



Quasi-Optimization Problems: Existence, Iterative Algorithms and Well-Posedness

Shengda Zeng¹ · Qing Nie² · Calogero Vetro³

Received: 26 February 2025 / Accepted: 21 April 2025 / Published online: 30 May 2025
© The Author(s), under exclusive licence to Springer Science+Business Media, LLC, part of Springer Nature 2025

Abstract

The aim of this article is to introduce and study a quasi-optimization problem (QOP, for short) on an infinite dimensional Banach space. First, under mild assumptions, we deliver three existence theorems for (QOP) by employing Kluge's fixed point principle for multivalued operators. Then, several sufficient and necessary conditions for a solution of (QOP) are proved. Furthermore, two iterative algorithms are proposed and the convergence results are obtained. Finally, the well-posedness and generalized well-posedness of (QOP) are introduced and its equivalent metric characteristic is established.

Keywords Quasi-optimization problem · Existence · Well-posedness · Iterative algorithm · Sufficient and necessary condition

Mathematics Subject Classification 49J53 · 49K99

Communicated by Hong-Kun Xu.

✉ Shengda Zeng
zengshengda@163.com

Qing Nie
1925948806@qq.com

Calogero Vetro
calogero.vetro@unipa.it

¹ National Center for Applied Mathematics in Chongqing, and School of Mathematical Sciences, Chongqing Normal University, Chongqing 401331, China

² College of Mathematics and Statistics, Chongqing Jiaotong University, Chongqing 400074, People's Republic of China

³ Department of Mathematics and Computer Science, Via Archirafi 34, 90123 Palermo, Italy

1 Introduction

In a wide range of intricate physical phenomena and engineering applications, mathematical models often result in inequalities rather than the more commonly seen equations. The classical mathematical models with inequalities include variational inequalities, hemivariational inequalities, equilibrium problems, optimization problems, optimal control problems and so forth. Most of the existed results are concentrated on the research of the inequalities in which the constraints are formulated by constant sets, namely, constraints are independent of the sought solution. Whereas, in numerous critical scenarios encountered in material sciences, aerospace industry, transportation and economic models, such as Nash equilibrium problems with shared constraints and transport optimization involving feedback control, the constraint sets explicitly depend on the unknown solution. Such dependence leads inherently to the concept of quasi-inequalities, quasi-variational inequalities, quasi-equilibrium problems, quasi-hemivariational inequalities and quasi-optimization problems. Due to the wide applications of quasi-inequalities, its research has attracted a multitude of scholars' attention to make contribution of strengthening the theory and applicability of quasi-inequalities. In the subsequent discussion, we give a concise overview of some of the relevant advancements concerning quasi-inequalities. Facchinei-Kanzow-Sagratta [7] applied the Karush-Kuhn-Tucker conditions to propose an approximating algorithm for a quasi-variational inequality, and established the globally convergent result for the proposed approximating algorithm. By employing the theory of Brézis pseudomonotone operators and Mosco-type continuity method, Kanzow-Steck [10] established a theoretical framework for analyzing the existence of solutions to a class of quasi-variational inequalities (QVIs) in a generic Banach space setting. Zeng-Migórski-Khan [20] utilized a fixed point theorem for multivalued mappings combined with the theory of nonsmooth analysis and the Minty technique to develop a framework for determining solvability of a large class of nonlinear quasi-hemivariational inequalities, and then the authors used the theoretical results to explore an approximation of an elastic contact problem with the constitutive law involving a convex subdifferential inclusion. Cotrina-Zuniga [6] introduced a new kind of quasi-equilibrium problems with non-self constraint maps and proved the existence of projection solutions for such kind of quasi-equilibrium problems. Alleche-Rădulescu [2] examined the existence of solutions to a quasi-equilibrium problem on a Banach space by applying the Michael's selection theorem and a fixed point argument. Han-Huang [8] invoked the Fan-KKM theorem and the scalarization method to establish three existence theorems for strongly efficient solutions, weakly efficient solutions and efficient solutions to a new sort of generalized vector quasi-equilibrium problems. For further results on this topic, readers are encouraged to refer to [4, 5, 11, 15, 17, 18] and the references therein.

Under the setting that $(\mathcal{W}, \|\cdot\|)$ is a reflexive Banach space with its dual space \mathcal{W}^* and $\langle \cdot, \cdot \rangle$ representing the duality pair of \mathcal{W}^* and \mathcal{W} , $\mathcal{C}: \mathcal{D} \rightrightarrows \mathcal{D}$ is a multivalued mapping with $\mathcal{D} \subset \mathcal{W}$ being nonempty, convex and closed, $\mathcal{J}: \mathcal{W} \rightarrow \mathbb{R}$ is a cost functional, in present article we are focused on the following quasi-optimization problem (QOP, for short)

Problem 1.1 Looking for $x^* \in \mathcal{C}(x^*)$ to satisfy

$$\mathcal{J}(x^*) := \min_{v \in \mathcal{C}(x^*)} \mathcal{J}(v).$$

Quasi-optimization as a particular inequality of quasi-inequality was firstly introduced by Nishimura-Friedman [16] in finite dimensional spaces for studying the Nash equilibrium with n person games. Because the constraint sets depend implicitly on the unknown solution and the cost function is nonlinear and in-differential in general. This leads to the invalidity of smooth methods, dual approaches and so on. Despite there are few works establishing the approximated solutions and generalized solutions (for example, α -approximating solutions and projected solutions) to a certain type of quasi-optimization problems under linear and smooth settings (see [1, 3, 9, 19]). But, until now, there is not a general framework to analyze the existence of solutions of quasi-optimization problems in infinite dimensional spaces in which the cost functional is nonlinear and the constraint map has not generalized differential. To fill this gap, in this paper, we develop a general framework to explore the solvability of the quasi-optimization problem, namely Problem 1.1, under very mild assumptions. On the other hand, well-posedness is a significant concept for the analysis of stability and sensitivity of a problem or a mathematical model. As far as we know that the research concerning the well-posedness and generalized well-posedness to quasi-optimization problems have not been studied yet, so the second contribution of this paper is to introduce the definition of approximating sequences of the quasi-optimization problem, Problem 1.1, and then we shall discuss the equivalent metric characteristic of well-posedness and generalized well-posedness of Problem 1.1.

The paper is structured as follows. In Section 2, we review some necessary and useful preliminaries which will be used in Section 3 from time to time. Section 3 contains three subsections. The first subsection is devoted to explore the existence of solutions of the quasi-optimization problem under the cases that the set \mathcal{C} is compact, bounded and unbounded. Whereas the second subsection is to propose two iterative algorithms and to analyze its global convergence. Finally, in the last subsection, the concepts of approximating sequence, well-posedness and generalized well-posedness for quasi-optimization problem, Problem 1.1, are introduced, and the equivalent metric characteristic of well-posedness and generalized well-posedness to quasi-optimization problem, Problem 1.1, are established.

2 Preliminaries

The section is going to recall some necessary notation and preliminary results concerning convex analysis and nonsmooth analysis (see [14], for instance).

It says that $l: \mathcal{W} \rightarrow \mathbb{R}$ is convex, if the following inequality is satisfied

$$l(\delta x + (1 - \delta)z) \leq \delta l(x) + (1 - \delta)l(z)$$

for every $x, z \in \mathcal{W}$ and $\delta \in [0, 1]$. For any $\mathcal{D} \subset \mathcal{W}$, $r : \mathcal{D} \rightarrow \mathbb{R}$ is called to be locally Lipschitz, if for each $z \in \mathcal{D}$, it has a neighborhood $O(z)$ of z and $L_z > 0$ with

$$|r(a) - r(b)| \leq L_z \|a - b\| \text{ for any } a, b \in O(z).$$

In the present article, it uses the notations ∂l and $\partial_C r$ to stand for the convex subdifferential of l and generalized Clarke subdifferential of r , respectively, namely,

$$\partial l(b) := \{\xi \in \mathcal{W}^* \mid \langle \xi, a - b \rangle \leq l(a) - l(b) \text{ for every } a \in \mathcal{W}\},$$

and

$$\partial_C r(y) := \{\xi \in \mathcal{W}^* \mid r^0(y; b) \geq \langle \xi, b \rangle \text{ for any } b \in \mathcal{W}\},$$

where r^0 is the generalized Clarke direction derivative of r , thus,

$$r^0(y; b) := \limsup_{\theta \downarrow 0, a \rightarrow y} \frac{r(a + \theta b) - r(a)}{\theta}.$$

l is said to be Gâteaux differentiable at $a \in \mathcal{W}$, if we can take an element $\nabla l(a) \in \mathcal{W}^*$ satisfying

$$\lim_{\theta \rightarrow 0} \frac{l(a + \theta b) - l(a)}{\theta} = l'(a; b) = \langle \nabla l(a), b \rangle \text{ for any } b \in \mathcal{W}.$$

It consolidates several essential properties pertaining to generalized Clarke directional derivatives and generalized Clarke subdifferentials for locally Lipschitz functions by the lemma.

