UNIVERSITY OF PALERMO
PHD JOINT PROGRAM:
UNIVERSITY OF CATANIA - UNIVERSITY OF MESSINA
XXXVII CYCLE

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCES

DOCTORAL THESIS

On the identities of 3×3 matrices with orthosymplectic superinvolution. Algebras with superautomorphism and codimension growth.

Author:
Sara ACCOMANDO

Coordinator:
Prof. Maria Carmela LOMBARDO

Supervisor:
Prof. Daniela LA MATTINA

Co-Supervisor:
Prof. Francesca Saviella BENANTI

A thesis submitted in fulfillment of the requirements for the degree of Doctor of Philosophy

in

Mathematics and Computational Sciences

ACADEMIC YEAR 2023/2024

Declaration of Authorship

I, Sara ACCOMANDO, declare that this thesis titled, "On the identities of 3×3 matrices with orthosymplectic superinvolution. Algebras with superautomorphism and codimension growth." and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed:

UNIVERSITY OF PALERMO

Abstract

Department of Mathematics and Computer Sciences

Doctor of Philosophy

On the identities of 3×3 matrices with orthosymplectic superinvolution. Algebras with superautomorphism and codimension growth.

by Sara ACCOMANDO

This thesis presents results concerning PI-algebras endowed with distinct additional structures.

First, we consider $M_{1,2}(F)$, the algebra of 3×3 matrices with orthosymplectic superinvolution $*$ over a field F of characteristic zero. We study the $*$-identities of this algebra through the representation theory of the group $\mathbb{H}_n = (\mathbb{Z}_2 \times \mathbb{Z}_2) \sim S_n$. To this end, we decompose the space of multilinear $*$-identities of degree n into the sum of irreducibles under the action of $\mathbb{H}_n$ and we study the irreducible characters appearing in this decomposition with non-zero multiplicity. Finally, by using the representation theory of the general linear group, we determine all the $*$-polynomial identities of $M_{1,2}(F)$ up to degree 3.

Next, we focus on superalgebras endowed with a superautomorphism of order ≤ 2. We characterize those superalgebras whose cocharacter multiplicities are bounded by a constant. Furthermore, we determine a characterization of the superalgebras with superautomorphism with polynomial growth of the codimensions and we give a classification of the subvarieties of the varieties of almost polynomial growth. Lastly, we characterize the superalgebras with superautomorphism with linear codimension growth.

Acknowledgements

First, my thankfulness goes to my tutor and my cotutor, prof. Daniela La Mattina and prof. Francesca Benanti, for my PhD experience: this thesis would have never been written without your help and support.

Then I want to thank all the people I met during this journey, both professors and colleagues, with whom I shared this path.

Lastly, my gratitude goes to my family and friends for always supporting me.

Contents

Chapter 1

Introduction

Let F be a field of characteristic zero and let A be an associative PI-algebra over F, i.e., an algebra satisfying at least one non trivial polynomial identity.

The set of all the polynomial identities satisfied by A, denoted by $Id(A)$, is a T-ideal of the free associative algebra $F\langle X\rangle$, where $X = \{x_1, x_2, \dots\}$ is a countable set. A T-ideal is an ideal invariant under all endomorphisms of $F\langle X\rangle$.

In characteristic zero, Specht [45] conjectured that every proper T-ideal of the free algebra is finitely generated and it was proved in the affirmative by Kemer in 1987 [30]. However explicit generators are known only in very few cases and the problem is still open for $M_k(F)$, with $k \geq 3$, where $M_k(F)$ is the algebra of $k \times k$ matrices over F.

Since in characteristic zero every T-ideal is generated by multilinear polynomials, one studies $P_n \cap Id(A)$, $n \geq 1$, where P_n is the space of multilinear polynomials of degree n. To this end, the representation theory of the symmetric group S_n is employed by defining an action on P_n as follows:

$$\sigma f(x_1, \dots, x_n) := f(x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

Since $P_n \cap Id(A)$ is invariant under this action, the space

$$\frac{P_n}{P_n \cap Id(A)}$$

has a structure of S_n-module and its character is called the n-th cocharacter of A. In [5] Benanti determined conditions under which the multiplicities are non-zero in the decomposition of the n-th cocharacter of $M_3(F)$ into irreducibles.

Another fundamental sequence in the study of polynomial identities is the sequence of codimensions, $c_n(A)$, $n = 1, 2, \dots$, introduced by Regev in 1972 [44]. He proved that if A is a PI-algebra, then its codimension sequence is exponentially bounded.

