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Dirichlet μ-Parametric Differential Problem with Multivalued Reaction Term

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Abstract: We study a Dirichlet μ-parametric differential problem driven by a variable competing exponent operator, given by the sum of a negative p-Laplace differential operator and a positive q-Laplace differential operator, with a multivalued reaction term in the sense of a Clarke subdifferential. The parameter $\mu \in \mathbb{R}$ makes it possible to distinguish between the cases of an elliptic principal operator ($\mu \leq 0$) and a non-elliptic principal operator ($\mu > 0$). We focus on the well-posedness of the problem in variable exponent Sobolev spaces, starting with energy functional analysis. Using a Galerkin approach with a priori estimate and embedding results, we show that the functional associated with the problem is coercive; hence, we prove the existence of generalized and weak solutions.

Keywords: competing operator; differential inclusion; existence of generalized and weak solutions; Galerkin method; (p,q)-Laplace differential inclusion

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) be a bounded domain with a smooth boundary $\partial\Omega$. In this paper we study a Dirichlet μ-parametric differential inclusion of the form

$$-\Delta_{p(x)}u(x) + \mu\Delta_{q(x)}u(x) \in \partial F(u) \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = 0. \tag{1}$$

In this problem, the principal operator is given by the sum of the negative p-Laplace differential operator $-\Delta_{p(x)}u = -\text{div}\,(|\nabla u|^{p(x)-2}\nabla u)$ for all $u \in W_0^{1,p(x)}(\Omega)$ and the positive q-Laplace differential operator $\Delta_{q(x)}u = \text{div}\,(|\nabla u|^{q(x)-2}\nabla u)$ for all $u \in W_0^{1,q(x)}(\Omega)$, weighted by the parameter $\mu \in \mathbb{R}$. This parameter separates the case of an elliptic principal operator ($\mu \leq 0$) and the case of a non-elliptic principal operator ($\mu > 0$). Further, we involve in the analysis the variable exponents $p, q \in C(\overline{\Omega})$ under the assumptions

$$\begin{aligned} 1 < q^- &= \inf_{x\in\overline{\Omega}} q(x) \leq q(x) \leq q^+ = \sup_{x\in\overline{\Omega}} q(x) \\ &< p^- = \inf_{x\in\overline{\Omega}} p(x) \leq p(x) \leq p^+ = \sup_{x\in\overline{\Omega}} p(x) < +\infty. \end{aligned}$$

On the right-hand side of the inclusion in (1), we consider the generalized gradient (Clarke subdifferential) ∂F of a locally Lipschitz function $F : \mathbb{R} \to \mathbb{R}$, and we denote with F° its generalized directional derivative. The basic assumption on the data is given as follows:

H_F: there exist positive constants $c, \widehat{c} > 0$ with $\widehat{c}\, p^+ C_{p^-}^{p^-} < p^-$ (see (6) for more information about the constant $C_{p^-} > 0$), such that

$$|z| \leq c + \widehat{c}\,|t|^{p^- - 1} \quad \text{for all } t \in \mathbb{R} \text{ and } z \in \partial F(t).$$

Based on Clarke's subdifferentiation theory (see Clarke [1]) for locally Lipschitz functions, in (1), we retrieve the following hemivariational inequality

$$\langle -\Delta_{p(x)} u(x), h(x) \rangle + \mu \langle \Delta_{q(x)} u(x), h(x) \rangle \leq \int_\Omega F^\circ(u(x); h(x))dx \tag{2}$$

for all $h \in W_0^{1,p(x)}(\Omega)$, where the brackets $\langle \cdot, \cdot \rangle$ refer to the duality between the Banach spaces $W_0^{1,p(x)}(\Omega)$ and $(W_0^{1,p(x)}(\Omega))^*$ introduced in the next section; see also Motreanu [2] for the case of constant exponents p and q. Such an inequality has relevant applications in the context of contact mechanics, and the corresponding theory was developed by Chang [3] and Panagiotopoulos [4]; see also the comprehensive monographies by Carl-Le-Motreanu [5], Motreanu-Panagiotopoulos [6] and Sofonea-Migorski [7]. Turning to the principal operator, for the elliptic case, we recall that the classical operator of the form $-\Delta_p - \Delta_q$ was studied by Faria et al. [8] and Zeng-Papageorgiou [9], in the case of positive solutions to Dirichlet differential equations. In detail, [8] follows a strategy based on a Schauder basis of $W_0^{1,p}(\Omega)$ (constant exponent Sobolev space), and the result is concluded by using a suitable version of strong maximum principle. In [9], the authors use the Leray–Schauder alternative principle in combination with the frozen variable method to control a right-hand side depending on the gradient. Albalawi et al. [10] considered the operator $-\Delta_{p(x)} - \mu\Delta_{q(x)}$, obtaining the existence, uniqueness and asymptotic behavior of weak solutions, as the parameter $\mu \geq 0$ varies. The strategy is based on the use of the surjectivity theorem for pseudomonotone operators in Banach spaces. An interesting generalization of this class of problems is when we assume that μ is a function; this way, we cover the case of the double-phase operator $-\Delta_p - \mu(x)\Delta_q$. Regarding this situation, using a similar approach to that employed in [10], we mention the work by Gasiński-Winkert [11] for the constant exponents p and q, and the very recent work by Vetro-Winkert [12] for the variable exponents p and q, where an additional logarithmic term is also considered. We remark that double-phase-type operators show strongly non-uniform ellipticity at the points where the function $\mu(\cdot)$ is zero; see also the regularity theory in Ragusa-Tachikawa [13]. For the case of a non-elliptic operator, we refer to the studies by Liu et al. [14] involving both the operators $-\Delta_p + \mu\Delta_q$ and $-\Delta_p + \mu(-\Delta)_q^s$, where $s \in (0,1)$. The authors focus on a unifying approach mainly based on Ekeland's variational principle, together with an approximation process involving a Galerkin basis of $W_0^{1,p}(\Omega)$. A key difference among the contributions mentioned above is that in the case of an elliptic principal operator, the authors establish existence results for weak solutions, in a classical sense. Conversely, in the case of a non-elliptic principal operator, the authors need to pose a notion of a suitable generalized solution. This is because the principal operator is not monotone; hence, the known methods to establish weak solutions (for example, the Minty–Browder surjectivity result for monotone-type operators) fail to account for such an operator. So, the notion of a generalized solution (see Definition 1 below) is a way to weaken the monotonicity requirement by dealing with a type of solution that can be constructed as a weak limit of a suitable minimizing sequence in finite-dimensional spaces. This finding depicts a strategy based on an approximation procedure, instead of on the solution of functional equations. The link between the two notions of solutions is properly discussed and commented on in the recent works by Motreanu [15] (constant exponent setting) and Vetro [16] (variable exponent setting). In addition, Ref. [16] considers a single operator $-\Delta_{p(x)}$ weighted by a non-local Kirchhoff-type term and with variable exponent $p \in C(\overline{\Omega})$; see also the references cited therein. Now, we follow a similar research direction with respect to the principal operator in (1) and obtain information about the existence of both weak solutions and generalized solutions to certain classes of differential inclusions. The corresponding constant exponent case is studied by Motreanu [15] (see also Liu et al. [14]) and, as mentioned above, is motivated by the hemivariational inequality context.