Lemma 2.1 *Given a locally Lipschitz function $r : \mathcal{D} \rightarrow \mathbb{R}$, then the statements hold*

- (i) *for each $a \in \mathcal{D}$, the function $\mathcal{W} \ni b \mapsto r^0(a; b) \in \mathbb{R}$ is positively homogeneous and subadditive with*

$$|r^0(a; b)| \leq L_a \|b\| \text{ for all } b \in \mathcal{W},$$

here $L_a > 0$ is the Lipschitz constant near a ;

- (ii) *$\mathcal{W}^2 \ni (a, b) \mapsto r^0(a; b) \in \mathbb{R}$ is upper semicontinuous;*
- (iii) *fixing $a \in \mathcal{D}$, it holds $r^0(a; b) = \max\{\langle \zeta, b \rangle \mid \zeta \in \partial_C r(a)\}$.*

An operator $A : \mathcal{W} \rightarrow \mathcal{W}^*$ is said to be a potential operator, if we can find a Gâteaux differentiable function $H_A : \mathcal{W} \rightarrow \mathbb{R}$ to satisfy $A = \nabla H_A$ with ∇H_A being the Gâteaux derivative of H_A . In such case, we can also call $H_A : \mathcal{W} \rightarrow \mathbb{R}$ as a potential of A .

In addition, we recall the concepts for closedness and continuity to multivalued mappings.

Definition 2.1 Let $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be a reflexive Banach space. $\Psi : \mathcal{W} \rightrightarrows \mathcal{Y}$ is called

- (a) (w, w) -closed, if $b \in \Psi(a)$ whenever $a_n \rightarrow a$ in \mathcal{W} , $b_n \rightarrow b$ in \mathcal{Y} with $b_n \in \Psi(a_n)$.
- (b) (w, s) -lower semicontinuous, if for each $b \in \Psi(a)$ it can take $b_n \in \Psi(a_n)$ such that $b_n \rightarrow b$ in \mathcal{Y} , whenever $a_n \rightarrow a \in \mathcal{W}$.

Definition 2.2 Let $Q, Q_1, Q_2 \subset \mathcal{W}$ be nonempty and bounded. Then,

- (a) $\text{diam}(Q)$ stands for the diameter of Q , i.e.,

$$\text{diam}(Q) = \sup\{\|e - f\| : e, f \in Q\}.$$

- (b) the distance from $e \in \mathcal{W}$ to $Q \subset \mathcal{W}$ is defined by

$$d(e, Q) = \inf\{\|e - q\| : q \in Q\}.$$

- (c) the Hausdorff metric between Q_1 and Q_2 is given by

$$\mathcal{H}(Q_1, Q_2) = \max\{e(Q_1, Q_2), e(Q_2, Q_1)\}$$

with

$$e(Q_1, Q_2) = \sup_{a \in Q_1} d(a, Q_2).$$

- (d) the noncompactness measure μ for Q is given by

$$\mu(Q) := \inf \left\{ \delta > 0 \mid Q = \bigcup_{i=1}^k Q_i, \text{diam}(Q_i) < \delta, i = 1, 2, \dots, k \right\}.$$

Definition 2.3 Let (X, τ) be a Hausdorff topological space and let $\{A_n\} \subset 2^X$ be a sequence of sets. We define the τ -Kuratowski lower limit of the sets A_n by

$$\tau\text{-}\liminf_{n \rightarrow \infty} A_n := \left\{ x \in X \mid x = \tau\text{-}\lim_{n \rightarrow \infty} x_n, x_n \in A_n \text{ for all } n \geq 1 \right\},$$

and the τ -Kuratowski upper limit of the sets A_n

$$\tau\text{-}\limsup_{n \rightarrow \infty} A_n := \left\{ x \in X \mid x = \tau\text{-}\lim_{k \rightarrow \infty} x_{n_k}, x_{n_k} \in A_{n_k}, n_1 < n_2 < \dots < n_k < \dots \right\}.$$

Finally, we recall the well-known Kluge’s fixed point principle for multivalued mappings (see [11, Theorem 3.3]).

Theorem 2.1 (Kluge’s fixed point theorem) *Suppose \mathcal{D} is a nonempty, convex, and closed subset of real reflexive Banach space \mathcal{W} . Assume that $\Psi : \mathcal{D} \rightrightarrows \mathcal{D}$ has nonempty, closed, and convex values such that its graph is sequentially weakly closed. If either \mathcal{D} is bounded or $\Psi(\mathcal{D})$ is bounded, then the fixed point set of Ψ is nonempty.*

3 Main Results

The section delivers the main results in this article. More precisely, it contains three subsections which give the existence of solutions as well as the sufficient and necessary conditions to a solution of Problem 1.1, two simple iterative algorithms and its convergence analysis, the well-posedness and generalized well-posedness as well as its metric characteristic to quasi-optimization problem, Problem 1.1.

3.1 Existence Results

This subsection is concerned with the research of existence theorems and the sufficient and necessary conditions to a solution of quasi-optimization problem. First, we deliver three existence theorems for the quasi-optimization problem under the cases that \mathcal{D} is compact, bounded and unbounded, respectively.

The first existence theorem is stated as follows, in which \mathcal{D} is assumed to be compact.

Theorem 3.1 *When $\mathcal{D} \subset \mathcal{W}$ is nonempty, compact and convex, $\mathcal{J} : \mathcal{D} \rightarrow \mathbb{R}$ is convex, continuous and bounded from below on \mathcal{D} , and $\mathcal{C} : \mathcal{D} \rightrightarrows \mathcal{D}$ is (w, w) -closed, (w, s) -lower semicontinuous and has nonempty, closed and convex values. Then, the quasi-optimization problem, Problem 1.1, has a nonempty and compact solution set.*

Proof To this end, for any $c \in \mathcal{D}$ fixed, it considers the intermediate optimization problem: taking $a_c \in \mathcal{C}(c)$ such that

$$\mathcal{J}(a_c) \leq \mathcal{J}(b) \text{ for any } b \in \mathcal{C}(c). \quad (1)$$

It introduces that the solution map $S : \mathcal{D} \rightrightarrows \mathcal{D}$ of problem (1), namely, $S(c)$ is the solution set of problem (1) corresponding to $c \in \mathcal{D}$. It is readily apparent that if $a \in \mathcal{D}$ is a fixed point of S , then one has $a \in \mathcal{C}(a)$ and

$$\mathcal{J}(a) \leq \mathcal{J}(b) \text{ for every } b \in \mathcal{C}(a).$$

So, $a \in \mathcal{D}$ is also a solution of quasi-optimization problem, Problem 1.1. Following this essential fact, our goal is to prove that S is well-defined and has at least a fixed point. Our method is based on the fixed point principle, Theorem 2.1, for obtaining the nonemptiness of fixed point set of $S : \mathcal{D} \rightrightarrows \mathcal{D}$.

Let $c \in \mathcal{D}$ be fixed. Since \mathcal{J} is continuous and bounded from below on \mathcal{D} and $\mathcal{C}(c) \subset \mathcal{D}$ with \mathcal{D} being compact, it invokes Weierstrass-Tonelli theorem to conclude that $S(c) \neq \emptyset$ and $S(\mathcal{D}) := \cup_{c \in \mathcal{D}} S(c) \subseteq \mathcal{D}$ is relatively compact. Taking any $a_1, a_2 \in S(c)$, it holds

$$\mathcal{J}(a_1) = \mathcal{J}(a_2) = \inf_{b \in \mathcal{C}(c)} \mathcal{J}(b).$$

The convexity of $\mathcal{C}(c)$ and \mathcal{J} shows

$$\inf_{b \in \mathcal{C}(c)} \mathcal{J}(b) \leq \mathcal{J}(\delta a_1 + (1 - \delta)a_2) \leq \delta \mathcal{J}(a_1) + (1 - \delta) \mathcal{J}(a_2) = \inf_{b \in \mathcal{C}(c)} \mathcal{J}(b)$$

for any $\delta \in [0, 1]$.