The class of all algebras satisfying a given set of polynomial identities $S \subseteq F\langle X\rangle$ is called the variety $\mathcal{V} = \mathcal{V}(S)$ determined by S. Given an algebra A, the variety generated by A is denoted by $\mathcal{V} = var(A)$ and is the set of all algebras satisfying the identities of A. In this context, an algebra B belongs to $var(A)$ if $\mathrm{Id}(A) \subseteq \mathrm{Id}(B)$. The growth of a variety of algebras is defined as the growth of the codimension sequence of any algebra A generating $\mathcal{V}$. In particular, we say that a variety $\mathcal{V}$ has polynomial growth if $c_n(\mathcal{V})$ is polynomially bounded, i.e., if there exist constants $a, p > 0$ such that $c_n(\mathcal{V}) \leq an^p$ for all n. Instead, we say that $\mathcal{V}$ has almost polynomial growth if $c_n(\mathcal{V})$ is not polynomially bounded but every proper subvariety of $\mathcal{V}$ has polynomial growth.

This dissertation is the result of the research I conducted over the past three years focusing on different problems related to PI-algebras with distinct additional structures. The results presented here are based on research papers I wrote during my PhD program. In particular, Chapter 3 is derived from the work presented in [1] while Chapter 4 is based on [2]. The structure of the thesis is as follows.

After this introduction, Chapter 2 provides the necessary backgrounds for our study, presenting the concepts and the results required to understand the subsequent chapters.

In Chapter 3, we focus on $M_3(F)$, the algebra of 3×3 matrices over F. The study of the polynomial identities satisfied by $M_3(F)$ with an additional structure was started in [6, 37]. In particular, we consider the superalgebra $M_3(F)$ endowed with a superinvolution, a structure that has been extensively studied in the last years (see [12, 43]). In the theory of polynomial identities, superalgebras with superinvolutions play a significant role. In [4] Aljadeff, Giambruno and Karasik proved that there exists a strict connection between algebras with involution and finite dimensional superalgebras with superinvolution.

Over the years, various results concerning the $*$-identities of these superalgebras have been obtained through the study of sequences attached to a PI-algebra (see [8, 16, 15, 23, 27, 24]).

A key milestone in this field was the classifications of superinvolutions on $M_n(F)$ by Racine in [43]. He proved that there are two types, the transpose and the orthosymplectic superinvolution. The study of the $*$-identities satisfied by these superalgebras with superinvolutions was started by Giambruno, Ioppolo and Martino in [14]. They focused on the standard polynomials and they determined the minimal degree of a standard polynomial vanishing on suitable subsets of symmetric or skew matrices for both types of superinvolutions.

A complete characterization of the cocharacter of $M_3(F)$ with transpose superinvolution was done in [9], where all the $*$-identities up to degree 3 were also determined.

Here we consider $M_3(F)$ with orthosymplectic superinvolution $*$ and we study the $*$-identities through the representation theory of the group $\mathbb{H}_n = (\mathbb{Z}_2 \times \mathbb{Z}_2) \sim S_n$. We decompose the space of multilinear $*$-identities of degree n into the sum of irreducibles under the $\mathbb{H}_n$-action in order to study the irreducible characters appearing in this decomposition with non-zero multiplicity. Moreover, by using the representation theory of the general linear group, we determine all the $*$-polynomial identities of $M_{1,2}(F)$ up to degree 3.

In Chapter 4, we present our results concerning superalgebras with superautomorphism.

In [29, 31] Kemer proved that $c_n(\mathcal{V})$, $n = 1, 2, \ldots$, is polynomially bounded if and only if $G, UT_2 \notin \mathcal{V}$, where G denotes the Grassmann algebra and UT_2 denotes the algebra of upper triangular matrices of size two. As a consequence, there exists no variety with intermediate growth, i.e., either its codimension sequence is polynomially bounded or grows exponentially. Some years later, in [39] and [38], La Mattina determined a complete list of finite dimensional algebras generating the subvarieties of $var(G)$ and $var(UT_2)$.

Characterizing algebras with polynomial codimension growth is of main interest. In literature, such a characterization has been determined for several algebras with additional structure, such as graded algebras [32], algebras with involution [21], algebras with superinvolution [27] and algebras with graded involution [22].

Here we study the codimension sequence of superalgebras with superautomorphism which are particular G-graded algebras, where G is a cyclic group of order

4. We focus our attention on superautomorphisms since they represent the connection link between graded involutions, superinvolutions and pseudoinvolutions. The study of these algebras in the case of polynomial growth was started in [25, 26]. It was proved that the codimensions of a superalgebra with superautomorphism are polynomially bounded if and only if the variety generated by this algebra does not contain the group algebra of $\mathbb{Z}_2$ and the algebra UT_2 with suitable superautomorphisms (see [25]). Motivated by this work, we determine additional characterizations involving the polynomial growth of the codimension sequence. Inspired by [24], we characterize the superalgebras with superautomorphism whose cocharacter multiplicities are bounded by a constant. Moreover, we classify the subvarieties of the varieties of almost polynomial growth. In the last part of the chapter we determine the superalgebras with superautomorphism with linear codimension growth.

Chapter 2

Backgrounds

This chapter is devoted to the introduction of the principal objects of our study. We give some definitions, examples and main results concerning the theory of PI-algebras. Finally, we introduce three sequences useful in order to study the PI-algebras and we give two examples of PI-algebras, the algebra of upper triangular matrices and the Grassmann algebra, whose sequences are explicitly known.