In this paper, we consider the following definitions of solutions to problem (1).

Definition 1. *We say that a function $u \in W_0^{1,p(x)}(\Omega)$ is a generalized solution to problem* (1) *if we can find a sequence $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,p(x)}(\Omega)$ satisfying*

(i) $u_n \xrightarrow{w} u$ *in* $W_0^{1,p(x)}(\Omega)$, *as* $n \to +\infty$;

(ii) $-\Delta_{p(x)} u_n + \mu \Delta_{q(x)} u_n - z_n \xrightarrow{w} 0$ *in* $(W_0^{1,p(x)}(\Omega))^*$, *as* $n \to +\infty$, *with* $z_n \in (W_0^{1,p(x)}(\Omega))^*$ *and* $z_n \in \partial F(u_n)$ *a.e. in* Ω;

(iii) $\langle -\Delta_{p(x)} u_n + \mu \Delta_{q(x)} u_n, u_n - u \rangle \to 0$, *as* $n \to +\infty$.

Now, to provide a notion of a weak solution to problem (1), we need to refer to the hemivariational inequality given in (2).

Definition 2. *We say that a function $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution to problem* (1) *if it solves the inequality* (2).

In summary, the first goal of this paper is to establish the existence of a generalized solution to Dirichlet μ-parametric differential inclusion, problem (1), in a general case when $\mu \in \mathbb{R}$, by employing the subdifferentiability theory of locally Lipschitz functions and the theory of local minimizers for functionals. The second goal of the paper is to obtain the existence of a weak solution to problem (1) in a case where $\mu \leq 0$, by using the theory of operators of the monotone type. To the best of our knowledge, this is the first study that considers the variable exponent setting together with an elliptic/non-elliptic differential operator and a multivalued reaction linked to a hemivariational inequality (2). We remark that the main difficulty to overcome is obtaining the boundedness of the energy functional (hence, to obtain the coercivity); for this, we use the crucial inequality $p^- > q^+$ (see the brief discussion in Section 6). The contributions of the paper can be stated as follows.

Theorem 1. *If the growth condition H_F holds, then for all $\mu \in \mathbb{R}$, problem* (1) *admits at least a generalized solution $u \in W_0^{1,p(x)}(\Omega)$.*

Theorem 2. *If the growth condition H_F holds, then for all $\mu \leq 0$, every generalized solution to problem* (1) *is a weak solution.*

These results are obtained by combining the following:

- A priori estimate and embedding results in the Sobolev space $W_0^{1,p(x)}(\Omega)$ (solution space).
- Construction of the sequence $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,p(x)}(\Omega)$ (see Definition 1) in a Galerkin context (approximation procedure).
- Analysis of the energy functional associated with (1) (well-posedness and regularity properties).

The paper is structured as follows. Section 2 collects background material including variable Lebesgue and Sobolev spaces, and nonsmooth analysis. In Section 3, we first introduce the energy functional associated with problem (1) and then examine its properties. In Section 4, we give the proofs of our existence theorems. Finally in Section 5 we show

how problem (1) simplifies in the case of continuous data or in the case of a single-valued reaction term. Section 6 provides some concluding remarks.

2. Preliminaries

The study of problem (1) needs the use of variable Lebesgue and Sobolev spaces. A comprehensive analysis of these spaces is given in the monography by Diening et al. [17]; see also Rădulescu and Repovš [18]. Let $B_1 = \left\{p \in C(\overline{\Omega}) : 1 < p^-\right\}$ and $M(\Omega) = \{u : \Omega \to \mathbb{R}$ be measurable$\}$; hence, we introduce the variable Lebesgue space $L^{p(x)}(\Omega)$ given by

$$L^{p(x)}(\Omega) = \left\{u \in M(\Omega) : \int_\Omega |u(x)|^{p(x)} dx < +\infty\right\}.$$

As usual, we endow the space with the Luxemburg norm

$$\|u\|_{p(x)} := \inf\left\{\lambda > 0 : \rho_{p(x)}\left(\frac{u}{\lambda}\right) \leq 1\right\},$$

where by

$$\rho_{p(x)}(u) := \int_\Omega |u(x)|^{p(x)} dx \quad \text{for all } u \in L^{p(x)}(\Omega), \tag{3}$$

we mean the modular function. So, we recall that the space $L^{p(x)}(\Omega)$ is a separable and reflexive Banach space. The dual space $(L^{p(x)}(\Omega))^*$ is denoted by $L^{p'(x)}(\Omega)$, where $p' \in B_1$ is the Hölder conjugate exponent to $p \in B_1$, namely $p'(x) = p(x)/(p(x)-1)$, for any $x \in \overline{\Omega}$. Then, we have the Hölder-type inequality

$$\int_\Omega |u\, h| dx \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right) \|u\|_{p(x)} \|h\|_{p'(x)} \leq 2\|u\|_{p(x)} \|h\|_{p'(x)}$$

for $u \in L^{p(x)}(\Omega)$ and $h \in L^{p'(x)}(\Omega)$. If $p_1, p_2 \in B_1$ with $p_1(x) \geq p_2(x)$ for all $x \in \overline{\Omega}$, then $L^{p_1(x)}(\Omega) \hookrightarrow L^{p_2(x)}(\Omega)$ is a continuous embedding. Given $p \in B_1$ and using $L^{p(x)}(\Omega)$, we can introduce the variable Sobolev space

$$W^{1,p(x)}(\Omega) := \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}.$$