This indicates that $S(c)$ is convex. Whereas, the continuity of \mathcal{J} guarantees that $S(c)$ is closed. Indeed, when $\{a_n\} \subset S(c)$ has $a_n \rightarrow a$ in \mathcal{W} , then $a \in \mathcal{C}(c)$ and it is true

$$\mathcal{J}(a) = \lim_{n \rightarrow \infty} \mathcal{J}(a_n) = \inf_{b \in \mathcal{C}(c)} \mathcal{J}(b).$$

It asserts that S has a weakly closed graph. Let us take sequences $\{a_n\}, \{b_n\} \subset \mathcal{D}$ with $a_n \rightarrow a$, $b_n \rightarrow b$ and $b_n \in S(a_n)$. Recall that \mathcal{C} is (w, w) -closed, it results in $b \in \mathcal{C}(a)$. Set $d \in \mathcal{C}(a)$. Whereas, the (w, s) -lower semicontinuity of \mathcal{C} permits us to take $\{d_n\}$ with $d_n \rightarrow d$ and $d_n \in \mathcal{C}(a_n)$. Hence,

$$\mathcal{J}(b_n) \leq \mathcal{J}(d_n).$$

Letting $n \rightarrow \infty$, it derives

$$\mathcal{J}(b) \leq \liminf_{n \rightarrow \infty} \mathcal{J}(b_n) \leq \liminf_{n \rightarrow \infty} \mathcal{J}(d_n) = \mathcal{J}(d). \quad (2)$$

Keeping in mind $d \in \mathcal{C}(a)$ is arbitrary, so, we infer $b \in S(a)$, namely, $\text{Gr}(S)$ is weakly closed. Consequently, we employ Theorem 2.1 by replacing S by Ψ to conclude that $\text{Fix}S \neq \emptyset$, that is, quasi-optimization problem, Problem 1.1, has a solution.

Taking any a solution sequence $\{a_n\} \subset \mathcal{D}$ of Problem 1.1, the compactness of \mathcal{D} allows us to assume that $a_n \rightarrow a$ for some $a \in \mathcal{D}$. Since $a_n \in \mathcal{C}(a_n)$ and \mathcal{C} is (w, w) -closed, then it yields $a \in \mathcal{C}(a)$. Let $d \in \mathcal{C}(a)$ be fixed. The assumption that \mathcal{C} is (w, s) -lower semicontinuous allows us to take $\{d_n\} \subset \mathcal{D}$ with $d_n \in \mathcal{C}(a_n)$ and $d_n \rightarrow d$. Using the fact that $\mathcal{J}(a_n) \leq \mathcal{J}(d_n)$ and \mathcal{J} is continuous and convex, it implies that $\mathcal{J}(a) \leq \inf_{b \in \mathcal{C}(a)} \mathcal{J}(b)$, thus, a is also a solution of quasi-optimization problem, Problem 1.1. This means that Problem 1.1, has a nonempty and compact solution set. \square

Particularly, when \mathcal{J} is strictly convex, then we have the corollary.

Corollary 3.1 Assume $\mathcal{D} \subset \mathcal{W}$ is nonempty, compact and convex, $\mathcal{J} : \mathcal{D} \rightarrow \mathbb{R}$ is strictly convex, continuous, bounded from below on \mathcal{D} , and $\mathcal{C} : \mathcal{D} \rightrightarrows \mathcal{D}$ is (w, w) -closed, (w, s) -lower semicontinuous and has nonempty, closed convex values, then quasi-optimization problem, Problem 1.1, has a nonempty and compact solution set.

Remark 3.1 The main difference of the proof between Theorem 3.1 and Corollary 3.1 is that in Corollary 3.1, $S : \mathcal{D} \rightarrow \mathcal{D}$ is a single-valued map not a multivalued operator (because of the strict convexity of \mathcal{J}), and the proof of nonemptiness of fixed point set is based on Schauder fixed point theorem.

Indeed, the compactness of \mathcal{D} could be relaxed by the boundedness assumption.

Theorem 3.2 *Assume $\mathcal{D} \subset \mathcal{W}$ is nonempty, bounded, closed and convex, $\mathcal{J} : \mathcal{D} \rightarrow \mathbb{R}$ is convex, continuous and bounded from below on \mathcal{D} , and $\mathcal{C} : \mathcal{D} \rightrightarrows \mathcal{D}$ is (w, w) -closed, (w, s) -lower semicontinuous and has nonempty, closed convex values. Then quasi-optimization problem, Problem 1.1, has a nonempty and weakly compact solution set.*

Proof Our main idea is also to invoke Kluge's fixed point principle, Theorem 2.1, to demonstrate that S admits a fixed point. The boundedness of \mathcal{D} guarantees the boundedness of $S(\mathcal{D}) := \cup_{a \in \mathcal{D}} S(a) \subseteq \mathcal{D}$. By the same line of reasoning can be applied as demonstrated in the proof of Theorem 3.1, it concludes that S has nonempty, closed and convex values. Therefore, it is necessary to verify that the graph of S is weakly closed.

Suppose $\{a_n\}, \{b_n\} \subset \mathcal{D}$ satisfy $a_n \rightarrow a, b_n \rightarrow b$ and $b_n \in S(a_n)$. Hence, $b \in \mathcal{C}(a)$, thanks to the (w, w) -closedness of \mathcal{C} . Take any $d \in \mathcal{C}(a)$, it utilizes the (w, s) -lower semicontinuity of \mathcal{C} to have $\{d_n\}$ with $d_n \rightarrow d$ and $d_n \in \mathcal{C}(a_n)$. Hence, we get

$$\mathcal{J}(b_n) \leq \mathcal{J}(d_n).$$

Letting $\liminf_{n \rightarrow \infty}$ to the inequality above, it derives (2) for all $d \in \mathcal{C}(a)$. So, Theorem 2.1 reveals that S has a fixed point.

Concerning the weak compactness of the solution set of quasi-optimization problem, Problem 1.1, we take any solution sequence $\{a_n\} \subset \mathcal{D}$. Note that \mathcal{W} is reflexive and \mathcal{D} is bounded, closed and convex, then \mathcal{D} is weakly compact. Therefore, it can say that $a_n \rightarrow a$. Also, we get $a \in \mathcal{C}(a)$ and for each $d \in \mathcal{C}(a)$ fixed, it could take $\{d_n\} \subset \mathcal{D}$ with $d_n \in \mathcal{C}(a_n)$ and $d_n \rightarrow d$ and $\mathcal{J}(a_n) \leq \mathcal{J}(d_n)$. Letting $n \rightarrow \infty$, one gives $\mathcal{J}(a) \leq \mathcal{J}(d)$ for all $d \in \mathcal{C}(a)$. We conclude that a is also a solution of quasi-optimization problem, Problem 1.1, namely, the solution set of quasi-optimization problem, Problem 1.1, is weakly compact. \square

While \mathcal{J} is strictly convex, it has the corollary.

Corollary 3.2 *Assume $\mathcal{D} \subset \mathcal{W}$ is nonempty, bounded, closed and convex, and $\mathcal{J} : \mathcal{D} \rightarrow \mathbb{R}$ is strictly convex, continuous and bounded from below on \mathcal{D} , and $\mathcal{C} : \mathcal{D} \rightrightarrows \mathcal{D}$ is (w, w) -closed, (w, s) -lower semicontinuous and has nonempty, closed convex values, then quasi-optimization problem, Problem 1.1, has a nonempty and weakly compact solution set.*

Proof The strict convexity of \mathcal{J} leads to that $S : \mathcal{D} \rightarrow \mathcal{D}$ is well-defined. Whereas, the proof of existence of fixed point to $S : \mathcal{D} \rightarrow \mathcal{D}$ is based on Brouwer fixed point theorem in weak topology of \mathcal{W} . Since \mathcal{D} is bounded, closed and convex, so, it is convex and weakly compact. Therefore, it needs to demonstrate that S is weakly continuous. Given a convergent sequence $\{a_n\} \subset \mathcal{D}$ with $a_n \rightarrow a$ in \mathcal{W} , set $b_n = S(a_n)$. Then, it yields

$$\mathcal{J}(b_n) \leq \mathcal{J}(c) \text{ for every } c \in \mathcal{C}(a_n).$$

Recall that $\{b_n\} \subset \mathcal{D}$ is bounded, so, $b_n \rightharpoonup b$ in \mathcal{W} with $b \in \mathcal{D}$. The closedness of \mathcal{C} implies that $b \in \mathcal{C}(a)$. If $d \in \mathcal{C}(a)$, then it has a sequence $\{d_n\}$ with $d_n \in \mathcal{C}(a_n)$ enjoying $d_n \rightarrow d$ in \mathcal{W} . Keeping in mind that $\mathcal{J}(b_n) \leq \mathcal{J}(d_n)$, we utilize the continuity and convexity of \mathcal{J} to find that $\mathcal{J}(b) \leq \mathcal{J}(d)$. Owing to the arbitrariness of $d \in \mathcal{C}(a)$ and \mathcal{J} is strictly convex, it admits that $b = S(a)$. Because every weakly convergent subsequence of $\{b_n\}$ tends to the same limit $b = S(a)$, then it can derive that $b_n \rightharpoonup b$ in the whole sequence. This means that $S(a_n) \rightharpoonup S(a)$ in \mathcal{W} . Employing Brouwer fixedpoint theorem in weak topology of \mathcal{W} , we infer that there is $a_0 \in \mathcal{D}$ such that $a_0 = S(a_0)$ which is a solution of quasi-optimization problem, Problem 1.1. □

Furthermore, we deliver the existence result to quasi-optimization problem, Problem 1.1, when \mathcal{D} is unbounded. In such case, an additional condition is required that the cost function \mathcal{J} satisfies a certain coercive condition.

