fact, equations with multivalued functions can be well applied in various problems of physics (see, Panagiotopoulos [42,43], Carl and Le [5], and their references). Moreover, the convection terms can be well used to model the convection effect of various fluid flows. In single or multiphase flows, convection arises naturally due to the coefficient of material heterogeneity and force influences (such as density and gravity). Generally, the standard variational tools and corresponding theory cannot be applied to problems involving convection terms due to its nonvariability. For more works dealing with problems with single-valued or multivalued convection terms, we refer to Dupaigne et al. [17], El Manouni et al. [18], Figueiredo and Madeira [21], Gasiński and Winkert [24], Liu et al. [34], and Papageorgiou et al. [44]. It is worth noting that, we do not assume any coercive condition throughout this article, but in general, some coercivity is required if we apply the surjective theorems. To overcome the lack of coercivity, usually, the sub- and supersolution method can be applied. However, instead of the sub-supersolution method, motivated by Carl [6], we will employ the penalty technique to construct an auxiliary penalty problem, which possesses the coercivity, and a properly structured penalty operator will be given in the sequel. As far as we know, this is the first work in addressing the multivalued variational inequality, incorporating a logarithmic perturbed double-phase operator and multivalued convection terms, utilizing a penalty-based approach.

Now, let us look at the main assumptions of data throughout this article:

(H_0) Let $p, q \in C(\overline{\Omega})$ be such that $1 < p(x) < N$ and $p(x) \le q(x) < p^*(x)$ for all $x \in \overline{\Omega}$ with $\Omega_< \coloneqq \{x \in \Omega : p(x) < q(x)\} \subsetneq \Omega_0 \coloneqq \{x \in \Omega : \vartheta(x) = 0\}$; also, ϑ is a non-negative function belonging to $L^\infty(\Omega)$.

(H_1) Suppose that $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to 2^{\mathbb{R}}\backslash\{\varnothing\}$, $f_1 : \Omega \times \mathbb{R} \to 2^{\mathbb{R}}\backslash\{\varnothing\}$, and $f_2 : \Gamma \times \mathbb{R} \to 2^{\mathbb{R}}\backslash\{\varnothing\}$ are graph measurable functions with $g(x, \cdot, \cdot) : \mathbb{R} \times \mathbb{R}^N \to 2^{\mathbb{R}}\backslash\{\varnothing\}$, and $f_1(x, \cdot) : \mathbb{R} \to 2^{\mathbb{R}}\backslash\{\varnothing\}$ being upper semicontinuous for a.a. $x \in \Omega$ and $f_2(x, \cdot) : \mathbb{R} \to 2^{\mathbb{R}}\backslash\{\varnothing\}$ being upper semicontinuous for a.a. $x \in \Gamma$.

(H_2) There exist $\sigma \in C(\overline{\Omega})$, $\sigma_1 \in C(\overline{\Omega})$, and $\sigma_2 \in C(\Gamma)$ fulfilling $1 < \sigma(x) < p^*(x)$, $1 < \sigma_1(x) < p^*(x)$, $1 < \sigma_2(x) < p_*(x)$, β, γ, β_1, and β_2 being positive constants and $0 < \alpha \in L^{\sigma'}(\Omega)$, $\alpha_1 \in L^{\sigma_1'}(\Omega)$, $\alpha_2 \in L^{\sigma_2'}(\Gamma)$ such that for all $t \in \mathbb{R}$, $y \in \mathbb{R}^N$ and for a.a. $x \in \Omega$:

$$\sup\{|\xi| : \xi \in g(x, t, y)\} \le \alpha(x) + \beta\, |t|^{\sigma(x)-1} + \gamma\, |y|^{\frac{p(x)}{\sigma'(x)}},$$
$$\sup\{|\eta| : \eta \in f_1(x, t)\} \le \alpha_1(x) + \beta_1\, |t|^{\sigma_1(x)-1},$$

also for all $t \in \mathbb{R}$ and a.a. $x \in \Gamma$:

$$\sup\{|\zeta| : \zeta \in f_2(x, t)\} \le \alpha_2(x) + \beta_2\, |t|^{\sigma_2(x)-1}.$$

In the sequel, we will use the embedding operator $i_{\sigma(\cdot)}$ mapping from $W^{1,A}(\Omega)$ to $L^{\sigma(\cdot)}(\Omega)$ and $i_{\sigma_1(\cdot)}$ mapping from $W^{1,A}(\Omega)$ to $L^{\sigma_1(\cdot)}(\Omega)$, as well as the trace operator $i_{\sigma_2(\cdot)}$ mapping from $W^{1,A}(\Omega)$ to $L^{\sigma_2(\cdot)}(\Gamma)$. According to (H2) together with Proposition 2.2(ii) and (iii) (Section 2), one can demonstrate that operators $i_{\sigma(\cdot)}$, $i_{\sigma_1(\cdot)}$, and $i_{\sigma_2(\cdot)}$ are compact. Let $i^*_{\sigma(\cdot)} : L^{\sigma'(\cdot)}(\Omega) \to W^{1,A}(\Omega)^*$, $i^*_{\sigma_1(\cdot)} : L^{\sigma_1'(\cdot)}(\Omega) \to W^{1,A}(\Omega)^*$, and $i^*_{\sigma_2(\cdot)} : L^{\sigma_2'(\cdot)}(\Gamma) \to W^{1,A}(\Omega)^*$ be the adjoint operators of $i_{\sigma(\cdot)}$, $i_{\sigma_1(\cdot)}$, and $i_{\sigma_2(\cdot)}$, respectively, with $W^{1,A}(\Omega)^*$ being the dual space of $W^{1,A}(\Omega)$. Let $M(\Omega)$ denote the space of measurable functions from Ω to $\mathbb{R}$. For any $\pi \in M(\Omega)$ and $y \in [M(\Omega)]^N$, we define the measurable selections of $g(\cdot, \pi, y)$ as

$$\tilde{g}(\pi) = \{\xi \in M(\Omega) \;:\; \xi(x) \in g(x, \pi(x), y(x)), \quad \text{for a.a. } x \text{ in } \Omega\},$$

invoking hypothesis (H1), we see that the aforementioned set is nonempty. For the same reason, for any $\pi \in M(\Omega)$ (resp., $\pi \in M(\Gamma)$), define measurable selections of $f_1(\cdot, \pi)$ as

$$\tilde{f}_1(\pi) = \{\eta \in M(\Omega) \;:\; \eta(x) \in f_1(x, \pi(x)), \quad \text{for a.a. } x \text{ in } \Omega\},$$

and of $f_2(\cdot, \pi)$:

$$\tilde{f}_2(\pi) = \{\zeta \in M(\Gamma) : \zeta(x) \in f_2(x, \pi(x)), \quad \text{for a.a. } x \text{ in } \Gamma\},$$

which are nonempty.

According to the growth conditions given in (H2), we deduce that for any $\pi \in L^{\sigma(\cdot)}(\Omega)$, and $\nabla\pi \in L^{p(\cdot)}$ (resp., $\pi \in L^{\sigma_1(\cdot)}(\Omega)$, $\pi \in L^{\sigma_2(\cdot)}(\Gamma)$), and $\tilde{g}(\pi) \subset L^{\sigma'(\cdot)}(\Omega)$ (resp., $\tilde{f}_1(\pi) \subset L^{\sigma_1'(\cdot)}(\Omega)$, $\tilde{f}_2(\pi) \subset L^{\sigma_2'(\cdot)}(\Gamma)$); thus, the mapping $\tilde{g}$

is from $L^{\sigma(\cdot)}(\Omega) \times [L^{p(\cdot)(\Omega)}]^N$ to $L^{\sigma'(\cdot)}(\Omega)$ with $\pi \mapsto \tilde{g}(\pi)$ (resp., $\tilde{f}_1$ is from $L^{\sigma_1(\cdot)}(\Omega)$ to $L^{\sigma_1'(\cdot)}(\Omega)$ with $\pi \mapsto \tilde{f}_1(\pi)$, $\tilde{f}_2$ from $L^{\sigma_2(\cdot)}(\Gamma)$ to $L^{\sigma_2'(\cdot)}(\Gamma)$ with $\pi \mapsto \tilde{f}_2(\pi)$). On this basis, we define the multivalued operator as $G = i^*_{\sigma(\cdot)}\tilde{g} : W^{1,A}(\Omega) \to 2^{W^{1,A}(\Omega)^*}$ (resp., $F_1 = i^*_{\sigma_1(\cdot)}\tilde{f}_1 i_{\sigma_1(\cdot)} : W^{1,A}(\Omega) \to 2^{W^{1,A}(\Omega)^*}$, $F_2 = i^*_{\sigma_2(\cdot)}\tilde{f}_2 i_{\sigma_2(\cdot)} : W^{1,A}(\Omega) \to 2^{W^{1,A}(\Omega)^*}$), i.e., $G(\pi, \nabla\pi) = \{\tilde{\xi} \in W^{1,A}(\Omega)^* : \tilde{\xi} \in \tilde{g}(\pi, \nabla\pi)\}$ (resp., $F_1(\pi) = \{\tilde{\eta} \in W^{1,A}(\Omega)^* : \tilde{\eta} \in \tilde{f}_1(\pi)\}$, $F_2(\pi) = \{\tilde{\zeta} \in W^{1,A}(\Omega)^* : \tilde{\zeta} \in \tilde{f}_2(\pi)\}$).

Now, we are ready to introduce the concept of weak solution for the multivalued variational inequality (1.1).

Definition 1.1. We define $\pi \in K$ a weak solution to problem (1.1), if there exist $\tau, \tau_1 \in C(\overline{\Omega})$ and $\tau_2 \in C(\Gamma)$ with $1 < \tau(x) < p^*(x)$, $1 < \tau_1(x) < p^*(x)$ for all $x \in \overline{\Omega}$ as well as $1 < \tau_2(x) < p_*(x)$ for all $x \in \Gamma$ and $\xi \in L^{\tau'(\cdot)}(\Omega)$, $\eta \in L^{\tau_1'(\cdot)}(\Omega)$, $\zeta \in L^{\tau_2'(\cdot)}(\Gamma)$ satisfying the conditions $\xi(x) \in g(x, \pi(x), \nabla\pi(x))$, $\eta(x) \in f_1(x, \pi(x))$ for a.a. $x \in \Omega$ and $\zeta(x) \in f_2(x, \pi(x))$ for a.a. $x \in \Gamma$ such that:

$$\int_\Omega \frac{A(x, |\nabla\pi|)}{|\nabla\pi|}\nabla\pi \cdot \nabla(v - \pi)\mathrm{d}x + \int_\Omega \xi(v - \pi)\mathrm{d}x + \int_\Omega \eta(v - \pi)\mathrm{d}x + \int_\Gamma \zeta(v - \pi)\mathrm{d}\varsigma \geq 0, \tag{1.7}$$

for all $v \in K$.

