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Research Paper

A characterization of varieties of algebras of proper central exponent greater than two ☆

Francesca S. Benanti [a], Angela Valenti [b,*]

[a] Dipartimento di Matematica e Informatica, Università di Palermo, Via Archirafi 34, 90123, Palermo, Italy
[b] Dipartimento di Ingegneria, Università di Palermo, Viale delle Scienze, 90128, Palermo, Italy

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ABSTRACT

Let F be a field of characteristic zero and let $\mathcal{V}$ be a variety of associative F-algebras. In [19] Regev introduced a numerical sequence measuring the growth of the proper central polynomials of a generating algebra of $\mathcal{V}$. Such sequence $c_n^\delta(\mathcal{V})$, $n \geq 1$, is called the sequence of proper central polynomials of $\mathcal{V}$ and in [12], [13] the authors computed its exponential growth. This is an invariant of the variety. They also showed that $c_n^\delta(\mathcal{V})$ either grows exponentially or is polynomially bounded.
The purpose of this paper is to characterize the varieties of associative algebras whose exponential growth of $c_n^\delta(\mathcal{V})$ is greater than two. As a consequence, we find a characterization of the varieties whose corresponding exponential growth is equal to two.
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* Corresponding author.
E-mail addresses: francescasaviella.benanti@unipa.it (F.S. Benanti), angela.valenti@unipa.it (A. Valenti).

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1. Introduction

Let A be an algebra over a field F of characteristic zero and $F\langle X\rangle$ the free associative algebra freely generated over F by the set $X = \{x_1, x_2, \dots\}$. A polynomial $f = f(x_1, \dots, x_n) \in F\langle X\rangle$ is a central polynomial of A if, for any $a_1, \dots, a_n \in A$, $f(a_1, \dots, a_n) \in Z(A)$, the center of A. In case f takes a non-zero value in A, we say that f is a proper central polynomial, otherwise we say that f is a polynomial identity of A. We denote by $\mathrm{Id}(A)$ the T-ideal of $F\langle X\rangle$ consisting of the polynomial identities of A and by $\mathrm{Id}^z(A)$ the T-space of central polynomials of A.

Central polynomials were first studied after a famous conjecture of Kaplansky asserting that $M_k(F)$, the algebra of $k \times k$ matrices over F, has proper central polynomials (see [14]). Later on, such a conjecture was proved independently by Formanek in [5] and Razmyslov in [17]. This result is highly non-trivial since even if an algebra has a non-zero center, the existence of proper central polynomials is not granted. For instance, it is well-known (see [12]) that the algebra of upper block triangular matrices has no proper central polynomials.

Here we are interested in a quantitative approach in order to get information about the growth of the proper central polynomials of a given algebra. To this end, for any $n \geq 1$, let P_n be the space of multilinear polynomials in $x_1, \dots, x_n$. Then define $P_n(A)$ as P_n modulo the identities of A and $P_n(A)^z$ as P_n modulo the central polynomials of A. The corresponding dimensions $c_n(A) = \dim P_n(A)$ and $c_n^z(A) = \dim P_n(A)^z$ are called the n-th codimension and the n-th central codimension of A, respectively. Clearly $c_n(A) - c_n^z(A) = c_n^\delta(A)$ is the dimension of the corresponding space of proper central polynomials.

The asymptotics of the codimensions of an algebra A have been extensively studied in the past, see for instance [18], [1], [3], [4] or [11] and [2] for a comprehensive overview of the main results. It turns out that if A satisfies a non-trivial identity then $\lim_{n\to\infty} \sqrt[n]{c_n(A)} = exp(A)$ exists and is an integer called the PI-exponent of A (see [8]). Recently in [12] and [13] it was proved that also $\lim_{n\to\infty} \sqrt[n]{c_n^z(A)} = exp^z(A)$ and $\lim_{n\to\infty} \sqrt[n]{c_n^\delta(A)} = exp^\delta(A)$ exist and are integers called the central exponent and the proper central exponent of A, respectively. Moreover all three codimensions either grow exponentially or are polynomially bounded. When $exp(A) \geq 2$, $exp(A)$ and $exp^z(A)$ coincide, whereas it can be shown that the difference $exp(A) - exp^\delta(A)$ can be any positive integer (see [12]). Hence $exp^\delta(A)$ is a second invariant of the T-ideal $\mathrm{Id}(A)$ that is worth investigating.

Let G denote the infinite dimensional Grassmann algebra and $UT_2(F)$ the algebra of 2×2 upper triangular matrices. Kemer in [15] proved that if an algebra A is such that $exp(A) \geq 2$, then either $\mathrm{Id}(G) \supseteq \mathrm{Id}(A)$ or $\mathrm{Id}(UT_2(F)) \supseteq \mathrm{Id}(A)$. Since it is well-known that $exp(G) = exp(UT_2) = 2$ (see [11]) and no intermediate growth is allowed, it follows that if B is an algebra such that $\mathrm{Id}(B) \supsetneq \mathrm{Id}(G)$ or $\mathrm{Id}(B) \supsetneq \mathrm{Id}(UT_2(F))$ then the codimensions of B grow polynomially. In the language of varieties of algebras, this says that G and $UT_2(F)$ generate the only two varieties of almost polynomial growth.

Recently in [6] an analogous result was proved in the setting of proper central polynomials. The authors constructed two finite dimensional algebras, named D and D_0, having exponential growth of the proper central codimensions and classified up to PI-equivalence the algebras A for which the sequence $c_n^\delta(A)$, $n \geq 1$, has almost polynomial growth. It follows that G, D and D_0 generate the only varieties of almost polynomial growth of the proper central polynomials.

In [9], Giambruno and Zaicev characterized the varieties of PI-exponent greater than two in terms of five suitable algebras of exponent 3 or 4.

The purpose of this paper is to characterize the varieties having proper central exponent greater than two. To this end, we shall explicitly exhibit a list of nine algebras $\mathcal{A}_i$ in order to prove the following result: a variety $\mathcal{V}$ has proper central exponent greater than two if and only if $\mathcal{A}_i \in \mathcal{V}$, for some i. By putting together this theorem with the results in [6], we shall obtain a characterization of the varieties of proper central exponent equal to two.

2. Preliminaries

Throughout this paper all algebras will be associative and over a field F of characteristic zero. Unless otherwise stated, we shall also assume that F is algebraically closed. Let $F\langle X\rangle$ be the free algebra over F on a countable set $X = \{x_1, x_2, \ldots\}$. Let $\mathrm{Id}(A)$ be the T-ideal of polynomial identities of the F-algebra A and let $\mathrm{Id}^z(A)$ be the space of central polynomials of A. Notice that $\mathrm{Id}^z(A)$ is a T-space, i.e., invariant under all endomorphisms of $F\langle X\rangle$. Clearly $\mathrm{Id}(A) \subseteq \mathrm{Id}^z(A)$ and the proper central polynomials correspond to the quotient space $\mathrm{Id}^z(A)/\mathrm{Id}(A)$.

Regev in [19] introduced the notion of central codimensions as follows. Let P_n be the space of multilinear polynomials in $x_1, \ldots, x_n$. Then, for $n = 1, 2, \ldots$,

$$c_n(A) = \dim \frac{P_n}{P_n \cap \mathrm{Id}(A)},\ c_n^z(A) = \dim \frac{P_n}{P_n \cap \mathrm{Id}^z(A)},\ c_n^\delta(A) = \dim \frac{P_n \cap \mathrm{Id}^z(A)}{P_n \cap \mathrm{Id}(A)}$$

are the sequences of codimensions, central codimensions and proper central codimensions of A, respectively.

It was proved in [7], [8], [12] and [13] that for any PI-algebra A the following three limits

$$exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)},\ exp^z(A) = \lim_{n\to\infty} \sqrt[n]{c_n^z(A)},\ exp^\delta(A) = \lim_{n\to\infty} \sqrt[n]{c_n^\delta(A)}$$

exist and are non negative integers called the PI-exponent, the central exponent and the proper central exponent (called also δ-exponent) of the algebra A, respectively. Clearly $exp^z(A), exp^\delta(A) \leq exp(A)$ and in [13, Theorem 3] it was shown that if $\exp(A) \geq 2$, then $exp^z(A) = exp(A)$. A basic property of the above three numerical sequences is that they either grow exponentially or are polynomially bounded. Hence no intermediate growth is allowed.

Let us introduce a useful tool in the theory of polynomial identities, the Grassmann envelope of an algebra. Let G be the infinite dimensional Grassmann algebra, that is the algebra with 1 generated by countably many elements $e_1, e_2, \ldots$, subject to the condition $e_ie_j = -e_je_i$. G has a natural structure of a superalgebra $G = G^{(0)} \oplus G^{(1)}$ where every monomial in the e_i's of even (odd resp.) length has homogeneous degree zero (one resp.).

If $B = B^{(0)} \oplus B^{(1)}$ is a superalgebra, one can construct the algebra $G(B) = (G^{(0)} \otimes B^{(0)}) \oplus (G^{(1)} \otimes B^{(1)})$ which is called the Grassmann envelope of B. The importance of the Grassmann envelope is highlighted in a well-known result of Kemer (see [16]) stating that if A is any PI-algebra there exists a finite dimensional superalgebra B such that $\mathrm{Id}(A) = \mathrm{Id}(G(B))$.

It is worth noticing that if A is a finite dimensional algebra, then regarding A as an algebra with trivial grading we get $\mathrm{Id}(G(A)) = \mathrm{Id}(G^{(0)} \otimes A) = \mathrm{Id}(A)$, and we may work with A instead of $G(A)$.

Now we recall how to compute the exponent and the proper central exponent of the algebra $G(B)$. By the Wedderburn-Malcev decomposition we write $B = \bar{B} + J$ where $\bar{B}$ is a maximal semisimple superalgebra inside B and J is the Jacobson radical of B (see [11, Theorem 3.4.3]). Also we can write $\bar{B} = B_1 \oplus \cdots \oplus B_q$, a decomposition into simple superalgebras.