Theorem 3.3 *Assume $\mathcal{D} \subset \mathcal{W}$ is nonempty, closed and convex, $\mathcal{J} : \mathcal{D} \rightarrow \mathbb{R}$ is coercive (i.e., $\mathcal{J}(a_n) \rightarrow +\infty$ as $n \rightarrow +\infty$ whenever $\{a_n\} \subset \mathcal{W}$ is such that $\|a_n\| \rightarrow +\infty$ as $n \rightarrow \infty$) convex, continuous and bounded from below on \mathcal{D} , and $\mathcal{C} : \mathcal{D} \rightrightarrows \mathcal{D}$ is (w, w) -closed, (w, s) -lower semicontinuous and has nonempty, closed and convex values with $\cap_{b \in \mathcal{D}} \mathcal{C}(b) \neq \emptyset$. Then quasi-optimization problem, Problem 1.1, has a nonempty and weakly compact solution set.*

Proof Keeping in mind that \mathcal{J} is bounded from below and coercive, so, it can apply Weierstrass-Tonelli theorem to conclude that $S : \mathcal{D} \rightrightarrows \mathcal{D}$ is well-defined. In the meanwhile, S has convex and closed values, thanks to the convexity and continuity of \mathcal{J} . Via employing the same reasoning as presented in the proof of Theorem 3.2, it admits that S has a weakly closed graph. We shall verify that $S(\mathcal{D})$ is bounded. If it was unbounded, then one has $\{b_n\} \subset S(\mathcal{D})$ with $\|b_n\| \rightarrow +\infty$, namely, there has $\{a_n\} \subset \mathcal{D}$ with $b_n \in S(a_n)$. By the definition of S and $\cap_{a \in \mathcal{D}} \mathcal{C}(a) \neq \emptyset$, it could take $a_0 \in \cap_{a \in \mathcal{D}} \mathcal{C}(a)$ such that

$$\mathcal{J}(b_n) \leq \mathcal{J}(a_0) < +\infty.$$

Letting $n \rightarrow \infty$ and using the coercivity of \mathcal{J} , it generates a contradiction

$$+\infty = \lim_{n \rightarrow \infty} \mathcal{J}(b_n) \leq \mathcal{J}(a_0) < +\infty.$$

This reveals that $S(\mathcal{D})$ is bounded.

Therefore, all conditions of Kluge’s fixed point theorem, Theorem 2.1, are fulfilled. Using this theorem, it admits that the fixed point set of S is nonempty, namely, quasi-optimization problem, Problem 1.1, is solvable. From the analysis above, it can see that $S(\mathcal{D})$ is bounded and S has a weakly closed graph. So, it uses a standard way to show that the solution set of quasi-optimization problem, Problem 1.1, is weakly compact. □

From Corollary 3.2 and Theorem 3.3, we have the following result.

Corollary 3.3 Assume $\mathcal{D} \subset \mathcal{W}$ is nonempty, closed and convex, $\mathcal{J} : \mathcal{D} \rightarrow \mathbb{R}$ is coercive (i.e., $\mathcal{J}(a_n) \rightarrow +\infty$ as $n \rightarrow \infty$ whenever $\{a_n\} \subset \mathcal{W}$ is such that $\|a_n\| \rightarrow +\infty$ as $n \rightarrow \infty$) convex, continuous and bounded from below on \mathcal{D} , and $\mathcal{C} : \mathcal{D} \rightrightarrows \mathcal{D}$ is (w, w) -closed, (w, s) -lower semicontinuous and has nonempty, closed convex values with $\bigcap_{a \in \mathcal{D}} \mathcal{C}(a) \neq \emptyset$. Then quasi-optimization problem, Problem 1.1, has a nonempty and weakly compact solution set.

We also deliver the sufficient and necessary conditions to a solution of Problem 1.1.

Theorem 3.4 Suppose that \mathcal{C} has nonempty, closed and convex values.

- (i) If \mathcal{J} is convex and Gâteaux differentiable, then $a \in \mathcal{D}$ is a solution of quasi-optimization problem, Problem 1.1, iff it solves the quasi-variational inequality: $a \in \mathcal{C}(a)$ and

$$\langle \nabla \mathcal{J}(a), b - a \rangle \geq 0 \text{ for every } b \in \mathcal{C}(a), \tag{3}$$

here $\nabla \mathcal{J}(a)$ represents the Gâteaux derivative of \mathcal{J} at $a \in \mathcal{D}$.

- (ii) If \mathcal{J} is convex, then $a \in \mathcal{D}$ is a solution of quasi-optimization problem, Problem 1.1, iff it solves the quasi-inclusion problem: $a \in \mathcal{D}$ such that

$$\partial \mathcal{J}(a) + \partial I_{\mathcal{C}(a)}(a) \ni 0 \text{ in } \mathcal{W}^*, \tag{4}$$

where $I_{\mathcal{C}(a)}$ is the indicator function of $\mathcal{C}(a)$, and $\partial \mathcal{J}$ (resp. $\partial I_{\mathcal{C}(a)}$) is the convex subdifferential of \mathcal{J} (resp. $I_{\mathcal{C}(a)}$).

- (iii) If \mathcal{J} is locally Lipschitz continuous and $a \in \mathcal{D}$ is a solution of quasi-optimization problem, Problem 1.1, then it solves the quasi-hemivariational inequality

$$\text{look for } a \in \mathcal{C}(a) \text{ such that } \mathcal{J}^0(a; b - a) \geq 0 \text{ for each } b \in \mathcal{C}(a), \tag{5}$$

and multivalued quasi-variational inequality

$$\text{look for } a \in \mathcal{C}(a) \text{ and } \zeta \in \partial_C \mathcal{J}(a) \text{ such that } \langle \zeta, b - a \rangle \geq 0 \text{ for all } b \in \mathcal{C}(a), \tag{6}$$

here \mathcal{J}^0 and $\partial_C \mathcal{J}$ represent the generalized Clarke directional derivative and generalized Clarke subdifferential of \mathcal{J} , respectively.

- (iv) When $\mathcal{J} = h + g + f$ with h being a convex and continuous function, g being a locally Lipschitz function, and f being a potential of $A: \mathcal{W} \rightarrow \mathcal{W}^*$, then a solution $a \in \mathcal{D}$ of quasi-optimization problem, Problem 1.1, solves the mixed quasi-variational-hemivariational inequality

$$\text{taking } a \in \mathcal{C}(a) \text{ to satisfy } \langle A(a), b - a \rangle + h(b) - h(a) + g^0(a; b - a) \geq 0 \text{ for all } b \in \mathcal{C}(a).$$

Proof

(i) Let $a \in \mathcal{D}$ be a solution of quasi-optimization problem, Problem 1.1. Because $\mathcal{C}(a)$ is convex, so, for any $b \in \mathcal{C}(a)$ and $\theta \in (0, 1)$, it derives

$$0 \leq \frac{\mathcal{J}(\theta b + (1 - \theta)a) - \mathcal{J}(a)}{\theta}.$$

Letting $\theta \rightarrow 0^+$, it derives

$$0 \leq \lim_{\theta \rightarrow 0^+} \frac{\mathcal{J}(a + \theta(b - a)) - \mathcal{J}(a)}{\theta} = \langle \nabla \mathcal{J}(a), b - a \rangle.$$

This means that a is a solution of quasi-variational inequality (3). Conversely, suppose $a \in \mathcal{D}$ is a solution of quasi-variational inequality (3). Note that \mathcal{J} is convex and Gâteaux differentiable. So, $\nabla \mathcal{J} = \partial \mathcal{J}$, where $\partial \mathcal{J}$ is the convex subdifferential of \mathcal{J} . By the definition of convex subgradient, one gives

$$\mathcal{J}(b) - \mathcal{J}(a) \geq \langle \nabla \mathcal{J}(a), b - a \rangle \geq 0 \text{ for every } b \in \mathcal{C}(a),$$

namely, $a \in \mathcal{D}$ is a solution of quasi-optimization problem, Problem 1.1.