Throughout this thesis F will denote a field of characteristic zero.

2.1 First definitions

We start with the definition of free associative algebra.

Definition 2.1.1. Let F be a field and X a countable set. The *free associative algebra on X over F* is the algebra $F\langle X\rangle$ of polynomials in the non-commuting indeterminates $x \in X$.

A linear basis of $F\langle X\rangle$ consists of all words in the alphabet X, including the empty word 1. These words are called *monomials* and the product of two monomials is defined by juxtaposition. The elements of $F\langle X\rangle$ are called *polynomials* and we write $f = f(x_1, \ldots, x_n) \in F\langle X\rangle$ if $x_1, \ldots, x_n$ are the only indeterminates occurring in f.

We denote by deg u, *the degree of the monomial u*, as the length of the word u. Furthermore $\deg_{x_i} u$ is the degree of u in the indeterminate x_i, as the number of the occurrences of x_i in u. The degree of the polynomial $f = f(x_1, \ldots, x_n)$ is the maximum degree of a monomial in f and we denote it by deg f. Finally we write $\deg_{x_i} f$, that is the degree of x_i in f, as the maximum of $\deg_{x_i} u$, where u is a monomial of f.

Up to isomorphism, the algebra $F\langle X\rangle$ is defined by the universal property, as follows: given an associative F-algebra A, any map $X \to A$ can be uniquely extended to a homomorphism of algebras $F\langle X\rangle \to A$. The cardinality of X is called the rank of $F\langle X\rangle$.

Now we are ready to define one of the main object of this thesis.

Definition 2.1.2. Let A be an associative F-algebra and $f = f(x_1, \ldots, x_n) \in F\langle X\rangle$. We say that $f \equiv 0$ is a *polynomial identity of A* if $f(a_1, \ldots, a_n) = 0$, $\forall\, a_1, \ldots, a_n \in A$.

We shall usually say that $f \equiv 0$ is an identity of A or that f itself is an identity of A or that A satisfies f.

Let Φ be the set of all the homomorphism $\varphi : F\langle X\rangle \to A$. Then $f \equiv 0$ is a polynomial identity of A if and only if $f \in \bigcap_{\varphi\in\Phi} Ker\varphi$.

Definition 2.1.3. Given an associative F-algebra A, we say that A is a PI-algebra if it satisfies at least a non-trivial polynomial identity.

Here are some examples of PI-algebras. Let $[x,y] = xy - yx$ be the *Lie commutator of x and y*, for $x, y \in A$.

Example 2.1.1. Let A be a commutative algebra. Then A is a PI-algebra since it satisfies the identity $[x,y] \equiv 0$.

Example 2.1.2. If A is a nilpotent algebra, then A is a PI-algebra since if $A^n = 0$, for some $n \geq 1$, then $x_1 \cdots x_n \equiv 0$ is a polynomial identity of A.

Example 2.1.3. Let A be a NIL algebra of bounded index. Then there exists $n \geq 1$ such that $a^n = 0$, $\forall\, a \in A$. Then $x^n \equiv 0$ is a polynomial identity of A and so A is a PI-algebra.

Example 2.1.4. Consider $UT_n(F)$, the algebra of all the upper triangular matrices of size n over F. Then $UT_n(F)$ is a PI-algebra since it satisfies $[x_1, x_2] \ldots [x_{2n-1}, x_{2n}] \equiv 0$.

Example 2.1.5. If $NT_n(F)$ is the algebra of the strictly upper bounded matrices of size n, then it satisfies the polynomial identity $x_1 \cdots x_n \equiv 0$ and so it is a PI-algebra.

Example 2.1.6. The algebra $M_2(F)$ of 2×2 matrices over F is a PI-algebra, since it satisfies the polynomial identity $[[x_1, x_2]^2, x_3] \equiv 0$, that is called the *Hall identity*.

Example 2.1.7. Let V be a vector space over F, with countable basis $\{e_1, e_2, \ldots\}$ and char$F \neq 2$. The *Grassmann algebra* G on V over F is the algebra generated by V over F, where the product between two elements is the juxtaposition, and satisfying the condition $e_i e_j = -e_j e_i$, $\forall\, i,\ j \geq 1$. We write $G = span_F\{e_{i_1} \cdots e_{i_k} |\ 1 \leq i_1 < \cdots < i_k,\ k \geq 0\}$. The Grassman algebra is a PI-algebra since $[x_1, x_2, x_3] = [[x_1, x_2], x_3] \equiv 0$.

2.2 T-ideals and varieties of algebras

In this section we introduce two of the most important objects in the PI-algebra theory. We start with the following definition.

Definition 2.2.1. Given an algebra A, we define

$$\mathrm{Id}(A) = \{f \in F\langle X\rangle \ :\ f \equiv 0 \text{ on } A\}$$

as the set of all the polynomial identities satisfied by A.

In particular $\mathrm{Id}(A)$ is a two sided ideal of $F\langle X\rangle$ that is invariant under all endomorphisms of $F\langle X\rangle$. It means that $\mathrm{Id}(A)$ is a T-ideal.

