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Research Article

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On semilinear inequalities involving the Dunkl Laplacian and an inverse-square potential outside a ball

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Abstract: Let Δ_k be the Dunkl generalized Laplacian operator associated with a root system R of $\mathbb{R}^N$, $N \geq 2$, and a nonnegative multiplicity function k defined on R and invariant by the finite reflection group W. In this study, we study the existence and nonexistence of weak solutions to the semilinear inequality $-\Delta_k u + \frac{\lambda}{|x|^2}u \geq |u|^p$ in $\mathbb{R}^N \backslash \overline{B_1}$ under the boundary condition $u \geq 0$ on ∂B_1, where $p > 1$, $\lambda \geq -(N-2+2\gamma)^2/4$, and B_1 is the open unit ball of $\mathbb{R}^N$. Namely, we show that the dividing line with respect to existence and nonexistence is given by a critical exponent that depends on λ, N, and $\gamma(k)$, where $\gamma(k) = \sum_{\alpha \in R^+} k(\alpha)$ and R^+ is the positive subsystem.

Keywords: semilinear inequalities, Dunkl operators, inverse-square potential, existence, nonexistence, critical exponent

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1 Introduction

We consider $\mathbb{R}^N$ with the inner product $(\cdot,\cdot)$ and its associated norm $|\cdot|$. For $\alpha \in \mathbb{R}^N \backslash \{0\}$, let σ_α be the reflection with respect to the hyperplane $\langle \alpha \rangle^\perp$ orthogonal to α, i.e.,

$$\sigma_\alpha(x) = x - 2\frac{(x,\alpha)}{|\alpha|^2}\alpha, \quad x \in \mathbb{R}^N.$$

Let $R \subset \mathbb{R}^N \backslash \{0\}$ be a root system, $W = W(R)$ be the reflection group of R generated by the reflections σ_α, $\alpha \in R$, and $k : R \to \mathbb{R}$ be a W-invariant function (multiplicity function). We denote

$$\gamma = \gamma(k) = \sum_{\alpha \in R^+} k(\alpha),$$

where R^+ is the positive subsystem. Throughout this article, we will assume that

$$k \geq 0, \quad \gamma > 0, \quad |\alpha|^2 = 2, \quad \text{for all } \alpha \in R.$$

The Dunkl Laplacian Δ_k associated with the root system R and the multiplicity function k is defined by

$$\Delta_k f = \Delta f + 2\sum_{\alpha \in R^+} k(\alpha)\delta_\alpha(f), \quad f \in C^2(\mathbb{R}^N),$$

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where

$$\delta_\alpha(f) = \frac{(\nabla f(x), \alpha)}{(\alpha, x)} - \frac{f(x) - f(\sigma_\alpha(x))}{(\alpha, x)^2}.$$

The objective of this work is to study the existence and nonexistence of weak solutions to the semilinear exterior problem

$$\begin{cases} -\Delta_k u + \dfrac{\lambda}{|x|^2} u \geq |u|^p, & x \in \mathbb{R}^N \backslash \overline{B_1}, \\ u \geq 0, & x \in \partial B_1, \end{cases} \tag{1.1}$$

where $N \geq 2$, $p > 1$, $\lambda \geq -\left(\frac{N-2+2\gamma}{2}\right)^2$, B_1 is the open unit ball of $\mathbb{R}^N$, $\overline{B_1}$ is the closure of B_1, and ∂B_1 is the boundary of B_1. Note that $\left(\frac{N-2+2\gamma}{2}\right)^2$ is the sharp constant in the Hardy-type inequality for Dunkl operators [36]. In the special case $k = 0$, the operator $-\Delta_k + \frac{\lambda}{|x|^2}$ reduces to

$$\mathcal{L}_\lambda \coloneqq -\Delta + \frac{\lambda}{|x|^2}.$$

Such an operator appears naturally in the nonlinear Wheeler-De Witt equation, which deals with the minisuperspace model in quantum cosmology (see Berestycki and Esteban [4] and references therein for a complete description of the model). Brezis et al. [7] studied the problem of existence and nonexistence of nonnegative solutions to the semilinear elliptic equation $\mathcal{L}_\lambda u = u^p$ ($u \geq 0$) in a ball of $\mathbb{R}^N$, $N \geq 3$, where $\lambda \geq -\left(\frac{N-2}{2}\right)^2$ and $p > 1$. We recall that $\left(\frac{N-2}{2}\right)^2$ is the best constant in the standard Hardy inequality. Namely, the existence of a critical exponent $p^+ > 1$ is shown, such that the problem admits nontrivial solutions if and only if $p < p^+$. This is a well-established theory for which we refer to the following partial list: [1–3,6,8–10,14,15,17,20] (see also references therein). Among other problems, Jleli et al. [26] considered the semilinear elliptic inequality

$$\mathcal{L}_\lambda u \geq |u|^p, \quad x \in \mathbb{R}^N \backslash \overline{B_1}, \tag{1.2}$$

under the Dirichlet-type boundary condition

$$u(x) \geq f(x), \quad x \in \partial B_1, \tag{1.3}$$

where $N \geq 2$, $\lambda \geq -\left(\frac{N-2}{2}\right)^2$, $p > 1$ and $f \in L^1(\partial B_1)$. Namely, it was proven that

$$p_{\mathrm{cr}}(\lambda, N) \coloneqq \begin{cases} \infty, & \text{if} \quad N - 2 + 2\lambda(N) = 0, \\ 1 + \dfrac{4}{N - 2 + 2\lambda(N)}, & \text{if} \quad N - 2 + 2\lambda(N) > 0, \end{cases}$$

where

$$\lambda(N) = \sqrt{\lambda + \left(\frac{N-2}{2}\right)^2}$$

is a critical exponent for (1.2)–(1.3) in the following sense:

(i) If $1 < p < p_{\mathrm{cr}}(\lambda, N)$, then (1.2)–(1.3) admit no weak solution, provided that

$$\int_{\partial B_1} f(x) \mathrm{d}\sigma(x) > 0.$$

(ii) If $p > p_{\mathrm{cr}}(\lambda, N)$, then (1.2)–(1.3) admit weak solutions for some $f > 0$.

We remark that the critical behavior of semilinear exterior problems has become a topic of intense study (see, e.g., [5,24,25,27,34,37]).

Dunkl operators have been introduced in 1989, in [13]. These operators have many applications to the study of special functions with root systems. Furthermore, the commutative algebra generated by these operators has been used in the study of certain exactly solvable models of quantum mechanics, namely, the Calogero-Sutherland-Moser models (see, e.g., [21,22,28]). Dunkl operators also find applications in harmonic analysis and integral transforms (see, for instance, [11,12,16,18,19,29,30,32,33,35]), as they naturally lead to the Laplace-Dunkl operators of type Δ_k, which are second-order differential/difference operators that generalize the standard Laplace operator.

Gallardo and Godefroy [18] extended the classical Liouville theorem for harmonic functions to Dunkl harmonic functions. Namely, it was shown that if u is a bounded Dunkl harmonic function in $\mathbb{R}^N$, then it is a constant. A Liouville theorem for Dunkl polyharmonic functions in $\mathbb{R}^N$ has been proved by Ren and Liu [31].