Let $B_{i_1}, \ldots, B_{i_k}$ be distinct simple subalgebras of $\bar{B}$. Then we say that $C = B_{i_1} \oplus \cdots \oplus B_{i_k}$ is an admissible subalgebra of B if $B_{i_1}JB_{i_2}J \cdots JB_{i_k} \neq 0$. Moreover we say that C is a centrally admissible subalgebra of B, if there exists a proper central polynomial $f(x_1, \ldots, x_n)$ of $G(B)$ having a non-zero evaluation involving at least one element of each $G(B_{i_h})$, $1 \leq h \leq k$.

Then $exp(G(B))$ is the maximal dimension of an admissible subalgebra of B and $exp^\delta(G(B))$ is the maximal dimension of a centrally admissible subalgebra of B (see [8], [12], [13]).

Recall that if $\mathcal{V} = var(A)$ is the variety of algebras generated by an algebra A, then we write $exp(\mathcal{V}) = exp(A)$ and $exp^\delta(\mathcal{V}) = exp^\delta(A)$. Hence the growth of $\mathcal{V}$ is the growth of the codimensions of the algebra A. Moreover a variety $\mathcal{V}$ has almost polynomial growth if $\mathcal{V}$ grows exponentially but any proper subvariety grows polynomially.

By [13, Remark 1] if two algebras have the same identities, they also have the same proper central polynomials. Hence the proper central exponent of an algebra A is an invariant of its T-ideal of identities and also of the corresponding variety of algebras. So we say that a variety of algebras $\mathcal{V}$ has almost polynomial δ-growth if $exp^\delta(\mathcal{V}) \geq 2$ and for any proper subvariety $\mathcal{W} \subsetneq \mathcal{V}$ we have that $exp^\delta(\mathcal{W}) \leq 1$. By abuse of notation we also say that an algebra A has almost polynomial δ-growth if $exp^\delta(A) \geq 2$ and for any algebra B such that $\mathrm{Id}(B) \supsetneq \mathrm{Id}(A)$, we have that $exp^\delta(B) \leq 1$.

Recently in [6] the authors classified the T-ideals whose corresponding varieties have almost polynomial growth of the proper central codimensions. More precisely, it was proved that if A is a PI-algebra with $exp^\delta(A) \geq 2$, then either $\mathrm{Id}(A) \subseteq \mathrm{Id}(D)$ or $\mathrm{Id}(A) \subseteq \mathrm{Id}(D_0)$ or $\mathrm{Id}(A) \subseteq \mathrm{Id}(G)$, where $D = \{(a_{ij}) \in UT_3(F) \mid a_{11} = a_{33}\}$ and $D_0 = \{(a_{ij}) \in$

$UT_4(F) \mid a_{11} = a_{44} = 0\}$. Here $UT_n(F)$ denotes the algebra of $n \times n$ upper triangular matrices over F.

3. Some algebras of small δ-exponent greater than two

The purpose of this section is to introduce some suitable algebras that will allow us to prove the main result of this paper. We start by recalling some notation and definitions.

For an ordered set of positive integers $(d_1, \ldots, d_m)$ let $UT(d_1, \ldots, d_m)$ denote the algebra of upper block-triangular matrices over F of size $d_1, \ldots, d_m$, i.e.,

$$UT(d_1, \ldots, d_m) = \begin{pmatrix} M_{d_1}(F) & \cdots & * & * \\ 0 & \ddots & & \\ \vdots & & & * \\ 0 & \cdots & 0 & M_{d_m}(F) \end{pmatrix},$$

where $M_{d_i}(F)$ is the algebra of $d_i \times d_i$ matrices over F.

Recall that if H is an arbitrary abelian group an H-grading on $M_n(F)$ is elementary if there is an n-tuple $\bar{h} = (h_1, \ldots, h_n)$ such that the matrix units e_{ij} are of homogeneous degree $h_i^{-1}h_j$. Clearly the algebra of upper block triangular matrices also admits elementary gradings. In fact, the embedding of such an algebra into a full matrix algebra with an elementary grading makes it a homogeneous subalgebra. We shall denote by $UT(d_1, \ldots, d_m)_{(h_1,\ldots,h_n)}$ the algebra of upper block triangular matrices with elementary grading induced by the n-tuple $\bar{h} = (h_1, \ldots, h_n)$ with $n = d_1 + \cdots + d_m$. In particular, for $H = \mathbb{Z}_2$, $UT(d_1, \ldots, d_m)_{(h_1,\ldots,h_n)}$ is a superalgebra and $G(UT(d_1, \ldots, d_m)_{(h_1,\ldots,h_n)})$ denotes its Grassmann envelope.

We also need to recall the classification of the finite dimensional simple superalgebras over F (see [16], pag. 21). Any such superalgebra is isomorphic to one of the following:

i) $M_{k,l}(F)$ is the algebra $M_{k+l}(F)$, $k \geq l \geq 0$, $k \neq 0$, endowed with the grading induced by the $(k+l)$-tuple $(\underbrace{0, \ldots, 0}_{k \text{ times}}, \underbrace{1, \ldots, 1}_{l \text{ times}})$;
ii) $M_k(F) + cM_k(F)$, where $c^2 = 1$, with grading $(M_k(F), cM_k(F))$.

We shall denote by $M_{k,l}(G)$ and $M_k(G)$, respectively, the corresponding Grassmann envelopes. In what follows we shall identify $F+cF$ with the graded subalgebra of $M_2(F)_{(0,1)}$ generated by the elements $c_0 = e_{11} + e_{22}$, and $c_1 = e_{12} + e_{21}$.

Next we define the following nine algebras over F that we shall use in the sequel.

$\mathcal{A}_1 = M_2(F)$,
$\mathcal{A}_2 = M_{1,1}(G)$,
$\mathcal{A}_3 = \{(a_{ij}) \in UT_4(F) \mid a_{11} = a_{44}\}$,
$\mathcal{A}_4 = \{(a_{ij}) \in UT_5(F) \mid a_{11} = a_{55} = 0\}$,

$\mathcal{A}_5 = \{(a_{ij}) \in UT_3(G) \mid a_{11} = a_{33},\ a_{22} \in G^{(0)}\}$,
$\mathcal{A}_6 = \{(a_{ij}) \in G(UT(2,3)_{(0,0,0,1,0)}) \mid a_{11} = a_{21} = a_{53} = a_{54} = a_{55} = 0,\ a_{33} = a_{44},\ a_{34} = a_{43}\}$,
$\mathcal{A}_7 = \{(a_{ij}) \in G(UT(3,2)_{(0,0,1,0,0)}) \mid a_{11} = a_{21} = a_{31} = a_{54} = a_{55} = 0,\ a_{22} = a_{33},\ a_{23} = a_{32}\}$,
$\mathcal{A}_8 = \{(a_{ij}) \in G(UT(1,2,1)_{(0,0,1,0)}) \mid a_{11} = a_{44},\ a_{22} = a_{33},\ a_{23} = a_{32}\}$,
$\mathcal{A}_9 = \{(a_{ij}) \in G(UT(1,2,1)_{(0,0,1,1)}) \mid a_{11} = a_{44},\ a_{22} = a_{33},\ a_{23} = a_{32}\}$.

It is well known that $exp(M_2(F)) = exp^{\delta}(M_2(F)) = 4$ and $exp(M_{1,1}(G)) = exp^{\delta}(M_{1,1}(G)) = 4$. For the remaining algebras we have the following

Lemma 3.1. *For $i = 3, \dots, 9$, $exp(\mathcal{A}_i) = exp^{\delta}(\mathcal{A}_i) = 3$.*

Proof. For the algebra $\mathcal{A}_3$, since $F(e_{11}+e_{44})Fe_{12}Fe_{22}Fe_{23}Fe_{33} \neq 0$, then $F(e_{11}+e_{44}) \oplus Fe_{22} \oplus Fe_{33}$ is a maximal admissible subalgebra. Hence $exp(\mathcal{A}_3) = 3$. The center of $\mathcal{A}_3$ is $Z(\mathcal{A}_3) = F(e_{11} + e_{22} + e_{33} + e_{44}) \oplus Fe_{14}$. Since $[e_{11} + e_{44}, e_{12}][e_{22}, e_{23}][e_{33}, e_{34}] = e_{14}$ we have that $[x_1, x_2][x_3, x_4][x_5, x_6]$ is a proper central polynomial of $\mathcal{A}_3$ and, so, $F(e_{11} + e_{44}) \oplus Fe_{22} \oplus Fe_{33}$ is a maximal centrally admissible subalgebra. It follows that $exp^{\delta}(\mathcal{A}_3) = 3$.

Regarding the algebra $\mathcal{A}_4$, it is easy to see that $Fe_{22} \oplus Fe_{33} \oplus Fe_{44}$ is a maximal admissible subalgebra and, so, $exp(\mathcal{A}_4) = 3$. The center of $\mathcal{A}_4$ is $Z(\mathcal{A}_4) = Fe_{15}$. Moreover $[x_1, x_2][x_3, x_4][x_5, x_6][x_7, x_8]$ is a proper central polynomial of $\mathcal{A}_4$ since $[e_{12}, e_{22}][e_{23}, e_{33}][e_{33}, e_{34}][e_{44}, e_{45}] = e_{15}$. It follows that $Fe_{22} \oplus Fe_{33} \oplus Fe_{44}$ is a maximal centrally admissible subalgebra and so $exp^{\delta}(\mathcal{A}_4) = 3$.