Definition 2.2.2. An ideal I of $F\langle X\rangle$ is a T-ideal if $\varphi(I) \subseteq I$ for all endomorphisms φ of $F\langle X\rangle$.

Moreover, if I is a T-ideal it is easy to prove that $\mathrm{Id}(F\langle X\rangle / I) = I$. Then all T-ideals of $F\langle X\rangle$ are of this type.

Now, we introduce the notion of variety of an algebra, since many algebras may correspond to the same T-ideal.

Definition 2.2.3. Given a non-empty set $S \subseteq F\langle X\rangle$, the class of all algebras A such that $f \equiv 0$ on A, for all $f \in S$, is called the *variety* $\mathcal{V} = \mathcal{V}(S)$ determined by S.

A variety $\mathcal{V}$ is called non-trivial if $S \neq 0$ and $\mathcal{V}$ is proper if it is non-trivial and contains a non-zero algebra.

Here we give some examples.

Example 2.2.1. The class of all commutative algebras forms a proper variety defined by $S = \{[x, y]\}$.

Example 2.2.2. If $S = \{x^n\}$, then $\mathcal{V}(S)$ is the class of all NIL algebras of exponent bounded by n.

If $\mathcal{V}$ is the variety determined by the set S and $\langle S \rangle_T$ is the T-ideal of $F\langle X \rangle$ generated by S, then $\mathcal{V}(S) = \mathcal{V}(\langle S \rangle_T)$ and $\langle S \rangle_T = \bigcap_{A \in \mathcal{V}} \mathrm{Id}(A)$. We write $\langle S \rangle_T = \mathrm{Id}(\mathcal{V})$. Then to each variety corresponds a T-ideal of $F\langle X \rangle$ and the converse is also true, indeed we have the following.

Theorem 2.2.1. *There is a one-to-one correspondence between T-ideals of $F\langle X \rangle$ and varieties of algebras. In this correspondence a variety $\mathcal{V}$ corresponds to the T-ideal of identities $Id(\mathcal{V})$ and a T-ideal I corresponds to the variety of algebras satisfying all the identities in I.*

Proof. If I_1 and I_2 are two T-ideals, $I_1 \neq I_2$, then there exists $f \in I_1 \setminus I_2$. But $\mathcal{V}(I_1) \neq \mathcal{V}(I_2)$, since $F\langle X \rangle / I_2$ does not satisfy f and so $F\langle X \rangle / I_2 \in \mathcal{V}(I_2)$, but $F\langle X \rangle / I_2 \notin \mathcal{V}(I_1)$. If $\mathcal{V}_1$ and $\mathcal{V}_2$ are two varieties, $\mathcal{V}_1 \neq \mathcal{V}_2$, then there exists $A \in \mathcal{V}_1 \setminus \mathcal{V}_2$. Hence there exists $f \in \mathrm{Id}(\mathcal{V}_2)$ such that $f \notin \mathrm{Id}(A)$. Since $\mathrm{Id}(A) \supseteq \mathrm{Id}(\mathcal{V}_1)$ it follows that $\mathrm{Id}(\mathcal{V}_2) \neq \mathrm{Id}(\mathcal{V}_1)$. □

If $\mathcal{V}$ is a variety and A is an F-algebra such that $\mathrm{Id}(A) = \mathrm{Id}(\mathcal{V})$, we say that $\mathcal{V}$ is the variety generated by A and we write $\mathcal{V} = var(A)$.

2.3 Homogeneous, multilinear and alternating polynomials

If the base field F is infinite, the study of the polynomial identities of a given algebra can be reduced to the study of homogeneous or multilinear polynomials. In this section we will show the reason why this reduction holds.

Let $F_n = F\langle x_1, \dots, x_n \rangle$ be the free algebra of rank $n \geq 1$ over F. This algebra can be naturally decomposed as

$$F_n = F_n^{(1)} \oplus F_n^{(2)} \oplus \dots,$$

where, $\forall\ k \geq 1$, $F_n^{(k)}$ is the subspace spanned by all monomials of total degree k. Since $F_n^{(i)} F_n^{(j)} \subseteq F_n^{(i+j)}$, for all $i, j \geq 1$, we say that F_n has a structure of graded algebra and the $F_n^{(i)}$'s are called the homogeneous components of F_n.

Moreover, such decomposition can be written as follows: for every $k \geq 1$,

$$F_n^{(k)} = \bigoplus_{i_1 + \cdots + i_n = k} F_n^{(i_1, \dots, i_n)},$$

where $F_n^{(i_1, \dots, i_n)}$ is the subspace spanned by all monomials of degree i_1 in $x_1, \dots, i_n$ in x_n. Clearly $F_n^{(i_1, \dots, i_n)} F_n^{(j_1, \dots, j_n)} \subseteq F_n^{(i_1 + j_1, \dots, i_n + j_n)}$ and we say that F_n is multigraded. Such decompositions naturally extend to $F\langle X \rangle$, for countable X.