(ii) It is obvious that $a \in \mathcal{D}$ is a solution of quasi-optimization problem, Problem 1.1, iff it solves the inequality

$$\mathcal{J}(b) + I_{\mathcal{C}(a)}(b) \geq \mathcal{J}(a) + I_{\mathcal{C}(a)}(a) \text{ for any } b \in \mathcal{W}.$$

By the definition of convex subgradient, the above inequality can be rewritten equivalently by the inclusion

$$0 \in \partial(\mathcal{J} + I_{\mathcal{C}(a)})(a) \subset \partial \mathcal{J}(a) + \partial I_{\mathcal{C}(a)}(a) \text{ in } \mathcal{W}^*.$$

When a is a solution of inclusion problem (4), then, there are $a^* \in \partial \mathcal{J}(a)$ and $\zeta \in \partial I_{\mathcal{C}(a)}(a)$ satisfying

$$0 = a^* + \zeta.$$

Hence,

$$0 = \langle a^*, b - a \rangle + \langle \zeta, b - a \rangle \leq \mathcal{J}(b) - \mathcal{J}(a) + I_{\mathcal{C}(a)}(b) - I_{\mathcal{C}(a)}(a) \text{ for all } b \in \mathcal{W}.$$

For any $b \in \mathcal{C}(a)$, one obtains

$$0 \leq \mathcal{J}(b) - \mathcal{J}(a) - I_{\mathcal{C}(a)}(a).$$

Because \mathcal{J} is finite on $\mathcal{C}(a)$, then $I_{\mathcal{C}(a)}(a)$ is finite, namely, $a \in \mathcal{C}(a)$. So, a solves quasi-optimization problem, Problem 1.1.

(iii) If $a \in \mathcal{D}$ is a solution of Problem 1.1, then it holds

$$0 \leq \frac{\mathcal{J}(a + \theta(b - a)) - \mathcal{J}(a)}{\theta}$$

for each $b \in \mathcal{C}(a)$ and $\theta \in (0, 1)$. Hence, it holds

$$\begin{aligned} 0 &\leq \limsup_{\theta \downarrow 0^+} \frac{\mathcal{J}(a + \theta(b - a)) - \mathcal{J}(a)}{\theta} \\ &\leq \limsup_{\theta \downarrow 0^+, e \rightarrow a} \frac{\mathcal{J}(e + \theta(b - a)) - J(e)}{\theta} = \mathcal{J}^0(a; b - a). \end{aligned}$$

This means that a solves hemivariational inequality (5). On the other hand, by Lemma 2.1(iii), we can see that

$$0 \leq \mathcal{J}^0(a; b - a) = \max_{\zeta \in \partial_C \mathcal{J}(a)} \langle \zeta, b - a \rangle$$

for every $b \in \mathcal{C}(a)$, because of the weak compactness of $\partial_C \mathcal{J}(a)$. Therefore, it holds

$$0 \leq \inf_{b \in \mathcal{C}(a)} \max_{\zeta \in \partial_C \mathcal{J}(a)} \langle \zeta, b - a \rangle.$$

Whereas, it utilizes Sion’s Minimax Theorem (see e.g., [12, Theorem 1]) to yield

$$0 \leq \max_{\zeta \in \partial_C \mathcal{J}(a)} \inf_{b \in \mathcal{C}(a)} \langle \zeta, b - a \rangle,$$

namely, there exists $\zeta \in \partial_C \mathcal{J}(a)$ to enjoy

$$0 \leq \langle \zeta, b - a \rangle \text{ for all } b \in \mathcal{C}(a).$$

(iv) Given a solution $a \in \mathcal{D}$ of quasi-optimization problem, Problem 1.1, with $\mathcal{J} = h + g + f$, then it holds

$$\begin{aligned} 0 &\leq \frac{h(a + \theta(b - a)) - h(a)}{\theta} + \frac{g(a + \theta(b - a)) - g(a)}{\theta} \\ &\quad + \frac{f(a + \theta(b - a)) - f(a)}{\theta} \\ &\leq h(b) - h(a) + \frac{g(a + \theta(b - a)) - g(a)}{\theta} + \frac{f(a + \theta(b - a)) - f(a)}{\theta} \end{aligned}$$

for every $b \in \mathcal{C}(a)$ and $\theta \in (0, 1)$. Hence, we get

$$0 \leq h(b) - h(a) + \limsup_{\theta \downarrow 0^+} \frac{g(a + \theta(b - a)) - g(a)}{\theta}$$

$$\begin{aligned}
 & + \lim_{\theta \rightarrow 0^+} \frac{f(a + \theta(b - a)) - f(a)}{\theta} \\
 \leq & h(b) - h(a) + \limsup_{\theta \downarrow 0^+, e \rightarrow a} \frac{g(e + \theta(b - a)) - g(e)}{\theta} \\
 & + \lim_{\theta \rightarrow 0^+} \frac{f(a + \theta(b - a)) - f(a)}{\theta} \\
 \leq & \langle A(a), b - a \rangle + h(b) - h(a) + g^0(a; b - a)
 \end{aligned}$$

for all $b \in \mathcal{C}(a)$. □

3.2 Iterative Algorithms and Convergence Analysis

In the current subsection, we are going to design two iterative algorithms for obtaining a solution of quasi-optimization problem, Problem 1.1. So, we denote by \mathbb{S} the solution set of Problem 1.1.

The first iterative algorithm is designed as follows.

Theorem 3.5 *Under the assumptions of Theorem 3.3, let $a_0 \in \mathcal{D}$. Then the sequence $\{a_n\}$ generated by the following way has a subsequence converging weakly to a solution of quasi-optimization problem, Problem 1.1,*

$$a_n \in S(a_{n-1}) \text{ for all } n \geq 1,$$

namely, set $\mathbb{S}_1 = S(x_0)$ and $\mathbb{S}_n = S(\mathbb{S}_{n-1})$ for $n \geq 2$ it holds

$$\emptyset \neq w - \limsup_{n \rightarrow \infty} \mathbb{S}_n \subset \mathbb{S}. \tag{7}$$

Proof For any $a_0 \in \mathcal{D}$ fixed, as demonstrated in the proof of Theorem 3.3, S is well-defined and $\mathbb{S}_1 = S(a_0)$ is nonempty. So, the sets \mathbb{S}_n for all $n \geq 2$ are nonempty. Recall that $S(\mathcal{D})$ is bounded (see the proof of Theorem 3.3), then the fact that $\mathbb{S}_n \subset S(\mathcal{D})$ indicates that $\cup_{n \in \mathbb{N}} \mathbb{S}_n$ is bounded as well. This leads to $\emptyset \neq w - \limsup_{n \rightarrow \infty} \mathbb{S}_n$.

For any $a \in w - \limsup_{n \rightarrow \infty} \mathbb{S}_n$, then, extracting a subsequence if required, one has $\{a_n\} \subset \mathcal{D}$ with

$$a_n \in \mathbb{S}_n \text{ (i.e., } a_n \in S(a_{n-1})) \text{ and } a_n \rightharpoonup a \text{ in } \mathcal{W}.$$

Owing to $a_n \in S(a_{n-1})$, so, $a_n \in \mathcal{C}(a_{n-1})$. The weakly-weakly closedness of \mathcal{C} implies $a \in \mathcal{C}(a)$. For any $d \in \mathcal{C}(a)$, it has $\{d_n\} \subset \mathcal{D}$ with

$$d_n \in \mathcal{C}(a_{n-1}) \text{ and } d_n \rightarrow d \text{ in } \mathcal{W}.$$

Hence, it yields

$$\mathcal{J}(a_n) \leq \mathcal{J}(d_n).$$

By passing to the lower limit in the preceding inequality, it gives

$$\mathcal{J}(a) \leq \liminf_{n \rightarrow \infty} \mathcal{J}(a_n) \leq \lim_{n \rightarrow \infty} \mathcal{J}(d_n) = \mathcal{J}(d).$$

Due to the arbitrariness of $d \in \mathcal{C}(a)$, it can see that $a \in \mathcal{D}$ is a solution of quasi-optimization problem, Problem 1.1. This means that (7) holds. \square

Assume that $G: \mathcal{W} \rightarrow \mathbb{R}$ is strictly convex, continuous and bounded from below on \mathcal{D} . For any $\varepsilon > 0$, we consider the perturbation quasi-optimization problem:

Problem 3.1 Find $a_\varepsilon \in \mathcal{D}$ to satisfy $a_\varepsilon \in \mathcal{C}(a_\varepsilon)$ and

$$\mathcal{J}(a_\varepsilon) + \varepsilon G(a_\varepsilon) \leq \mathcal{J}(b) + \varepsilon G(b) \text{ for all } b \in \mathcal{C}(a_\varepsilon).$$

Because of convexity of \mathcal{J} and G , it is clear that for any $d \in \mathcal{D}$ given, if the following problem is solvable, then it has a unique solution

Problem 3.2 Find $a_\varepsilon \in \mathcal{C}(d)$ to fulfill

$$\mathcal{J}(a_\varepsilon) + \varepsilon G(a_\varepsilon) \leq \mathcal{J}(b) + \varepsilon G(b) \text{ for all } b \in \mathcal{C}(d).$$

So, we denote by $P_\varepsilon: \mathcal{D} \rightarrow \mathcal{D}$ the solution mapping of Problem 3.2.