A maximal admissible subalgebra of $\mathcal{A}_5$ is $(F + cF)(e_{11} + e_{33}) \oplus Fe_{22}$, hence $exp(\mathcal{A}_5) = 3$. The center of $\mathcal{A}_5$ is $Z(\mathcal{A}_5) = G^{(0)}(e_{11} + e_{22} + e_{33}) + G^{(0)}e_{13}$. Moreover $[[x_1, x_2, x_3][x_4, x_5, x_6], x_7]$ is a proper central polynomial of $\mathcal{A}_5$ and, so, $exp^{\delta}(\mathcal{A}_5) = 3$.

It is easy to check that a maximal admissible subalgebra of $\mathcal{A}_6$ is $Fe_{22} \oplus F(e_{33}+e_{44}) \oplus F(e_{34} + e_{43})$. Moreover, $Z(\mathcal{A}_6) = G^{(0)}e_{15}$, $[x_1, x_2][x_3, x_4][x_5, x_6, x_7]$ is a proper central polynomial and, so, $exp(\mathcal{A}_6) = exp^{\delta}(\mathcal{A}_6) = 3$.

A maximal admissible subalgebra of the algebra $\mathcal{A}_7$ is $F(e_{22}+e_{33}) \oplus F(e_{23}+e_{32}) \oplus Fe_{44}$. Moreover $Z(\mathcal{A}_7) = G^{(0)}e_{15}$ and $[x_1, x_2, x_3][x_4, x_5][x_6, x_7]$ is a proper central polynomial. Thus $exp(\mathcal{A}_7) = exp^{\delta}(\mathcal{A}_7) = 3$.

Finally, a maximal admissible subalgebra for both algebras $\mathcal{A}_8$ and $\mathcal{A}_9$ is $F(e_{11} + e_{44}) \oplus F(e_{22} + e_{33}) \oplus F(e_{23} + e_{32})$. Hence $exp(\mathcal{A}_8) = exp(\mathcal{A}_9) = 3$. The center of $\mathcal{A}_8$ is $Z(\mathcal{A}_8) = G^{(0)}(e_{11} + e_{22} + e_{33} + e_{44}) + G^{(0)}e_{14}$ and the center of $\mathcal{A}_9$ is $Z(\mathcal{A}_9) = G^{(0)}(e_{11} + e_{22} + e_{33} + e_{44}) + G^{(1)}e_{14}$. Moreover $[x_1, x_2, x_3][x_4, x_5, x_6]$ is a proper central polynomial for both $\mathcal{A}_8$ and $\mathcal{A}_9$. Hence $exp^{\delta}(\mathcal{A}_8) = exp^{\delta}(\mathcal{A}_9) = 3$. □

Next we will determine generators for the T-ideal of identities of the algebras $\mathcal{A}_6$ and $\mathcal{A}_7$. To this end we first need to prove a preliminary lemma.

Let $\langle f_1, \dots, f_r\rangle_T$ denote the T-ideal of the free algebra generated by the polynomials $f_1, \dots, f_r$. We have

Lemma 3.2.

1) *If* $C_1 = \{(a_{ij}) \in M_3(G) \mid a_{11} = a_{22}, a_{13} \in G^{(0)}, a_{12} = a_{21}, a_{23} \in G^{(1)}, a_{31} = a_{32} = a_{33} = 0\}$, *then* $Id(C_1) = \langle [x_1, x_2, x_3]x_4\rangle_T$.
2) *If* $C_2 = \{(a_{ij}) \in M_3(G) \mid a_{22} = a_{33}, a_{12} \in G^{(0)}, a_{23} = a_{32}, a_{13} \in G^{(1)}, a_{11} = a_{21} = a_{31} = 0\}$, *then* $Id(C_2) = \langle x_1[x_2, x_3, x_4]\rangle_T$.

Proof. It is easily checked that C_1 satisfies the identity $[x_1, x_2, x_3]x_4 \equiv 0$. Hence $\mathrm{Id}(C_1) \supseteq \langle [x_1, x_2, x_3]x_4\rangle_T$.

Next we shall work modulo the identity $[x_1, x_2, x_3]x_4 \equiv 0$. First we claim that

$$[x_1, x_2]x_3x_4 \equiv x_3[x_1, x_2]x_4. \tag{1}$$

Indeed

$$[x_1, x_2]x_3x_4 = ([[x_1, x_2], x_3] + x_3[x_1, x_2])x_4 = [x_1, x_2, x_3]x_4 + x_3[x_1, x_2]x_4 \equiv x_3[x_1, x_2]x_4.$$

Moreover from the equality

$$0 \equiv [x_1, x_2x_3, x_4]x_5 = [[x_1, x_2]x_3 + x_2[x_1, x_3], x_4]x_5 =$$
$$([x_1, x_2, x_4]x_3 + [x_1, x_2][x_3, x_4] + [x_2, x_4][x_1, x_3] + x_2[x_1, x_3, x_4])x_5$$

we get

$$[x_1, x_2][x_3, x_4]x_5 + [x_2, x_4][x_1, x_3]x_5 \equiv 0.$$

Since by (1), $[x_2, x_4][x_1, x_3]x_5 \equiv [x_1, x_3][x_2, x_4]x_5$, we obtain

$$[x_1, x_2][x_3, x_4]x_5 + [x_1, x_3][x_2, x_4]x_5 \equiv 0. \tag{2}$$

From the relations (1) and (2) it follows that any monomial

$$w = x_{u_1} \cdots x_{u_{n-1}} x_a \in P_n$$

can be written modulo the identity $[x_1, x_2, x_3]x_4$ as a linear combination of products of the type

$$x_{i_1} \cdots x_{i_k} [x_{j_1}, x_{j_2}] \cdots [x_{j_{2m-1}}, x_{j_{2m}}] x_a$$

with $i_1 < \cdots < i_k$, $j_1 < \cdots < j_{2m}$, $k + 2m = n - 1$.

Let $f(x_1,\dots,x_n)$ be a multilinear polynomial identity of C_1, we shall prove that it vanishes modulo the identity $[x_1,x_2,x_3]x_4$. Suppose that $f(x_1,\dots,x_n)\neq 0$ modulo $[x_1,x_2,x_3]x_4$ and write

$$f(x_1,\dots,x_n)\equiv\sum\beta_{\underline{i},\underline{j},a}x_{i_1}\cdots x_{i_{n-2m-1}}[x_{j_1},x_{j_2}]\cdots[x_{j_{2m-1}},x_{j_{2m}}]x_a,$$

where $\beta_{\underline{i},\underline{j},a}\in F$, $\underline{i}=(i_1,\dots,i_{n-2m-1})$, $i_1<\cdots<i_{n-2m-1}$, $\underline{j}=(j_1,\dots,j_{2m})$ and $j_1<\cdots<j_{2m}$. We choose the minimal m such that for some $(n-2m-1,2m)$-tuple $(i_1,\dots,i_{n-2m-1};j_1,\dots,j_{2m})$ we have that $\beta_{\underline{i},\underline{j},a}\neq 0$. We make the substitution $x_{i_1}=\cdots=x_{i_{n-2m-1}}=e_{11}+e_{22}$, $x_{j_t}=g_t(e_{12}+e_{21})$ for all $t=1,\dots,2m$, $x_a=ge_{13}$, where $g\in G^{(0)}$, the g_t's are distinct generators of $G^{(1)}$ and $gg_t\neq 0$. Clearly we obtain, for a suitable coefficient $\beta_{\underline{i},\underline{j},a}\in F$,

$$\beta_{\underline{i},\underline{j},a}g_1\cdots g_{2m}ge_{13}\neq 0.$$

The minimality of m gives that all other coefficients vanish and we get a contradiction.

Similar considerations show that $\mathrm{Id}(C_2)=\langle x_1[x_2,x_3,x_4]\rangle_T$. □

Recall that the free supercommutative algebra $S=F[U,V]$ is the algebra with 1 generated by two countable sets $U=\{u_1,u_2,\dots\}$ and $V=\{v_1,v_2,\dots\}$ over F, subject to the condition that the elements of U are central and the elements of V anticommute among them. The algebra S has a natural $\mathbb{Z}_2$-grading $S=S^{(0)}\oplus S^{(1)}$ if we require the variables of U to be even and those of V to be odd. We shall call the elements of V Grassmann variables.

If $B=B^{(0)}\oplus B^{(1)}$ is a superalgebra, the algebra $S(B)=(S^{(0)}\otimes B^{(0)})+(S^{(1)}\otimes B^{(1)})$ is called the superenvelope of B. It has the following basic property (see [11, Proposition 3.8.5]).

Proposition 3.1. *Let $B=B^{(0)}\oplus B^{(1)}$ be a finite-dimensional superalgebra. If $\{a_1,\dots,a_k\}$ and $\{b_1,\dots,b_t\}$ are bases of $B^{(0)}$ and $B^{(1)}$ respectively, then the subalgebra of $S(B)$ generated by the elements*

$$\xi_i=u_{i_1}\otimes a_1+\cdots+u_{i_k}\otimes a_k+v_{i_1}\otimes b_1+\cdots+v_{i_t}\otimes b_t,\ i=1,2,\dots$$

is a relatively free algebra of the variety $var(G(B))=var(S(B))$ with free generators ξ_1, $\xi_2,\dots$.

Now, we can prove the following.

Proposition 3.2.