Among (1.7), the boundary integral $\int_\Gamma \zeta(v - \pi)\mathrm{d}\varsigma$ means that

$$\int_\Gamma \zeta(i_{\tau_2(\cdot)}v\,|_\Gamma - i_{\tau_2(\cdot)}\pi\,|_\Gamma)\mathrm{d}\varsigma,$$

with $i_{\tau_2(\cdot)} : W^{1,A}(\Omega) \to L^{\tau_2(\cdot)}(\partial\Omega)$ representing the trace operator, where $i_{\tau_2(\cdot)}v\,|_\Gamma$ denotes the restriction of $i_{\tau_2(\cdot)}v$ on the boundary Γ.

The structure of this article is outlined as follows. In Section 2, some critical preliminaries will be given, including essential findings related to variable exponent Sobolev spaces and Musielak-Orlicz spaces in the context of the functional A given in (1.4), as well as the definition of penalty operators related to set K. Section 3 is aimed at showing our main existence results with the aid of penalty technique and a surjective theorem related to multivalued pseudomonotone operators. Then, in Section 4, we establish the compactness of the solution set pertaining to (1.1) under suitable assumptions. Finally, in Section 5, we consider the special instances to multivalued variational inequality (1.1), i.e., the set K is defined as a bilateral constraint set, and we choose the multivalued terms F_1 and F_2 defined, respectively, in the domain and its boundary as the Clarke's subdifferential of locally Lipschitz functions.

2 Preliminaries

In this section, let us introduce basic and useful notations and prove conclusions involving the variable exponent Lebesgue space and Musielak-Orlicz spaces formulated by (1.4) (i.e., $L^A(\Omega)$ and $W^{1,A}(\Omega)$), which are pivotal to our proof of the primary findings, most of them taken from Diening et al. [16], Fan and Zhao [19], Harjulehto and Hästö [28], Kováčik and Rákosník [30], and Lu et al. [36].

In the sequel, we choose Ω a bounded domain in $\mathbb{R}^N (N \geq 2)$ with Lipschitz boundary $\partial\Omega$, for a given continuous function ℓ such that $1 < \ell(x)$ for all $x \in \overline{\Omega}$, we define ℓ^- and ℓ^+ by

$$\ell^- \coloneqq \min_{x\in\overline{\Omega}} \ell(x) \quad \text{and} \quad \ell^+ \coloneqq \max_{x\in\overline{\Omega}} \ell(x).$$

Also, let $\ell' \in C(\overline{\Omega})$ be the conjugate of ℓ, i.e., $\frac{1}{\ell(x)} + \frac{1}{\ell'(x)} = 1$ for all $x \in \overline{\Omega}$. Given $\ell \in C(\overline{\Omega})$ with $1 < \ell(x)$ for all $x \in \overline{\Omega}$, we introduce the variable exponent Lebesgue space related to ℓ:

$$L^{\ell(\cdot)}(\Omega) = \{\pi \in M(\Omega) : \rho_{\ell(\cdot)}(\pi) < \infty\},$$

where $\rho_{\ell(\cdot)}$ represents the corresponding modular function:

$$\rho_{\ell(\cdot)}(\pi) = \int_{\Omega} |\pi|^{\ell(x)} \mathrm{d}x \quad \text{for all } \pi \in L^{\ell(x)}(\Omega).$$

Generally, the variable exponent Lebesgue space $L^{\ell(x)}(\Omega)$ is endowed with the Luxemburg norm, which is defined as follows:

$$\|\pi\|_{\ell(\cdot)} = \inf\left\{\lambda > 0 : \int_{\Omega} \left(\frac{|\pi|}{\lambda}\right)^{\ell(x)} \mathrm{d}x \leq 1\right\},$$

and $(L^{\ell(\cdot)}(\Omega), \|\pi\|_{\ell(\cdot)})$ is shown to be a separable and reflexive Banach space. Similarly, one can give the definition of space $(L^{\ell(\cdot)}(\partial\Omega), \|\pi\|_{\ell(\cdot),\partial\Omega})$. The next Hölder-type inequality with respect to $L^{\ell(\cdot)}(\Omega)$ and its dual space $L^{\ell'(\cdot)}(\Omega)$ is a useful tool for inequality estimation:

$$\int_{\Omega} |\pi v| \mathrm{d}x \leq \left[\frac{1}{\ell^{-}} + \frac{1}{(\ell')^{-}}\right] \|\pi\|_{\ell(\cdot)} \|v\|_{\ell'(\cdot)} \leq 2\|\pi\|_{\ell(\cdot)} \|v\|_{\ell'(\cdot)},$$

for all $\pi \in L^{\ell(x)}(\Omega)$, and all $v \in L^{\ell'(x)}(\Omega)$. In addition, if $\ell_1, \ell_2 \in C(\overline{\Omega})$ fulfilling $1 < \ell_1(x) \leq \ell_2(x)$ for all $x \in \overline{\Omega}$, then there holds the continuous embedding:

$$L^{\ell_2(\cdot)}(\Omega) \hookrightarrow L^{\ell_1(\cdot)}(\Omega).$$

Furthermore, we give the definition of variable exponent Sobolev space

$$W^{1,\ell(\cdot)}(\Omega) = \{\pi \in L^{\ell(\cdot)}(\Omega) : |\nabla\pi| \in L^{\ell(\cdot)}(\Omega)\}$$

equipped with the norm

$$\|\pi\|_{1,\ell(\cdot)} = \|\pi\|_{\ell(\cdot)} + \|\nabla\pi\|_{\ell(\cdot)},$$

with $\|\nabla\pi\|_{\ell(\cdot)} \coloneqq \||\nabla\pi|\|_{\ell(\cdot)}$. Hereafter, the completion of $C_0^{\infty}(\Omega)$ under the norm $\|\cdot\|_{1,\ell(\cdot)}$ is given as

$$W_0^{1,\ell(\cdot)}(\Omega) = \overline{C_0^{\infty}(\Omega)}^{\|\cdot\|_{1,\ell(\cdot)}},$$

which is a subspace of $W^{1,\ell(\cdot)}(\Omega)$. Both $W^{1,\ell(\cdot)}(\Omega)$ and $W_0^{1,\ell(\cdot)}(\Omega)$ are separable, reflexive, and uniformly convex Banach spaces. Moreover, for all $\pi \in W_0^{1,\ell(\cdot)}(\Omega)$ and $c_0 > 0$, we have the next Poincaré inequality:

$$\|\pi\|_{\ell(\cdot)} \leq c_0 \|\nabla\pi\|_{\ell(\cdot)}.$$

Thus, we can replace the norm of space $W_0^{1,\ell(\cdot)}(\Omega)$ with the equivalent norm $\|\nabla\pi\|_{\ell(\cdot)}$, i.e.,

$$\|\pi\|_{1,\ell(\cdot),0} = \|\nabla\pi\|_{\ell(\cdot)}, \quad \text{for all } \pi \in W_0^{1,\ell(\cdot)}(\Omega).$$

Now, let us recall the definitions and some crucial properties to the Musielak-Orlicz Lebesgue space $L^{A}(\Omega)$, as well as its associated Musielak-Orlicz Sobolev spaces $W^{1,A}(\Omega)$. To start with, we give the definition for $L^{A}(\Omega)$ as follows:

$$L^{A}(\Omega) = \{\pi \in M(\Omega) : \rho_A(\pi) < +\infty\}.$$

We point out that A is a generalized N-function (see [36, Definition 2.7]); thus, according to Harjulehto-Hästö [28], the related modular function is defined as

$$\rho_A(\pi) = \int_{\Omega} A(x, |\pi|) \mathrm{d}x, \quad \text{for all } \pi \in L^{A}(\Omega).$$

If $L^{A}(\Omega)$ is equipped with the following Luxemburg norm

$$\|\pi\|_A = \inf\left\{\lambda > 0 : \rho_A\left(\frac{\pi}{\lambda}\right) \leq 1\right\},$$

then $(L^{A}(\Omega), \|\pi\|_A)$ is a reflexive and separable Banach space (the precise proof can be found in [36]).

Now, we are ready to examine the relationship with respect to modular ρ_A and the Luxemburg norm $\|\cdot\|_A$, referring to [36, Proposition 2.20].

Proposition 2.1. *If hypothesis (H0) is satisfied, and for all $\pi \in L^A(\Omega)$, the modular function ρ_A is formulated by*

$$\rho_A = \int_\Omega [|\pi|^{p(x)} + \vartheta(x)|\pi|^{q(x)}]\log(e + k|\pi|)\mathrm{d}x,$$

then

(i) $\|\pi\|_A = \lambda \Leftrightarrow \rho_A(\frac{\pi}{\lambda}) = 1$, *with* $\pi \neq 0$;
(ii) $\|\pi\|_A < 1$ (*resp.* , =1, >1) $\Leftrightarrow \rho_A(\pi) < 1$ (*resp.* , =1, >1);
(iii) $\|\pi\|_A < 1 \Rightarrow \|\pi\|_A^{q^+ +1} \leq \rho_A(\pi) \leq \|\pi\|_A^{p^-}$;
(iv) $\|\pi\|_A > 1 \Rightarrow \|\pi\|_A^{p^-} \leq \rho_A(\pi) \leq \|\pi\|_A^{q^+ +1}$;
(v) $\|\pi_n\|_A \to 0 \Leftrightarrow \rho_A(\pi_n) \to 0$;
(vi) $\|\pi_n\|_A \to \infty \Leftrightarrow \rho_A(\pi_n) \to \infty$;
(vii) $\|\pi_n\|_A \to 1 \Leftrightarrow \rho_A(\pi_n) \to 1$;
(viii) $\pi_n \to \pi \in L^A(\Omega) \Rightarrow \rho_A(\pi_n) \to \rho_A(\pi)$.