To the best of our knowledge, the study of existence and nonexistence for nonlinear problems involving Dunkl operators has not been previously considered in the literature. Motivated by this fact and the aforementioned results, the problem of existence and nonexistence for (1.1) is investigated in this work. More precisely, the proof of the nonexistence part is based on nonlinear capacity estimates specifically adapted to the operator $-\Delta_k + \frac{\lambda}{|x|^2}$, the considered domain, and the boundary condition. The existence results are established by the construction of explicit solutions.

The rest of this article is organized as follows. In Section 2, we recall some basic notions and properties related to Dunkl operators. The main results are stated in Section 3. Section 4 is devoted to some preliminary estimates. In Section 5, we prove the obtained results.

We fix below some notations that will be used throughout this article:

- For $\tau > 0$, the open ball of $\mathbb{R}^N$ with center 0 and radius τ is denoted by B_τ. Its closure is denoted by $\overline{B_\tau}$.
- The support of a function $\varphi : \mathbb{R}^N \ni x \mapsto \varphi(x) \in \mathbb{R}$ is denoted by $\operatorname{supp}(\varphi)$.
- We mean by $t \gg 1$ that $t > 1$ is sufficiently large.
- By C and C_i, we mean generic positive constants independent of the scaling parameter τ and the solution u. Their values are not necessarily the same from one line to another.

2 Dunkl operators

For the reader's convenience, we recall in this section some basic notions and properties related to Dunkl operators. For more details, we refer to [30,32,33].

For any $x \in \mathbb{R}^N$, $N \geq 1$, we denote by $\langle x \rangle^\perp$ the hyperplane orthogonal to x, i.e.,

$$\langle x \rangle^\perp = \{z \in \mathbb{R}^N : (z, x) = 0\}.$$

For $\alpha \in \mathbb{R}^N \backslash \{0\}$, let σ_α be the reflection in the hyperplane $\langle x \rangle^\perp$, i.e.,

$$\sigma_\alpha(x) = x - 2\frac{(x, \alpha)}{|\alpha|^2}\alpha, \quad x \in \mathbb{R}^N.$$

We remark that every reflexion σ_α is an orthogonal transformation (it preserves the inner product).

A finite subset $R \subset \mathbb{R}^N \backslash \{0\}$ is said to be a root system (see the monography of Humphreys [23] for details on root systems), if for all $\alpha \in R$, we have

(i) $\sigma_\alpha R = R$;
(ii) $R \cap \mathbb{R}\alpha = \{\pm\alpha\}$.

For a given root system R, the subgroup of $O(N, \mathbb{R})$ (the orthogonal group with respect to function composition) generated by σ_α, $\alpha \in R$, is denoted by $W = W(R)$, and called the reflection group (or the Coxeter-Weyl group) associated with R. Note that (see Rösler [33, Lemma 2.2]) W is finite and the set of reflections contained in W is exactly $\{\sigma_\alpha, \alpha \in R\}$.

Let R be a root system. For a given $\beta \in \mathbb{R}^N \backslash \cup_{\alpha \in R} \langle \alpha \rangle^{\perp}$, we denote by R^+ the positive subsystem

$$R^+ = \{\alpha \in R : (\alpha, \beta) > 0\}.$$

We know that for every $\alpha \in R$, we have $\alpha \in R^+$ or $-\alpha \in R^+$.

Let $k : R \to \mathbb{R}$ be a multiplicity function on the root system R, i.e., k is invariant under the natural action of W on R, i.e.,

$$k(\alpha) = k(w(\alpha)),$$

for all $w \in W$ and $\alpha \in R$. As mentioned in the previous section, we set

$$\gamma = \gamma(k) = \sum_{\alpha \in R^+} k(\alpha),$$

and we assume that

$$k \geq 0, \quad \gamma > 0, \quad |\alpha|^2 = 2, \qquad \text{for all } \alpha \in R. \tag{2.1}$$

Furthermore, we denote by ω_k the weight function given as

$$\omega_k(x) = \prod_{\alpha \in R^+} |(\alpha, x)|^{2k(\alpha)}.$$

For $f \in L^1(\mathbb{R}^N, \omega_k(x)\mathrm{d}x)$, we have the relation

$$\int_{\mathbb{R}^N} f(x)\omega_k(x)\mathrm{d}x = \int_0^{\infty} \left(\int_{S^{N-1}} f(r\theta)\omega_k(r\theta)\mathrm{d}\sigma(\theta) \right) r^{N-1}\mathrm{d}r,$$

which can be written, due to the homogeneity of ω_k, as

$$\int_{\mathbb{R}^N} f(x)\omega_k(x)\mathrm{d}x = \int_0^{\infty} \left(\int_{S^{N-1}} f(r\theta)\omega_k(\theta)\mathrm{d}\sigma(\theta) \right) r^{2\gamma+N-1}\mathrm{d}r,$$

where $\mathrm{d}\sigma$ is the normalized surface measure on the unit sphere S^{N-1} of $\mathbb{R}^N$. In particular, if f is radial, i.e., $f(x) = F(|x|)$, then we obtain

$$\int_{\mathbb{R}^N} f(x)\omega_k(x)\mathrm{d}x = c_k \int_0^{\infty} F(r) r^{2\gamma+N-1}\mathrm{d}r, \tag{2.2}$$

where

$$c_k = \int_{S^{N-1}} \omega_k(\theta)\mathrm{d}\sigma(\theta).$$

For $j = 1, 2, \ldots, N$, the Dunkl operators T_j associated with the root system R and the multiplicity function k, are given by

$$T_j f(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{\alpha \in R^+} k(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle}, \quad f \in C^1(\mathbb{R}^N),$$

where $\alpha = (\alpha_1, \ldots, \alpha_N)$. Note that by the W-invariance of k, the definition of the Dunkl operator does not depend on the special choice of R^+. Also, the length of the roots is irrelevant in the formula for T_j.

We note some results from Mejjaoli and Trimèche, more precisely, referring to [30, Proposition 2.1], we have

If $f \in C_c(\mathbb{R}^N)$ (f is continuous and compactly supported in $\mathbb{R}^N$) and $g \in C^1(\mathbb{R}^N)$, then

$$\int_{\mathbb{R}^N} T_i f(x) g(x)\omega_k(x)\mathrm{d}x = -\int_{\mathbb{R}^N} T_i g(x) f(x)\omega_k(x)\mathrm{d}x.$$

If $f, g \in C^1(\mathbb{R}^N)$ and g is W-invariant, then

$$T_j(fg) = gT_j f + fT_j g.$$

The Dunkl Laplacian Δ_k associated with the root system R and the multiplicity function k is defined by

$$\Delta_k f = \sum_{j=1}^{N} T_j^2 f = \Delta f + 2\sum_{\alpha\in R^+} k(\alpha)\delta_\alpha(f), \quad f \in C^2(\mathbb{R}^N),$$

where

$$\delta_\alpha(f) = \frac{(\nabla f(x), \alpha)}{(\alpha, x)} - \frac{f(x) - f(\sigma_\alpha(x))}{(\alpha, x)^2}.$$

We remark that in the case $k = 0$, the Dunkl Laplacian Δ_k reduces to the standard Laplacian.