1) *$Id(\mathcal{A}_6)=\langle x_1[x_2,x_3][x_4,x_5,x_6]x_7\rangle_T$.*
2) *$Id(\mathcal{A}_7)=\langle x_1[x_2,x_3,x_4][x_5,x_6]x_7\rangle_T$.*

Proof. First we consider the algebra $\mathcal{A}_6$. We notice that $\mathcal{A}_6 = G(B)$ where $B = B^{(0)} \oplus B^{(1)}$ is a finite-dimensional superalgebra with $\dim_F B^{(0)} = 8$ and $\dim_F B^{(1)} = 4$. We write

$$\mathcal{A}_6 = \begin{pmatrix} B_1 & J \\ 0 & B_2 \end{pmatrix}$$

where $B_1 = \begin{pmatrix} 0 & G^{(0)} \\ 0 & G^{(0)} \end{pmatrix}$, $B_2 = \left\{ \begin{pmatrix} g_1 & h_1 & G^{(0)} \\ h_1 & g_1 & G^{(1)} \\ 0 & 0 & 0 \end{pmatrix} \mid g_1 \in G^{(0)}, h_1 \in G^{(1)} \right\}$ and $J = \begin{pmatrix} G^{(0)} & G^{(1)} & G^{(0)} \\ G^{(0)} & G^{(1)} & G^{(0)} \end{pmatrix}$.

We shall prove that $\mathrm{Id}(\mathcal{A}_6) = \mathrm{Id}(B_1)\mathrm{Id}(B_2)$ by making use of Proposition 3.1. We choose $8n$ polynomial variables ξ_j^i, $1 \le i \le n$, $i \le j \le 8$, and $4n$ Grassmann variables η_l^i, $1 \le i \le n$, $1 \le l \le 4$. Denote by $\mathcal{G}$ the subalgebra generated by the elements

$$Z^i = \sum_{j=1}^{8} \xi_j^i a_j + \sum_{l=1}^{4} \eta_l^i b_l \in F[\xi_j^i, \eta_l^i] \otimes B,$$

where $a_1, \ldots, a_8$ are a basis of $B^{(0)}$ and $b_1, \ldots, b_4$ a basis of $B^{(1)}$. Since we choose a basis of B made of matrix units, we shall use a double index for the commutative and Grassmann variables.

Consider three infinite disjoint sets of commutative and Grassmann variables and denote them as follows:

$$X = \{\xi_{1,2}^i, \xi_{2,2}^i\},\ Y = \{\xi_{3,3}^i, \xi_{3,5}^i, \eta_{3,4}^i, \eta_{4,5}^i\}$$

and

$$U = \{\xi_{1,3}^i, \xi_{1,5}^i, \xi_{2,3}^i, \xi_{2,5}^i, \eta_{1,4}^i, \eta_{2,4}^i\}.$$

For $i = 1, \ldots, n$, consider the following matrices:

$$X^i = \xi_{1,2}^i e_{12} + \xi_{2,2}^i e_{22},$$
$$Y^i = \xi_{3,3}^i (e_{33} + e_{44}) + \xi_{3,5}^i e_{35} + \eta_{3,4}^i (e_{34} + e_{43}) + \eta_{4,5}^i e_{45},$$
$$U^i = \xi_{1,3}^i e_{13} + \xi_{1,5}^i e_{15} + \xi_{2,3}^i e_{23} + \xi_{2,5}^i e_{25} + \eta_{1,4}^i e_{14} + \eta_{2,4}^i e_{24}.$$

Then, by Proposition 3.1, the generic matrices $X^1, \ldots, X^n$ generate the relatively free algebra $\tilde{B}_1$ of rank n of the variety $var(B_1)$ and $Y^1, \ldots, Y^n$ generate the relatively free algebra $\tilde{B}_2$ of rank n of the variety $var(B_2)$.

If we prove that $U^1, \ldots, U^n$ generate a free $(\tilde{B}_1, \tilde{B}_2)$-bimodule then, by Lewin theorem [11, Corollary 1.8.2], we obtain that the set $\{Z^i = X^i + Y^i + U^i \mid 1 \leq i \leq n\}$ generates a relatively free algebra of the variety corresponding to the product of the T-ideals $\mathrm{Id}(B_1)\mathrm{Id}(B_2)$. This will complete the proof of the proposition.

Hence we need to show that $U^1, \ldots, U^n$ generate a free $(\tilde{B}_1, \tilde{B}_2)$-bimodule. Let $a_1^j, a_2^j, \ldots$, $1 \leq j \leq n$, be non-zero elements of $\tilde{B}_1$ and let $b_1, b_2, \ldots$ be linearly independent elements of $\tilde{B}_2$. We want to prove that

$$\sum_i a_i^1 U^1 b_i + \cdots + a_i^n U^n b_i \neq 0.$$

Since $U^1, \ldots, U^n$ depend on disjoint sets of variables it is enough to check that

$$a_1^1 U^1 b_1 + \cdots + a_t^1 U^1 b_t \neq 0,$$

for all $t \geq 1$. Since $a_1^1 \neq 0$, it has a non zero value in B_1 under some specialization of the variables in X. This value is of the type $\lambda_{1,2}^1 e_{12} + \lambda_{2,2}^1 e_{22} \neq 0$, with $\lambda_{1,2}^1, \lambda_{2,2}^1 \in G^{(0)}$. Suppose now that $a_1^1 U^1 b_1 + \cdots + a_t^1 U^1 b_t = 0$. Then

$$\sum_{i=1}^t a_i^1 (\xi_{2,3}^i e_{23} + \xi_{2,5}^i e_{25} + \eta_{2,4}^i e_{24}) b_i = 0$$

and, so,

$$\sum_{i=1}^t (\lambda_{1,2}^i e_{12} + \lambda_{2,2}^i e_{22})(\xi_{2,3}^i e_{23} + \xi_{2,5}^i e_{25} + \eta_{2,4}^i e_{24}) b_i = 0.$$

Assume for instance that $\lambda_{1,2}^1 \neq 0$. If we consider the following specialization $\xi_{2,3}^i \to 1$, $\xi_{2,5}^i \to 0$, $\eta_{2,4}^i \to 0$, then

$$\sum_{i=1}^t (\lambda_{1,2}^i e_{13} + \lambda_{2,2}^i e_{23}) b_i = 0$$

and in particular

$$e_{13}(\sum_{i=1}^t \lambda_{1,2}^i b_i) = 0.$$

Let $f = \sum_{i=1}^t \lambda_{1,2}^i b_i$. This element is not an identity for B_2 but for any specialization of the variables in Y, i.e. for any homomorphism $\phi : \tilde{B}_2 \to B_2$, we have $e_{13}\phi(f) = 0$. This means that in $\phi(f)$ all elements of the first row are zero. By [10, Lemma 4], the linear span $\langle \phi(f) \rangle$ of all values of f on B_2 is a Lie ideal of B_2 and this is false. The contradiction just obtained proves that $\mathrm{Id}(\mathcal{A}_6) = \mathrm{Id}(B_1)\mathrm{Id}(B_2)$. Since, by Lemma 3.2, $\mathrm{Id}(B_1) = \langle x_1[x_2, x_3]\rangle_T$ and $\mathrm{Id}(B_2) = \langle [x_4, x_5, x_6]x_7\rangle_T$, we obtain that $\mathrm{Id}(\mathcal{A}_6) = \langle x_1[x_2, x_3][x_4, x_5, x_6]x_7\rangle_T$.

The proof of the second part of the proposition is very similar to the above proof and we omit it. □

Remark 3.1. Consider the following algebras

- $\mathcal{A}_6^1 = \{(a_{ij}) \in G(UT(2,3)_{(0,0,0,1,1)}) \mid a_{11} = a_{21} = a_{53} = a_{54} = a_{55} = 0,\ a_{33} = a_{44},\ a_{34} = a_{43}\}$,
- $\mathcal{A}_6^2 = \{(a_{ij}) \in G(UT(2,3)_{(0,1,1,0,1)}) \mid a_{11} = a_{21} = a_{53} = a_{54} = a_{55} = 0,\ a_{33} = a_{44},\ a_{34} = a_{43}\}$,
- $\mathcal{A}_6^3 = \{(a_{ij}) \in G(UT(2,3)_{(0,1,1,0,0)}) \mid a_{11} = a_{21} = a_{53} = a_{54} = a_{55} = 0,\ a_{33} = a_{44},\ a_{34} = a_{43}\}$,
- $\mathcal{A}_7^1 = \{(a_{ij}) \in G(UT(3,2)_{(0,1,0,1,1)}) \mid a_{11} = a_{21} = a_{31} = a_{54} = a_{55} = 0,\ a_{22} = a_{33},\ a_{23} = a_{32}\}$,
- $\mathcal{A}_7^2 = \{(a_{ij}) \in G(UT(3,2)_{(0,0,1,0,1)}) \mid a_{11} = a_{21} = a_{31} = a_{54} = a_{55} = 0,\ a_{22} = a_{33},\ a_{23} = a_{32}\}$,
- $\mathcal{A}_7^3 = \{(a_{ij}) \in G(UT(3,2)_{(0,1,0,1,0)}) \mid a_{11} = a_{21} = a_{31} = a_{54} = a_{55} = 0,\ a_{22} = a_{33},\ a_{23} = a_{32}\}$.

By Proposition 3.2 and its proof it follows that

$$Id(\mathcal{A}_6^1) = Id(\mathcal{A}_6^2) = Id(\mathcal{A}_6^3) = Id(\mathcal{A}_6) = \langle x_1[x_2, x_3][x_4, x_5, x_6]x_7\rangle_T$$

and

$$Id(\mathcal{A}_7^1) = Id(\mathcal{A}_7^2) = Id(\mathcal{A}_7^3) = Id(\mathcal{A}_7) = \langle x_1[x_2, x_3, x_4][x_5, x_6]x_7\rangle_T.$$

4. Building up algebras of small δ-exponent

Throughout this section $B = B^{(0)} \oplus B^{(1)}$ will be a finite dimensional superalgebra over an algebraically closed field F of characteristic zero.

We start with the following.