Definition 2.3.1. A polynomial $f \in F_n^{(k)}$ is called *homogeneous of degree k*, for some $k \geq 1$. If $f \in F_n^{(k_1, \dots, k_n)}$, then we say that f is *multihomogeneous of multidegree* $(k_1, \dots, k_n)$. Moreover, a polynomial f is *homogeneous in the variable* x_i if x_i appears with the same degree in every monomial of f.

If F is an infinite field and $f \in F\langle X\rangle$, we can always write

$$f = \sum_{i_1\geq 0,\dots,i_n\geq 0} f^{(i_1,\dots,i_n)},$$

where $f^{(i_1,\dots,i_n)} \in F_n^{(i_1,\dots,i_n)}$ is the sum of all monomials in f where the variables $x_1,\dots,x_n$ appear with degree $i_1,\dots,i_n$, respectively. We call the non-zero polynomials $f^{(i_1,\dots,i_n)}$ *multihomogeneous components of* f. It means that each polynomial has a decomposition in multihomogeneous polynomials.

Theorem 2.3.1. *Let F be an infinite field. If $f \equiv 0$ is a polynomial identity for the algebra A, then every multihomogeneous component of f is still a polynomial identity for A.*

Proof. Let $f \in F_n$. For every variable x_t, $1 \leq t \leq n$, we can decompose $f = \sum_{i=0}^{m} f_i$, where f_i is the sum of all monomials of f in which x_t appears at degree i and $m = \deg_{x_t} f$ is the degree of f in x_t. By induction, it suffices to prove that for every variable x_t, $f_i \equiv 0, \forall\, i \geq 0$. Let $\alpha_0,\dots,\alpha_m$ be distinct elements of F. For every $j = 0,\dots,m$, $f(x_1,\dots,\alpha_j x_t,\dots,x_n) \equiv 0$ is still an identity of A. Since each f_i is homogeneous in x_t of degree i, $f_i(x_1,\dots,\alpha_j x_t,\dots,x_n) = \alpha_j^i f_i(x_1,\dots,x_t,\dots,x_n)$. Then

$$f(x_1,\dots,\alpha_j x_t,\dots,x_n) = \sum_{i=0}^{m} \alpha_j^i f_i(x_1,\dots,x_t,\dots,x_n) \equiv 0 \tag{2.1}$$

over A, $\forall\, j = 0,\dots,m$. We write the Vandermonde matrix,

$$\Delta = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \alpha_0 & \alpha_1 & \dots & \alpha_m \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ \alpha_0^m & \alpha_1^m & \dots & \alpha_m^m \end{pmatrix}.$$

Then (2.1) says that for every $a_1,\dots,a_n \in A$, if we write $f_i(a_1,\dots,a_n) = \overline{f}_i$, then $(\overline{f}_0,\dots,\overline{f}_m)\Delta = 0$. Since $det(\Delta) = \prod_{0\leq i<j\leq m}(\alpha_j - \alpha_i) \neq 0$, then $f_0 \equiv 0,\dots,f_m \equiv 0$ are polynomial identities of A. □

Remark 2.3.1. The previous theorem is true even if the field F is finite such that $|F| > \deg f$.

One of the most important consequence of the previous theorem is that over an infinite field every T-ideal is generated by its multihomogeneous polynomials.

We introduce a special type of multihomogeneous polynomials.

Definition 2.3.2. A polynomial f is *linear* in the variable x_i if x_i has degree 1 in each monomial of f. If a polynomial f is linear in each of its variables, it is called *multilinear*, i.e., it is multihomogeneous of multidegree $(1,\dots,1)$. In this case, we write $f(x_1,\dots,x_n) = \sum_{\sigma\in S_n} \alpha_\sigma x_{\sigma(1)}\cdots x_{\sigma(n)}$, where $\alpha_\sigma \in F$ and S_n is the symmetric group over $\{1,\dots,n\}$.

If $f(x_1,\dots,x_n)$ is a linear polynomial in one variable, say x_1, then we have that $f(\sum \alpha_i y_i, x_2,\dots,x_n) = \sum \alpha_i f(y_i,x_2,\dots,x_n)$, for all $\alpha_i \in F, y_i \in F\langle X\rangle$.

Remark 2.3.2. Let A be an F-algebra spanned by a set B over F. If a multilinear polynomial f vanishes on B, then f is a polynomial identity of A.

Proof. Let $a_1 = \sum \alpha_{1_i} u_i, \ldots, a_n = \sum \alpha_{n_i} u_i$ be elements of A, with $u_i \in B$. Since $f = f(x_1, \ldots, x_n)$ is linear in every variable, then we get that $f(a_1, \ldots, a_n) = \sum \alpha_{1_{i_1}} \cdots \alpha_{n_{i_n}} f(u_{i_1}, \ldots, u_{i_n}) = 0$. □

Now, we introduce the process of multilinearization of a polynomial.

Definition 2.3.3. Let S be a set of polynomials in $F\langle X\rangle$ and $f \in F\langle X\rangle$. We say that f is a consequence of the polynomials in S (or f follows from the polynomials in S) if $f \in \langle S\rangle_T$, the T-ideal generated by the set S.