If f is a radial function, i.e., $f(x) = F(r)$, $r = |x|$, then (see [30, Proposition 4.15])

$$\Delta_k f(x) = F''(r) + \frac{2\gamma + N - 1}{r} F'(r). \tag{2.3}$$

Finally, we recall below Green's formula for Δ_k (see [30, Theorem 4.11]):

Let Ω be an open bounded regular set of $\mathbb{R}^N$, which is W-invariant, and $f, g \in C^2(\overline{\Omega})$. Then,

$$\int_\Omega (f\Delta_k g - g\Delta_k f)\omega_k(x)\mathrm{d}x = \int_{\partial\Omega} \left(f\frac{\partial g}{\partial \nu} - g\frac{\partial f}{\partial \nu} \right) \omega_k(y)\mathrm{d}\sigma(y), \tag{2.4}$$

where $\frac{\partial}{\partial \nu}$ denotes the normal derivative on $\partial\Omega$.

3 Main results

Let $k : R \to \mathbb{R}$ be a multiplicity function on the root system $R \subset \mathbb{R}^N \backslash \{0\}$ and invariant by the finite reflection group $W = W(R)$. Throughout this study, it is assumed that (2.1) holds. We consider (1.1), where $N \geq 2$, $p > 1$ and $\lambda \geq -\left(\frac{N-2+2\gamma}{2}\right)^2$. Before stating our main results, we first define the weak solutions to (1.1).

Let $Q = \mathbb{R}^N \backslash B_1$. We introduce the set of functions

$$\Phi = \left\{ \varphi \in C^2(Q) : \operatorname{supp}(\varphi) \Subset Q, \varphi \geq 0, \varphi|_{\partial B_1} = 0, \frac{\partial \varphi}{\partial \nu}|_{\partial B_1} \leq 0 \right\},$$

where ν is the outward unit normal vector on ∂B_1, relative to Q, and $\frac{\partial}{\partial \nu}$ denotes the normal derivative on ∂B_1. Note that $\partial B_1 \subset Q$.

Definition 1. We say that $u \in L^p_{\mathrm{loc}}(Q, \omega_k \mathrm{d}x)$ is a weak solution to (1.1), if

$$\int_Q |u|^p \varphi \omega_k \mathrm{d}x \leq \int_Q u \left(-\Delta_k \varphi + \frac{\lambda}{|x|^2}\varphi \right) \omega_k \mathrm{d}x, \tag{3.1}$$

for every $\varphi \in \Phi$.

Remark 1. Note that any regular solution to (1.1) is a weak solution in the sense of Definition 1. Namely, let u be a regular solution to (1.1) and $\varphi \in \Phi$. We can take $\rho > 1$ sufficiently large so that $\operatorname{supp}(\varphi) \subsetneq A(1, \rho) := \{x \in \mathbb{R}^N : 1 \leq |x| < \rho\}$. Note that $A(1, \rho)$ is W-invariant since $W \subset O(N, \mathbb{R})$. This allows us to use the Green's formula (2.4) that requires the domain of integration is invariant with respect to the reflexion group. Hence, multiplying the first inequality in (1.1) by $\varphi\omega_k$ and integrating over $A(1, \rho)$, we obtain (3.1).

Let us introduce the parameters

$$\lambda(N,\gamma) = \sqrt{\lambda + \left(\frac{N-2+2\gamma}{2}\right)^2} \tag{3.2}$$

and

$$\mu = \lambda(N,\gamma) - \frac{N-2+2\gamma}{2}. \tag{3.3}$$

Our first main result is stated in the following theorem.

Theorem 3.1. *We distinguish the following cases*:

(*I*) *For all*

$$1 < p < 1 + \frac{2}{\mu + 2\gamma + N - 2}, \tag{3.4}$$

(1.1) *admits no nontrivial weak solution.*

(*II*) *For all*

$$p > 1 + \frac{2}{\mu + 2\gamma + N - 2}, \tag{3.5}$$

(1.1) *admits nontrivial solutions.*

Remark 2. We point out the following:

(i) By (2.1), we have

$$\mu + 2\gamma + N - 2 = \lambda(N,\gamma) + \frac{2\gamma + N - 2}{2} > 0.$$

(ii) From Theorem 3.1, the dividing line with respect to existence and nonexistence of weak solutions to (1.1) is given by the critical exponent

$$p_{\mathrm{cr}}(\lambda, N, \gamma) := 1 + \frac{2}{\mu + 2\gamma + N - 2}. \tag{3.6}$$

(iii) If $\lambda = 0$ (so $\mu = 0$), we deduce from Theorem 3.1 that the exterior problem

$$\begin{cases} -\Delta_k u \geq |u|^p, & x \in \mathbb{R}^N \backslash \overline{B_1}, \\ u \geq 0, & x \in \partial B_1, \end{cases}$$

admits as critical exponent the real number

$$p_{\mathrm{cr}}(0, N, \gamma) := 1 + \frac{2}{2\gamma + N - 2}.$$

Our second main result is concerned with the critical case $p = p_{\mathrm{cr}}(\lambda, N, \gamma)$.

Theorem 3.2. *Let* $p = p_{\mathrm{cr}}(\lambda, N, \gamma)$, *where* $p_{\mathrm{cr}}(\lambda, N, \gamma)$ *is given by* (3.6). *Assume that one of the following conditions hold*:

(*I*) $\lambda > -\left(\frac{N-2+2\gamma}{2}\right)^2$;

(*II*) $\lambda = -\left(\frac{N-2+2\gamma}{2}\right)^2$ *and* $N + 2\gamma \geq 6$.

Then, (1.1) *admits no nontrivial weak solution.*

Remark 3. At this moment, when $\lambda = -\left(\frac{N-2+2\gamma}{2}\right)^2$, $N + 2\gamma < 6$, and $p = p_{\mathrm{cr}}(\lambda, N, \gamma)$, we do not know whether we have existence or nonexistence of weak solutions to (1.1). We leave this problem as an open question.

4 Preliminary estimates

This section will discuss the properties of families of test functions. Furthermore, we will construct the preliminary estimates related to our results. In this section, we construct some estimates. Let $p > 1$ and $\lambda \geq -\left(\frac{N-2+2\gamma}{2}\right)^2$.

4.1 *A priori* estimate

For $\varphi \in \Phi$, we introduce the integral term

$$\chi(\varphi) = \int_Q \varphi^{\frac{-1}{p-1}} \left| -\Delta_k \varphi + \frac{\lambda}{|x|^2}\varphi \right|^{\frac{p}{p-1}} \omega_k \mathrm{d}x. \tag{4.1}$$

We have the following *a priori* estimate.

Lemma 4.1. *Assume that* $u \in L^p_{\mathrm{loc}}(Q, \omega_k \mathrm{d}x)$ *is a weak solution to* (1.1). *Then,*

$$\int_Q |u|^p \varphi \omega_k \mathrm{d}x \leq \chi(\varphi), \tag{4.2}$$

for every $\varphi \in \Phi$, *provided that* $\chi(\varphi) < \infty$.