Lemma 4.1. *If $B^{(0)}$ contains three orthogonal idempotents e_1, e_2, e_3 such that $e_1j_1e_2j_2e_3j_3e_1 \neq 0$, for some $j_1, j_2, j_3 \in J(B)$, then there exists a $\mathbb{Z}_2$-graded subalgebra $\bar{B}$ of B such that $Id(G(\bar{B})) \subseteq Id(\mathcal{A}_3)$.*

Proof. By linearity we assume, as we may, that the elements j_1, j_2, j_3 are homogeneous. Let $\bar{B}$ be the $\mathbb{Z}_2$-graded subalgebra of B generated by the homogeneous elements

$$e_1, e_2, e_3, e_1j_1e_2, e_2j_2e_3, e_3j_3e_1$$

and let I be the ideal of $\bar{B}$ generated by the element $e_3j_3e_1j_1e_2$. We build a homomorphism of superalgebras $\varphi : \bar{B} \to UT_4(F)$ by setting $\varphi(e_1) = e_{11} + e_{44}$, $\varphi(e_2) = e_{22}$, $\varphi(e_3) = e_{33}$, $\varphi(e_1j_1e_2) = e_{12}$, $\varphi(e_2j_2e_3) = e_{23}$ and $\varphi(e_3j_3e_1) = e_{34}$. Then $I = Ker\varphi$ and $\bar{B}/I \simeq \varphi(\bar{B})$ is isomorphic to the algebra $\mathcal{A}_3$ with a suitable $\mathbb{Z}_2$-grading.

Now, considering images under the mapping φ, we see that the elements

$$e_1, e_2, e_3, e_1j_1e_2, e_2j_2e_3, e_3j_3e_1, e_1j_1e_2j_2e_3, e_2j_2e_3j_3e_1, e_1j_1e_2j_2e_3j_3e_1$$

are linearly independent and form a basis of $\bar{B}$ mod I. By abuse of notation we identify these representatives with the corresponding cosets.

Let g_1, g_2, g_3 be distinct homogeneous elements of G having the same homogeneous degree as j_1, j_2, j_3, respectively. Then the elements

$$1 \otimes e_1,\ 1 \otimes e_2,\ 1 \otimes e_3,\ g_1 \otimes e_1j_1e_2,\ g_2 \otimes e_2j_2e_3,\ g_3 \otimes e_3j_3e_1,$$
$$g_1g_2 \otimes e_1j_1e_2j_2e_3,\ g_2g_1 \otimes e_2j_2e_3j_3e_1,\ g_1g_2g_3 \otimes e_1j_1e_2j_2e_3j_3e_1,$$

form a basis of a subalgebra $\mathcal{C}$ of $G(\bar{B}/I)$ isomorphic to $\mathcal{A}_3$ with the trivial grading.

It follows that $Id(\mathcal{A}_3) = Id(\mathcal{C}) \supseteq Id(G(\bar{B}/I)) \supseteq Id(G(\bar{B}))$, and the proof is complete. □

Lemma 4.2. *If $B^{(0)}$ contains three orthogonal idempotents e_1, e_2, e_3 such that $j_1e_1j_2e_2j_3e_3j_4 \neq 0$ for some $j_1, j_2, j_3, j_4 \in J(B)$, then there exists a $\mathbb{Z}_2$-graded subalgebra $\bar{B}$ of B such that $Id(G(\bar{B})) \subseteq Id(\mathcal{A}_4)$.*

Proof. As in the previous lemma we assume that the elements j_1, j_2, j_3, j_4 are homogeneous. Let $\bar{B}$ be the $\mathbb{Z}_2$-graded subalgebra of B generated by the homogeneous elements

$$e_1, e_2, e_3, j_1e_1, e_1j_2e_2, e_2j_3e_3, e_3j_4$$

and let I be the homogeneous ideal of $\bar{B}$ generated by the elements $e_3j_4e_1$, $e_3j_4e_2$, $e_3j_4e_3$, $e_3j_4j_1e_1$, $e_3j_1e_1$, $e_2j_1e_1$, $e_1j_1e_1$. We build a homomorphism of superalgebras

$$\psi : \bar{B} \to UT_5(F)$$

defined by setting $\psi(e_1) = e_{22}$, $\psi(e_2) = e_{33}$, $\psi(e_3) = e_{44}$, $\psi(j_1e_1) = e_{12}$, $\psi(e_1j_2e_2) = e_{23}$, $\psi(e_2j_3e_3) = e_{34}$ and $\psi(e_3j_4) = e_{45}$. Then $I = Ker\psi$ and $\bar{B}/I \simeq \psi(\bar{B})$ is isomorphic to the algebra $\mathcal{A}_4$ with a suitable $\mathbb{Z}_2$-grading.

The elements

$$e_1, e_2, e_3,\ j_1e_1,\ e_1j_2e_2,\ e_2j_3e_3,\ e_3j_4,\ j_1e_1j_2e_2,\ e_1j_2e_2j_3e_3,$$
$$e_2j_3e_3j_4,\ j_1e_1j_2e_2j_3e_3,\ e_1j_2e_2j_3e_3j_4,\ j_1e_1j_2e_2j_3e_3j_4$$

are linearly independent and form a basis of $\bar{B}$ mod I.

By abuse of notation we identify these representatives with the corresponding cosets. It follows that, for distinct $g_i \in G^{(0)} \cup G^{(1)}$ having the same homogeneous degree of the corresponding j_i, the elements

$$1 \otimes e_1, 1 \otimes e_2, 1 \otimes e_3,\; g_1 \otimes j_1 e_1,\; g_2 \otimes e_1 j_2 e_2,\; g_3 \otimes e_2 j_3 e_3,\; g_4 \otimes e_3 j_4,$$
$$g_1 g_2 \otimes j_1 e_1 j_2 e_2,\; g_2 g_3 \otimes e_1 j_2 e_2 j_3 e_3,\; g_3 g_4 \otimes e_2 j_3 e_3 j_4,$$
$$g_1 g_2 g_3 \otimes j_1 e_1 j_2 e_2 j_3 e_3,\; g_2 g_3 g_4 \otimes e_1 j_2 e_2 j_3 e_3 j_4,\; g_1 g_2 g_3 g_4 \otimes j_1 e_1 j_2 e_2 j_3 e_3 j_4,$$

are a basis of a subalgebra $\mathcal{C}$ of $G(\bar{B}/I)$ isomorphic to $\mathcal{A}_4$ with the trivial grading.

We have that $Id(\mathcal{A}_4) = Id(\mathcal{C}) \supseteq Id(G(\bar{B}/I)) \supseteq Id(G(\bar{B}))$, a desired conclusion. □

Lemma 4.3. *If $B^{(0)}$ contains two orthogonal idempotents e_1, e_2, with $e_2 \in F \oplus cF$, such that $e_2 j_2 e_1 j_1 e_2 \neq 0$, for some $j_1, j_2 \in J(B)$, then there exists a $\mathbb{Z}_2$-graded subalgebra $\bar{B}$ of B such that $Id(G(\bar{B})) \subseteq Id(\mathcal{A}_5)$.*

Proof. Let $\bar{B}$ be the $\mathbb{Z}_2$-graded subalgebra of B generated by the homogeneous elements

$$e_1, e_2, ce_2, e_1 j_1 e_2, e_2 j_2 e_1$$

and let I be the ideal of $\bar{B}$ generated by $e_1 j_1 e_2 j_2 e_1, e_1 j_1 c e_2 j_2 e_1$.

We consider the homomorphism of superalgebras $\varphi : \bar{B} \to UT_3(F + cF)$ by setting $\varphi(e_1) = e_{22}$, $\varphi(e_2) = e_{11} + e_{33}$, $\varphi(ce_2) = c(e_{11} + e_{33})$, $\varphi(e_1 j_1 e_2) = \alpha e_{23}$ and $\varphi(e_2 j_2 e_1) = \alpha e_{12}$, where $\alpha \in \{1, c\}$ according to the homogeneous degree of j_1 and j_2. Then $I = Ker\varphi$ and $\bar{B}/I \simeq \varphi(\bar{B}) = \bar{A}$ where

$$\bar{A} = \left\{ \begin{pmatrix} a & x & y \\ 0 & b & z \\ 0 & 0 & a \end{pmatrix} : b \in F, a, x, y, z \in F + cF \right\}.$$

The following elements

$$e_1, e_2, ce_2, e_1 j_1 e_2, e_2 j_2 e_1, e_1 j_1 c e_2, c e_2 j_2 e_1, e_2 j_2 e_1 j_1 e_2, c e_2 j_2 e_1 j_1 e_2$$

are linearly independent and form a basis of $\bar{B}$ mod I. We identify these representatives with the corresponding cosets. Then the elements

$$1 \otimes e_1, 1 \otimes e_2, 1 \otimes ce_2, 1 \otimes e_1 j_1 e_2, 1 \otimes e_2 j_2 e_1,$$
$$1 \otimes e_1 j_1 c e_2, 1 \otimes c e_2 j_2 e_1, 1 \otimes e_2 j_2 e_1 j_1 e_2, 1 \otimes c e_2 j_2 e_1 j_1 e_2$$

form a basis of $G(\bar{B}/I) \simeq G(\bar{A}) = \mathcal{A}_5$ and $Id(\mathcal{A}_5) \supseteq Id(G(\bar{B}))$. □

Lemma 4.4. *If $B^{(0)}$ contains two orthogonal idempotents e_1, e_2, with $e_2 \in F \oplus cF$, such that $j_1 e_1 j_2 e_2 j_3 \neq 0$ for some $j_1, j_2, j_3 \in J(B)$, then there exists a $\mathbb{Z}_2$-graded subalgebra $\bar{B}$ of B such that $Id(G(\bar{B})) \subseteq Id(\mathcal{A}_6)$.*