Theorem 3.6 *Under the setting of Theorem 3.3, $G: \mathcal{W} \rightarrow \mathbb{R}$ is continuous, bounded, strictly convex and bounded from below on \mathcal{D} . Let $\varepsilon_n > 0$ and $\varepsilon_n \rightarrow 0^+$, and let $P_n = P_{\varepsilon_n}$. Then, for any $a_0 \in \mathcal{D}$ fixed, $\{a_n\}$ generated by the following way has a subsequence converging weakly to a solution of quasi-optimization problem, Problem 1.1,*

$$a_n = P_n(a_{n-1}) \text{ for all } n \geq 1,$$

that is, there is $\{a_{n_k}\} \subset \{a_n\}$ having

$$a_{n_k} \rightharpoonup a \in \mathbb{S}.$$

Proof From the hypotheses, it can see that for each $\varepsilon > 0$ the mapping P_ε is well-defined. For any $a_0 \in \mathcal{D}$, let $\varepsilon_n > 0$ and $\varepsilon_n \rightarrow 0^+$, and $a_n = P_n(a_{n-1})$. Hence, it has

$$\mathcal{J}(a_n) + \varepsilon_n G(a_n) \leq \mathcal{J}(b) + \varepsilon_n G(b) \text{ for all } b \in \mathcal{C}(a_{n-1}).$$

Let $b_0 \in \bigcap_{d \in \mathcal{D}} \mathcal{C}(d)$. Then, it derives

$$\mathcal{J}(a_n) + \varepsilon_n c_G \leq \mathcal{J}(a_n) + \varepsilon_n G(a_n) \leq \mathcal{J}(b_0) + \varepsilon_n |G(b_0)| \leq \mathcal{J}(b_0) + \varepsilon_1 |G(b_0)|,$$

for any $n \in \mathbb{N}$, where $c_G \in \mathbb{R}$ and $c_G \leq G(d)$ for each $d \in \mathcal{D}$ (i.e., c_G is the lower bound of G). If $\{a_n\}$ is unbounded, then it can say $\|a_n\| \rightarrow +\infty$ as $n \rightarrow \infty$. Letting

$n \rightarrow \infty$ for the above inequalities and using the coercivity of \mathcal{J} , it finds

$$+\infty \leq \lim_{n \rightarrow \infty} [\mathcal{J}(a_n) + \varepsilon_n c_G] \leq \mathcal{J}(b_0) + \varepsilon_1 |G(b_0)| < +\infty.$$

This implies a contradiction. Therefore, $\{a_n\}$ is bounded in \mathcal{W} . This permits us to take a subsequence, denoted by the same way, with $a_n \rightarrow a$ in \mathcal{W} . Keeping in mind that $a_n = P_n(a_{n-1})$, one has $a_n \in \mathcal{C}(a_{n-1})$. It admits that $a \in \mathcal{C}(a)$ owing to the (w, w) -closedness of \mathcal{C} . For any $b \in \mathcal{C}(a)$, there exists $b_n \in \mathcal{C}(a_{n-1})$ such that $b_n \rightarrow b$. So, we have

$$\mathcal{J}(a_n) + \varepsilon_n G(a_n) \leq \mathcal{J}(d_n) + \varepsilon_n G(d_n).$$

Hence,

$$\mathcal{J}(a) \leq \liminf_{n \rightarrow \infty} [\mathcal{J}(a_n) + \varepsilon_n G(a_n)] \leq \lim_{n \rightarrow \infty} [\mathcal{J}(d_n) + \varepsilon_n G(d_n)] \leq \mathcal{J}(d).$$

This means that $a \in \mathbb{S}$. □

3.3 Well-posedness and Generalized Well-Posedness

The subsection is concerned with the study of well-posedness and generalized well-posedness to quasi-optimization problem, Problem 1.1.

For every $\varepsilon > 0$, let us consider the ε -approximating set $\mathcal{M}(\varepsilon)$ of quasi-optimization problem, Problem 1.1, given by

$$\begin{aligned} \mathcal{M}(\varepsilon) &:= \left\{ a \in \mathcal{D} \mid d(a, \mathcal{C}(a)) \leq \varepsilon \text{ and } \mathcal{J}(a) \leq \inf_{b \in \mathcal{C}(a)} \mathcal{J}(b) + \varepsilon \right\} \\ &= \{ a \in \mathcal{D} \mid a \in \mathcal{C}(a) + O(\varepsilon) \text{ and } \mathcal{J}(a) \leq \mathcal{J}(b) + \varepsilon \text{ for each } b \in \mathcal{C}(a) \}, \end{aligned} \tag{8}$$

where $O(\varepsilon) = \{b \in \mathcal{W} \mid \|b\| < \varepsilon\}$.

Definition 1 $\{a_n\}$ in \mathcal{D} is called to be an approximating sequence of quasi-optimization problem, Problem 1.1, if it can find a sequence $\{\varepsilon_n\} \subset \mathbb{R}$ having

$$\varepsilon_n > 0, \varepsilon_n \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ and } a_n \in \mathcal{M}(\varepsilon_n).$$

Also, we give the definitions of well-posedness and generalized well-posedness to quasi-optimization problem, Problem 1.1.

Definition 3.1 We say that

- (i) Quasi-optimization problem, Problem 1.1, is strongly (resp. weakly) well-posed, if it is uniquely solvable and every approximating sequence converges strongly (resp. weakly) to the unique solution.

- (ii) Quasi-optimization problem, Problem 1.1, is strongly (resp. weakly) generalized well-posed, if it is solvable and every approximating sequence has a subsequence converging strongly (resp. weakly) to a solution of quasi-optimization problem, Problem 1.1.

Referring to \mathbb{S} (the solution set of quasi-optimization problem, Problem 1.1), it is not hard to see that $\mathbb{S} \subset \mathcal{M}(\varepsilon)$ for any $\varepsilon > 0$. The first theorem in this subsection gives the sufficient and necessary conditions for determining the strong well-posedness of quasi-optimization problem, Problem 1.1.

Theorem 3.7 *Quasi-optimization problem, Problem 1.1, is strongly well-posed iff $\mathbb{S} \neq \emptyset$ and*

$$\lim_{\varepsilon \rightarrow 0} \text{diam}(\mathcal{M}(\varepsilon)) = 0. \quad (9)$$

Proof It assumes that quasi-optimization problem, Problem 1.1, is strongly well-posed, then $\mathbb{S} = \{a^*\}$. We shall verify the identity (9). By contradiction, if (9) is not true, then one can take $\tau > 0$, $\{\varepsilon_n\}$ with $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$ and points $a_n^{(1)}, a_n^{(2)} \in \mathcal{M}(\varepsilon_n)$ having

$$\|a_n^{(1)} - a_n^{(2)}\| > \tau - \frac{1}{n} > 0 \text{ for any } n \in \mathbb{N}. \quad (10)$$

Recall that $\{a_n^{(1)}\}, \{a_n^{(2)}\}$ are two approximating sequences to quasi-optimization problem, Problem 1.1, and quasi-optimization problem, Problem 1.1, is strongly well-posed. This means that

$$\lim_{n \rightarrow \infty} a_n^{(1)} = \lim_{n \rightarrow \infty} a_n^{(2)} = a^*.$$

Whereas (10) implies a contradiction

$$0 < \tau \leq \|a_n^{(1)} - a_n^{(2)}\| + \frac{1}{n} \leq \|a_n^{(1)} - a^*\| + \|a_n^{(2)} - a^*\| + \frac{1}{n} \rightarrow 0.$$

Consequently, $\lim_{\varepsilon \rightarrow 0} \text{diam}(\mathcal{M}(\varepsilon)) = 0$ holds.

Conversely, when \mathbb{S} is nonempty and (9) holds. Now, (9) and the fact that $\mathbb{S} \subset \mathcal{M}(\varepsilon)$ for any $\varepsilon > 0$, deduce that $\mathbb{S} = \{a^*\}$. Suppose $\{a_n\}$ is an approximating sequence of quasi-optimization problem, Problem 1.1, then it has $\{\varepsilon_n\}$ with $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$ and $a_n \in \mathcal{M}(\varepsilon_n)$. It follows from (9) that

$$\lim_{n \rightarrow \infty} \|a_n - a^*\| \leq \lim_{n \rightarrow \infty} \text{diam}(\mathcal{M}(\varepsilon_n)) = 0.$$

This reveals that $a_n \rightarrow a^*$, thus, quasi-optimization problem, Problem 1.1, is strongly well-posed. \square

In Theorem 3.7, it requires that \mathbb{S} is nonempty. Since \mathbb{S} is a subset of $\mathcal{M}(\varepsilon)$ for any ε , so, whether we can replace $\mathcal{M}(\varepsilon) \neq \emptyset$ for $\varepsilon > 0$ by $\mathbb{S} \neq \emptyset$ is an interesting question. The following theorem gives a positive answer.