Proof. Assume that $u \in L^p_{\mathrm{loc}}(Q, \omega_k \mathrm{d}x)$ is a weak solution to (1.1). Let $\varphi \in \Phi$ be such that $\chi(\varphi) < \infty$. By (3.1), we have

$$\int_Q |u|^p \varphi \omega_k \mathrm{d}x \leq \int_Q |u| \left| -\Delta_k \varphi + \frac{\lambda}{|x|^2}\varphi \right| \omega_k \mathrm{d}x. \tag{4.3}$$

Making use of Young's inequality, we obtain

$$\begin{aligned} \int_Q |u| \left| -\Delta_k \varphi + \frac{\lambda}{|x|^2}\varphi \right| \omega_k \mathrm{d}x &= \int_Q \left(|u| (\varphi\omega_k)^{\frac{1}{p}} \right) \left(\varphi^{\frac{-1}{p}} \left| -\Delta_k \varphi + \frac{\lambda}{|x|^2}\varphi \right| \omega_k^{\frac{p-1}{p}} \right) \mathrm{d}x \\ &\leq \frac{1}{p} \int_Q |u|^p \varphi \omega_k \mathrm{d}x + \frac{p-1}{p} \chi(\varphi). \end{aligned} \tag{4.4}$$

Then, (4.2) follows from (4.3) and (4.4). □

4.2 Construction of families of test functions

Let us introduce the function F defined in Q by

$$F(x) = \begin{cases} |x|^{\mu}(1 - |x|^{-2\lambda(N,\gamma)}), & \text{if} \quad \lambda > -\left(\frac{N-2+2\gamma}{2}\right)^2, \\ |x|^{\mu} \ln|x|, & \text{if} \quad \lambda = -\left(\frac{N-2+2\gamma}{2}\right)^2, \end{cases} \tag{4.5}$$

where $\lambda(N, \gamma)$ and μ are the parameters defined, respectively, by (3.2) and (3.3). Some useful properties of F are provided by the following two lemmas.

Lemma 4.2. *The function F satisfies the following properties*:

(*i*) $F \in C^2(Q)$;

(*ii*) $F \geq 0$;

(*iii*) *For all* $x \in \partial B_1$, *we have*

$$F(x) = 0, \quad \frac{\partial F}{\partial \nu}(x) = \begin{cases} -2\lambda(N,\gamma) & \text{if} \quad \lambda > -\left(\frac{N-2+2\gamma}{2}\right)^2, \\ -1 & \text{if} \quad \lambda = -\left(\frac{N-2+2\gamma}{2}\right)^2; \end{cases}$$

(*iv*) $-\Delta_k F + \frac{\lambda}{|x|^2} F = 0$, *in* Q.

Proof. Properties (i)–(iii) follow immediately from the definition of F. On the other hand, since F is radial, making use of (2.3), we obtain (iv). ☐

Lemma 4.3. The following hold:

(*i*) *Let* $\lambda > -\left(\frac{N-2+2\gamma}{2}\right)^2$. *For* $\tau \gg 1$ *and* $\frac{\tau}{2} < |x| < \tau$, *we have*

$$C_1 \tau^\mu \leq F(x) \leq C_2 \tau^\mu, \ |\nabla F(x)| \leq C \tau^{\mu-1}.$$

(*ii*) *Let* $\lambda = -\left(\frac{N-2+2\gamma}{2}\right)^2$. *For* $\tau \gg 1$ *and* $\frac{\tau}{2} < |x| < \tau$, *we have*

$$C_1 \tau^\mu \ln \tau \leq F(x) \leq C_2 \tau^\mu \ln \tau, \ |\nabla F(x)| \leq C \tau^{\mu-1} \ln \tau.$$

Proof. (i) Let $\lambda > -\left(\frac{N-2+2\gamma}{2}\right)^2$. By (4.5), we have

$$F(x) = |x|^\mu (1 - |x|^{-2\lambda(N,\gamma)}).$$

Since $F(x) \sim |x|^\mu$ as $|x| \to \infty$, for $|x| \gg 1$, it holds that

$$C_1 \, |x|^\mu \leq F(x) \leq C_2 \, |x|^\mu.$$

In particular, for $\tau \gg 1$ and $\frac{\tau}{2} < |x| < \tau$, we obtain

$$C_1 \tau^\mu \leq F(x) \leq C_2 \tau^\mu.$$

On the other hand, we have

$$\nabla F(x) = |x|^{\mu-1} [\mu - (\mu - 2\lambda(N,\gamma)) |x|^{-2\lambda(N,\gamma)}] \frac{x}{|x|},$$

which yields

$$|\nabla F(x)| = |x|^{\mu-1} |\mu - (\mu - 2\lambda(N,\gamma)) |x|^{-2\lambda(N,\gamma)}|.$$

Observing that $\frac{|\nabla F(x)|}{|x|^{\mu-1}} \to |\mu|$ as $|x| \to \infty$, we deduce that for $|x| \gg 1$, we have

$$|\nabla F(x)| \leq C \, |x|^{\mu-1}.$$

In particular, for $\tau \gg 1$ and $\frac{\tau}{2} < |x| < \tau$, we obtain

$$|\nabla F(x)| \leq C \tau^{\mu-1}.$$

(ii) Let $\lambda = -\left(\frac{N-2+2\gamma}{2}\right)^2$. By (4.5), we have

$$F(x) = |x|^\mu \ln |x|.$$

On the other hand, by (3.3), we have

$$\mu = -\frac{N-2+2\gamma}{2} < 0,$$

which implies that for $s \gg 1$, the function $s \mapsto s^\mu \ln s$ is decreasing. Hence, for $\tau \gg 1$ and $\frac{\tau}{2} < |x| < \tau$, we obtain

$$\tau^\mu \ln \tau \le F(x) \le \left(\frac{\tau}{2}\right)^\mu \ln\left(\frac{\tau}{2}\right) \le C_2 \tau^\mu \ln \tau.$$

Furthermore, we have

$$\nabla F(x) = |x|^{\mu-1} \ln|x| \left(\mu + \frac{1}{\ln|x|}\right) \frac{x}{|x|},$$

which implies that for $|x| \gg 1$,

$$|\nabla F(x)| \le C\,|x|^{\mu-1} \ln|x|.$$

In particular, for $\tau \gg 1$ and $\frac{\tau}{2} < |x| < \tau$, we obtain

$$|\nabla F(x)| \le C\tau^{\mu-1} \ln \tau. \qquad \square$$

4.2.1 Test functions of the first type

Let $\xi : [0,\infty) \to [0,1]$ be a smooth function such that

$$\xi(s) = \begin{cases} 1, & \text{if} \quad 0 \le s \le \frac{1}{2}, \\ 0, & \text{if} \quad s \ge 1, \end{cases}$$

and for $\tau \gg 1$, set

$$\xi_\tau(x) = \xi\left(\frac{|x|}{\tau}\right), \quad x \in Q. \tag{4.6}$$

Lemma 4.4. *For $\tau \gg 1$, we have*

$$0 \le \xi_\tau \le 1, \quad \xi_\tau \in C^\infty(Q), \quad \operatorname{supp}(\xi_\tau) = \{x \in \mathbb{R}^N : 1 \le |x| \le \tau\} \tag{4.7}$$

and

$$|\nabla \xi_\tau| \le C\tau^{-1}, \quad |\Delta_k \xi_\tau| \le C\tau^{-2}. \tag{4.8}$$