Proof. Let us assume that the elements j_1, j_2, j_3 are homogeneous of degree zero and let $\bar{B}$ be the $\mathbb{Z}_2$-graded subalgebra of B generated by the homogeneous elements

$$e_1, e_2, ce_2, j_1e_1, e_1j_2e_2, e_2j_3$$

such that $j_1e_1j_2e_2j_3 \neq 0$. Let I be the homogeneous ideal of $\bar{B}$ generated by the elements $e_1j_1e_1$, $e_2j_1e_1$, $e_2j_3e_1$, $e_2j_3e_2$, $e_2j_3j_1e_1$. We consider the homomorphism of superalgebras $\varphi : \bar{B} \to UT(2,3)_{(0,0,0,1,0)}$ by setting $\varphi(e_1) = e_{22}$, $\varphi(e_2) = e_{33} + e_{44}$, $\varphi(ce_2) = e_{34} + e_{43}$, $\varphi(j_1e_1) = e_{12}$, $\varphi(e_1j_2e_2) = e_{23}$ and $\varphi(e_2j_3) = e_{35}$. Then $I = Ker\varphi$ and $\bar{B}/I \simeq \varphi(\bar{B}) = \bar{A}$ where

$$\bar{A} = \begin{pmatrix} 0 & a & b & d & e \\ 0 & f & g & h & l \\ 0 & 0 & u & v & m \\ 0 & 0 & v & u & n \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & b & 0 & e \\ 0 & f & g & 0 & l \\ 0 & 0 & u & 0 & m \\ 0 & 0 & 0 & u & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & h & 0 \\ 0 & 0 & 0 & v & 0 \\ 0 & 0 & v & 0 & n \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The following elements

$$e_1, e_2,\ ce_2,\ j_1e_1,\ e_1j_2e_2,\ e_2j_3,\ j_1e_1j_2e_2,$$
$$j_1e_1j_2ce_2,\ j_1e_1j_2e_2j_3,\ e_1j_2ce_2,\ e_1j_2e_2j_3,\ ce_2j_3$$

are linearly independent mod I and form a basis of $\bar{B}/I$. By the same procedure of the previous lemmas it follows that $G(\bar{A}) = \mathcal{A}_6$ and so $Id(\mathcal{A}_6) \supseteq Id(G(\bar{B}))$.

The same result is obtained when $j_1 \in J^{(0)}$ and $j_2, j_3 \in J^{(1)}$. Moreover if $j_1 \in J^{(0)}$ and j_2, j_3 are of different homogeneous degree we obtain that $Id(\mathcal{A}_6^1) \supseteq Id(G(\bar{B}))$.

In case $j_1 \in J^{(1)}$ we get that either $Id(\mathcal{A}_6^2) \supseteq Id(G(\bar{B}))$ or $Id(\mathcal{A}_6^3) \supseteq Id(G(\bar{B}))$ according to whether j_2, j_3 are of the same or of different homogeneous degree. Since, by Remark 3.1, $Id(\mathcal{A}_6^1) = Id(\mathcal{A}_6^2) = Id(\mathcal{A}_6^3) = Id(\mathcal{A}_6)$, we get the desired result. □

Lemma 4.5. *If $B^{(0)}$ contains two orthogonal idempotents e_1, e_2, with $e_2 \in F \oplus cF$, such that $j_1e_2j_2e_1j_3 \neq 0$ for some $j_1, j_2, j_3 \in J(B)$, then there exists a $\mathbb{Z}_2$-graded subalgebra $\bar{B}$ of B such that $Id(G(\bar{B})) \subseteq Id(\mathcal{A}_7)$.*

Proof. By following the proof of Lemma 4.4 we obtain that, if $j_3 \in J^{(0)}$ and j_1, j_2 are of the same homogeneous degree then $Id(\mathcal{A}_7) \supseteq Id(G(\bar{B}))$. Otherwise, if $j_3 \in J^{(0)}$ and j_1, j_2 are of different homogeneous degree then $Id(\mathcal{A}_7^1) \supseteq Id(G(\bar{B}))$. Similarly, if $j_3 \in J^{(1)}$, we get either $Id(\mathcal{A}_7^2) \supseteq Id(G(\bar{B}))$ or $Id(\mathcal{A}_7^3) \supseteq Id(G(\bar{B}))$.

Recalling Remark 3.1 we have that $Id(\mathcal{A}_7^1) = Id(\mathcal{A}_7^2) = Id(\mathcal{A}_7^3) = Id(\mathcal{A}_7)$ and we are done. □

Lemma 4.6. *If $B^{(0)}$ contains two orthogonal idempotents e_1, e_2, with $e_2 \in F \oplus cF$, such that $e_1j_1e_2j_2e_1 \neq 0$, for some $j_1, j_2 \in J(B)$, then there exists a $\mathbb{Z}_2$-graded subalgebra $\bar{B}$ of B such that $Id(G(\bar{B})) \subseteq Id(\mathcal{A}_8)$ or $Id(G(\bar{B})) \subseteq Id(\mathcal{A}_9)$.*

Proof. First we suppose that the elements j_1, j_2 are homogeneous of degree zero. Let $\bar{B}$ be the $\mathbb{Z}_2$-graded subalgebra of B generated by the homogeneous elements

$$e_1, e_2,\ ce_2,\ e_1j_1e_2,\ e_2j_2e_1$$

and let I be the ideal of $\bar{B}$ generated by $e_2j_2e_1j_1e_2, e_1j_1ce_2j_2e_1$. We consider the homomorphism of superalgebras $\varphi : \bar{B} \to UT(1,2,1)_{(0,0,1,0)}$ by setting $\varphi(e_1) = e_{11} + e_{44}$, $\varphi(e_2) = e_{22} + e_{33}$, $\varphi(ce_2) = e_{23} + e_{32}$ and $\varphi(e_1j_1e_2) = e_{12}$, $\varphi(e_2j_2e_1) = e_{24}$. Then $I = Ker\varphi$ and $\bar{B}/I \simeq \varphi(\bar{B}) = \bar{A}$ where

$$\bar{A} = \begin{pmatrix} a & b & d & e \\ 0 & u & v & h \\ 0 & v & u & m \\ 0 & 0 & 0 & a \end{pmatrix} = \begin{pmatrix} a & b & 0 & e \\ 0 & u & 0 & h \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & d & 0 \\ 0 & 0 & v & 0 \\ 0 & v & 0 & m \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The following elements

$$e_1, e_2,\ ce_2,\ e_1j_1e_2,\ e_2j_2e_1,\ e_1j_1ce_2,\ ce_2j_2e_1,\ e_1j_1e_2j_2e_1$$

are linearly independent and form a basis of $\bar{B}$ mod I. We identify these representatives with the corresponding cosets. The elements

$$1 \otimes e_1, 1 \otimes e_2, 1 \otimes ce_2, 1 \otimes e_1j_1e_2, 1 \otimes e_2j_2e_1, 1 \otimes e_1j_1ce_2, 1 \otimes ce_2j_2e_1, 1 \otimes e_1j_1e_2j_2e_1$$

form a basis of $G(\bar{B}/I)$ isomorphic to $\mathcal{A}_8$. Thus $Id(\mathcal{A}_8) \supseteq Id(G(\bar{B}))$.

If $j_1, j_2 \in J^{(1)}$ we obtain the same result. If j_1 and j_2 are of different homogeneous degree we get a homomorphism of superalgebras $\varphi : \bar{B} \to UT(1,2,1)_{(0,0,1,1)}$ such that $\bar{B}/I \simeq \varphi(\bar{B}) = \bar{A}$ where

$$\bar{A} = \begin{pmatrix} a & b & d & e \\ 0 & u & v & h \\ 0 & v & u & m \\ 0 & 0 & 0 & a \end{pmatrix} = \begin{pmatrix} a & b & 0 & 0 \\ 0 & u & 0 & 0 \\ 0 & 0 & u & m \\ 0 & 0 & 0 & a \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & d & e \\ 0 & 0 & v & h \\ 0 & v & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Its Grassmann envelope is $G(\bar{A}) = \mathcal{A}_9$. Thus $Id(\mathcal{A}_9) \supseteq Id(G(\bar{B}))$ and we are done. □

5. The main results

In this section we characterize the varieties with proper central exponent greater or equal to two.

Theorem 5.1. *Let F be a field of characteristic zero and $\mathcal{V}$ a variety of algebras over F. Then $exp^{\delta}(\mathcal{V}) > 2$ if and only if $\mathcal{A}_i \in \mathcal{V}$ for some $1 \le i \le 9$.*

Proof. As we recalled in Section 2, we may assume that $\mathcal{V} = var(G(B))$ where $G(B)$ is the Grassmann envelope of a finite dimensional superalgebra $B = B^{(0)} + B^{(1)}$. Write $B = B_1 \oplus \cdots \oplus B_q + J$, with the B_i's simple superalgebras, and let $\exp^\delta(G(B)) = d > 2$.

If, for some $i \leq q$, $B_i = M_{k,l}(F)$ then $G(B) \supseteq G(B_i) = M_{k,l}(G)$. If $l = 0$ and $k \geq 2$, $M_{k,0}(G) = G^{(0)} \otimes M_k(F) \supseteq M_2(G^{(0)})$ and, so, $Id(G(B)) \subseteq Id(M_2(G^{(0)})) \subseteq Id(M_2(F)) = Id(\mathcal{A}_1)$, as wished. If $l > 0$, $Id(G(B)) \subseteq Id(G(B_i)) = Id(M_{k,l}(G)) \subseteq Id(M_{1,1}(G)) = Id(\mathcal{A}_2)$ and we are done also in this case.

Suppose now that, for some $i \leq q$, $B_i = M_k(F) \oplus cM_k(F)$. Then if $k \geq 2$, $G(B_i) = G(M_k(F) \oplus cM_k(F)) \supseteq G^{(0)} \otimes M_k(F) \supseteq G^{(0)} \otimes M_2(F)$, and as above we get $Id(G(B)) \subseteq Id(M_2(F)) = Id(\mathcal{A}_1)$.

Therefore we may assume that, for $1 \leq i \leq q$, either $B_i = F$ with trivial grading or $B_i = F \oplus cF$.