Analogously, the definition of Musielak-Orlicz Sobolev space $W^{1,A}(\Omega)$ is as follows:

$$W^{1,A}(\Omega) = \{\pi \in L^A(\Omega) : |\nabla\pi| \in L^A(\Omega)\},$$

with the norm given by

$$\|\pi\|_{1,A} = \|\pi\|_A + \|\nabla\pi\|_A, \tag{2.1}$$

where $\|\nabla\pi\|_A = \||\nabla\pi|\|_A$. For the convenience, we use the notation $\|\pi\|$ representing the norm given in (2.1). In addition, the space $W_0^{1,A}(\Omega)$ denotes the completion of $C_0^\infty(\Omega)$ within $W^{1,A}(\Omega)$, and $W_0^{1,A}(\Omega)$ can be endowed with the equivalent norm

$$\|\pi\|_{1,A,0} = \|\nabla\pi\|_A, \quad \text{for all } \pi \in W_0^{1,A}(\Omega),$$

due to the Poincaré inequality given by [36, Proposition 2.23]. Note that $W^{1,A}(\Omega)$ and its subspace $W_0^{1,A}(\Omega)$ are reflexive and separable Banach spaces (proved by [36]). According to [36, Proposition 2.21], if we take the equivalent norm in space $W^{1,A}(\Omega)$, i.e.,

$$\|\pi\|_{\hat{\rho}_A} := \inf\left\{\lambda > 0 : \hat{\rho}_A\left(\frac{\pi}{\lambda}\right) \leq 1\right\},$$

where

$$\hat{\rho}_A(\pi) = \int_\Omega (|\nabla\pi|^{p(x)} + \vartheta(x)|\nabla\pi|^{q(x)})\log(e + k|\nabla\pi|)\mathrm{d}x + \int_\Omega (|\pi|^{p(x)} + \vartheta(x)|\pi|^{q(x)})\log(e + k|\pi|)\mathrm{d}x,$$

for $\pi \in W^{1,A}(\Omega)$; then, the conclusions in Proposition 2.1 still hold true with ρ_A replaced by $\hat{\rho}_A$ and $L^A(\Omega)$ replaced with $W^{1,A}(\Omega)$.

The next proposition gives some embedding results with respect to the variable exponent Lebesgue space, $L^A(\Omega)$ as well as $W^{1,A}(\Omega)$, that can be found in [36, Propositions 2.22 and 2.23]. Recall that $X \hookrightarrow\hookrightarrow Y$ means space X is compactly embedded into space Y.

Proposition 2.2. *Suppose hypothesis (H0) hold, then*

(i) *there hold the continuous embeddings*: $L^A(\Omega) \hookrightarrow L^{\ell(\cdot)}(\Omega)$, $W^{1,A}(\Omega) \hookrightarrow W^{1,\ell(\cdot)}(\Omega)$, *and* $W_0^{1,A}(\Omega) \hookrightarrow W_0^{1,\ell(\cdot)}(\Omega)$ *for all* $\ell \in C(\overline{\Omega})$ *such that* $1 \leq \ell(x) \leq p(x)$ *for all* $x \in \Omega$;
(ii) *there hold the compact embeddings*: $W^{1,A}(\Omega) \hookrightarrow\hookrightarrow L^{\ell(\cdot)}(\Omega)$ *and* $W_0^{1,A}(\Omega) \hookrightarrow\hookrightarrow L^{\ell(\cdot)}(\Omega)$ *for all* $\ell \in C(\overline{\Omega})$ *such that* $1 \leq \ell(x) < p^*(x)$ *for all* $x \in \overline{\Omega}$;

(*iii*) *there hold the compact embeddings*: $W^{1,A}(\Omega) \hookrightarrow\hookrightarrow L^{\ell(\cdot)}(\partial\Omega)$ *and* $W_0^{1,A}(\Omega) \hookrightarrow\hookrightarrow L^{\ell(\cdot)}(\partial\Omega)$ *for all* $\ell \in C(\overline{\Omega})$ *such that* $1 \leq \ell(x) < p_*(x)$ *for all* $x \in \overline{\Omega}$;
(*iv*) *the compact embedding* $W^{1,A}(\Omega) \hookrightarrow\hookrightarrow L^A(\Omega)$ *holds true.*

In the following parts, we define $t_+ \coloneqq \max\{t, 0\}$ for any $t \in \mathbb{R}$ (in the same time, define $t_- \coloneqq -\min\{t, 0\}$ for any $t \in \mathbb{R}$), and let $\pi_\pm(\cdot) \coloneqq [\pi(\cdot)]_\pm$ for any function $\pi : \Omega \to \mathbb{R}$.

Proposition 2.3. [36, Proposition 2.24] *Let* $\pi \in W^{1,A}(\Omega)$, $v \in W_0^{1,A}(\Omega)$, *and a sequence* $\{\pi_n\} \subset W^{1,A}(\Omega)$. *Assume that the hypothesis (H0) hold; thus,*
(*i*) $\pm\pi_\pm \in W^{1,A}(\Omega)$, *and* $\nabla(\pm\pi_\pm) = \nabla\pi 1_{\{\pm\pi>0\}}$;
(*ii*) *if* $\pi_n \to \pi$ in $W^{1,A}(\Omega)$, *then* $\pm(\pi_n)_\pm \to \pm\pi_\pm$ in $W^{1,A}(\Omega)$;
(*iii*) $\pm v_\pm \in W_0^{1,A}(\Omega)$.

The dual spaces of $W^{1,A}(\Omega)$ and $W_0^{1,A}(\Omega)$ are denoted as $W^{1,A}(\Omega)^*$ and $W_0^{1,A}(\Omega)^*$, respectively. Given a Banach space X with its dual space X^*, then define $\mathcal{K}(X^*)$ as

$$\mathcal{K}(X^*) = \{U \subset X^* : U \neq \emptyset, U \text{ is closed and convex}\}.$$

Definition 2.4. Given a real reflexive Banach space X with its dual space X^*, let $\langle\cdot, \cdot\rangle$ be their duality pairing. An operator $B : X \to X^*$ satisfies the following properties:
(i) hemicontinuous if for any $\pi, v, w \in X$, the real-valued function $t \mapsto \langle B(\pi + tv), w\rangle$ is continuous over $\mathbb{R}$;
(ii) monotone (resp., strictly monotone) if $\langle B\pi - Bv, \pi - v\rangle \geq$ (resp., $>$)0 for all $\pi, v \in X$ with $\pi \neq v$;
(iii) (S_+)-property if $\pi_n \rightharpoonup \pi$ in X and $\limsup_{n\to\infty}\langle B\pi_n, \pi_n - \pi\rangle \leq 0$ imply $\pi_n \to \pi$ in X.

Now, let us recall some important properties concerning the logarithmic perturbed variable exponent double-phase operator $J : X \to X^*$ defined as (see also (1.3)):

$$\langle J(\pi), v\rangle \coloneqq \int_\Omega \frac{A'(x, |\nabla\pi|)}{|\nabla\pi|}\nabla\pi \cdot \nabla v \mathrm{d}x, \tag{2.2}$$

for all $\pi, v \in W^{1,A}(\Omega)$, where $X = W^{1,A}(\Omega)$ or $X = W_0^{1,A}(\Omega)$ and $\langle\cdot, \cdot\rangle$ represents the dual pairing between X and X^*. According to [36], we have the following proposition.

Proposition 2.5. *Supposing that hypothesis (H1) holds true, and it can be demonstrated that* A (*given by* (2.2)) *is continuous, bounded, strictly monotone* (*which implies maximal monotonicity*) *and possesses* (S_+) *property.*

In what follows, there are some frequently used definitions with respect to multivalued operators.

Definition 2.6. Let X be a real reflexive Banach space with dual space X^*, and let $\langle\cdot, \cdot\rangle$ denote their duality pairing. Then, operator $B : X \to 2^{X^*}$ is said to be
(i) pseudomonotone iff
  (a) the set $B(\pi)$ is nonempty, bounded, closed, and convex for all $\pi \in X$;
  (b) B is upper semicontinuous from each finite dimensional subspace of X to the weak topology on X^*;
  (c) $(\pi_n) \subset X$ with $\pi_n \rightharpoonup \pi$, and $\pi_n^* \in B(\pi_n)$ being such that $\limsup\langle\pi_n^*, \pi_n - \pi\rangle \leq 0$, imply that there exists $\pi^*(v) \in B(\pi)$ such that

$$\liminf\langle\pi_n^*, \pi_n - v\rangle \geq \langle\pi^*(v), \pi - v\rangle,$$

  for each element $v \in X$.
(ii) generalized pseudomonotone iff $(\pi_n) \subset X$ and $(\pi_n^*) \subset X^*$ with $\pi_n^* \in B(\pi_n)$ being such that $\pi_n \rightharpoonup \pi$ in X, $\pi_n^* \rightharpoonup \pi^*$ in X^* and $\limsup\langle\pi_n^*, \pi_n - \pi\rangle \leq 0$ imply that the element π^* lies in $B(\pi)$ and

$$\langle \pi_n^*, \pi_n \rangle \to \langle \pi^*, \pi \rangle.$$

(iii) coercive iff for $\pi \in D(B)$ satisfying $\|\pi\|_X \to \infty$, there hold

$$\frac{\inf\{\langle \pi^*, \pi \rangle : \pi^* \in B(\pi)\}}{\|\pi\|_X} \to +\infty.$$

(iv) coercive related to K ($K \subset X$ being a closed convex set) if there exists $v_0 \in K$ satisfying

$$\frac{\inf\{\langle \pi^*, \pi - v_0 \rangle : \pi^* \in B(\pi)\}}{\|\pi\|_X} \to +\infty \quad \text{as } \|\pi\|_X \to \infty, \ \pi \in D(B).$$

Remark 2.7. If $B : X \to 2^{X^*}$ is pseudomonotone, then B automatically possesses the generalized pseudomonotonicity. Moreover, if $D(B) = X$, the maximal monotonicity of B guarantees that B is also pseudomonotone.

In addition, if B is a generalized pseudomonotone multivalued operator, then B becomes a pseudomonotone multivalued operator under the following sufficient conditions (see also [5, Proposition 2.18]).

Proposition 2.8. *Suppose that X is a real reflexive Banach space and $B : X \to 2^{X^*}$ is an operator fulfilling*

(i) *For each $\pi \in X$, the set $B(\pi)$ is a non-empty, closed, and convex subset of X^*;*

(ii) *$B : X \to 2^{X^*}$ is bounded;*

(iii) *$B : X \to 2^{X^*}$ is generalized pseudomonotone.*

Then, B is a pseudomonotone operator.

To prove the existence results in Section 3, a penalty operator related to K will be constructed, so we recall the precise definition as follows.

Definition 2.9. Let $K \neq 0$ be a closed and convex subset of a reflexive Banach space X. A bounded, hemicontinuous, and monotone operator $P : X \to X^*$ is called a penalty operator associated with K if

$$P(\pi) = 0 \Leftrightarrow \pi \in K.$$

The next surjective theorem is the critical tool to be applied for showing the main existence theorem in this article (see also [5]).

Theorem 2.10. *Let X be a real reflexive Banach space, and let $T : X \to 2^{X^*}$ be a bounded and pseudomonotone operator, which is coercive in the sense that there exists a real-valued function $c : \mathbb{R}_+ \to \mathbb{R}$ with*

$$c(r) \to +\infty, \quad \text{as } r \to +\infty,$$

such that for all $(\pi, \pi^*) \in Gr(T)$, one has

$$\langle \pi^*, \pi - v_0 \rangle \geq c(\|\pi\|_X)\|\pi\|_X,$$

for some $v_0 \in X$. Then, T is subjective, i.e., range $(T) = X$.