Proof. The properties (4.7) follow immediately from the definition of ξ_τ. On the other hand, for all $x \in Q$, we have

$$|\nabla \xi_\tau(x)| = \tau^{-1} \left|\xi'\left(\frac{|x|}{\tau}\right)\right| \le C\tau^{-1}.$$

Furthermore, since ξ_τ is a radial function, it follows from (2.3) that

$$\Delta_k \xi_\tau(x) = \tau^{-2} \xi''\left(\frac{|x|}{\tau}\right) + \frac{2\gamma + N - 1}{|x|} \tau^{-1} \xi'\left(\frac{|x|}{\tau}\right),$$

which implies by the properties of ξ that

$$\operatorname{supp}(\Delta_k \xi_\tau) \subset \left\{x \in Q : \frac{\tau}{2} \le |x| \le \tau\right\},$$

and for all $x \in Q$ with $\frac{\tau}{2} \le |x| \le \tau$, we obtain

$$\begin{aligned}|\Delta_k \xi_\tau(x)| &\le \tau^{-2}\left(\left|\xi''\left(\frac{|x|}{\tau}\right)\right| + C\tau\,|x|^{-1}\left|\xi'\left(\frac{|x|}{\tau}\right)\right|\right)\\ &\le \tau^{-2}\left(\left|\xi''\left(\frac{|x|}{\tau}\right)\right| + C\left|\xi'\left(\frac{|x|}{\tau}\right)\right|\right)\\ &\le C\tau^{-2}.\end{aligned}$$

Then, (4.8) is proved. □

For $\tau, t \gg 1$, we introduce test functions of the form

$$\varphi(x) = F(x)\xi_\tau^t(x), \quad x \in Q, \tag{4.9}$$

where F and ξ_τ are defined, respectively, by (4.5) and (4.6). On this basis, we establish the following result.

Lemma 4.5. *For $\tau, t \gg 1$, the function $\varphi(x) = F(x)\xi_\tau^t(x)$, given in (4.9), belongs to Φ.*

Proof. Clearly, $\varphi \in C^2(Q)$ and $\varphi \ge 0$. By the definition of F and (4.7), we have

$$\operatorname{supp}(\varphi) = \{x \in \mathbb{R}^N : 1 \le |x| \le \tau\}.$$

On the other hand, since $F = 0$ on ∂B_1, $\varphi = 0$ on ∂B_1. Furthermore, we have

$$\nabla\varphi(x) = \nabla F(x)\xi_\tau^t(x) + F(x)\nabla(\xi_\tau^t(x)).$$

Since $F = 0$ on ∂B_1 and $\xi_\tau = 1$ for $1 \le |x| \le \frac{\tau}{2}$, we deduce that

$$\frac{\partial\varphi}{\partial\nu}(x) = \frac{\partial F}{\partial\nu}(x), \quad x \in \partial B_1.$$

Then, by Lemma 4.2 (iii), we have $\frac{\partial\varphi}{\partial\nu}\,|_{\partial B_1} \le 0$. Consequently, $\varphi \in \Phi$. □

4.2.2 Test functions of the second type

Let $\zeta : \mathbb{R} \to [0,1]$ be a smooth function such that

$$\zeta(s) = \begin{cases}1, & \text{if} \quad s \le 0,\\ 0, & \text{if} \quad s \ge 1.\end{cases}$$

For $\tau \gg 1$, let

$$\zeta_\tau(x) = \zeta\left(\frac{\ln|x|}{\ln\sqrt{\tau}} - 1\right), \quad x \in Q,$$

i.e.,

$$\zeta_\tau(x) = \begin{cases}1, & \text{if} \quad 1 \le |x| \le \sqrt{\tau},\\ \zeta\left(\frac{\ln|x|}{\ln\sqrt{\tau}} - 1\right)0, & \text{if} \quad \sqrt{\tau} \le |x| \le \tau,\\ 0, & \text{if} \quad |x| \ge \tau.\end{cases} \tag{4.10}$$

For these families of test functions, we note the following properties.

Lemma 4.6. *For $\tau \gg 1$, we have*

$$0 \le \zeta_\tau \le 1, \quad \zeta_\tau \in C^\infty(Q), \quad \operatorname{supp}(\zeta_\tau) = \{x \in \mathbb{R}^N : 1 \le |x| \le \tau\} \tag{4.11}$$

and

$$|\nabla\zeta_\tau| \leq \frac{C}{|x|\ln\tau}, \quad |\Delta_k\zeta_\tau| \leq \frac{C}{|x|^2\ln\tau}. \tag{4.12}$$

Proof. The properties (4.11) follow immediately from the definition of ζ_τ. On the other hand, for all $x \in Q$, we have

$$|\nabla\zeta_\tau(x)| = \frac{1}{|x|\ln\sqrt{\tau}}\left|\zeta'\left(\frac{\ln|x|}{\ln\sqrt{\tau}} - 1\right)\right| \leq \frac{C}{|x|\ln\tau}.$$

Furthermore, since ζ_τ is a radial function, it follows from (2.3) that

$$\begin{aligned}\Delta_k\zeta_\tau(x) &= \frac{1}{|x|^2[\ln(\sqrt{\tau})]^2}\zeta''\left(\frac{\ln|x|}{\ln\sqrt{\tau}} - 1\right) + \frac{2\gamma + N - 1}{|x|^2\ln\sqrt{\tau}}\zeta'\left(\frac{\ln|x|}{\ln\sqrt{\tau}} - 1\right)\\ &\leq \frac{C}{|x|^2\ln\sqrt{\tau}}.\end{aligned}$$

Then, (4.12) is proved. □

Combining the previous families of test functions, for $\tau, t \gg 1$, we can introduce a new test function of the form

$$\psi(x) = F(x)\zeta_\tau^t(x), \quad x \in Q, \tag{4.13}$$

where F and ζ_τ are defined, respectively, by (4.5) and (4.10). We remark now a key feature of ψ in the following result.

Lemma 4.7. *For $\tau, t \gg 1$, the function $\psi(x) = F(x)\zeta_\tau^t(x)$, given in* (4.13), *belongs to* Φ.

Proof. Clearly, $\psi \in C^2(Q)$ and $\psi \geq 0$. By the definition of F and (4.11), we have

$$\operatorname{supp}(\psi) = \{x \in \mathbb{R}^N : 1 \leq |x| \leq \tau\}.$$

On the other hand, since $F = 0$ on ∂B_1, $\psi = 0$ on ∂B_1. Furthermore, we have

$$\nabla\psi(x) = \nabla F(x)\zeta_\tau^t(x) + F(x)\nabla(\zeta_\tau^t(x)).$$

Since $F = 0$ on ∂B_1 and $\zeta_\tau = 1$ for $1 \leq |x| \leq \sqrt{\tau}$, we deduce that

$$\frac{\partial\psi}{\partial\nu}(x) = \frac{\partial F}{\partial\nu}(x), \quad x \in \partial B_1.$$

Then, by Lemma 4.2 (iii), we have $\frac{\partial\psi}{\partial\nu}|_{\partial B_1} \leq 0$. Consequently, we deduce that $\psi \in \Phi$. □

4.3 Estimates of $\chi(\varphi)$

For $\tau, t \gg 1$, let φ be the test function defined by (4.9). The following two lemmas provide estimates of $\chi(\varphi)$.