We endow this space with the norm

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)},$$

where we set $\|\nabla u\|_{p(x)} = \||\nabla u|\|_{p(x)}$, and ∇u is the weak gradient of u. If $p \in B_1$, then we obtain the anisotropic Sobolev space $W_0^{1,p(x)}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{1,p(x)}}$. We know that the Sobolev spaces $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and uniformly convex (hence, reflexive) Banach spaces. Also, for some constant $C_P > 0$, we have the following version of Poincaré inequality:

$$\|u\|_{p(x)} \leq C_P \|\nabla u\|_{p(x)} \text{ for all } u \in W_0^{1,p(x)}(\Omega). \tag{4}$$

Consequently, on $W_0^{1,p(x)}(\Omega)$, we can use the norm

$$\|u\| = \|\nabla u\|_{p(x)} \text{ for all } u \in W_0^{1,p(x)}(\Omega).$$

The norm $\|\cdot\|_{p(x)}$ and the modular function $\rho_{p(x)}(\cdot)$ (see (3)) are closely related by the following proposition.

Proposition 1 ([19], Theorem 1.3)**.** *If $p \in B_1$ and $u \in L^{p(x)}(\Omega)$, then the following hold:*

(i) $\|u\|_{p(x)} < 1\ (= 1, > 1) \Leftrightarrow \rho_{p(x)}(u) < 1\ (= 1, > 1)$;

(ii) *if* $\|u\|_{p(x)} > 1$, *then* $\|u\|_{p(x)}^{p^-} \le \rho_{p(x)}(u) \le \|u\|_{p(x)}^{p^+}$;

(iii) *if* $\|u\|_{p(x)} < 1$, *then* $\|u\|_{p(x)}^{p^+} \le \rho_{p(x)}(u) \le \|u\|_{p(x)}^{p^-}$.

According to Proposition 1, we deduce that

$$\|u\|_{p(x)}^{p^+} + 1 \ge \rho_{p(x)}(u) \ge \|u\|_{p(x)}^{p^-} - 1 \text{ for all } u \in L^{p(x)}(\Omega). \tag{5}$$

Given $p \in B_1$, we introduce the critical Sobolev exponent p^* corresponding to p as follows:

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } N \le p(x), \end{cases} \text{ for all } x \in \overline{\Omega}.$$

The following anisotropic embedding result holds true.

Proposition 2. *If $p, \alpha \in B_1$ with $\alpha(x) < p^*(x)$ for all $x \in \overline{\Omega}$, then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{\alpha(x)}(\Omega)$ is compact.*

According to Proposition 2, we can find a constant $C_\alpha > 0$ satisfying the inequality

$$\|u\|_{\alpha(x)} \le C_\alpha \|\nabla u\|_{p(x)} \text{ for all } u \in W_0^{1,p(x)}(\Omega). \tag{6}$$

Next we introduce the operator $-\Delta_{p(x)} : W_0^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))^*$ defined by

$$\langle -\Delta_{p(x)} u, h \rangle = \int_\Omega |\nabla u(x)|^{p(x)-2} \nabla u(x) \cdot \nabla h(x) dx \quad \text{for all } u, h \in W_0^{1,p(x)}(\Omega),$$

and the operator $\Delta_{q(x)} : W_0^{1,q(x)}(\Omega) \to (W_0^{1,q(x)}(\Omega))^*$ defined by

$$\langle \Delta_{q(x)} u, h \rangle = -\int_\Omega |\nabla u(x)|^{q(x)-2} \nabla u(x) \cdot \nabla h(x) dx \quad \text{for all } u, h \in W_0^{1,q(x)}(\Omega).$$

Both operators $-\Delta_{p(x)}$ and $\Delta_{q(x)}$ (resp. the negative p-Laplace differential operator and the positive q-Laplace differential operator) are bounded (i.e., bounded sets are mapped to bounded sets)and continuous. Further, $-\Delta_{p(x)}$ and $-\Delta_{q(x)}$ are strictly monotone and of type $(S)_+$ (i.e., for the negative p-Laplace differential operator, if $u_n \xrightarrow{w} u$ in $W_0^{1,p(x)}(\Omega)$ and $\limsup_{n\to+\infty} \langle -\Delta_{p(x)} u_n, u_n - u \rangle \le 0$, then $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$).

Now, we need some notions from nonsmooth critical point theory; see, for example, Gasiński-Papageorgiou [20] and Chang [3].

Let X be a Banach space and X^* its topological dual. So, a function $\psi : X \to \mathbb{R}$ is locally Lipschitz if for every $u \in X$, we can find an open neighborhood U of u and a constant $C_U > 0$ satisfying

$$|\psi(x) - \psi(y)| \le C_U \|x - y\| \quad \text{for all } x, y \in U.$$

We know that every continuous convex function $\psi : X \to \mathbb{R}$ is locally Lipschitz. Now, for a locally Lipschitz function $\psi : X \to \mathbb{R}$, we introduce the generalized directional derivative at $u \in X$ in the direction $h \in X$ given by

$$\psi^\circ(u; h) = \limsup_{x \to u, t \downarrow 0} \frac{\psi(x + th) - \psi(x)}{t}.$$

So, $h \to \psi^{\circ}(u;h)$ is convex and sublinear continuous, and hence, according to the Hahn–Banach theorem, one can introduce the set

$$\partial\psi(u) = \{u^* \in X^* : \langle u^*, h\rangle \leq \psi^{\circ}(u;h), \text{ for all } h \in X\},$$

where by $\langle\cdot,\cdot\rangle$, we mean the duality brackets for the pair (X, X^*). The multifunction $u \to \partial\psi(u)$ is the Clarke subdifferential of $\psi(\cdot)$. We remark that if $\psi \in C^1(X)$, then the function $\psi : X \to \mathbb{R}$ is locally Lipschitz with $\partial\psi(u) = \{\psi'(u)\}$ for all $u \in X$. For a locally Lipschitz $\psi(\cdot)$, we say that $u \in X$ is a critical point whenever $0 \in \partial\psi(u)$.

The following proposition summarizes the most important properties of the generalized directional derivative and the Clarke subdifferential for locally Lipschitz functions.