Definition 2.3.4. Two sets of polynomials are *equivalent* if they generate the same T-ideal.

Theorem 2.3.2. *If the algebra A satisfies an identity of degree k, then it satisfies a multilinear identity of degree less than or equal to k.*

Proof. Let $f(x_1, \ldots, x_n) \in F\langle X\rangle$ be a polynomial identity of the algebra A. If every variable x_i has degree ≤ 1 in every monomial of f, then we get a multilinear polynomial. We assume that there exists a variable, say x_1, such that $\deg_{x_1} f = d > 1$. We write the polynomial

$$h(y_1, y_2, x_2, \ldots, x_n) = f(y_1 + y_2, x_2, \ldots, x_n) - f(y_1, x_2, \ldots, x_n) - f(y_2, x_2, \ldots, x_n).$$

We remark that h is still a polynomial identity of A. We claim that h is a non-zero polynomial. Indeed, suppose the $h = 0$. Since every application $X \longrightarrow X$ can be extended to an endomorphism of $F\langle X\rangle$, we substitute y_1 and y_2 with x_1 in h and we still obtain the zero polynomial: $h(x_1, x_1, x_2, \ldots, x_n) = f(2x_1, x_2, \ldots, x_n) - 2f(x_1, \ldots, x_n) = 0$. If we decompose f in the sum $f = f_0 + f_1 + \cdots + f_d$, where f_k is the sum of all monomials of degree k in x_1, then, by the previous one, we get $-f_0 + (2^2 - 2)f_2 + \cdots + (2^d - 2)f_d = 0$, that is a contradiction since $d > 1$. Then h is a non-zero polynomial. Since $\deg_{y_1} h = d - 1 < \deg_{x_1} f$, by the definition of h, by induction, we get a multilinear polynomial which is an identity of A. □

We remark that the proof of the previous theorem shows how the process of multilinearization of a polynomial works. It has some important consequences.

Theorem 2.3.3. *If $charF = 0$, every non-zero polynomial $f \in F\langle X\rangle$ is equivalent to a finite set of multilinear polynomials.*

Proof. By Theorem 2.3.1, f is equivalent to the set of its multihomogeneous components. Suppose that $f = f(x_1, \ldots, x_n)$ is multihomogeneous. We apply the multilinearization process to f: if $\deg_{x_1} f = d > 1$, then we write $f(y_1 + y_2, x_2, \ldots, x_n) = \sum_{i=0}^{d} g_i(y_1, y_2, x_2, \ldots, x_n)$, where $\deg_{y_1} g_i = i$, $\deg_{y_2} g_i = d - i$ and $\deg_{x_t} g_i = \deg_{x_t} f$, for all $t = 2, \ldots, n$. Every polynomial $g_i = g_i(y_1, y_2, x_2, \ldots, x_n)$, $i = 1, \ldots, d-1$, is a consequence of f. We remark that $g_i(y_1, y_1, x_2, \ldots, x_n) = \binom{d}{i} f(y_1, x_2, \ldots, x_n)$, for all i. Since $charF = 0$, $\binom{d}{i} \neq 0$, then f is a consequence of every g_i, $i = 1, \ldots, d-1$. By induction, we get the desired result. □

Corollary 2.3.1. *If $charF = 0$, every T-ideal is generated, as a T-ideal, by all the multilinear polynomials it contains.*

We remark that it is easy to check if a multilinear element $f(x_1, \ldots, x_m) \in F\langle X\rangle$ vanishes on an algebra A: if we fix a basis $\{s_1, s_2, \ldots\}$ of A, then $f(x_1, \ldots, x_m) \equiv 0$ is an identity of A if and only if $f(s_{i_1}, \ldots, s_{i_m}) = 0$ for every m-ple of basis elements.

Now we introduce the alternating polynomials.

Definition 2.3.5. Let $f = f(x_1, \dots, x_n, y_1, \dots, y_t)$ a polynomial that is linear in each variables $x_1, \dots, x_n$. We say that f is *alternating in the variables* $x_1, \dots, x_n$ if, for all $1 \leq i < j \leq n$, the polynomial f vanishes when we substitute x_i instead of x_j. In case f is alternating in all of its variables, we simply say that f is alternating.

Proposition 2.3.1. *Let $f(x_1, \dots, x_n, y_1, \dots, y_t)$ be a polynomial alternating in $x_1, \dots, x_n$ and let A be an F-algebra. If $a_1, \dots, a_n \in A$ are linearly dependent over F, then, for all $b_1, \dots, b_t \in A$, $f(a_1, \dots, a_n, b_1, \dots, b_t) = 0$.*

Proof. By the hypothesis, one of the a_i's, say a_1, can be written as a linear combination of the others, $a_1 = \sum_{i=2}^{n} \alpha_i a_i$, $\alpha_i \in F$. But then $f(a_1, \dots, a_n, b_1, \dots, b_t) = \sum_{i=2}^{n} \alpha_i f(a_i, a_2, \dots, a_n, b_1, \dots, b_t) = 0$, since f is alternating on $x_1, \dots, x_n$ and in each term $f(a_i, a_2, \dots, a_n, b_1, \dots, b_t)$ two arguments coincide. □

An example of alternating polynomial is the standard one.