Lemma 4.8. *If $\lambda > -\left(\frac{N-2+2\gamma}{2}\right)^2$, then*

$$\chi(\varphi) \leq C\tau^{\frac{(\mu-2+2\gamma+N)p-\mu-2\gamma-N}{p-1}}. \tag{4.14}$$

Proof. For all $x \in Q$, we have

$$\Delta_k\varphi(x) = \Delta_k(F(x)\xi_\tau^t(x)).$$

Since F and ξ_τ are the radial functions, we obtain by (2.3) that

$$\begin{aligned}-\Delta_k\varphi(x) + \frac{\lambda}{|x|^2}\varphi(x) &= -F(x)\Delta_k(\xi_\tau^t) - \xi_\tau^t\Delta_k F(x) - 2(\nabla F(x), \nabla\xi_\tau^t(x)) + \frac{\lambda}{|x|^2}F(x)\xi_\tau^t(x)\\ &= \xi_\tau^t(x)\left(-\Delta_k F(x) + \frac{\lambda}{|x|^2}F(x)\right) - tF(x)\xi_\tau^{t-2}(x)(\xi_\tau(x)\Delta_k\xi_\tau(x) + (t-1)|\nabla\xi_\tau(x)|^2)\\ &\quad - 2t\xi_\tau^{t-1}(x)(\nabla F(x), \nabla\xi_\tau(x)).\end{aligned}$$

From Lemma 4.2 (iv), we deduce that

$$-\Delta_k\varphi(x) + \frac{\lambda}{|x|^2}\varphi(x) = -tF(x)\xi_\tau^{t-2}(x)(\xi_\tau(x)\Delta_k\xi_\tau(x) + (t-1)|\nabla\xi_\tau(x)|^2) - 2t\xi_\tau^{t-1}(x)(\nabla F(x), \nabla\xi_\tau(x)), \tag{4.15}$$

which implies by (4.1), (4.8), Lemma 4.3(i), and the properties of ξ that

$$\chi(\varphi) = \int\limits_{\frac{\tau}{2}<|x|<\tau} \varphi^{\frac{-1}{p-1}}\left|-\Delta_k\varphi + \frac{\lambda}{|x|^2}\varphi\right|^{\frac{p}{p-1}}\omega_k\mathrm{d}x \tag{4.16}$$

and

$$\left|-\Delta_k\varphi(x) + \frac{\lambda}{|x|^2}\varphi(x)\right| \leq C(\tau^{\mu-2}\xi_\tau^{t-2}(x) + \tau^{\mu-2}\xi_\tau^{t-1}(x)) \leq C\tau^{\mu-2}\xi_\tau^{t-2}(x),\ \frac{\tau}{2} < |x| < \tau.$$

The aforementioned estimate together with Lemma 4.3(i) yields (we recall that $0 \leq \xi_\tau \leq 1$)

$$\begin{aligned}\varphi^{\frac{-1}{p-1}}(x)\left|-\Delta_k\varphi(x) + \frac{\lambda}{|x|^2}\varphi(x)\right|^{\frac{p}{p-1}} &\leq C\tau^{\frac{(\mu-2)p}{p-1}}\xi_\tau^{\frac{(t-2)p}{p-1}}F^{\frac{-1}{p-1}}(x)\xi_t^{\frac{-t}{p-1}}(x)\\ &\leq C\tau^{\frac{(\mu-2)p-\mu}{p-1}}\xi_\tau^{t-\frac{2p}{p-1}}(x)\\ &\leq C\tau^{\frac{(\mu-2)p-\mu}{p-1}}.\end{aligned}$$

Then, by (2.2) and (4.16), we obtain

$$\chi(\varphi) \leq C\tau^{\frac{(\mu-2)p-\mu}{p-1}}\int\limits_{r=\frac{\tau}{2}}^{\tau} r^{2\gamma+N-1}\mathrm{d}r = C\tau^{\frac{(\mu-2+2\gamma+N)p-\mu-2\gamma-N}{p-1}},$$

which proves (4.14). □

Proceeding as in the proof of Lemma 4.8 and using Lemma 4.3(ii), we obtain the following estimate.

Lemma 4.9. *If* $\lambda = -\left(\frac{N-2+2\gamma}{2}\right)^2$, *then*

$$\chi(\varphi) \leq C\tau^{\frac{(N-2+2\gamma)p-N-2\gamma-2}{2(p-1)}}\ln\tau.$$

4.4 Estimates of $\chi(\psi)$

For $\tau, t \gg 1$, let ψ be the test function defined by (4.13).

Lemma 4.10. *If* $\lambda = -\left(\frac{N-2+2\gamma}{2}\right)^2$ *and* $p = p_{\mathrm{cr}}(\lambda, N, \gamma)$, *where* $p_{\mathrm{cr}}(\lambda, N, \gamma)$ *is given by* (3.6), *then*

$$\chi(\psi) \leq C(\ln\tau)^{\frac{p-2}{p-1}}. \tag{4.17}$$

Proof. Proceeding as in the proof of Lemma 4.8, we obtain that for all $x \in Q$,

$$-\Delta_k\psi(x) + \frac{\lambda}{|x|^2}\psi(x) = -tF(x)\zeta_\tau^{t-2}(x)(\zeta_\tau(x)\Delta_k\zeta_\tau(x) + (t-1)|\nabla\zeta_\tau(x)|^2) - 2t\zeta_\tau^{t-1}(x)(\nabla F(x), \nabla\zeta_\tau(x)), \tag{4.18}$$

which implies by (4.1) and (4.10) that

$$\chi(\psi) = \int\limits_{\sqrt{\tau}<|x|<\tau} \psi^{\frac{-1}{p-1}} \left| -\Delta_k\psi + \frac{\lambda}{|x|^2}\psi \right|^{\frac{p}{p-1}} \omega_k \mathrm{d}x. \tag{4.19}$$