Proposition 3. *Assume that $\psi, \phi : X \to \mathbb{R}$ are locally Lipschitz functions at $u \in X$. Then, the following are true:*

(P1) The function $h \to \psi^{\circ}(u;h)$ is finite, positively homogeneous, subadditive, and satisfies

$$|\psi^{\circ}(u;h)| \leq k\|h\| \quad \textit{for all } h \in X,$$

where k is the locally Lipschitz constant of ψ.

(P2) $(\psi+\phi)^{\circ}(u;h) \leq \psi^{\circ}(u;h) + \phi^{\circ}(u;h)$.

(P3) The sum rules are as follows:

$$\partial(\psi+\phi)(u) \subset \partial\psi(u) + \partial\phi(u).$$

(P4) For every $\alpha \in \mathbb{R}$, one has $(\alpha\psi)^{\circ}(u;h) = \alpha\psi^{\circ}(u;h)$, and hence, $\partial(\alpha\psi)(u) = \alpha\partial\psi(u)$.

(P5) If ψ has a local minimum or maximum at $u \in X$, then $0 \in \partial\psi(u)$.

(P6) The Clarke subdifferential $\partial\psi(u)$ is a nonempty, convex, weak*-compact subset of X^*.

(P7) If $\{u_n\}_{n\in\mathbb{N}}$ and $\{\zeta_n\}_{n\in\mathbb{N}}$ are two sequences in X and X^*, respectively, such that $\zeta_n \in \partial\psi(u_n)$ and $u_n \to u$ in X and $\zeta_n \xrightarrow{w^*} \zeta$, then we have $\zeta \in \partial\psi(u)$.

(P8) Mean-value theorem: If ψ is locally Lipschitz on an open neighborhood containing the segment $[u,v]$, then there exist $w \in (u,v)$ and $\zeta \in \partial\psi(w)$, satisfying

$$\psi(v) - \psi(u) = \langle\zeta, v-u\rangle.$$

Finally, in the existence result, we will use the Galerkin basis of the anisotropic Sobolev space $W_0^{1,p(x)}(\Omega)$ (recall that $W_0^{1,p(x)}(\Omega)$ is a separable Banach space). So, let us introduce this notion as follows.

Definition 3. *We say that a sequence $\{X_n\}_{n\in\mathbb{N}}$ of vector subspaces of $W_0^{1,p(x)}(\Omega)$ is a Galerkin basis of $W_0^{1,p(x)}(\Omega)$ if the following conditions hold:*

(i) $\dim(X_n) < +\infty$ *for all $n \in \mathbb{N}$;*

(ii) $X_n \subseteq X_{n+1}$ *for all $n \in \mathbb{N}$;*

(iii) $\overline{\cup_{n=1}^{\infty} X_n} = W_0^{1,p(x)}(\Omega)$.

3. Energy Functional Analysis

Since we have discussed the framework space in the previous section, now, we introduce the energy functional $J : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ associated with problem (1), and follow a variational approach to show its properties. More precisely, we define the functional

$$J(u) = \int_{\Omega} \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \mu\int_{\Omega} \frac{1}{q(x)}|\nabla u|^{q(x)}dx - \int_{\Omega} F(u(x))dx \tag{7}$$

for all $u \in W_0^{1,p(x)}(\Omega)$. To apply our strategy, we need the functional $\Psi : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\Psi(u) = \int_\Omega F(u(x))dx \quad \text{for all } u \in L^{p(x)}(\Omega) \tag{8}$$

to be Lipschitz continuous on the bounded subsets of $L^{p(x)}(\Omega)$. So, we note that $\Psi : L^{p(x)}(\Omega) \to \mathbb{R}$ given in (8) is Lipschitz on the bounded subsets of $L^{p(x)}(\Omega)$ whenever $F : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function satisfying the growth condition H_F. This way, $\Psi(\cdot)$ has a well-defined Clarke subdifferential $\partial\Psi : L^{p(x)}(\Omega) \to 2^{L^{p'(x)}(\Omega)}$ everywhere. Since $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$ compactly, we can consider $\partial\Psi$ from $W_0^{1,p(x)}(\Omega)$ to $2^{(W_0^{1,p(x)}(\Omega))^*}$.

Now, we see that $J : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ is locally Lipschitz and coercive, namely $J(u) \to +\infty$ as $\|u\| \to +\infty$.

Proposition 4. *If the growth condition H_F holds, then the functional $J(\cdot)$ in (7) is locally Lipschitz and we have*

$$\partial J(u) = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla h dx - \mu \int_\Omega |\nabla u|^{q(x)-2}\nabla u \cdot \nabla h dx - \partial\Psi(u) \tag{9}$$

for all $u, h \in W_0^{1,p(x)}(\Omega)$. Further, $J(\cdot)$ is coercive on $W_0^{1,p(x)}(\Omega)$.

Proof. We first note that $J : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ defined by (7) is locally Lipschitz since the same property holds for $\Psi(\cdot)$. So, we just have to obtain the Clarke subdifferential of $J(\cdot)$ (hence, of $\Psi(\cdot)$), to conclude (9). We still need to show the coercivity of $J(\cdot)$. Hence, using (5) and the continuous embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow W_0^{1,q^+}(\Omega)$, we have

$$\begin{aligned}\int_\Omega |\nabla u|^{q(x)}dx &\le \|\nabla u\|_{q(x)}^{q^+} + 1 \\ &\le C\|\nabla u\|_{p(x)}^{q^+} + 1 \quad (\text{for some } C > 0),\end{aligned} \tag{10}$$

where the positive constant $C > 0$ refers to the embedding of $W_0^{1,p(x)}(\Omega)$ in $W_0^{1,q^+}(\Omega)$. Using assumption H_F and the mean-value theorem (see (P8) of Proposition 3), we obtain

$$|F(t)| \le |F(0)| + c|t| + \frac{\widehat{c}}{p^-}|t|^{p^-} \quad \text{for all } t \in \mathbb{R}. \tag{11}$$