To conclude this section, we recall some important definitions and properties related to Clarke's generalized directional derivative and gradient. Given a real Banach space X and its dual space X^*, then a function $F : X \to \mathbb{R}$ is locally Lipschitz continuous at $\pi \in X$, if it holds

$$|F(u) - F(v)| \leq L_\pi \|u - v\|_X, \quad \text{for all } u, v \in N(\pi),$$

where $L_\pi > 0$ and $N(\pi)$ is the neighborhood of π.

Definition 2.11. Let $B : X \to \mathbb{R}$ be a locally Lipschitz continuous function. We denote by $B^\circ(\pi; v)$ the Clarke's generalized directional derivative of B at the point $\pi \in X$ in the direction $v \in X$, and there hold

$$B^\circ(\pi;\,v)=\limsup_{w\to\pi,t\downarrow 0}\frac{B(w+tv)-B(w)}{t}.$$

Furthermore, the generalized subdifferential operator $\partial B : X \to 2^{X^*}$ of $B : X \to \mathbb{R}$ is formulated as

$$\partial B(\pi)=\{\xi\in X^*: B^\circ(\pi;\,v)\geq\langle\xi,v\rangle_{X^*\times X}\quad\text{for all } v\in X\}\quad\text{for all } \pi\in X.$$

Finally, the proposition involving some critical properties concerning the Clarke's generalized directional derivative and gradient given below will be applied from time to time, and it can be found in [12,45].

Proposition 2.12. *Let $F : X \to \mathbb{R}$ be locally Lipschitz continuous at $\pi \in X$, $L_\pi > 0$ denoting corresponding Lipschitz constant, and there hold the following conclusions*:
(*i*) *function $v \mapsto F^\circ(\pi;\,v)$ is positively homogeneous, subadditive, and satisfies*:

$$|F^\circ(\pi;\,v)|\leq L_\pi\|v\|_X\quad\text{for all } v\in X;$$

(*ii*) *function $(\pi, v) \mapsto F^\circ(\pi;\,v)$ is upper semicontinuous*;
(*iii*) *$\partial F(\pi)$ is a nonempty, convex, and weakly* compact subset of X^* with $\|\xi\|_{X^*} \leq L_\pi$ for all $\xi \in \partial F(\pi)$*;
(*iv*) *$F^\circ(\pi;\,v) = \max\{\langle\xi, v\rangle_{X^*\times X} : \xi \in \partial F(\pi)\}$ for all $v \in X$*;
(*v*) *the multivalued function $X \ni \pi \mapsto \partial F(\pi) \subset X^*$ is upper semicontinuous from X into X^*.*

3 Existence result

Next, we will show the existence result of the multivalued variational inequality (1.1) with the aid of penalty technique under a noncoercive framework, where a proper penalty operator will be constructed to overcome the lack of coercivity.

The following hypothesis states some properties on the penalty operator:

(H_3) Assume there exists a penalty operator P mapping from $W^{1,A}(\Omega)$ to $W^{1,A}(\Omega)^*$ associated with K, also satisfying: for each $\pi \in W^{1,A}(\Omega)$, it holds

$$\langle P(\pi),\pi-v_0\rangle\geq c_1\int_\Omega|\pi|^{\theta(x)}\log(e+k|\pi|)\mathrm{d}x+c_2\int_\Gamma|\pi|^{\sigma_2(x)}\mathrm{d}\varsigma-c_3,\tag{3.1}$$

for all $v_0 \in K$, where c_1, c_2, and c_3 represent positive constants and $\theta \in C(\overline{\Omega})$ is such that $\theta(x) = \max\{q(x), \sigma(x), \sigma_1(x)\}$ for all $x \in \overline{\Omega}$.

For instance, the bilateral constraint set given in Section 5 satisfies assumptions of (H3).

Next, we discuss some useful properties of the considered operators. First, similar to the proof of [7, Proposition 3.1], one can show that $J + G + F_1 + F_2 : W^{1,A}(\Omega) \to 2^{W^{1,A}(\Omega)^*}$ is bounded and pseudomonotone. Moreover, since operator $P : W^{1,A}(\Omega) \to 2^{W^{1,A}(\Omega)^*}$ possesses properties of boundedness, hemicontinuity, and monotonicity, we know that μP is pseudomontone and bounded where $0 < \mu \in \mathbb{R}$. Consequently, the following proposition arises naturally from these properties.

Proposition 3.1. *For any $0 < \mu \in \mathbb{R}$, the multivalued operator $J + G + F_1 + F_2 + \mu P : W^{1,A}(\Omega) \to 2^{W^{1,A}(\Omega)^*}$ is bounded and pseudomonotone.*

We establish the central theorem of this section, namely, a key existence result. We point out that the proof is mainly inspired by Carl [6].

Theorem 3.2. *If hypotheses ($H0$)–($H3$) hold, we obtain that there exists at least one weak solution for multivalued variational inequality* (1.1).

Proof. To begin with, we consider the following penalty problem: find $\pi \in W^{1,A}(\Omega)$ such that

$$0 \in J(\pi) + G(\pi, \nabla\pi) + F_1(\pi) + F_2(\pi) + \mu P(\pi) + \frac{1}{\varepsilon}P(\pi), \quad \text{in } W^{1,A}(\Omega)^*, \tag{3.2}$$

where $\mu, \varepsilon > 0$.

First, we show that penalty problem (3.2) has solutions. To this end, since Proposition (3.1) implies that the penalty operator $J + G + F_1 + F_2 + \mu P : W^{1,A}(\Omega) \to 2^{W^{1,A}(\Omega)^*}$ is bounded and pseudomonotone, we need to show it is also coercive related to K for large enough $\mu > 0$, namely, find $v_0 \in K$ such that

$$\lim_{\|\pi\|\to\infty}\left[\inf_{\substack{\tilde{\xi}\in G(\pi,\nabla\pi)\\ \tilde{\eta}\in F_1(\pi)\\ \tilde{\zeta}\in F_2(\pi)}} \frac{\langle J(\pi) + \tilde{\xi} + \tilde{\eta} + \tilde{\zeta} + \mu P(\pi) + \frac{1}{\varepsilon}P(\pi), \pi - v_0\rangle}{\|\pi\|}\right] = +\infty. \tag{3.3}$$

We point out that in the following parts of this article $\varepsilon_i > 0$, $c_i > 0$, and $C_i > 0$ for $i \in \mathbb{N}^+$; also, C(d) denotes a positive constant depending on argument d.

Under hypothesis (H2), for $\tilde{\xi} \in G(\pi, \nabla\pi) = i^*_{\sigma(\cdot)}\tilde{g}(\pi, \nabla\pi)$, we calculate that

$$\begin{aligned}
|\langle\tilde{\xi}, \pi - v_0\rangle| &= \left|\int_\Omega \tilde{\xi}(\pi - v_0)\mathrm{d}x\right| \\
&\le \int_\Omega\left(\alpha(x) + \beta\,|\pi|^{\sigma(x)-1} + \gamma\,|\nabla\pi|^{\frac{p(x)}{\sigma'(x)}}\right)(|\pi| + |v_0|)\mathrm{d}x \\
&\le \varepsilon_1\int_\Omega|\nabla\pi|^{p(x)}\mathrm{d}x + C(\varepsilon_1, \gamma, \sigma')\int_\Omega|\pi|^{\sigma(x)} + |v_0|^{\sigma(x)}\mathrm{d}x \\
&\quad + \beta\int_\Omega|\pi|^{\sigma(x)}\mathrm{d}x + \varepsilon_2\int_\Omega|\pi|^{\sigma(x)}\mathrm{d}x + C(\varepsilon_2, \sigma)\int_\Omega[\alpha(x)]^{\sigma'(x)}\mathrm{d}x \\
&\quad + \varepsilon_3\int_\Omega|\pi|^{\sigma(x)}\mathrm{d}x + C(\varepsilon_3, \beta, \sigma)\int_\Omega|v_0|^{\sigma(x)}\mathrm{d}x + \int_\Omega\alpha(x)|v_0|\mathrm{d}x \\
&\le \varepsilon_1\int_\Omega|\nabla\pi|^{p(x)}\mathrm{d}x + [C(\varepsilon_1, \gamma, \sigma') + \beta + \varepsilon_2 + \varepsilon_3]\int_\Omega|\pi|^{\sigma(x)}\mathrm{d}x + c_4 \\
&\le \varepsilon_1\int_\Omega|\nabla\pi|^{p(x)}\mathrm{d}x + C_1\int_\Omega|\pi|^{r(x)}\mathrm{d}x + c_5.
\end{aligned} \tag{3.4}$$

For $\tilde{\eta} \in F_1(\pi) = i^*_{\sigma_1(\cdot)}\tilde{f}_1 i_{\sigma_1(\cdot)}(\pi)$, there hold

$$\begin{aligned}
|\langle\tilde{\eta}, \pi - v_0\rangle| &= \left|\int_\Omega \tilde{\eta}(\pi - v_0)\mathrm{d}x\right| \\
&\le \int_\Omega(\alpha_1(x) + \beta_1\,|\pi|^{\sigma_1(x)-1})(|\pi| + |v_0|)\mathrm{d}x \\
&\le \beta_1\int_\Omega|\pi|^{\sigma_1(x)}\mathrm{d}x + \varepsilon_4\int_\Omega|\pi|^{\sigma_1(x)}\mathrm{d}x + C(\varepsilon_4, \sigma)\int_\Omega[\alpha_1(x)]^{\sigma_1'(x)}\mathrm{d}x \\
&\quad + \varepsilon_5\int_\Omega|\pi|^{\sigma_1(x)}\mathrm{d}x + C(\varepsilon_5, \beta_1, \sigma_1)\int_\Omega|v_0|^{\sigma_1(x)}\mathrm{d}x + \int_\Omega\alpha_1(x)|v_0|\mathrm{d}x \\
&\le (\beta_1 + \varepsilon_4 + \varepsilon_5)\int_\Omega|\pi|^{\sigma_1(x)}\mathrm{d}x + c_6 \\
&\le C_2\int_\Omega|\pi|^{\sigma_1(x)}\mathrm{d}x + c_7.
\end{aligned} \tag{3.5}$$

Similarly, for $\tilde{\zeta} \in F_2(\pi) = i^*_{\sigma_2(\cdot)}\tilde{f}_2 i_{\sigma_2(\cdot)}(\pi)$, we have

$$|\langle\tilde{\zeta}, \pi - v_0\rangle| \le (\beta_2 + \varepsilon_6 + \varepsilon_7)\int_\Gamma |\pi|^{\sigma_2(x)}\mathrm{d}\varsigma + c_8 \le C_3\int_\Gamma |\pi|^{\sigma_2(x)}\mathrm{d}\varsigma + c_9. \tag{3.6}$$