On the other hand, by the definition of F, using Lemma 4.6 and (4.18), we obtain that for all $x \in \mathbb{R}^N$ with $\sqrt{\tau} < |x| < \tau$,

$$\begin{aligned} \left| -\Delta_k\psi(x) + \frac{\lambda}{|x|^2}\psi(x) \right| &\le C_1 F(x)\zeta_\tau^{t-2}(x)(|\Delta_k\zeta_\tau(x)| + |\nabla\zeta_\tau(x)|^2) + C_2\zeta_\tau^{t-2}(x)|\nabla F(x)||\nabla\zeta_\tau(x)| \\ &\le C\zeta_\tau^{t-2}(x)|x|^{\mu-2}\ln|x|(\ln\tau)^{-1}, \end{aligned}$$

which implies by (2.2) and (4.19) that

$$\begin{aligned} \chi(\psi) &\le C(\ln\tau)^{\frac{-p}{p-1}} \int\limits_{\sqrt{\tau}<|x|<\tau} \zeta_t^{t-\frac{2p}{p-1}}(x)|x|^{\mu-\frac{2p}{p-1}}\ln|x|\omega_k(x)\mathrm{d}x \\ &\le C(\ln\tau)^{\frac{-p}{p-1}} \int\limits_{\sqrt{\tau}<|x|<\tau} |x|^{\mu-\frac{2p}{p-1}}\ln|x|\omega_k(x)\mathrm{d}x \\ &= C(\ln\tau)^{\frac{-p}{p-1}} \int\limits_{r=\sqrt{\tau}}^{\tau} r^{\mu-\frac{2p}{p-1}+2\gamma+N-1}\ln r\mathrm{d}r. \end{aligned}$$

Furthermore, from $\lambda = -\left(\frac{N-2+2\gamma}{2}\right)^2$ and $p = p_{\mathrm{cr}}(\lambda, N, \gamma)$, we have

$$\mu - \frac{2p}{p-1} + 2\gamma + N = 0.$$

Therefore, we obtain

$$\chi(\psi) \le C(\ln\tau)^{\frac{-p}{p-1}} \int\limits_{r=\sqrt{\tau}}^{\tau} r^{-1}\ln r\mathrm{d}r = C(\ln\tau)^{\frac{p-2}{p-1}},$$

which proves (4.17). □

5 Proofs of the main results

This section describes the proofs of our main theorems in detail.

Proof of Theorem 3.1. The proof consists in two parts.

(I) We argue by contradiction. Namely, let us suppose that $u \in L^p_{\mathrm{loc}}(Q, \omega_k\mathrm{d}x)$ is a nontrivial weak solution to (1.1). Then, from Lemmas 4.1 and 4.5, we have

$$\int\limits_Q |u|^p\varphi\omega_k\mathrm{d}x \le \chi(\varphi), \tag{5.1}$$

where for $\tau, t \gg 1$, the function φ is given by (4.9). On the other hand, by the definition of φ, the properties of ξ and (4.6), we have

$$\int\limits_Q |u|^p\varphi\omega_k\mathrm{d}x = \int\limits_Q |u|^p F\xi_\tau^t\omega_k\mathrm{d}x \ge \int\limits_{1<|x|<\frac{\tau}{2}} |u|^p F\xi_\tau^t\omega_k\mathrm{d}x = \int\limits_{1<|x|<\frac{\tau}{2}} |u|^p F\omega_k\mathrm{d}x. \tag{5.2}$$

Then, it follows from Lemmas 4.8, 4.9, (5.1), and (5.2) that

$$\int\limits_{1<|x|<\frac{\tau}{2}} |u|^p F\omega_k \mathrm{d}x \leq \begin{cases} C\tau^{\frac{(\mu-2+2\gamma+N)p-\mu-2\gamma-N}{p-1}}, & \text{if} \quad \lambda > -\left(\dfrac{N-2+2\gamma}{2}\right)^2, \\ C\tau^{\frac{(N-2+2\gamma)p-N-2\gamma-2}{2(p-1)}} \ln\tau, & \text{if} \quad \lambda = -\left(\dfrac{N-2+2\gamma}{2}\right)^2. \end{cases} \tag{5.3}$$

Note that if $\lambda = -\left(\frac{N-2+2\gamma}{2}\right)^2$, then we have

$$\frac{(N-2+2\gamma)p-N-2\gamma-2}{2(p-1)} = \frac{(\mu-2+2\gamma+N)p-\mu-2\gamma-N}{p-1}.$$

So, (5.3) yields

$$\int\limits_{1<|x|<\frac{\tau}{2}} |u|^p F\omega_k \mathrm{d}x \leq C\tau^{\frac{(\mu-2+2\gamma+N)p-\mu-2\gamma-N}{p-1}} \ln\tau. \tag{5.4}$$

Observe also that by (3.4), we obtain

$$\frac{(\mu-2+2\gamma+N)p-\mu-2\gamma-N}{p-1} < 0.$$

Now, passing to the limit as $\tau \to \infty$ in (5.4), we conclude that

$$\int\limits_{Q} |u|^p F\omega_k \mathrm{d}x = 0, \tag{5.5}$$

which contradicts the fact that u is nontrivial. This proves part (I) of Theorem 3.1; hence, we pass to the second part of the statement.

(II) This time, we discuss two cases.

Case 1: $\lambda > -\left(\frac{N-2+2\gamma}{2}\right)^2$.

For

$$\mu - 2\lambda(N,\gamma) < \theta < \min\left\{\frac{-2}{p-1}, \mu\right\}, \tag{5.6}$$

let

$$P(\theta) = -\theta^2 + (2-2\gamma-N) + \lambda.$$

Observe that $\mu - 2\lambda(N,\gamma) < \mu$ (since $\lambda(N,\gamma) > 0$) and (3.5) is equivalent to

$$\mu - 2\lambda(N,\gamma) < \frac{-2}{p-1},$$

which show that the set of θ satisfying (5.6) is nonempty. On the other hand, the polynomial function $P(\theta)$ admits two distinct roots θ_1 and θ_2, given by

$$\theta_1 = \mu - 2\lambda(N,\gamma) < \mu = \theta_2,$$

which implies that $P(\theta) > 0$ for all θ satisfying (5.6). We now consider solutions of the form

$$u(x) = \varepsilon\,|x|^\theta, \quad x \in Q, \tag{5.7}$$

where

$$0 < \varepsilon \leq [P(\theta)]^{\frac{1}{p-1}}. \tag{5.8}$$

Since u is a radial function, we obtain by (2.3) that

$$-\Delta_k u + \frac{\lambda}{|x|^2} u = \varepsilon P(\theta)|x|^{\theta-2}. \tag{5.9}$$

Then, making use of (5.6), (5.7), (5.8), and (5.9), we obtain

$$\left(-\Delta_k u + \frac{\lambda}{|x|^2} u\right) |u|^{-p} = \varepsilon^{1-p} P(\theta)|x|^{-\theta(p-1)-2} \geq 1,$$

for all $x \in Q$ and $u_{|\partial B_1} = \varepsilon > 0$. This shows that for all θ and ε satisfying, respectively, (5.6) and (5.8), the function u defined by (5.7) is a solution to (1.1).