Starting from (7), inequalities (5) and (6), together with the estimates (10) and (11), lead to the following estimate

$$\begin{aligned}J(u) &\ge \frac{1}{p^+}\int_\Omega |\nabla u|^{p(x)}dx - \frac{|\mu|}{q^-}\int_\Omega |\nabla u|^{q(x)}dx - \int_\Omega (c\,|u| + \frac{\widehat{c}}{p^-}|u|^{p^-})dx - |F(0)||\Omega| \\ &\ge \frac{1}{p^+}(\|\nabla u\|_{p(x)}^{p^-} - 1) - \frac{|\mu|}{q^-}(C\|\nabla u\|_{p(x)}^{q^+} + 1) - c\,C_1\|\nabla u\|_{p(x)} \\ &\quad - \frac{\widehat{c}}{p^-}C_{p^-}^{p^-}\|\nabla u\|_{p(x)}^{p^-} - |F(0)||\Omega| \\ &\ge \frac{1}{p^+}(1 - \frac{\widehat{c}\,p^+}{p^-}C_{p^-}^{p^-})\|\nabla u\|_{p(x)}^{p^-} - \frac{|\mu|}{q^-}C\|\nabla u\|_{p(x)}^{q^+} - c\,C_1\|\nabla u\|_{p(x)} \\ &\quad - \frac{1}{p^+} - \frac{|\mu|}{q^-} - |F(0)||\Omega|,\end{aligned}$$

where $|\Omega|$ is the Lebesgue measure of Ω. According to H_F, we know that $\widehat{c}\,p^+C_{p^-}^{p^-} < p^-$, which equivalently gives us

$$1 - \frac{\widehat{c}\,p^+}{p^-}C_{p^-}^{p^-} > 0.$$

Since $p^- > q^+$, we conclude that the functional $J(\cdot)$ is coercive. □

Next, involving a Galerkin basis $\{X_n\}_{n\in\mathbb{N}}$ of $W_0^{1,p(x)}(\Omega)$, we obtain the following result for local minimizers of $J(\cdot)$.

Proposition 5. *If the growth condition H_F holds, then for every $n \in \mathbb{N}$, the functional $J(\cdot)$ in (7) satisfies $J(u_n) = \inf\{J(v) : v \in X_n\}$ for some $u_n \in X_n$ and $z_n \in (W_0^{1,p(x)}(\Omega))^*$ with $z_n \in \partial F(u_n)$ a.e. on Ω satisfying*

$$\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \cdot \nabla h dx - \mu \int_\Omega |\nabla u_n|^{q(x)-2}\nabla u_n \cdot \nabla h dx - \int_\Omega z_n h dx = 0 \tag{12}$$

for all $h \in X_n$.

Proof. According to the definition of the Galerkin basis, we know that

$$\dim(X_n) < +\infty \quad \text{for all } n \in \mathbb{N}.$$

Further, Proposition 4 ensures that $J(\cdot)$ is locally Lipschitz and coercive. It follows that we can find $u_n \in X_n$ such that

$$J(u_n) = \inf\{J(v) : v \in X_n\}, \tag{13}$$

and hence, we have the necessary condition for local minimizer u_n (see (P5) of Proposition 3), namely

$$0 \in \partial J(u_n). \tag{14}$$

By (14), we deduce that there exist $z_n' \in \partial\Psi(u_n)$ such that

$$\langle -\Delta_{p(x)}u_n + \mu\Delta_{q(x)}u_n - z_n', h\rangle = 0. \tag{15}$$

Theorem 2.7.5 and Remark 2.7.6 of Clarke [1] ensure that $\partial\Psi(u_n) \subset \int_\Omega \partial F(u_n)dx$, that is, corresponding to $z_n' \in \partial\Psi(u_n)$, we can find $z_n \in \partial F(u_n)$ a.e. on Ω satisfying

$$\langle z_n', h\rangle = \int_\Omega z_n h dx. \tag{16}$$

Now, using (15) in (16), we conclude that (12) holds true. □

Now, we give a result concerning the boundedness of the minimizer u_n, $n \in \mathbb{N}$, obtained in Proposition 5.

Proposition 6. *If the growth condition H_F holds and $u_n \in W_0^{1,p(x)}(\Omega)$ is a solution of (12), then we have the a priori estimate*

$$\|\nabla u_n\|_{p(x)} \le M_1 \text{ for some } M_1 > 0, \text{ all } n \in \mathbb{N}. \tag{17}$$

Proof. If in (12), we use the test function $h = u_n \in X_n$, (10) and H_F, then

$$\int_\Omega |\nabla u_n|^{p(x)}dx = \mu \int_\Omega |\nabla u_n|^{q(x)}dx + \int_\Omega z_n\, u_n dx$$

$$\leq |\mu|(C\|\nabla u\|_{p(x)}^{q^+}+1)+c\,C_1\|\nabla u_n\|_{p(x)}+\frac{\widehat{c}}{p^-}C_{p^-}^{p^-}\|\nabla u_n\|_{p(x)}^{p^-}.$$

Consequently, we easily obtain

$$(1-\frac{\widehat{c}}{p^-}C_{p^-}^{p^-})\|\nabla u_n\|_{p(x)}^{p^-}\leq |\mu|(C\|\nabla u\|_{p(x)}^{q^+}+1)+c\,C_1\|\nabla u_n\|_{p(x)}$$

for all $n\in\mathbb{N}$. According to H_F, we know that $\widehat{c}\,C_{p^-}^{p^-}<1$, which equivalently gives us

$$1-\frac{\widehat{c}}{p^-}C_{p^-}^{p^-}>0.$$

Since $p^->q^+>1$, we conclude that the norm of ∇u_n is bounded by a constant in $W_0^{1,p(x)}(\Omega)$, namely the a priori estimate on the gradient of u_n given in (17) is true. □

Now, we discuss boundedness of the operator involved in (12).

Proposition 7. *If the growth condition H_F holds and $u_n\in W_0^{1,p(x)}(\Omega)$ is a solution of (12), then we can find a positive constant $M_2>0$ such that*

$$\|-\Delta_{p(x)}u_n+\mu\Delta_{q(x)}u_n-z_n\|_{(W_0^{1,p(x)}(\Omega))^*}\leq M_2 \tag{18}$$

for all $n\in\mathbb{N}$.