We see that the potential functional with respect to J defined by

$$I(\pi) = \int_\Omega (|\nabla\pi|^{p(x)} + \vartheta(x)|\nabla\pi|^{q(x)})\log(e + k|\nabla\pi|)\mathrm{d}x = \int_\Omega A(x, |\nabla\pi|)\mathrm{d}x$$

is convex and satisfies

$$\langle J(\pi), \pi - v_0\rangle \ge I(\pi) - I(v_0) = I(\pi) - c_9. \tag{3.7}$$

Taking (3.1), (3.4), (3.5), (3.6), and (3.7) into account, let $\varepsilon_1 < \frac{1}{2}$ and μ be large enough, and we see that

$$\begin{aligned}
&\langle J(\pi) + \tilde{\xi} + \tilde{\eta} + \tilde{\zeta} + \mu P(\pi) + \frac{1}{\varepsilon}P(\pi), \pi - v_0\rangle \\
&\ge (1 - \varepsilon_1)\int_\Omega (|\nabla\pi|^{p(x)} + \vartheta(x)|\nabla\pi|^{q(x)})\log(e + k|\nabla\pi|) + (|\pi|^{p(x)} + \vartheta(x)|\pi|^{q(x)})\log(e + k|\pi|)\mathrm{d}x \\
&\quad - 2(1 - \varepsilon_1)\max\{1, \|\vartheta\|_\infty\}\int_\Omega |\pi|^{q(x)}\log(e + k|\pi|)\mathrm{d}x - C_1\int_\Omega |\pi|^{\sigma(x)}\mathrm{d}x - C_2\int_\Omega |\pi|^{\sigma_1(x)}\mathrm{d}x \\
&\quad - C_3\int_\Gamma |\pi|^{\sigma_2(x)}\mathrm{d}\varsigma + \mu c_1\int_\Omega |\pi|^{\theta(x)}\log(e + k|\pi|)\mathrm{d}x + \mu c_2\int_\Gamma |\pi|^{\sigma_2(x)}\mathrm{d}\varsigma - c_{10} \\
&\ge \frac{1}{2}\int_\Omega A(x, |\nabla\pi|) + A(x, |\pi|)\mathrm{d}x + C_4\int_\Omega |\pi|^{\theta(x)}\log(e + k|\pi|)\mathrm{d}x + C_5\int_\Gamma |\pi|^{\sigma_2(x)}\mathrm{d}\varsigma - c_{11}, \\
&\ge \frac{1}{2}\int_\Omega A(x, |\nabla\pi|) + A(x, |\pi|)\mathrm{d}x + C_6\|\pi\|^{\theta^-}_{\theta(\cdot)} + C_7\|\pi\|^{\sigma_2^-}_{\sigma_2(\cdot),\Gamma} - c_{12},
\end{aligned} \tag{3.8}$$

for all $\pi \in K$ with $\tilde{\xi} \in G(\pi, \nabla\pi)$, $\tilde{\eta} \in F_1(\pi)$, and $\tilde{\zeta} \in F_2(\pi)$.

Furthermore, we introduce the operator $\mathcal{T}: W^{1,A}(\Omega) \to W^{1,A}(\Omega)^*$ formulated by

$$\langle\mathcal{T}(\pi), v\rangle = \int_\Omega \frac{A'(x, |\nabla\pi|)}{|\nabla\pi|}\nabla\pi \cdot \nabla v\mathrm{d}x + \int_\Omega \frac{A'(x, |\pi|)}{|\pi|}\pi \cdot v\mathrm{d}x,$$

for all $\pi, v \in W^{1,A}(\Omega)$. Referring to [36, Proposition 3.5], we obtain the coercivity of $\mathcal{T}$, i.e.,

$$\lim_{\|\pi\|_{1,A}\to\infty} \frac{1}{\|\pi\|_{1,A}}\int_\Omega A(x, |\nabla\pi|) + A(x, |\pi|)\mathrm{d}x = \infty.$$

This associated with (3.8) derives (3.3). Hence, it is not hard to discover operator $J + G + F_1 + F_2 + \mu P + \frac{1}{\varepsilon}P : W^{1,A}(\Omega) \to 2^{W^{1,A}(\Omega)^*}$ is bounded, pseudomonotone, and coercive related to K. Hence, by applying Theorem 2.10, we infer that the penalty problem admits at least one solution $\hat{\pi}$, i.e., we can find $\hat{\xi} \in G(\hat{\pi}, \nabla\hat{\pi})$, $\hat{\eta} \in F_1(\hat{\pi})$, and $\hat{\zeta} \in F_2(\hat{\pi})$ such that

$$J(\hat{\pi}) + \hat{\xi} + \hat{\eta} + \hat{\zeta} + \mu P(\hat{\pi}) + \frac{1}{\varepsilon}P(\hat{\pi}) = 0, \quad \text{in } W^{1,A}(\Omega)^*. \tag{3.9}$$

Second, we show that the solutions for penalty problem are bounded in $W^{1,A}(\Omega)$. Utilizing the monotonicity of penalty operator P, taking $v_0 \in K$ (implies $P(v_0) = 0$), and testing (3.9) with $\hat{\pi} - v_0$, we obtain

$$\begin{aligned}
0 &= \frac{1}{\|\pi\|}\langle J(\hat{\pi}) + \hat{\xi} + \hat{\eta} + \hat{\zeta} + \mu P(\hat{\pi}) + \frac{1}{\varepsilon}P(\hat{\pi}), \hat{\pi} - v_0\rangle \\
&\ge \frac{1}{\|\pi\|}\langle J(\hat{\pi}) + \hat{\xi} + \hat{\eta} + \hat{\zeta} + \mu P(\hat{\pi}), \hat{\pi} - v_0\rangle.
\end{aligned}$$

By the aforementioned inequality and note that multivalued operator $J + G + F_1 + F_2 + \mu P$ is coercive related to K, we deduce that

$$\|\hat{\pi}\| \le C \quad \text{for any } \varepsilon > 0.$$

Additionally, by the boundedness of $\mathcal{A} : J + G + F_1 + F_2 + \mu P$, we obtain $\|\mathcal{A}\hat{\pi}\| \le C$ for all $\varepsilon > 0$. Therefore, from (3.9), we have

$$P(\hat{\pi}) = \varepsilon(\|\mathcal{A}\hat{\pi}\|) \to 0 \quad \text{in } W^{1,A}(\Omega)^*, \tag{3.10}$$

as $\varepsilon \to 0$.

Finally, choosing a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \downarrow 0$ and letting $\{\hat{\pi}_n\}$ be the sequence of solutions to (3.9) with ε being replaced by ε_n such that $\hat{\pi}_n \rightharpoonup \pi$ in $W^{1,A}(\Omega)$ as $n \to \infty$, we aimed at demonstrating that π solves the multivalued variational inequality (1.1). Utilizing the monotonicity of P, we obtain $\langle P(\hat{\pi}_n) - P(v), \hat{\pi}_n - v\rangle \ge 0$ for any fixed $v \in W^{1,A}(\Omega)$, and then according to (3.10), there hold

$$\lim_{n\to\infty} \langle P(v) - P(\hat{\pi}_n), \hat{\pi}_n - v\rangle = \langle P(v), \hat{\pi} - v\rangle \le 0, \quad \text{for all } v \in W^{1,A}(\Omega). \tag{3.11}$$

Therefore, thanks to the continuity of P, we obtain $\langle P(\pi), z\rangle \le 0$ for any $z \in W^{1,A}(\Omega)$ by taking $v = \pi - \alpha z$ with $\alpha > 0$ and passing to the limit as $\alpha \downarrow 0$ among (3.11). Hence, $P(\pi) = 0$, which indicates $\pi \in K$.

Let $\{\hat{\xi}_n\}$, $\{\hat{\eta}_n\}$, and $\{\hat{\zeta}_n\}$ solve (3.2) with $\varepsilon \coloneqq \varepsilon_n$. For $v \in K$ (indicates $P(v) = 0$), testing (3.2) with $v - \hat{\pi}_n$, by the monotonicity of P, we obtain

$$\langle J(\hat{\pi}_n) + \hat{\xi}_n + \hat{\eta}_n + \hat{\zeta}_n, v - \hat{\pi}_n\rangle = \left(\mu + \frac{1}{\varepsilon}\right)\langle P(v) - P(\hat{\pi}_n), v - \hat{\pi}_n\rangle \ge 0, \tag{3.12}$$

where $\hat{\xi}_n \in G(\hat{\pi}_n, \nabla\hat{\pi}_n)$, $\hat{\eta}_n \in F_1(\hat{\pi}_n)$, and $\hat{\zeta}_n \in F_2(\hat{\pi}_n)$. Note that $\pi \in K$, and (3.12) holds if we take $v = \pi$. Therefore, for $\hat{\pi}_n^* \coloneqq J(\hat{\pi}_n) + \hat{\xi}_n + \hat{\eta}_n + \hat{\zeta}_n \in J(\hat{\pi}_n) + G(\hat{\pi}_n, \nabla\hat{\pi}_n) + F_1(\hat{\pi}_n) + F_2(\hat{\pi}_n)$, there holds

$$\limsup_{n\to\infty}\langle \hat{\pi}_n^*, \hat{\pi}_n - \pi\rangle \le 0.$$

By the pseudomonotonicity of $J + G + F_1 + F_2$, we see that for each $z \in W^{1,A}(\Omega)$ and the corresponding $\pi^*(z) \in J(\pi) + G(\pi, \nabla\pi) + F_1(\pi) + F_2(\pi)$, there hold

$$\liminf_{n\to\infty}\langle \hat{\pi}_n^*, \hat{\pi}_n - z\rangle \ge \langle \pi^*(z), \pi - z,\rangle,$$

namely

$$\langle \pi^*(z), z - \pi,\rangle \ge -\liminf_{n\to\infty}\langle \hat{\pi}_n^*, \hat{\pi}_n - z\rangle = \limsup_{n\to\infty}\langle \hat{\pi}_n^*, z - \hat{\pi}_n\rangle. \tag{3.13}$$

Particularly, for $z = v \in K$, $\hat{\pi}_n^* \coloneqq J(\hat{\pi}_n) + \hat{\xi}_n + \hat{\eta}_n + \hat{\zeta}_n \in J(\hat{\pi}_n) + G(\hat{\pi}_n, \nabla\hat{\pi}_n) + F_1(\hat{\pi}_n) + F_2(\hat{\pi}_n)$, and $\pi^*(v) = J(\pi) + \xi_v + \eta_v + \zeta_v$ with $\xi_v \in G(\pi, \nabla\pi)$, $\eta_v \in F_1(\pi)$ and $\zeta_v \in F_2(\pi)$, (3.13) still holds true, and then we obtain

$$\langle J(\pi) + \xi_v + \eta_v + \zeta_v, v - \pi\rangle \ge \limsup_{n\to\infty}\langle J(\hat{\pi}_n) + \hat{\xi}_n + \hat{\eta}_n + \hat{\zeta}_n, v - \hat{\pi}_n\rangle, \quad \text{for all } v \in K. \tag{3.14}$$

Recalling that $\hat{\pi}_n$ solves (3.2) with $\varepsilon = \varepsilon_n$ and taking $v \in K$ we obtain

$$\langle J(\hat{\pi}_n) + \hat{\xi}_n + \hat{\eta}_n + \hat{\zeta}_n, v - \hat{\pi}_n\rangle = \left(\mu + \frac{1}{\varepsilon}\right)\langle P(v) - P(\hat{\pi}_n), v - \hat{\pi}_n\rangle \ge 0,$$

which in association with (3.14) implies

$$\langle J(\pi_n) + \xi_v + \eta_v + \zeta_v, v - \pi\rangle \ge 0, \quad \text{for all } v \in K,$$

with $\xi_v \in G(\pi, \nabla\pi)$, $\eta_v \in F_1(\pi)$ and $\zeta_v \in F_2(\pi)$. □

4 Compactness result

In this section, we discuss the compactness of the solution set of the multivalued variational inequality (1.1), where the S_+-property of logarithmic perturbed double-phase operator will be utilized.