Theorem 3.8 *Suppose \mathcal{J} continuous, and \mathcal{C} is (s, s) -closed and (s, s) -lower semi-continuous. Then, quasi-optimization problem, Problem 1.1, is strongly well-posed iff*

$$\mathcal{M}(\varepsilon) \neq \emptyset \text{ for any } \varepsilon > 0, \text{ and } \lim_{\varepsilon \rightarrow 0} \text{diam}(\mathcal{M}(\varepsilon)) = 0. \tag{11}$$

Proof From Theorem 3.7 and the fact that $\mathbb{S} \subset \mathcal{M}(\varepsilon)$ for each $\varepsilon > 0$, it can see that (11) is true when quasi-optimization problem, Problem 1.1, is well-posed.

While (11) is fulfilled, set $\{a_n\}$ be an approximating sequence of quasi-optimization problem, Problem 1.1. It can find a positive sequence $\{\varepsilon_n\}$ with $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$ and $a_n \in \mathcal{M}(\varepsilon_n)$ for each $n \in \mathbb{N}$. Because of the monotonicity of $\mathcal{M}(\varepsilon)$ and (11), one has that $\{a_n\}$ is a Cauchy sequence. So, it can say that $a_n \rightarrow a^*$ in \mathcal{W} . We assert that a^* is the unique solution of quasi-optimization problem, Problem 1.1. Recall that $d(a_n, \mathcal{C}(a_n)) \leq \varepsilon_n$, it can take $\bar{a}_n \in \mathcal{C}(a_n)$ with

$$\|a_n - \bar{a}_n\| \leq \varepsilon_n + \frac{1}{n}.$$

Letting $n \rightarrow \infty$, it yields that $\bar{a}_n \rightarrow a^*$. The closedness of \mathcal{C} implies that $a^* \in \mathcal{C}(a^*)$. For any $b \in \mathcal{C}(a^*)$, the lower semicontinuity of \mathcal{C} allows to find a sequence $\{b_n\} \subset \mathcal{D}$ with $b_n \in \mathcal{C}(a_n)$ and $b_n \rightarrow b$ in \mathcal{W} . Owing to $\mathcal{J}(a_n) \leq \mathcal{J}(b_n) + \varepsilon_n$, it utilizes the continuity of \mathcal{J} to conclude that

$$\mathcal{J}(a^*) = \lim_{n \rightarrow \infty} \mathcal{J}(a_n) \leq \lim_{n \rightarrow \infty} \mathcal{J}(b_n) \leq \mathcal{J}(b).$$

This indicates that $a^* \in \mathbb{S}$. But, (11) derives that $\mathbb{S} = \{a^*\}$. Therefore, the whole sequence $\{a_n\}$ converges strongly to a^* , that is, quasi-optimization problem, Problem 1.1, is strongly well-posed. □

Next, we shall discuss the generalized well-posedness to quasi-optimization problem, Problem 1.1. To this end, it needs to show that $\mathcal{M}(\varepsilon)$ is closed under the continuity of \mathcal{J} and closedness of \mathcal{C} .

Lemma 3.1 *Assume \mathcal{J} is continuous, and \mathcal{C} has closed values and is (s, w) -closed and (s, s) -lower semicontinuity. Then, for each $\varepsilon > 0$, $\mathcal{M}(\varepsilon)$ is closed.*

Proof Let $\{a_n\} \in \mathcal{M}(\varepsilon)$ satisfy $a_n \rightarrow a^*$ as $n \rightarrow \infty$. Then, it has

$$d(a_n, \mathcal{C}(a_n)) \leq \varepsilon \text{ and } \mathcal{J}(a_n) \leq \mathcal{J}(b) + \varepsilon \text{ for all } b \in \mathcal{C}(a_n).$$

Since \mathcal{C} has closed values, so, it has $\bar{a}_n \in \mathcal{C}(a_n)$ such that

$$\|a_n - \bar{a}_n\| \leq \varepsilon + \frac{1}{n}.$$

It can see that $\{\bar{a}_n\}$ is bounded. Then, it can assume that $\bar{a}_n \rightarrow \bar{a}$ in \mathcal{W} . The (s, w) -closedness of \mathcal{C} implies that $\bar{a} \in \mathcal{C}(a^*)$. By considering the lower limit as $n \rightarrow \infty$

in the inequality above, it derives

$$d(a^*, \mathcal{C}(a)) \leq \|a^* - \bar{a}\| \leq \lim_{n \rightarrow \infty} \|a_n - \bar{a}_n\| \leq \varepsilon.$$

For each $b \in \mathcal{C}(a^*)$, it has $\{b_n\} \subset \mathcal{D}$ with $b_n \in \mathcal{C}(a_n)$ such that $b_n \rightarrow b$ in \mathcal{W} . Hence, we get

$$\mathcal{J}(a_n) \leq \mathcal{J}(b_n) + \varepsilon.$$

Letting $n \rightarrow \infty$, one has $\mathcal{J}(a^*) \leq \mathcal{J}(b) + \varepsilon$. By the arbitrariness of $b \in \mathcal{C}(a^*)$, it admits that $a^* \in \mathcal{M}(\varepsilon)$, namely, $\mathcal{M}(\varepsilon)$ is closed. \square

The first result concerning the generalized well-posedness to quasi-optimization problem, Problem 1.1, is stated as the following theorem.

Theorem 3.9 *Quasi-optimization problem, Problem 1.1, is strongly well-posed in the generalized sense iff its solution set \mathbb{S} is nonempty compact and*

$$\lim_{\varepsilon \rightarrow 0^+} e(\mathcal{M}(\varepsilon), \mathbb{S}) = 0. \tag{12}$$

Proof If quasi-optimization problem, Problem 1.1, is strongly well-posed in generalized sense, then $\mathbb{S} \neq \emptyset$. For any sequence $\{a_n\} \subset \mathbb{S}$, it is an approximating sequence to quasi-optimization problem, Problem 1.1. Therefore, $\{a_n\}$ has a subsequence converging strongly to a solution of quasi-optimization problem, Problem 1.1. This means that \mathbb{S} is compact. On the other side, suppose (12) is not true. So, it has a positive sequence $\{\varepsilon_n\}$ with $\varepsilon_n \rightarrow 0$ and $\theta > 0$ and $\bar{a}_n \in \mathcal{M}(\varepsilon_n)$ having

$$d(\bar{a}_n, \mathbb{S}) > \theta - \frac{1}{n} \text{ for any } n \in \mathbb{N}. \tag{13}$$

Evidently, $\{\bar{a}_n\}$ is an approximating sequence of Problem 1.1. Then, passing to a subsequence if necessary, it can say that $\bar{a}_n \rightarrow a^*$ with $a^* \in \mathbb{S}$. Letting $n \rightarrow \infty$ for inequality (13), it yields

$$0 = d(a^*, \mathbb{S}) \geq \theta > 0.$$

This leads to a contradiction, so, (12) is valid.

While \mathbb{S} is nonempty compact and (12) is satisfied, we take any approximating sequence $\{a_n\}$ to quasi-optimization problem, Problem 1.1, thus, it has $\{\varepsilon_n\}$ with $0 < \varepsilon_n$ and $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$ and $a_n \in \mathcal{M}(\varepsilon_n)$. Now, (12) permits us to take $\{a_n^*\} \subset \mathbb{S}$ having

$$\|a_n - a_n^*\| \rightarrow 0.$$

Because of compactness of \mathbb{S} , it can suppose that $a_n^* \rightarrow a^*$ in \mathcal{W} for some $a^* \in \mathbb{S}$. So, one has $a_n \rightarrow a^*$. This shows that quasi-optimization problem, Problem 1.1, is generalized well-posed. \square

The following theorem provides the metric characteristic to the approximating set of quasi-optimization problem, Problem 1.1, when it is generalized well-posed.