Case 2: $\lambda = -\left(\frac{N-2+2\gamma}{2}\right)^2$. For $0 < \theta < 1$ and $\varepsilon > 0$, we consider functions of the form

$$u(x) = \varepsilon\, |x|^{\mu} [\ln(2|x|)]^{\theta}, \quad x \in Q. \tag{5.10}$$

Since u is a radial function, we obtain by (2.3) that

$$-\Delta_k u + \frac{\lambda}{|x|^2} u = \varepsilon\theta(1-\theta)|x|^{\mu-2}[\ln(2|x|)]^{\theta-2},$$

which implies that

$$-\Delta_k u + \frac{\lambda}{|x|^2} u - |u|^p = \varepsilon\, |x|^{\mu-2}[\ln(2|x|)]^{\theta-2} \left(\theta(1-\theta) - \varepsilon^{p-1}\, |x|^{\mu(p-1)+2}[\ln(2|x|)]^{\theta p-\theta+2}\right). \tag{5.11}$$

Remark that (3.5) is equivalent to

$$\mu(p-1) + 2 < 0,$$

which implies that

$$\lim_{|x|\to\infty} |x|^{\mu(p-1)+2}[\ln(2|x|)]^{\theta p-\theta+2} = 0.$$

Then, by continuity of the function $Q \ni x \mapsto |x|^{\mu(p-1)+2}[\ln(2|x|)]^{\theta p-\theta+2}$, we deduce that there exists a constant $\iota > 0$ such that

$$|x|^{\mu(p-1)+2}[\ln(2|x|)]^{\theta p-\theta+2} \leq \iota, x \in Q.$$

Then, taking

$$0 < \varepsilon \leq \left(\frac{\theta(1-\theta)}{\iota}\right)^{\frac{1}{p-1}}, \tag{5.12}$$

we deduce from (5.10) and (5.11) that

$$-\Delta_k u + \frac{\lambda}{|x|^2} u - |u|^p \geq 0,$$

for all $x \in Q$ and $u_{|\partial B_1} = \varepsilon[\ln 2]^{\theta} > 0$. This shows that for all $0 < \theta < 1$ and ε satisfying (5.12), the function u defined by (5.10) is a solution to (1.1). The proof of part (II) of Theorem 3.1 is then complete. ☐

Now, we give the proof of our second main result.

Proof of Theorem 3.2. We also use the contradiction argument. Namely, let us suppose that $u \in L^p_{\text{loc}}(Q, \omega_k \mathrm{d}x)$ is a nontrivial weak solution to (1.1). The order of presentation of arguments is as follows:

(I) Let $\lambda > -\left(\frac{N-2+2\gamma}{2}\right)^2$. Taking $p = p_{\text{cr}}(\lambda, N, \gamma)$, we obtain by Lemma 4.8 and (5.3) that

$$\chi(\varphi) \leq C \tag{5.13}$$

and

$$\int_{1<|x|<\frac{\tau}{2}} |u|^p F\omega_k \mathrm{d}x \le C.$$

The aforementioned estimate implies that

$$u \in L^p(Q, F(x)\omega_k(x)\mathrm{d}x). \tag{5.14}$$

On the other hand, by the properties of ξ, (4.6), and (4.15), we have

$$\operatorname{supp}\left(-\Delta_k\varphi + \frac{\lambda}{|x|^2}\varphi\right) \subset \left\{x \in \mathbb{R}^N : \frac{\tau}{2} \le |x| \le \tau\right\},$$

which implies by Hölder's inequality that

$$\begin{aligned}
\int_Q |u| \left|-\Delta_k\varphi + \frac{\lambda}{|x|^2}\varphi\right| \omega_k \mathrm{d}x &= \int_{\frac{\tau}{2}<|x|<\tau} |u| \left|-\Delta_k\varphi + \frac{\lambda}{|x|^2}\varphi\right| \omega_k \mathrm{d}x \\
&= \int_{\frac{\tau}{2}<|x|<\tau} \left(|u|(\varphi\omega_k)^{\frac{1}{p}}\right)\left(\varphi^{\frac{-1}{p}} \left|-\Delta_k\varphi + \frac{\lambda}{|x|^2}\varphi\right| \omega_k^{\frac{p-1}{p}}\right)\mathrm{d}x \\
&\le \left(\int_{\frac{\tau}{2}<|x|<\tau} |u|^p F\xi_\tau^t \omega_k \mathrm{d}x\right)^{\frac{1}{p}} [\chi(\varphi)]^{\frac{p-1}{p}}.
\end{aligned}$$

Then, using (5.13) and that $0 \le \xi_\tau \le 1$, we obtain

$$\int_Q |u| \left|-\Delta_k\varphi + \frac{\lambda}{|x|^2}\varphi\right| \omega_k \mathrm{d}x \le C\left(\int_{\frac{\tau}{2}<|x|<\tau} |u|^p F\omega_k \mathrm{d}x\right)^{\frac{1}{p}},$$

which implies by (3.1) and (5.2) that

$$\int_{1<|x|<\frac{\tau}{2}} |u|^p F\omega_k \mathrm{d}x \le C\left(\int_{\frac{\tau}{2}<|x|<\tau} |u|^p F\omega_k \mathrm{d}x\right)^{\frac{1}{p}}.$$

Finally, passing to the limit as $\tau \to \infty$ in the aforementioned inequality, using (5.14) and the dominated convergence theorem, we obtain (5.5), which contradicts the fact that u is nontrivial. This proves part (I) of Theorem 3.2. It remains to prove the second part of the statement.

(II) Let $\lambda = -\left(\frac{N-2+2\gamma}{2}\right)^2$ and $N + 2\gamma \ge 6$. From Lemmas 4.1 and 4.7, we have

$$\int_Q |u|^p \psi\omega_k \mathrm{d}x \le \chi(\psi), \tag{5.15}$$

where for $\tau, t \gg 1$, the function ψ is given by (4.13). On the other hand, by the definition of ψ and (4.10), we have

$$\int_Q |u|^p \psi\omega_k \mathrm{d}x \ge \int_{1<|x|<\sqrt{\tau}} |u|^p F\omega_k \mathrm{d}x. \tag{5.16}$$

Note that from $N + 2\gamma \ge 6$ and $p = p_{\mathrm{cr}}(\lambda, N, \gamma)$, we have $p \le 2$, which implies by Lemma 4.10 that

$$\chi(\psi) \le C. \tag{5.17}$$

Hence, in view of (5.15), (5.16), and (5.17), we have

$$\int_{1<|x|<\sqrt{\tau}} |u|^p F\omega_k \mathrm{d}x \le C,$$

which implies (5.14). Furthermore, from (4.10) and (4.18), we have

$$\operatorname{supp}\left(-\Delta_k\psi + \frac{\lambda}{|x|^2}\psi\right) \subset \{x \in \mathbb{R}^N : \sqrt{\tau} \le |x| \le \tau\},$$

which implies by (5.17) and Hölder's inequality that

$$\int_Q |u| \left|-\Delta_k\psi + \frac{\lambda}{|x|^2}\psi\right| \omega_k \mathrm{d}x \le C\left(\int_{\sqrt{\tau}<|x|<\tau} |u|^p F\omega_k \mathrm{d}x\right)^{\frac{1}{p}}.$$

Then, by (3.1) and (5.16), it holds that

$$\int_{1<|x|<\sqrt{\tau}} |u|^p F\omega_k \mathrm{d}x \le C\left(\int_{\sqrt{\tau}<|x|<\tau} |u|^p F\omega_k \mathrm{d}x\right)^{\frac{1}{p}}.$$

Therefore, passing to the limit as $\tau \to \infty$ in the aforementioned inequality, using (5.14) and the dominated convergence theorem, we obtain (5.5), which contradicts the fact that u is nontrivial. This proves part (II) of Theorem 3.2. □

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