To guarantee the compactness of the solutions set corresponding to (1.1), we assume the following hypothesis:

(H_4) Let $\mathcal{U}$ be a nonempty set of solutions to (1.1), and suppose that there exist $u_1, u_2 \in W^{1,A}(\Omega)$ such that $\mathcal{U} \subseteq [u_1, u_2]$.

Theorem 4.1. *Let* $(H0)$–$(H4)$ *be satisfied, then* $\mathcal{U}$ *is compact in* $W^{1,A}(\Omega)$.

Proof. Let $\{\pi_n\}$ be a sequence in $\mathcal{U}$, then we see that π_n is bounded in $L^A(\Omega)$. Take $\{\xi_n\} \subset L^{\sigma'(\cdot)}(\Omega)$, $\{\eta_n\} \subset L^{\sigma_1'(\cdot)}(\Omega)$, $\{\zeta_n\} \subset L^{\sigma_2'(\cdot)}(\Gamma)$ as the corresponding sequence satisfying (1.1) with $\pi \coloneqq \pi_n$, i.e., $\pi_n \in K$ such that

$$\langle J(\pi_n) + \xi_n + \eta_n + \zeta_n, v - \pi_n\rangle \geq 0, \quad \text{for all } v \in K,$$

i.e., for all $v \in K$, there hold

$$\langle J(\pi_n), v - \pi_n\rangle + \int_\Omega \xi_n(v - \pi_n)\mathrm{d}x + \int_\Omega \eta_n(v - \pi_n)\mathrm{d}x + \int_\Gamma \zeta_n(v - \pi_n)d\varsigma \geq 0, \tag{4.1}$$

where $\xi_n \in G(\pi_n, \nabla\pi_n)$, $\eta_n \in F_1(\pi_n)$, and $\zeta_n \in F_2(\pi_n)$. The aforementioned inequality along with (3.7) implies that

$$\begin{aligned}
&\int_\Omega \left(|\nabla\pi_n|^{p(x)} + \vartheta\, |\nabla\pi_n|^{q(x)}\right)\log(e + k|\nabla\pi_n|)\mathrm{d}x \\
&\leq I(v) + \int_\Omega \xi_n(v - \pi_n)\mathrm{d}x + \int_\Omega \eta_n(v - \pi_n)\mathrm{d}x + \int_\Gamma \zeta_n(v - \pi_n)d\varsigma \\
&\leq I(v) + \frac{1}{2}\int_\Omega |\nabla\pi_n|^{p(x)}\mathrm{d}x + \left[\beta + \varepsilon_8 + C\left(\frac{1}{2}, l, p\right)\right]\int_\Omega |\pi_n|^{\sigma(x)}\mathrm{d}x + \left(\beta_1, \varepsilon_9\right)\int_\Omega |\pi_n|^{\sigma_1(x)}\mathrm{d}x \\
&\quad + \left(\beta_2, \varepsilon_{10}\right)\int_\Gamma |\pi_n|^{\sigma_2(x)}\mathrm{d}x + c_{13} \\
&\leq \frac{1}{2}\int_\Omega \left(|\nabla\pi_n|^{p(x)} + \vartheta\, |\nabla\pi_n|^{q(x)}\right)\log(e + k|\nabla\pi_n|)\mathrm{d}x + C_8\int_\Omega \max\{|u_1|, |u_2|\}^{\sigma(x)}\mathrm{d}x + C_9\int_\Omega \max\{|u_1|, |u_2|\}^{\sigma_1(x)}\mathrm{d}x \\
&\quad + C_{10}\int_\Gamma \max\{|u_1|, |u_2|\}^{\sigma_2(x)}\mathrm{d}x + c_{13} \\
&\leq \frac{1}{2}\int_\Omega \left(|\nabla\pi_n|^{p(x)} + \vartheta\, |\nabla\pi_n|^{q(x)}\right)\log(e + k|\nabla\pi_n|)\mathrm{d}x + C_{11}\max\{\|u_1\|_{\sigma(\cdot)}^{\sigma^+}, \|u_2\|_{\sigma(\cdot)}^{\sigma^+}\} \\
&\quad + C_{12}\max\{\|u_1\|_{\sigma_1(\cdot)}^{\sigma_1^+}, \|u_2\|_{\sigma_1(\cdot)}^{\sigma_1^+}\} + C_{13}\max\{\|u_1\|_{\sigma_2(\cdot),\Gamma}^{\sigma_2^+}, \|u_2\|_{\sigma_2(\cdot),\Gamma}^{\sigma_2^+}\} + c_{14} \\
&\leq \frac{1}{2}\int_\Omega \left(|\nabla\pi_n|^{p(x)} + \vartheta\, |\nabla\pi_n|^{q(x)}\right)\log(e + k|\nabla\pi_n|)\mathrm{d}x + c_{15}.
\end{aligned} \tag{4.2}$$

Therefore, from (4.2), we know that

$$\|\nabla\pi_n\|_A^{p^-} \leq c_{16}.$$

Thus, $\{\pi_n\}$ is bounded in $W^{1,A}(\Omega)$. Therefore, we can find $\pi_0 \in W^{1,A}(\Omega)$ such that $\pi_n \rightharpoonup \pi_0$ in $W^{1,A}(\Omega)$ in the sense of subsequence, and by the weakly closeness of set K (note that K is closed and convex), we obtain $\pi_0 \in K$. In addition, $\pi_n \to \pi_0$ in $L^A(\Omega)$ and in $L^{\sigma_1(\cdot)}(\Omega)$ (hence, $\pi_n \to \pi_0$ a.e. in Ω); also, $\pi_n|_\Gamma \to \pi_0|_\Gamma$ in $L^{\sigma(\cdot)}(\Gamma)$ (hence, $\pi_n|_\Gamma \to \pi_0|_\Gamma$ a.e. in Γ).

From (H2), we obtain the boundedness of $\{\xi_n\}$ in $L^{\sigma'(\cdot)}(\Omega)$, as well as the boundedness of $\{\eta_n\}$ in $L^{\sigma_1'(\cdot)}(\Omega)$ and $\{\zeta_n\}$ is bounded in $L^{\sigma_2'(\cdot)}(\Gamma)$. Then, we can find some $\xi_0 \in L^{\sigma'(\cdot)}(\Omega)$, $\eta_0 \in L^{\sigma_1'(\cdot)}(\Omega)$, and $\zeta_0 \in L^{\sigma_2'(\cdot)}(\Gamma)$ such that $\xi \rightharpoonup \xi_0$ in $L^{\sigma'(\cdot)}(\Omega)$, $\eta \rightharpoonup \eta_0$ in $L^{\sigma_1'(\cdot)}(\Omega)$, and $\zeta \rightharpoonup \zeta_0$ in $L^{\sigma_2'(\cdot)}(\Gamma)$. Along with the compactness of operators $i_{\sigma(\cdot)}, i_{\sigma_1(\cdot)}, i_{\sigma_2(\cdot)}$, and hence of $i^*_{\sigma(\cdot)}, i^*_{\sigma_1(\cdot)}, i^*_{\sigma_2(\cdot)}$, we obtain $i^*_{\sigma(\cdot)}\xi_n \to i^*_{\sigma(\cdot)}\xi_0$, $i^*_{\sigma_1(\cdot)}\eta_n \to i^*_{\sigma_1(\cdot)}\eta_0$, and $i^*_{\sigma_2(\cdot)}\zeta_n \to i^*_{\sigma_2(\cdot)}\zeta_0$ in $W^{1,A}(\Omega)^*$. Then as $n \to \infty$, there hold

$$\int_\Omega \xi_n(\pi_n - \pi_0)\mathrm{d}x \to 0, \quad \int_\Omega \eta_n(\pi_n - \pi_0)\mathrm{d}x \to 0, \quad \text{and} \quad \int_\Gamma \zeta_n(\pi_n - \pi_0)\mathrm{d}\varsigma \to 0,$$

which in association with (4.1) where $v := \pi_0$ indicate that

$$\limsup_{n\to\infty}\int_\Omega (|\nabla\pi_n|^{p(x)-2}\nabla\pi_n + \vartheta\,|\nabla\pi_n|^{q(x)-2}\nabla\pi_n)\log(e + k|\nabla\pi_n|)\nabla(\pi_n - \pi_0)\mathrm{d}x \le 0. \tag{4.3}$$

Thus, applying the (S_+)-property with respect to operator $J : W^{1,A}(\Omega) \to W^{1,A}(\Omega)^*$, we obtain $\pi_n \to \pi_0$ in $W^{1,A}(\Omega)$. By (4.3) together with the generalized pseudomonotonicity of $G = i^*_{\sigma(\cdot)}\tilde{g}$, $F_1 = i^*_{\sigma_1(\cdot)}\tilde{f}_1 i_{\sigma_1(\cdot)}$, and $F_2 = i^*_{\sigma_2(\cdot)}\tilde{f}_2 i_{\sigma_2(\cdot)}$ from $W^{1,A}(\Omega)$ to $W^{1,A}(\Omega)^*$, one can obtain that $\xi_0 \in i^*_{\sigma(\cdot)}\tilde{g}(\pi_0, \nabla\pi_0)$, $\eta_0 \in i^*_{\sigma_1(\cdot)}\tilde{f}_1 i_{\sigma_1(\cdot)}(\pi_0)$, and $\zeta_0 \in i^*_{\sigma_2(\cdot)}\tilde{f}_2 i_{\sigma_2(\cdot)}(\pi_0)$, as well as

$$\int_\Omega \xi_n\pi_n\mathrm{d}x \to \int_\Omega \xi_0\pi_0\mathrm{d}x, \quad \int_\Omega \eta_n\pi_n\mathrm{d}x \to \int_\Omega \eta_0\pi_0\mathrm{d}x, \quad \text{and} \quad \int_\Gamma \zeta_n\pi_n\mathrm{d}\varsigma \to \int_\Gamma \zeta_0\pi_0\mathrm{d}\varsigma.$$