Definition 2.3.6. The polynomial $St_m(x_1, \dots, x_m) = \sum_{\sigma \in S_m} (sgn\sigma) x_{\sigma(1)} \cdots x_{\sigma(m)}$ is called *standard polynomial of degree m*.

The symbol ^ means omission. As an example, $f(x_1, \dots, \hat{x}_i, \dots, x_m)$ is a polynomial in which the variable x_i does not appear.

Proposition 2.3.2. *[20, Proposition 1.5.7]*

1. *If $f(x_1, \dots, x_m)$ is a multilinear alternating polynomial of degree m, then, for some $\alpha \in F$, $f = \alpha St_m(x_1, \dots, x_m)$.*

2. *$St_{m+1}(x_1, \dots, x_{m+1}) = \sum_{i=1}^{m+1} (-1)^{i+1} x_i St_m(x_1, \dots, \hat{x}_i, \dots, x_{m+1})$. So, if $St_m \equiv 0$ is an identity for an algebra A, St_{m+1} is still an identity of A.*

The following theorem follows from the fact that the standard polynomial is alternating ([20, Theorem 1.5.8].

Theorem 2.3.4. *Let A be an F-algebra. If $dim_F A = n < \infty$, then $St_{n+1} \equiv 0$ in A.*

2.4 Representation theory

In this section we introduce the representation theory of the symmetric group by the theory of Young tableaux. Then we shall present the permutation action of the symmetric group S_n on the space of multilinear polynomials in n variables, since it will be a useful tool in order to study T-ideals.

Let V be a vector space over a field F and let $GL(V)$ be the group of invertible endomorphisms of V. From now on, we suppose that the characteristic of the base field F is zero.

Definition 2.4.1. A *representation of a group G on V* is a homomorphism of groups $\rho : G \to GL(V)$.

Let $End(V)$ be the algebra of F-endomorphisms of V. If FG is the group algebra of G over F and ρ is a representation of G on V, then ρ induces an homomorphism of F-algebras, $\rho' : FG \to End(V)$ such that $\rho'(1_G) = 1$.

We deal with the case $\dim_F V = n < \infty$. In this case n is called *the dimension* or *the degree of the representation* ρ.

A representation of a group G uniquely determines a finite dimensional FG-module: if $\rho : G \to GL(V)$ is a representation of G, V becomes a left G-module, with $gv = \rho(g)(v)$, $\forall\, g \in G,\ v \in V$. And, if M is a G-module which is finite dimensional as a vector space over F, then $\rho : G \to GL(M)$ such that $\rho(g)(m) = gm$, for all $g \in G, m \in M$, defines a representation of G on M.

Definition 2.4.2. A representation $\rho : G \to GL(V)$ is *irreducible* if V is an irreducible G-module. We say that ρ is *completely reducible* if V is the direct sum of its irreducible submodules.

An algebra A is semisimple if $J(A) = 0$, where $J(A)$ is the Jacobson radical of A.

Theorem 2.4.1 (Maschke)**.** *Let G be a finite group and $charF = 0$ or $charF = p > 0$ and $p \nmid |G|$. Then the group algebra FG is semisimple.*

It follows that every G-module V is completely reducible. Then, if $\dim_F V < \infty$, V is direct sum of a finite number of irreducible G-module.

We recall that an element $e \in FG$ is an *idempotent* if $e^2 = e$. Since FG is semisimple of finite dimension, then every ideal is generated by a central idempotent, as it can be proved in the following proposition (see [20, Proposition 2.1.7]).

Proposition 2.4.1. *If M is an irreducible representation of G, then $M \cong J_i$, a minimal left ideal of $M_{n_i}(D^{(i)})$, for some $i = 1, \dots, k$. So there exists a minimal idempotent $e \in FG$ such that $M \cong FGe$.*

Then, if F is algebraically closed, the group algebra FG can be decomposed as $FG \cong \bigoplus_{i=1}^{k} e_i FG$, where $\forall\, i = 1, \dots, k,\ e_i FG \cong M_{n_i}(F)$ and e_i is the minimal central idempotent of FG.

Let us introduce the character of a representation.

First, we recall that $tr : M_k(F) \to F$ is the usual trace function.

Definition 2.4.3. Let $\rho : G \to GL(V)$ be a representation of G. Then the map $\chi_\rho : G \to F$ such that $\chi_\rho(g) = tr(\rho(g))$ is called *character* of the representation ρ and dim $V = \deg \chi_\rho$ is the *degree* of the character χ_ρ.

We say that χ_ρ is irreducible if ρ is irreducible.

2.4.1 S_n-representations

In this subsection we describe the S_n-representation. We start defining the Young tableaux.