Proof. For every function $h\in W_0^{1,p(x)}(\Omega)$, we have

$$\begin{aligned}&|\langle -\Delta_{p(x)}u_n+\mu\Delta_{q(x)}-z_n,h\rangle|\\ =&\left|\int_\Omega|\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla h dx-\mu\int_\Omega|\nabla u_n|^{q(x)-2}\nabla u_n\cdot\nabla h dx-\int_\Omega z_n h dx\right|\\ \leq&\int_\Omega|\nabla u_n|^{p(x)-1}|\nabla h|dx+|\mu|\int_\Omega|\nabla u_n|^{q(x)-1}|\nabla h|dx+\int_\Omega|z_n||h|dx.\end{aligned}\tag{19}$$

With respect to the modular function $\rho_{p(x)}(\cdot)$ in (3), we obtain $\rho_{p'(x)}(|\nabla u_n|^{p(x)-1})=\rho_{p(x)}(|\nabla u_n|)$, and hence, we can find $\alpha>0$ such that

$$\||\nabla u_n|^{p(x)-1}\|_{p'(x)}\leq\|\nabla u_n\|_{p(x)}^{\alpha}.$$

Using this inequality, we estimate the first two terms on the right-hand side of (19); hence, we obtain

$$\begin{aligned}\int_\Omega|\nabla u_n|^{p(x)-1}|\nabla h|dx&\leq 2\||\nabla u_n|^{p(x)-1}\|_{p'(x)}\|\nabla h\|_{p(x)}\\ &\leq 2\|\nabla u_n\|_{p(x)}^{\alpha}\|\nabla h\|_{p(x)}\\ &\leq\widehat{M}\|\nabla h\|_{p(x)}\end{aligned}\tag{20}$$

for some $\widehat{M}>0$, recall (17).

Further, we have

$$\begin{aligned}\int_\Omega|\nabla u_n|^{q(x)-1}|\nabla h|dx&\leq\int_\Omega(1+|\nabla u_n|^{p(x)-1})|\nabla h|dx\\ &\leq C_1\|\nabla h\|_{p(x)}+2\|\nabla u_n\|_{p(x)}^{\alpha}\|\nabla h\|_{p(x)}\\ &\leq\widetilde{M}\|\nabla h\|_{p(x)}\end{aligned}\tag{21}$$

for some $\widetilde{M} > 0$, see (17).

We must now estimate the third term in the right-hand side of (19); hence, the Poincaré inequality (4) gives us

$$\begin{aligned}\int_\Omega |z_n||h|dx &\leq \int_\Omega (c + \widehat{c}|u_n|^{p^- -1})|h|dx \\ &\leq \int_\Omega (c + \widehat{c} + \widehat{c}|u_n|^{p(x)-1})|h|dx \\ &\leq (c + \widehat{c})C_1\|\nabla h\|_{p(x)} + 2\widehat{c}\,C_P^{\beta+1}\|\nabla u_n\|^\beta_{p(x)}\|\nabla h\|_{p(x)} \quad (\text{for some } \beta > 0) \\ &\leq \bar{M}\|\nabla h\|_{p(x)} \end{aligned} \tag{22}$$

for some $\bar{M} > 0$, see again (17).

Now, turning to the inequality (19), and hence, combining the estimates (20)–(22), we obtain the following:

$$|\langle -\Delta_{p(x)}u_n + \mu\Delta_{q(x)} - z_n, h\rangle| \leq (\widehat{M} + \widetilde{M} + \bar{M})\|\nabla h\|_{p(x)}.$$

We conclude that (18) holds with a positive constant $M_2 = \widehat{M} + \widetilde{M} + \bar{M}$. □

Next, we establish the existence of a minimizing sequence for the functional $J(\cdot)$ defined by (7).

Proposition 8. *If the growth condition H_F holds and $u_n \in X_n \subset W_0^{1,p(x)}(\Omega)$ is a solution of (12), then the corresponding sequence $\{u_n\}_{n\in\mathbb{N}}$ satisfies*

$$\lim_{n\to+\infty} J(u_n) = \inf\{J(v) : v \in W_0^{1,p(x)}(\Omega)\}. \tag{23}$$

Proof. From the proof of Proposition 5, we already have $u_n \in X_n$ satisfying

$$J(u_n) = \inf\{J(v) : v \in X_n\} \quad (\text{i.e., } (13)),$$

and we know that $X_n \subseteq X_{n+1}$ for all $n \in \mathbb{N}$ (i.e., property (ii) of the Galerkin basis $\{X_n\}_{n\in\mathbb{N}}$). We deduce that $\{J(u_n)\}_{n\in\mathbb{N}}$ is a nonincreasing bounded sequence, so it admits limit $\ell \in \mathbb{R}$, namely

$$\ell := \lim_{n\to+\infty} J(u_n).$$

Now, in contradiction to (23), we can assume that

$$\begin{aligned} &\ell > \inf\{J(v) : v \in W_0^{1,p(x)}(\Omega)\}, \\ \Rightarrow\quad &\exists\, v^* \in W_0^{1,p(x)}(\Omega) \;:\; J(v^*) < \ell. \end{aligned}$$

Since $J(v)$ is a continuous functional, we can find U an open neighborhood of v^* in $W_0^{1,p(x)}(\Omega)$ so that

$$J(v) < \ell \text{ for all } v \in U \subset W_0^{1,p(x)}(\Omega). \tag{24}$$

Again, according to Definition 3(iii), we obtain

$$\begin{aligned} &W_0^{1,p(x)}(\Omega) = \overline{\bigcup_{n=1}^{\infty} X_n}, \\ \Rightarrow\quad &U \cap \left(\bigcup_{n=1}^{+\infty} X_n\right) \neq \emptyset, \\ \Rightarrow\quad &\exists\, \widehat{v} \in U \cap X_{n_0} \text{ for some } n_0 \in \mathbb{N}, \text{ such that (24) holds.} \end{aligned}$$

This way, we can merge the results in (13) and (24) to obtain

$$\inf\{J(v) : v \in X_{n_0}\} \leq J(\widehat{v}) < \ell \leq \inf\{J(v) : v \in X_{n_0}\}.$$

So, we reach a contradiction, which shows that (23) holds true. □

4. Existence Theorems

In this section, using the auxiliary results in Section 3, we establish our existence results. More precisely, we first prove Theorem 1, which says that for all $\mu \in \mathbb{R}$, problem (1) admits at least a generalized solution in $W_0^{1,p(x)}(\Omega)$.