It along with (4.1) as well as the continuity of J implies that

$$\begin{aligned}&\int_\Omega (|\nabla\pi_0|^{p(x)-2}\nabla\pi_n + \vartheta\,|\nabla\pi_0|^{q(x)-2}\nabla\pi_0)\log(e + k|\nabla\pi_0|)\nabla(v - \pi_0)\mathrm{d}x\\ &\quad + \int_\Omega \xi_n(v - \pi_0)\mathrm{d}x + \int_\Omega \eta_n(v - \pi_0)\mathrm{d}x + \int_\Gamma \zeta_n(v - \pi_0)\mathrm{d}\varsigma \ge 0,\end{aligned}$$

for all $v \in K$, namely,

$$\langle J(\pi_0) + \xi_0 + \eta_0 + \zeta_0, v - \pi_0\rangle \ge 0, \quad \text{for all } v \in K,$$

which indicates $\pi_0 \in \mathcal{U}$, thus demonstrating the compactness for $\mathcal{U}$. □

5 Special cases

In this section, we consider some special instances of the multivalued variational inequality (1.1).

Thereafter, we let K be a set of bilateral constraints given by

$$K = \{\pi \in W^{1,A}(\Omega) : \varphi_1(x) \le \pi(x) \le \varphi_2(x), \quad \text{for a.a. } x \in \mathbb{R}^N\}, \tag{5.1}$$

where $\varphi_1, \varphi_2 \in W^{1,A}(\Omega)$, and take $\Gamma = \partial\Omega$. Apparently, K fulfills the lattice condition, i.e.,

$$K \vee K \subset K \quad \text{and} \quad K \wedge K \subset K.$$

Note that $K \vee K = \{\omega \vee z : \omega \in K, z \in K\}$ with $\omega \vee z = \max\{\omega, z\}$, and similarly, $K \wedge K = \{\omega \wedge z : \omega \in K, z \in K\}$ with $\omega \wedge z = \min\{\omega, z\}$. The penalty operator associated with K can be formulated as a mapping P from $W^{1,A}(\Omega)$ to $W^{1,A}(\Omega)^*$:

$$\begin{aligned}\langle P(\pi), v\rangle = &\int_\Omega ([(\pi - \varphi_2)_+]^{\theta(x)-1} - [(\varphi_1 - \pi)_+]^{\theta(x)-1})\log(e + k|\pi|)v\mathrm{d}x\\ &+ \int_\Gamma ([(\pi - \varphi_2)_+]^{\sigma_2(x)-1} - [(\varphi_1 - \pi)_+]^{\sigma_2(x)-1})v\mathrm{d}\varsigma,\end{aligned} \tag{5.2}$$

in which $v \in W^{1,A}(\Omega)$, $\theta(x) = \max\{p(x), \sigma(x), \sigma_1(x)\}$ for all $x \in \overline{\Omega}$. Actually, the aforementioned integrand can be formulated by $\psi_1(x,t) = [(t-\varphi_2)_+]^{\theta(x)-1}\log(e+k|t|) - [(\varphi_1 - t)_+]^{\theta(x)-1}\log(e+k|t|)$ and $\psi_2(x,t) = [(t-\varphi_2)_+]^{\sigma_2(x)-1} - [(\varphi_1-t)_+]^{\sigma_2(x)-1}$, which is equivalent to

$$\psi_1(x,t) = \begin{cases} (t-\varphi_2)^{\theta(x)-1}\log(e+k|t|), & \text{if } t > \varphi_2,\\ 0, & \text{if } \varphi_1 < t < \varphi_2,\\ -(\varphi_1 - t)^{\theta(x)-1}\log(e+k|t|), & \text{if } t < \varphi_1, \end{cases}$$

and

$$\psi_2(x,t) = \begin{cases} (t-\varphi_2)^{\sigma_2(x)-1}, & \text{if } t > \varphi_2,\\ 0, & \text{if } \varphi_1 < t < \varphi_2,\\ -(\varphi_1 - t)^{\sigma_2(x)-1}, & \text{if } t < \varphi_1. \end{cases}$$

We see that $\psi_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function; also, it turns out to be monotone nondecreasing and satisfying

$$|\psi_1(x,t)| \le c_{17}\,|t|^{\theta(x)-1}\log(e+k|t|) + a_1(x),$$

for a.e. $x \in \Omega$, for all $t \in \mathbb{R}^N$, some $c_{17} > 0$ and $a_1(x) \in L^{\theta(x)+m}$ with $m > 0, \theta(x) + m < p^*(x)$, since $\varphi_1, \varphi_2 \in W^{1,A}(\Omega)$, $W^{1,A}(\Omega) \hookrightarrow L^{\theta(\cdot)+m}(\Omega)$ and there exists small enough $C(m) > 0$ fulfilling $\log(e+k|t|) \le c_{18} + |t|^{C(m)}$ with c_{18} being a positive constant.

Analogously, $\psi_2 : \Omega \times \mathbb{R} \to \mathbb{R}$ is a monotone nondecreasing Carathéodory function fulfilling

$$|\psi_2(x,t)| \le a_2(x) + c_{19}\,|t|^{\sigma_2(x)-1},$$

for a.e. $x \in \Gamma$ and all $t \in \mathbb{R}^N$ with $c_{19} > 0$ and $a_2(x) \in L^{q'(\cdot)}(\Omega)$, since $\varphi_1, \varphi_2 \in W^{1,A}(\Omega)$, in addition, $W^{1,A}(\Omega) \hookrightarrow L^{\sigma_2(\cdot)}(\Omega)$. Thus, the operator P is also bounded, continuous, and monotone. By simple calculation, we obtain $\int_\Omega \psi_1(x,\pi)\pi \mathrm{d}x \ge \int_\Omega |\pi|^{\theta(x)}\log(e+k|\pi|)\mathrm{d}x - c_{20}$ and $\int_\Omega \psi_2(x,\pi)\pi\mathrm{d}x \ge \int_\Omega |\pi|^{\sigma_2(x)}\mathrm{d}x - c_{21}$ for any $\pi \in W^{1,A}(\Omega)$ (see for instance [33, Theorem 3.4]). Moreover, let us check that $P(\pi) = 0$ if and only if $\pi \in K$. First, if $\pi \in K$, then $\psi_1(x,\pi) = 0$ and $\psi_2(x,\pi) = 0$, it is easy to see that $P(\pi) = 0$. Conversely, if $P(\pi) = 0$, then we obtain $\langle P(\pi), v\rangle = 0$ for all $v \in W^{1,A}(\Omega)$. Testing $\langle P(\pi), v\rangle = 0$ with $v = (\pi - \varphi_2)_+$, we obtain

$$\int_\Omega [(\pi-\varphi_2)_+]^{\theta(x)-1}\log(e+k|\pi|)\mathrm{d}x + \int_\Gamma [(\pi-\varphi_2)_+]^{\theta(x)-1}\log(e+k|\pi|)\mathrm{d}\varsigma = 0.$$

Thus, $(\pi - \varphi_2)_+ = 0$, which means $\pi \le \varphi_2$ a.e. in Ω. Analogously, taking $v = (\varphi_1 - \pi)_+$ in $\langle P(\pi), v\rangle = 0$, we can obtain $\pi \ge \varphi_1$ a.e. in Ω. So we infer that P is a penalty operator associated with K.

In addition, we are going to verify that the penalty operator satisfies hypothesis (H3). By the definition of P, using the same method in [33], we have

$$\begin{aligned}\langle P(\pi), \pi - v_0\rangle &= \int_\Omega \psi_1(x,\pi)\pi\mathrm{d}x - \int_\Omega \psi_1(x,t)v_0\mathrm{d}x + \int_\Gamma \psi_2(x,\pi)\pi\mathrm{d}\varsigma - \int_\Gamma \psi_2(x,\pi)v_0\mathrm{d}\varsigma\\ &\ge c_{22}\int_\Omega |\pi|^{\theta(x)}\log(e+k|\pi|)\mathrm{d}x + c_{23}\int_\Gamma |\pi|^{\sigma_2(x)}\mathrm{d}\varsigma - c_{24},\end{aligned}$$

with $c_{22}, c_{23}, c_{24} > 0$.

Furthermore, we defined the multivalued operators F_1 and F_2 with two Clarke's generalized gradients of two locally Lipschitz functions j_1 and j_2:

$$j_1 : \Omega \times \mathbb{R} \to \mathbb{R}, \quad \text{with } (x,t) \mapsto j_1(x,t),$$

and

$$j_2 : \Gamma \times \mathbb{R} \to \mathbb{R} \quad \text{with } (x,t) \mapsto j_2(x,t),$$

namely, $f_1(\pi)(x) \coloneqq \partial j_1(x,\pi)$ and $f_2(\pi)(x) \coloneqq \partial j_2(x,\pi)$.

We suppose j_1 and j_2 satisfy the following assumptions:

(H_5) For every $t \in \mathbb{R}$, the mappings $x \mapsto j_1(x,t)$ and $x \mapsto j_2(x,t)$ are measurable within Ω and on Γ, respectively. Also, for almost every $x \in \Omega$ and $x \in \Gamma$, the functions $t \mapsto j_1(x,t)$ and $t \mapsto j_2(x,t)$ satisfy the local Lipschitz condition in $\mathbb{R}$. Moreover, for $t \in \mathbb{R}$, $\sigma_1 \in C(\overline{\Omega})$, $\sigma_2 \in C(\Gamma)$ with $1 < \sigma_1(x) < p^*(x)$ for all $x \in \overline{\Omega}$ and $1 < \sigma_2(x) < p_*(x)$ for all $x \in \Gamma$, we make the following growth assumptions:

$$\sup\{|\eta| : \eta \in \partial j_1(x,t)\} \leq \alpha_1(x) + \beta_1 |t|^{\sigma_1(x)-1}, \quad \text{for a.a. } x \text{ in } \Omega,$$

and

$$\sup\{|\zeta| : \zeta \in \partial j_2(x,t)\} \leq \alpha_2(x) + \beta_2 |t|^{\sigma_2(x)-1} \quad \text{for a.a. } x \text{ in } \Gamma,$$

where $\beta_1, \beta_2 \geq 0$ and $\alpha_1(x) \in L^{\sigma_1'(\cdot)}(\Omega)$, $\alpha_2 \in L^{\sigma_2'(\cdot)}(\Gamma)$.