Recall that $d = \dim D$ where D is a centrally admissible subalgebra of B of maximal dimension. We may clearly assume that $D = B_1 \oplus \cdots \oplus B_r$, and, so, by definition there exists a proper central polynomial f of $G(B)$ and an evaluation $0 \neq f(b_1, \ldots, b_r, b_{r+1}, \ldots, b_n) \in Z(G(B))$, where $b_1 \in G(B_1), \ldots, b_r \in G(B_r)$, $b_{r+1}, \ldots, b_n \in G(B)$.

Since $exp^\delta(G(B)) = \dim D = d \geq 3$ one of the following three cases occurs:

1) D contains three simple components of the type F;
2) D contains one copy of F and one of $F \oplus cF$;
3) D contains two copies of $F \oplus cF$.

Suppose first that Case 1 occurs. D contains three orthogonal idempotents e_1, e_2, e_3 homogeneous of degree zero. Recalling that f is the proper central polynomial corresponding to D we have that in a non-zero evaluation $\bar{f}$ of f three variables are evaluated in

$$g_1 \otimes e_1,\ g_2 \otimes e_2,\ g_3 \otimes e_3,$$

where $g_1, g_2, g_3 \in G^{(0)}$ are distinct elements. Since $G^{(0)}$ is central in G and f is multilinear, we may assume that $g_1 = g_2 = g_3 = 1$ and, so, the elements $1 \otimes e_1,\ 1 \otimes e_2,\ 1 \otimes e_3$ appear in the evaluation of f.

Suppose first that a monomial of f can be evaluated into a product of the type

$$(1 \otimes e_1)(h_1 \otimes j_1)(1 \otimes e_2)(h_2 \otimes j_2)(1 \otimes e_3)(h_3 \otimes j_3)(1 \otimes e_1) \neq 0,$$

for some $h_1 \otimes j_1, h_2 \otimes j_2, h_3 \otimes j_3 \in G^{(0)} \otimes J^{(0)} \cup G^{(1)} \otimes J^{(1)}$. Hence

$$h_1 h_2 h_3 \otimes e_1 j_1 e_2 j_2 e_3 j_3 e_1 \neq 0$$

which says that $e_1 j_1 e_2 j_2 e_3 j_3 e_1 \neq 0$, for some homogeneous elements $j_1, j_2, j_3 \in J$. By Lemma 4.1 we get that there exists a $\mathbb{Z}_2$-graded subalgebra $\bar{B}$ of B such that $Id(G(\bar{B})) \subseteq$

$Id(\mathcal{A}_3)$ and we are done in this case. This clearly holds for any permutation of the e_i's, $i = 1, 2, 3$.

Hence, if m is a monomial of f whose evaluation $\bar{m}$ in $\bar{f}$ is non-zero, we may assume that $(1 \otimes e_i)\bar{m}(1 \otimes e_i) = 0$, $1 \leq i \leq 3$. This also says that $(1 \otimes e_i)\bar{f}(1 \otimes e_i) = 0$ and, since $\bar{f}$ is central, $(1 \otimes e_i)\bar{f} = \bar{f}(1 \otimes e_i) = 0$.

Next suppose that there exists an evaluation of a monomial of f containing a product of the type

$$(h_1 \otimes j_1)(1 \otimes e_1)(h_2 \otimes j_2)(1 \otimes e_2)(h_3 \otimes j_3)(1 \otimes e_3)(h_4 \otimes j_4) \neq 0,$$

for some $h_1 \otimes j_1, h_2 \otimes j_2, h_3 \otimes j_3, h_4 \otimes j_4 \in G^{(0)} \otimes J^{(0)} \cup G^{(1)} \otimes J^{(1)}$. Then

$$h_1 h_2 h_3 h_4 \otimes j_1 e_1 j_2 e_2 j_3 e_3 j_4 \neq 0,$$

and, so, $j_1 e_1 j_2 e_2 j_3 e_3 j_4 \neq 0$ for some homogeneous elements $j_1, j_2, j_3, j_4 \in J$. By Lemma 4.2 we have that there exists a $\mathbb{Z}_2$-graded subalgebra $\bar{B}$ of B such that $Id(G(\bar{B})) \subseteq Id(\mathcal{A}_4)$ and we are done.

Hence we may assume that in $\bar{f}$, the evaluation of a monomial of f never starts and ends with elements of J. As remarked above, we may also assume that $(1 \otimes e_i)\bar{f} = \bar{f}(1 \otimes e_i) = 0$, $1 \leq i \leq 3$. Let us set $E_i = 1 \otimes e_i$, $1 \leq i \leq 3$, and write

$$\bar{f} = E_1\bar{f}_1 + \bar{f}_2 E_1 + E_2\bar{f}_3 + \bar{f}_4 E_2 + E_3\bar{f}_5 + \bar{f}_6 E_3,$$

where $E_1\bar{f}_1$ is the sum of the evaluations of the monomials of f starting with E_1, $\bar{f}_2 E_1$ is the sum of the evaluations of the remaining monomials of f ending with E_1, $E_2\bar{f}_3 + \bar{f}_4 E_2$ is the sum of the evaluations of the remaining monomials starting or ending with E_2, and so on.

Since $E_1\bar{f} = \bar{f}E_2 = \bar{f}E_3 = 0$ and we may assume that $E_i\bar{f}_j E_i = 0$ for $i \neq j$, we get

$$0 = E_1\bar{f} + \bar{f}E_2 + \bar{f}E_3 =$$
$$E_1\bar{f}_1 + E_1\bar{f}_4 E_2 + E_1\bar{f}_6 E_3 + E_1\bar{f}_1 E_2 + \bar{f}_4 E_2 + E_3\bar{f}_5 E_2 + E_1\bar{f}_1 E_3 + E_2\bar{f}_3 E_3 + \bar{f}_6 E_3.$$

Moreover

$$0 = E_1\bar{f}E_2 = E_1\bar{f}_1 E_2 + E_1\bar{f}_4 E_2,$$
$$0 = E_1\bar{f}E_3 = E_1\bar{f}_1 E_3 + E_1\bar{f}_6 E_3.$$

Then

$$0 = E_1\bar{f} + \bar{f}E_2 + \bar{f}E_3 = E_1\bar{f}_1 + \bar{f}_4 E_2 + \bar{f}_6 E_3 + E_3\bar{f}_5 E_2 + E_2\bar{f}_3 E_3$$

and so

$$E_1\bar{f}_1 + \bar{f}_4E_2 + \bar{f}_6E_3 = -E_3\bar{f}_5E_2 - E_2\bar{f}_3E_3.$$

Similarly, since $\bar{f}E_1 = E_2\bar{f} = E_3\bar{f} = 0$, we get

$$\bar{f}_2E_1 + E_2\bar{f}_3 + E_3\bar{f}_5 = -E_2\bar{f}_6E_3 - E_3\bar{f}_4E_2.$$

In conclusion

$$\bar{f} = E_1\bar{f}_1+\bar{f}_2E_1+E_2\bar{f}_3+\bar{f}_4E_2+E_3\bar{f}_5+\bar{f}_6E_3 = -E_3\bar{f}_5E_2-E_2\bar{f}_3E_3-E_2\bar{f}_6E_3-E_3\bar{f}_4E_2.$$

Since $\bar{f}E_2 = 0$, it follows that $0 = \bar{f}E_2 = -E_3\bar{f}_5E_2 - E_3\bar{f}_4E_2$. Then

$$\bar{f} = -E_2\bar{f}_3E_3 - E_2\bar{f}_6E_3$$

and we reach a contradiction $0 = \bar{f}E_3 = \bar{f}$. This completes Case 1.

We remark that, if D contains two copies of $F \oplus cF$, then it also contains one copy of F and one copy of $F \oplus cF$. This says that Case 3 can be deduced from Case 2. Hence throughout we shall assume that D contains one copy of F and one of $F \oplus cF$ and we only deal with Case 2.

Hence let $D \supseteq Fe_1 \oplus (Fe_2 \oplus cFe_2)$. Then in the non-zero evaluation $\bar{f}$ of the central polynomial f at least one variable is evaluated in $1 \otimes e_1$ and one variable in $G^{(0)} \oplus cG^{(1)}$. Suppose first that there exists an evaluation of a monomial of f such that

$$(1 \otimes e_1)(h_1 \otimes j_1)(g_1 \otimes e_2)(h_2 \otimes j_2)(1 \otimes e_1) \neq 0,$$

for some $h_1 \otimes j_1, h_2 \otimes j_2, \in G^{(0)} \otimes J^{(0)} \cup G^{(1)} \otimes J^{(1)}$ and $g_1 \in G^{(0)}$. Then

$$h_1g_1h_2 \otimes e_1j_1e_2j_2e_1 \neq 0$$

and by Lemma 4.6 we obtain that there exists a $\mathbb{Z}_2$-graded subalgebra $\bar{B}$ of B such that either $Id(G(\bar{B})) \subseteq Id(\mathcal{A}_8)$ or $Id(G(\bar{B})) \subseteq Id(\mathcal{A}_9)$.

If there exists an evaluation of a monomial of f such that

$$(1 \otimes e_1)(h_1 \otimes j_1)(g_1 \otimes ce_2)(h_2 \otimes j_2)(1 \otimes e_1) \neq 0,$$

for some $h_1 \otimes j_1, h_2 \otimes j_2, \in G^{(0)} \otimes J^{(0)} \cup G^{(1)} \otimes J^{(1)}$ and $g_1 \in G^{(1)}$, then $e_1j_1ce_2j_2e_1 \neq 0$. If we put $j_1' = j_1c$ we are done by the previous case.