Proof of Theorem 1. From Propositions 5 and 6 we infer that

$$\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,p(x)}(\Omega) \text{ is bounded.}$$

So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p(x)}(\Omega) \text{ and } u_n \to u \text{ in } L^{p(x)}(\Omega), \text{ as } n \to +\infty.$$

So, Definition 1(i) is established.

From Propositions 5 and 7, we infer that

$$\{-\Delta_{p(x)}u_n + \mu\Delta_{q(x)}u_n - z_n\}_{n\in\mathbb{N}} \subset (W_0^{1,p(x)}(\Omega))^* \text{ is bounded,}$$

where $z_n \in (W_0^{1,p(x)}(\Omega))^*$. So, we may assume that

$$-\Delta_{p(x)}u_n + \mu\Delta_{q(x)}u_n - z_n \xrightarrow{w} y \text{ in } (W_0^{1,p(x)}(\Omega))^* \text{ as } n \to +\infty \tag{25}$$

for some $y \in (W_0^{1,p(x)}(\Omega))^*$.

In (12), we choose $h \in \cup_{n=1}^{+\infty} X_n$, pass to the limit as $n \to +\infty$ and use (25). Since $h \in X_n$ for $n > n_0$ for some $n_0 \in \mathbb{N}$, we obtain

$$\begin{aligned}
&\langle -\Delta_{p(x)}u_n, h\rangle + \langle \mu\Delta_{q(x)}u_n, h\rangle - \int_\Omega z_n h dx = 0,\\
\Rightarrow\quad &\lim_{n\to+\infty} \langle -\Delta_{p(x)}u_n + \mu\Delta_{q(x)}u_n - z_n, h\rangle = 0,\\
\Rightarrow\quad &\langle y, h\rangle = 0,\\
\Rightarrow\quad &y = 0 \quad \text{(according to Definition 3 (iii)).}
\end{aligned}$$

So, we conclude that

$$-\Delta_{p(x)}u_n + \mu\Delta_{q(x)}u_n - z_n \xrightarrow{w} 0 \quad \text{in } (W_0^{1,p(x)}(\Omega))^* \text{ as } n \to +\infty, \tag{26}$$

Hence, Definition 1(ii) is established too.

In (12), we choose $h = u_n - u \in W_0^{1,p(x)}(\Omega)$, pass to the limit as $n \to +\infty$ and use (26). Due to the properties of the sequences $\{u_n\}_{n\in\mathbb{N}}$ and $\{z_n\}_{n\in\mathbb{N}}$, we deduce that

$$\lim_{n\to+\infty} \int_\Omega z_n(u_n - u)dx = 0,$$

; then, we obtain

$$\lim_{n\to+\infty} \left[\langle -\Delta_{p(x)}u_n, u_n - u\rangle + \mu\Delta_{q(x)}u_n, u_n - u\rangle - \int_\Omega z_n(u_n - u)dx\right] = 0,$$

$$\Rightarrow \quad \lim_{n\to+\infty}\Big[\langle -\Delta_{p(x)}u_n, u_n-u\rangle + \mu\langle \Delta_{q(x)}u_n, u_n-u\rangle\Big] = 0,$$

$$\Rightarrow \quad \text{Definition 1(iii) holds true.}$$

So, we have proved that for all $\mu \in \mathbb{R}$, problem (1) admits a generalized solution $u \in W_0^{1,p(x)}(\Omega)$. $\square$

Next, we establish Theorem 2, which shows that in a case where $\mu \le 0$ (principal elliptic operator), the generalized solution $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution in the sense of Definition 2.

Proof of Theorem 2. Since $-\Delta_{q(x)}$ is a monotone operator (that is, $\langle -\Delta_{q(x)}u + \Delta_{q(x)}v, u-v\rangle \ge 0$ for all $u, v \in W_0^{1,q(x)}(\Omega)$), using Definition 1(i,iii) and $\mu \le 0$, we obtain

$$\begin{aligned}
&\limsup_{n\to+\infty}\langle -\Delta_{p(x)}u_n, u_n-u\rangle \\
&= \limsup_{n\to+\infty}\big[\langle -\Delta_{p(x)}u_n + \mu\Delta_{q(x)}u_n, u_n-u\rangle + \mu\langle -\Delta_{q(x)}u_n + \Delta_{q(x)}u, u_n-u\rangle \\
&\quad + \mu\langle -\Delta_{q(x)}u, u_n-u\rangle\big] \\
&\le \limsup_{n\to+\infty}\langle -\Delta_{p(x)}u_n + \mu\Delta_{q(x)}u_n, u_n-u\rangle + \mu \lim_{n\to+\infty}\langle -\Delta_{q(x)}u, u_n-u\rangle = 0.
\end{aligned}$$

Invoking the $(S)_+$-property and the continuity of the negative p-Laplace differential operator $-\Delta_{p(x)} : W_0^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))^*$, we infer that

$$\begin{aligned}
&u_n \to u \quad \text{in } W_0^{1,p(x)}(\Omega), \\
\Rightarrow \quad &-\Delta_{p(x)}u_n \to -\Delta_{p(x)}u \text{ in } (W_0^{1,p(x)}(\Omega))^* \text{ as } n \to +\infty.
\end{aligned}$$

Further, we have

$$\begin{aligned}
&u_n \to u, \\
\Rightarrow \quad &\lim_{n\to+\infty} -\Delta_{q(x)}u_n = -\Delta_{q(x)}u \quad \text{in } (W_0^{1,q(x)}(\Omega))^*.
\end{aligned}$$

From Proposition 5 and Proposition 7, we infer that

$$\{z_n\}_{n\in\mathbb{N}} \subset (W_0^{1,p(x)}(\Omega))^* \text{ is bounded.}$$

So, we may assume that

$$z_n \xrightarrow{w} z \text{ in } (W_0^{1,p(x)}(\Omega))^* \text{ as } n \to +\infty.$$

Now, we note that $z \in \partial F(u)$ a.e. on Ω (see (P7) of Proposition 3). So, the weak convergence in (26), $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$ and $z_n \in \partial F(u_n) \subset (W_0^{1,p(x)}(\Omega))^*$ imply that

$$-\Delta_{p(x)}u + \mu\Delta_{q(x)}u - z = 0 \quad \text{in } (W_0^{1,p(x)}(\Omega))^*,$$

where $z \in \partial F(u) \subset (W_0^{1,p(x)}(\Omega))^*$ a.e. on Ω.