We point out that, under the assumption (H5), if we choose $f_1(\pi)(x) = \partial j_1(x,\pi)$ and $f_2(\pi)(x) = \partial j_2(x,\pi)$, then $\pi \in K$ constitutes a solution to (1.6) if and only if it also serves a solution to (1.1), i.e., find $\xi \in G(\pi, \nabla\pi)$, $\eta(x) \in \partial j_1(x,\pi(x))$, and $\zeta(x) \in \partial j_2(x,\pi(x))$ such that

$$\langle J(\pi), v-\pi\rangle + \int_\Omega \xi(v-\pi)\mathrm{d}x + \int_\Omega \eta(v-\pi)\mathrm{d}x + \int_\Gamma \zeta(v-\pi)\mathrm{d}\varsigma \geq 0, \tag{5.3}$$

for all v in K.

Under hypothesis (H5), as done in [6, Lemma 2.5], one can prove that $f_1(x,t) \coloneqq \partial j_1(x,t)$ and $f_2(x,t) \coloneqq \partial j_2(x,t)$ fulfill (H1) and (H2).

Furthermore, to show the equivalence of (1.6) and (5.3), we present the definitions pertaining to sub- and supersolutions for the problem stated in (1.1), thereby clarifying the concepts that will be utilized in our subsequent analysis.

Definition 5.1. Function $\underline{\pi} \in W^{1,A}(\Omega)$ is a subsolution of problem (1.1), provided that there exist $\tau, \tau_1 \in C(\overline{\Omega})$ and $\tau_2 \in C(\Gamma)$ with $1 < \tau(x) < p^*(x)$, $1 < \tau_1(x) < p^*(x)$ for all $x \in \overline{\Omega}$ as well as $1 < \tau_2(x) < p_*(x)$ for all $x \in \Gamma$ and $\underline{\xi} \in L^{\tau'(\cdot)}(\Omega)$, $\underline{\eta} \in L^{\tau_1'(\cdot)}(\Omega)$, $\underline{\zeta} \in L^{\tau_2'(\cdot)}(\Gamma)$ such that:

(i) $\underline{\pi} \vee K \subset K$;

(ii) $\underline{\xi}(x) \in g(x, \underline{\pi}(x), \nabla\underline{\pi}(x))$, $\underline{\eta}(x) \in f_1(x, \underline{\pi}(x))$ for a.a. $x \in \Omega$ and $\underline{\zeta}(x) \in f_2(x, \underline{\pi}(x))$ for a.a. $x \in \Gamma$;

(iii) the inequality

$$\int_\Omega \frac{A'(x, |\nabla\underline{\pi}|)}{|\nabla\underline{\pi}|}\nabla\underline{\pi} \cdot \nabla(v-\underline{\pi})\mathrm{d}x + \int_\Omega \underline{\xi}(v-\underline{\pi})\mathrm{d}x + \int_\Omega \underline{\eta}(v-\underline{\pi})\mathrm{d}x + \int_\Gamma \underline{\zeta}(v-\underline{\pi})\mathrm{d}\varsigma \geq 0$$

holds for all $v \in \underline{\pi} \wedge K$.

Definition 5.2. Function $\overline{\pi} \in W^{1,A}(\Omega)$ is a supersolution of problem (1.1), provided that there exist $\tau, \tau_1 \in C(\overline{\Omega})$ and $\tau_2 \in C(\Gamma)$ with $1 < \tau(x) < p^*(x)$, $1 < \tau_1(x) < p^*(x)$ for all $x \in \overline{\Omega}$ as well as $1 < \tau_2(x) < p_*(x)$ for all $x \in \Gamma$ and $\overline{\xi} \in L^{\tau'(\cdot)}(\Omega)$, $\overline{\eta} \in L^{\tau_1'(\cdot)}(\Omega)$, $\overline{\zeta} \in L^{\tau_2'(\cdot)}(\Gamma)$ such that:

(i) $\overline{\pi} \vee K \subset K$;

(ii) $\overline{\xi}(x) \in g(x, \overline{\pi}(x), \nabla\overline{\pi}(x))$, $\overline{\eta}(x) \in f_1(x, \overline{\pi}(x))$ for a.a. $x \in \Omega$ and $\overline{\zeta}(x) \in f_2(x, \overline{\pi}(x))$ for a.a. $x \in \Gamma$;

(iii)

$$\int_\Omega \frac{A'(x, |\nabla\overline{\pi}|)}{|\nabla\overline{\pi}|}\nabla\overline{\pi} \cdot \nabla(v-\overline{\pi})\mathrm{d}x + \int_\Omega \overline{\xi}(v-\overline{\pi})\mathrm{d}x + \int_\Omega \overline{\eta}(v-\overline{\pi})\mathrm{d}x + \int_\Gamma \overline{\zeta}(v-\overline{\pi})\mathrm{d}\varsigma \geq 0,$$

for all $v \in \overline{\pi} \wedge K$.

Theorem 5.3. *Let* (*H*0) *and* (*H*5) *be satisfied, then* π *is the solution to multivalued variational inequality* (5.3) *if and only if it is a solution to the variational-hemivariational inequality* (1.6).

Proof. Let u be a solution of (1.1). Then, we can find $\eta \in L^{\tau_1'(\cdot)}(\Omega)$ and $\zeta \in L^{\tau_2'(\cdot)}(\Gamma)$ such that

$$\eta(x) \in f_1(x, \pi(x)) = \partial j_1(x, \pi(x)), \quad \text{for a.a. } x \in \Omega,$$
$$\zeta(x) \in f_2(x, \pi(x)) = \partial j_2(x, \pi(x)), \quad \text{for a.a. } x \in \Gamma,$$

and inequality (5.3) holds. According to the definition of ∂j and ∂j_Γ, we see that

$$\begin{aligned} j_1^\circ(x, \pi; v - \pi) &\geq \eta(x)(v - \pi), \quad \text{in } \Omega, \\ j_2^\circ(x, \pi; v - \pi) &\geq \zeta(x)(v - \pi), \quad \text{on } \Gamma, \end{aligned} \tag{5.4}$$

for all $v \in K$. Since ∂j_1 and ∂j_2 fulfill (H1) and (H2), we infer that $j_1^\circ(x, \pi; v - \pi)$ belongs to $L^1(\Omega)$ and $j_2^\circ(x, \pi; v - \pi)$ belongs to $L^1(\Gamma)$. Taking (5.4) and (5.3) into account, we see that (1.6) holds true.

Conversely, let u be a solution of (1.6). We need to verify that π is a subsolution and in the same time a supersolution of inequality (5.3); thus, by [5, Theorem 6.9], we know that π is solution of (5.3). Next, we prove that π is a supersolution to problem (5.3). Since set K fulfills the lattice condition, $v \in \pi \vee K$ means $v = \pi \vee \psi = \pi + (\psi - \pi)_+$ with $\psi \in K$; taking v into (1.6), we obtain

$$\langle J(\pi), (\psi - \pi)_+\rangle + \int_\Omega \xi(\psi - \pi)_+ \mathrm{d}x + \int_\Omega j_1^\circ(x, \pi; (\psi - \pi)_+)\mathrm{d}x + \int_\Gamma j_2^\circ(x, \pi; (\psi - \pi)_+)\mathrm{d}\varsigma \geq 0.$$

Due to the positively homogeneous property (recall Proposition 2.12) of $\varrho \mapsto j_1^\circ(x, t; \varrho)$ (resp., $\varrho \mapsto j_2^\circ(x, t; \varrho)$), the aforementioned inequality is equivalent to

$$\langle J(\pi), (\psi - \pi)_+\rangle + \int_\Omega \xi(\psi - \pi)_+ \mathrm{d}x + \int_\Omega j_1^\circ(x, \pi; 1)(\psi - \pi)_+\mathrm{d}x + \int_\Gamma j_2^\circ(x, \pi; 1)(\psi - \pi)_+\mathrm{d}\varsigma \geq 0,$$

for all $\psi \in K$. Furthermore, take $v \in \pi \wedge K$, then it means $v - \pi = (\psi - \pi)_+$, where $\psi \in K$; hence, the aforementioned inequality means for all $v \in \pi \wedge K$, it holds

$$\langle J(\pi), v - \pi\rangle + \int_\Omega \xi(\psi - \pi)_+ \mathrm{d}x + \int_\Omega j_1^\circ(x, \pi; 1)(v - \pi)\mathrm{d}x + \int_\Gamma j_2^\circ(x, \pi; 1)(v - \pi)\mathrm{d}\varsigma \geq 0. \tag{5.5}$$

Note that Proposition 2.12(iv) as well as hypothesis (H2) guarantees that there exist functions $\bar{\eta} \in L^{\sigma_1'(\cdot)}(\Omega)$, $\bar{\zeta} \in L^{\sigma_2'(\cdot)}(\Gamma)$ such that $\bar{\eta} \in \partial j_1(x, \pi(x))$ in Ω and $\bar{\zeta}(x) \in \partial j_2(x, \pi(x))$ on $\partial\Omega$ satisfying

$$j_1^\circ(x, \pi(x); 1) = \max\{h_1(x) : h_1(x) \in \partial j_1(x, \pi(x))\} =: \bar{\eta}(x) \tag{5.6}$$

for a.a. $x \in \Omega$, and

$$j_2^\circ(x, \pi(x); 1) \max\{h_2(x) : h_2(x) \in \partial j_2(x, \pi(x))\} =: \bar{\zeta}(x) \tag{5.7}$$

for a.a. $x \in \Gamma$. Applying (5.5)–(5.7), we obtain

$$\langle J(\pi), v - \pi\rangle + \int_\Omega \bar{\xi}(\psi - \pi)_+ \mathrm{d}x + \int_\Omega \bar{\eta}(v - \pi)\mathrm{d}x + \int_\Gamma \bar{\zeta}(v - \pi)\mathrm{d}\varsigma \geq 0, \quad \text{for all } v \in \pi \wedge K,$$

and among the aforementioned inequality $\bar{\xi} =: \xi$, so, π is proved to be a supersolution to problem (5.3).

Likewise, one can verify π is also a subsolution of problem (5.3), so the proof is complete. □

Finally, by the definition of the bilateral constraint set K (recall (5.1)), if $\pi \in K$, then we can find a nonempty solution set $\mathcal{U}$ wit respect to (5.3) fulfilling $\mathcal{U} \subseteq [\varphi_1, \varphi_2]$, so (H4) is satisfied. Thus, the following compactness result holds true.

Corollary 5.4. *If hypotheses* (*H*0) *and* (*H*5) *are fulfilled, then the solution set of* (5.3) *is compact.*

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