Next let us consider the case when there exists an evaluation of a monomial of f such that

$$(g_1 \otimes e_2)(h_1 \otimes j_1)(1 \otimes e_1)(h_2 \otimes j_2)(g_2 \otimes e_2) \neq 0,$$

for some $h_1 \otimes j_1, h_2 \otimes j_2, \in G^{(0)} \otimes J^{(0)} \cup G^{(1)} \otimes J^{(1)}, g_1, g_2 \in G^{(0)}$. By Lemma 4.3, there exists a $\mathbb{Z}_2$-graded subalgebra $\bar{B}$ of B such that $Id(\mathcal{A}_5) \supseteq Id(G(\bar{B}))$.

Hence we may assume that if $\bar{m} \neq 0$ is the evaluation in $\bar{f}$ of a monomial of f, $(1 \otimes e_i)\bar{m}(1 \otimes e_i) = 0$, $1 \leq i \leq 2$. This also says that $(1 \otimes e_i)\bar{f}(1 \otimes e_i) = 0$ and $(1 \otimes e_i)\bar{f} = \bar{f}(1 \otimes e_i) = 0$, $1 \leq i \leq 2$.

Next suppose that a monomial of f has an evaluation of the type

$$(h_1 \otimes j_1)(1 \otimes e_1)(h_2 \otimes j_2)(g_1 \otimes e_2)(h_3 \otimes j_3) \neq 0,$$

for some $h_1 \otimes j_1, h_2 \otimes j_2, h_3 \otimes j_3 \in G^{(0)} \otimes J^{(0)} \cup G^{(1)} \otimes J^{(1)}, g_1 \in G^{(0)}$.

Suppose $j_1e_1j_2e_2j_3 \neq 0$. By Lemma 4.4 there exists a $\mathbb{Z}_2$-graded subalgebra $\bar{B}$ of B such that $Id(G(\bar{B})) \subseteq Id(\mathcal{A}_6)$. The same result holds when $j_1e_1j_2ce_2j_3 \neq 0$.

Now we consider the case when a monomial of f has an evaluation of the type

$$(h_1 \otimes j_1)(g_1 \otimes e_2)(h_2 \otimes j_2)(1 \otimes e_1)(h_3 \otimes j_3) \neq 0,$$

for some $h_1 \otimes j_1, h_2 \otimes j_2, h_3 \otimes j_3 \in G^{(0)} \otimes J^{(0)} \cup G^{(1)} \otimes J^{(1)}$ and $g_1 \in G^{(0)}$. By Lemma 4.5 there exists a $\mathbb{Z}_2$-graded subalgebra $\bar{B}$ of B such that $Id(G(\bar{B})) \subseteq Id(\mathcal{A}_7)$. The same result holds when $j_1ce_2j_2e_1j_3 \neq 0$.

As remarked before, if $\bar{m} \neq 0$, then $(1 \otimes e_i)\bar{m}(1 \otimes e_i) = 0$ and also $(1 \otimes e_i)\bar{f} = \bar{f}(1 \otimes e_i) = 0$, $\forall i = 1, 2$. If we denote $(1 \otimes e_i) = E_i$, then we may write, as in the previous case,

$$\bar{f} = E_1\bar{f}_1 + \bar{f}_2E_1 + E_2\bar{f}_3 + \bar{f}_4E_2. \tag{3}$$

From $E_1\bar{f} = 0$ we get $E_1\bar{f}_1 + E_1\bar{f}_4E_2 = 0$ and so $E_1\bar{f}_1E_2 + E_1\bar{f}_4E_2 = 0$. It follows that $E_1\bar{f}_1 = E_1\bar{f}_1E_2$.

Moreover, since $\bar{f}E_2 = 0$, from (3) similarly we get $E_1\bar{f}_1E_2 + \bar{f}_4E_2 = 0$ and so $E_1\bar{f}_1 + \bar{f}_4E_2 = 0$. As a consequence we obtain that

$$\bar{f} = \bar{f}_2E_1 + E_2\bar{f}_3.$$

Since $0 = E_2\bar{f} = E_2\bar{f}_2E_1 + E_2\bar{f}_3$, we get $\bar{f} = \bar{f}_2E_1 + E_2\bar{f}_3 = \bar{f}_2E_1 - E_2\bar{f}_2E_1$. It follows that $0 = \bar{f}E_1 = \bar{f}_2E_1 - E_2\bar{f}_2E_1 = \bar{f} \neq 0$, a contradiction. □

Now we are able to characterize the varieties with proper central exponent equal to two.

Corollary 5.1. *Let $\mathcal{V}$ be a variety of PI-algebras over a field F of characteristic zero. Then $exp^\delta(\mathcal{V}) = 2$ if and only if $A_i \notin \mathcal{V}$ for every $i \in \{1, \ldots, 9\}$ and either $D \in \mathcal{V}$ or $D_0 \in \mathcal{V}$ or $G \in \mathcal{V}$.*

Proof. By [6] $\exp^\delta(\mathcal{V}) \geq 2$ if and only if either $D \in \mathcal{V}$ or $D_0 \in \mathcal{V}$ or $G \in \mathcal{V}$. Moreover, by Theorem 5.1, $\exp^\delta(\mathcal{V}) \leq 2$ if and only if $A_i \notin \mathcal{V}$ for $i \in \{1, \ldots, 9\}$ and so the proof is completed. □

Next we recall the following.

Definition 5.1. Let $\mathcal{V}$ be a variety of algebras. We say that $\mathcal{V}$ is minimal of proper central exponent $d \geq 2$ if $\exp^{\delta}(\mathcal{V}) = d$ and for every proper subvariety $\mathcal{U} \subset \mathcal{V}$ we have that $\exp^{\delta}(\mathcal{U}) < d$.

We denote $var(\mathcal{A}_i) = \mathcal{V}_i$, for $1 \leq i \leq 9$.

Proposition 5.1. *For $1 \leq i, j \leq 7$, $i \neq j$, we have that $\mathcal{V}_i \not\subseteq \mathcal{V}_j$. Moreover $\mathcal{V}_8, \mathcal{V}_9 \not\subseteq \mathcal{V}_i$, $1 \leq i \leq 7$.*

Proof. Since $exp(\mathcal{V}_1) = exp(\mathcal{V}_2) = 4$ and, for all $i \geq 3$, $exp(\mathcal{V}_i) = 3$, then, for $j = 1, 2$, $\mathcal{V}_j \not\subseteq \mathcal{V}_i$.

Since St_4, the standard polynomial of degree four, is an identity of $\mathcal{A}_1$ but $St_4 \notin Id(\mathcal{A}_i)$, for all $i \geq 3$, then $\mathcal{V}_i \not\subseteq \mathcal{V}_1$. Moreover, $[[x_1, x_2]^2, x_3] \in Id(\mathcal{A}_1)$ but $[[x_1, x_2]^2, x_3] \notin Id(\mathcal{A}_2)$ hence $\mathcal{V}_2 \not\subseteq \mathcal{V}_1$. Also, $[[x_1, x_2], [x_3, x_4], x_5] \in Id(\mathcal{A}_2)$ but it is not in $Id(A_i)$, $i = 3, \ldots, 9$, then $\mathcal{V}_i \not\subseteq \mathcal{V}_2$. Since $[[x_1, x_2], [x_3, x_4], x_5] \in Id(\mathcal{A}_2)$ and $[[x_1, x_2], [x_3, x_4], x_5] \notin Id(\mathcal{A}_1)$, then $\mathcal{V}_1 \not\subseteq \mathcal{V}_2$.

The proof is completed by an easy check of the following statements.

- $[x_1, x_2][x_3, x_4][x_5, x_6][x_7, x_8] \in Id(\mathcal{A}_3)$ but $\notin Id(\mathcal{A}_i)$, $4 \leq i \leq 9$,
- $x_1[x_2, x_3][x_4, x_5][x_6, x_7]x_8 \in Id(\mathcal{A}_4)$ but $\notin Id(\mathcal{A}_i)$, $3 \leq i \leq 9$, $i \neq 4$
- $[x_1, x_2, x_3][x_4, x_5][x_6, x_7, x_8] \in Id(\mathcal{A}_5)$ but $\notin Id(A_i)$, $3 \leq i \leq 9$, $i \neq 5$,
- $x_1[x_2, x_3][x_4, x_5, x_6]x_7 \in Id(\mathcal{A}_6)$ but $\notin Id(\mathcal{A}_i)$ $i = 3, \ldots, 9$, $i \neq 6$,
- $x_1[x_2, x_3, x_4][x_5, x_6]x_7 \in Id(\mathcal{A}_7)$ but $\notin Id(\mathcal{A}_i)$, $3 \leq i \leq 9$, $i \neq 7$. □

As a consequence we get the following.

Corollary 5.2. *The varieties $\mathcal{V}_1$ and $\mathcal{V}_2$ are minimal of proper central exponent equal to 4. The varieties $\mathcal{V}_i$, $3 \leq i \leq 7$, are minimal of proper central exponent equal to 3.*

Proof. By Lemma 3.1, $\exp^{\delta}(\mathcal{V}_1) = \exp^{\delta}(\mathcal{V}_2) = 4$ and $\exp^{\delta}(\mathcal{V}_i) = 3$, for $3 \leq i \leq 7$. Now, let $\mathcal{U}$ be a proper subvariety of $\mathcal{V}_i$ for some $i \in \{1, \ldots, 7\}$, and suppose that $\exp^{\delta}(\mathcal{U}) = \exp^{\delta}(\mathcal{V}_i) > 2$.

By Theorem 5.1, there exist $j \in \{1, \ldots, 9\}$ such that $\mathcal{A}_j \in \mathcal{U}$. Hence $\mathcal{V}_j \subseteq \mathcal{U} \subset \mathcal{V}_i$ and by Proposition 5.1, we obtain that $\mathcal{V}_j = \mathcal{U} = \mathcal{V}_i$, a contradiction. □

Data availability

No data was used for the research described in the article.

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