Theorem 3.10 *Assume \mathcal{J} is continuous, and \mathcal{C} has closed values and is (s, w) -closed and (s, s) -lower semicontinuous. Quasi-optimization problem, Problem 1.1, is generalized strongly well-posed iff \mathbb{S} is nonempty and*

$$\mathcal{M}(\varepsilon) \neq \emptyset \text{ for any } \varepsilon > 0 \text{ and } \lim_{\varepsilon \rightarrow 0} \mu(\mathcal{M}(\varepsilon)) = 0. \tag{14}$$

Proof Suppose that quasi-optimization problem, Problem 1.1, is generalized strongly well-posed. Then, \mathbb{S} is nonempty, so does $\mathcal{M}(\varepsilon)$. Hence,

$$\mathcal{H}(\mathcal{M}(\varepsilon), \mathbb{S}) = \max\{e(\mathcal{M}(\varepsilon), \mathbb{S}), e(\mathbb{S}, \mathcal{M}(\varepsilon))\} = e(\mathcal{M}(\varepsilon), \mathbb{S}) \text{ for each } \varepsilon > 0 \tag{15}$$

because of $\mathbb{S} \subset \mathcal{M}(\varepsilon)$ for each $\varepsilon > 0$. Theorem 3.9 indicates that \mathbb{S} is compact. Hence, we have

$$\mu(\mathbb{S}) = 0. \tag{16}$$

Using (15) and (16), it gives

$$\mu(\mathcal{M}(\varepsilon)) \leq 2\mathcal{H}(\mathcal{M}(\varepsilon), \mathbb{S}) + \mu(\mathbb{S}) = 2e(\mathcal{M}(\varepsilon), \mathbb{S}).$$

The latter together with (12) lead to conclude

$$\mu(\mathcal{M}(\varepsilon)) = 0,$$

i.e., (14) is verified.

Let (14) hold. Lemma 3.1 says that $\mathcal{M}(\varepsilon)$ is closed for any $\varepsilon > 0$. So, it can set $S = \bigcap_{\varepsilon > 0} \mathcal{M}(\varepsilon)$. By the generalized Cantor theorem (see [13]) and the fact that $\mathbb{S} \subset \mathcal{M}(\varepsilon)$, then S is closed, $\lim_{\varepsilon \rightarrow 0} \mathcal{H}(\mathcal{M}(\varepsilon), S) = 0$ and $\emptyset \neq \mathbb{S} \subset S$. It will illustrate that $S \subseteq \mathbb{S}$. For any $a^* \in S$ and $\varepsilon_n > 0$, it can take $a_n \in \mathcal{M}(\varepsilon_n)$ such that

$$\|a^* - a_n\| \leq \varepsilon_n.$$

This implies $a_n \rightarrow a^*$ in \mathcal{W} . So, we deduce

$$d(a_n, \mathcal{C}(a_n)) \leq \varepsilon_n \text{ and } \mathcal{J}(a_n) \leq \mathcal{J}(b) + \varepsilon_n \text{ for every } b \in \mathcal{C}(a_n).$$

The closedness of \mathcal{C} admits that $a^* \in \mathcal{C}(a^*)$. For any $d \in \mathcal{C}(a^*)$, it can take $\{d_n\} \subset \mathcal{D}$ with $d_n \in \mathcal{C}(a_n)$ and $d_n \rightarrow d$ in \mathcal{W} . One uses the continuity of \mathcal{J} to get

$$\mathcal{J}(a^*) \leq \lim_{n \rightarrow \infty} \mathcal{J}(a_n) \leq \lim_{n \rightarrow \infty} [\mathcal{J}(d_n) + \varepsilon_n] = \mathcal{J}(d).$$

We obtain $a^* \in \mathbb{S}$, thus, $S = \mathbb{S}$. From (14), it can see that $\mathbb{S} = S$ is compact. Also, the fact $\lim_{\varepsilon \rightarrow 0} e(\mathcal{M}(\varepsilon), \mathbb{S}) = \mathcal{H}(\mathcal{M}(\varepsilon), S) = 0$ and Theorem 3.9 reveal that quasi-optimization problem, Problem 1.1, is generalized well-posed. \square

4 Conclusion

Motivated by potential applications to mathematical models, we focused on a quasi-optimization problem, Problem 1.1, to obtain new abstract results that are not examined by other scholars. The obtained results are given following a unifying strategy which gives direct and accessible proofs to specialists and non-specialists. In summary, the main results of the present paper contain the following perspectives:

- We applied the Kluge's fixed point theorem of multivalued version to prove three existence theorems for the quasi-optimization problem, Problem 1.1, under the setting that C is compact, or bounded, or unbounded.
- Theorem 3.4 delivers several sufficient and necessary conditions to a solution of the quasi-optimization problem, Problem 1.1.
- We proposed two iterative algorithms to find approximating sequences of Problem 1.1, and proved that the iterative algorithms are globally convergent.
- The concept of approximating sequences, well-posedness and generalized well-posedness of the quasi-optimization problem, Problem 1.1, are introduced, and the corresponding equivalent metric characteristic is obtained.

Acknowledgements This project has received funding from the Natural Science Foundation of Guangxi Grant Nos. 2021GXNSFFA196004 and 2024GXNSFBA010337, the NNSF of China Grant No. 12371312, the Natural Science Foundation of Chongqing Grant No. CSTB2024NSCQ-JQX0033.

Author Contributions The authors contributed equally to this paper.

Data Availability Data sharing not applicable to this article as no data sets were generated or analysed during the current study.

Declarations

Competing interests There is no conflict of interests.

Ethical Approval Not applicable.

References

1. Ait Mansour, M., Bahraoui, M.A., El Bekkali, A.: Sharp estimates for approximate and exact solutions to quasi-optimization problems. *Optim.* **71**, 1331–1355 (2022)
2. Alleche, B., Radulescu, V.D.: Solutions and approximate solutions of quasi-equilibrium problems in Banach spaces. *J. Optim. Theory Appl.* **170**, 629–649 (2016)
3. Anh, L.Q., Khanh, P.Q., Van My, D.T.: On well-posedness for perturbed quasi-equilibrium and quasi-optimization problems. *Numer. Func. Anal. Optim.* **42**, 583–607 (2021)
4. Aussel, D., Chaipunya, P.: Variational and quasi-variational inequalities under local reproducibility: Solution concept and applications. *J. Optim. Theory Appl.* **203**, 1531–1563 (2024)
5. Cen, J., Khan, A.A., Motreanu, D., Zeng, S.D.: Inverse problems for generalized quasi-variational inequalities with application to elliptic mixed boundary value systems. *Inverse Problems.* **38**, 065006 (2022)
6. Cotrina, J., Zuniga, J.: Quasi-equilibrium problems with non-self constraint map. *J. Global Optim.* **75**, 177–197 (2019)
7. Facchinei, F., Kanzow, C., Sagratella, S.: Solving quasi-variational inequalities via their KKT conditions. *Math. Program.* **144**, 369–412 (2014)

8. Han, Y., Huang, N.J.: Existence and connectedness of solutions for generalized vector quasi-equilibrium problems. *J. Optim. Theory Appl.* **179**, 65–85 (2018)
9. Hung, N.V., Keller, A.A.: Painlevé-Kuratowski convergence of the solution sets for controlled systems of fuzzy vector quasi-optimization problems with application to controlling traffic networks under uncertainty. *Comput. Appl. Math.* **40**, 28 (2021)
10. Kanzow, C., Steck, D.: Quasi-variational inequalities in Banach spaces: Theory and augmented Lagrangian methods. *SIAM J. Optim.* **29**, 3174–3200 (2019)
11. Khan, A.A., Motreanu, D.: Existence theorems for elliptic and evolutionary variational and quasi-variational inequalities. *J. Optim. Theory Appl.* **167**, 1136–1161 (2015)
12. Kindler, J.: A simple proof of Sion's minimax theorem. *American Math. Mon.* **112**, 356–358 (2005)
13. Kuratowski, C.: *Topology. Vol. I and II.* Academic Press, New York, NY, USA (1966)
14. Migórski, S., Ochal, A., Sofonea, M.: *Nonlinear inclusions and hemivariational inequalities: models and analysis of contact problems.* Springer Science and Business Media (2012)
15. Nguyen, L.V., Qin, X.: Some results on strongly pseudomonotone quasi-variational inequalities. *Set-Valued Var. Anal.* **28**, 239–257 (2020)
16. Nishimura, K., Friedman, J.: Existence of Nash equilibrium in n person games without quasi-concavity. *International Economic Review.* 637–648 (1981)
17. Ravat, U., Shanbhag, U.V.: On the existence of solutions to stochastic quasi-variational inequality and complementarity problems. *Math. Program.* **165**, 291–330 (2017)
18. Rodrigues, J.F., Santos, L.: On nonlocal variational and quasi-variational inequalities with fractional gradient. *Appl. Math. Optim.* **80**, 835–852 (2019)
19. Tuan, L.A.: Existence of solutions of set quasi-optimization problems involving Minkowski difference. *Numer. Func. Anal. Optim.* **44**, 1129–1152 (2023)
20. Zeng, S.D., Migórski, S., Khan, A.A.: Nonlinear quasi-hemivariational inequalities: Existence and optimal control. *SIAM J. Control Optim.* **59**, 1246–1274 (2021)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.