Consequently, the hemivariational inequality (2) is solved and so the generalized solution $u \in W_0^{1,p(x)}(\Omega)$ is indeed a weak solution to problem (1). $\square$

5. Special Case

According to Clarke's theory [1] and Chang [3] (Example 1) (see also Motreanu [2]), we consider the data $f \in L^{\infty}_{\text{loc}}(\mathbb{R})$. Hence, we introduce the locally Lipschitz function $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(t) = \int_0^t f(s)ds \quad \text{for all } t \in \mathbb{R}. \tag{27}$$

Involving the notation

$$\underline{f}_{\delta}(t) = \text{essinf}_{|\tau - t| < \delta} f(\tau) \quad \text{for all } t \in \mathbb{R}$$

and

$$\overline{f}_{\delta}(t) = \text{esssup}_{|\tau - t| < \delta} f(\tau) \quad \text{for all } t \in \mathbb{R}$$

then, the generalized directional derivative at u in the direction h is as follows:

$$\begin{aligned} F^{\circ}(u;h) = \limsup_{x \to u, t \downarrow 0} \frac{F(x+th) - F(x)}{t} &= \limsup_{x \to u, t \downarrow 0} \frac{1}{t} \int_x^{x+th} f(s)ds \\ &\leq \begin{cases} \lim\limits_{\delta \to 0} \overline{f}_{\delta}(x)h & \text{if } h > 0, \\ \lim\limits_{\delta \to 0} \underline{f}_{\delta}(x)h & \text{if } h < 0. \end{cases} \end{aligned}$$

So, the Clarke subdifferential $\partial F(t)$ is given as

$$\partial F(t) = \left[\lim_{\delta \to 0} \underline{f}_{\delta}(t), \lim_{\delta \to 0} \overline{f}_{\delta}(t) \right], \tag{28}$$

which is a compact real interval. This way, we obtain the μ-parametric inclusion

$$-\Delta_{p(x)} u(x) + \mu \Delta_{q(x)} u(x) \in \left[\lim_{\delta \to 0} \underline{f}_{\delta}(t), \lim_{\delta \to 0} \overline{f}_{\delta}(t) \right] \quad \text{in } \Omega. \tag{29}$$

Now, for continuous data f, from (29), we deduce the μ-parametric equation

$$-\Delta_{p(x)} u(x) + \mu \Delta_{q(x)} u(x) = f(u(x)) \quad \text{in } \Omega, \tag{30}$$

Indeed, in this case, $[\lim_{\delta \to 0} \underline{f}_{\delta}(t), \lim_{\delta \to 0} \overline{f}_{\delta}(t)]$ turns down to the single value $f(u(x))$.

Note that in the described situations, every weak solution $u \in W_0^{1,p(x)}(\Omega)$ to the problem is indeed a generalized solution. We conclude with the following example, which illustrates the growth condition H_F.

Example 1. *Let $c > 0$ and consider the (locally Lipschitz) function $F : \mathbb{R} \to \mathbb{R}$ defined by*

$$F(t) = \begin{cases} t\left(1 + c\, t \cos \frac{1}{t}\right) & \text{for } t \neq 0, \\ 0 & \text{for } t = 0. \end{cases}$$

So, the Clarke subdifferential $\partial F(t)$ is as follows:

$$\partial F(t) = \begin{cases} 1 + \sin \frac{1}{t} + 2c\, t \cos \frac{1}{t} & \text{for } t \neq 0, \\ [0, 2] & \text{for } t = 0. \end{cases}$$

Now, for all $t \in \mathbb{R}$ *and* $z \in \partial F(t)$*, one has*

$$|z| \leq 2(1 + c|t|),$$

namely the growth condition H_F *holds true for every* $c > 0$ *such that* $2c\, p^+ C_{p^-}^{p^-} < 2 < p^-$ *(recall (6)). Using such a subdifferential, we obtain the corresponding differential problem*

$$-\Delta_{p(x)} u(x) + \mu \Delta_{q(x)} u(x) \in \partial F(u) \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = 0,$$

where we impose $p^- > 2$*. Finally, we can apply Theorem 1 to conclude that this problem admits a weak solution for all* $\mu \in \mathbb{R}$*. Further, Proposition 5 implies that the same problem admits a weak solution in a case where* $\mu \leq 0$*.*

6. Conclusions

We note the following remarks. In the proof of coercivity (Proposition 4), a fundamental assumption is strict inequality $p^- > q^+$. However, it is also possible to consider a case where $q^+ = p^-$ by imposing the additional condition

$$\frac{1}{p^+}\left(1 - \frac{\widehat{c}\, p^+ C_{p^-}^{p^-}}{p^-}\right) - \frac{|\mu|}{q^+} C \quad \text{for some } C > 0.$$

But this condition is restrictive in the sense that it holds only for sufficiently small $|\mu|$.

On the other hand, it is possible to change assumption H_F through a more general growth condition of the following form:

H_F': there exist $\alpha \in C(\overline{\Omega})$ and positive constants $c, \widehat{c} > 0$ with $\widehat{c}\, p^+ C_{p^-}^{p^-} < p^-$ (recall (6)), such that

$$|z| \leq c + \widehat{c}\,|t|^{\alpha(x)} \quad \text{for all } t \in \mathbb{R} \text{ and } z \in \partial F(t).$$

This setting requires the additional inequalities

$$1 \leq \alpha(x) \leq \alpha^+ \leq p^- \quad \text{for all } x \in \overline{\Omega}$$

to conclude that the functional is coercive (Proposition 4). However, if $\alpha^+ < p^-$ (strictly), there is no need to assume the inequality $\widehat{c}\, p^+ C_{p^-}^{p^-} < p^-$ in H_F'. The proofs of the results remain substantially the same. This way, one can try to investigate the qualitative properties of the solution set, as well as to refine the a priori estimates.

Further investigations may lead to generalization of the principal operator. For example, in recent work by Galewski-Motreanu [21], the authors consider the existence problem to certain equations driven by a non-elliptic operator with constant exponents p and q and a Carathéodory single-valued reaction term. In place of the parameter $\mu \in \mathbb{R}$, the authors consider a positive continuous-weight function depending on the gradient and acting on the negative p-Laplace differential operator. Their approach is also based on the approximation procedure in finite-dimensional spaces.

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