Definition 2.4.4. Let $n \geq 1$ an integer. A *partition λ of n* is a finite sequence of integers $\lambda = (\lambda_1, \dots, \lambda_r)$ such that $\lambda_1 \geq \cdots \geq \lambda_r > 0$ and $\sum_{i=1}^{r} \lambda_i = n$. We write $\lambda \vdash n$ or $|\lambda| = n$.

If $r = 1$, then $\lambda_1 = n$ and $\lambda = (n)$. If $\lambda = (k, \dots, k)$ and $n = kd$, then $\lambda = (k^d)$.

The group algebra FS_n has a decomposition into simple components which are algebras of matrices over the field F. Moreover, the number of irreducible non equivalent representations equals the number of the conjugacy class of S_n. These conjugacy class are indexed by the partitions of n: given $\sigma \in S_n$ there exists a partition $\lambda \vdash n$ that uniquely determine the conjugacy class of σ.

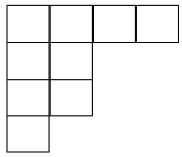
We denote by χ_λ the irreducible S_n-character corresponding to $\lambda \vdash n$ and we write $d_\lambda = \chi_\lambda(1)$ as the degree of χ_λ. It follows that FS_n has the following decomposition:

$$FS_n = \bigoplus_{\lambda \vdash n} I_\lambda \cong \bigoplus_{\lambda \vdash n} M_{d_\lambda}(F),$$

where $I_\lambda = e_\lambda FS_n \cong M_{d_\lambda}(F)$ is the minimal ideal of FS_n, corresponding to $\lambda \vdash n$, and $e_\lambda = \sum_{\sigma \in S_n} \chi_\lambda(\sigma)\sigma$ is, up to a scalar, a central idempotent.

Definition 2.4.5. If $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$, the *Young diagram associated to* λ is the finite subset of $\mathbb{Z} \times \mathbb{Z}$, defined as $D_\lambda = \{(i,j) \in \mathbb{Z} \times \mathbb{Z} : i = 1, \ldots, r, j = 1, \ldots, \lambda_i\}$.

Example 2.4.1. The diagram $D_{(4,2,2,1)}$ is represented by



Given a partition $\lambda \vdash n$, we denote by λ' the conjugate partition of λ, $\lambda' = (\lambda'_1, \ldots, \lambda'_s)$, such that $\lambda'_1, \ldots, \lambda'_s$ are the lengths of the columns of D_λ.

Definition 2.4.6. Consider $\lambda \vdash n$. A *Young tableau* T_λ of the diagram D_λ is a filling of the boxes of D_λ with the integers $1, 2, \ldots, n$. We shall also say that T_λ is a tableau of shape λ.

Definition 2.4.7. A tableau T_λ of shape λ is *standard* if the integers in each row and in each column of T_λ, increase from left to right and from top to bottom, respectively.

There is a connection between standard tableaux and the degree of the irreducible S_n-characters.

Theorem 2.4.2. *Given a partition $\lambda \vdash n$, the number of standard tableaux of shape λ equals d_λ, the degree of χ_λ, the irreducible character corresponding to λ.*

As in [28], we introduce an important formula which characterize the degree d_λ of the irreducible character χ_λ.

Given a diagram D_λ, $\lambda \vdash n$, we identify a box of D_λ with the corresponding point (i,j). For instance, the third box of the first row has coordinate $(1,3)$.

Definition 2.4.8. For any box $(i,j) \in D_\lambda$, we define the *hook number* of (i,j) as $h_{ij} = \lambda_i + \lambda'_j - i - j + 1$, where $\lambda' = (\lambda'_1, \ldots, \lambda'_s)$ is the conjugate partition of λ.

It is easy to see that h_{ij} counts the number of boxes in the "hook" with edge in (i,j), i.e., the boxes to the right and below (i,j).

Proposition 2.4.2 (The Hook Formula)**.**

$$d_\lambda = \frac{n!}{\Pi_{i,j} h_{ij}},$$

where the product runs over all boxes of D_λ.

We are able to compute the complete number of left minimal ideals of FS_n as follows. Given a tableau T_λ of shape $\lambda \vdash n$, we write $T_\lambda = D_\lambda(a_{ij})$, where a_{ij} is the integer in the (i,j) box.

Definition 2.4.9. The *row stabilizer* of T_λ is

$$R_{T_\lambda} = S_{\lambda_1}(a_{11}, a_{12}, \ldots, a_{1\lambda_1}) \times \cdots \times S_{\lambda_r}(a_{r1}, a_{r2}, \ldots, a_{r\lambda_r}),$$

where $S_{\lambda_i}(a_{i1}, a_{i2}, \ldots, a_{i\lambda_i})$ denotes the symmetric group acting on the integers $a_{i1}, a_{i2}, \ldots, a_{i\lambda_i}$, for $i = 1, \ldots, r$. Then R_{T_λ} is the subgroup of S_n consisting of all the permutation stabilizing the rows of T_λ.

Definition 2.4.10. The *column stabilizer* of $T_\lambda = D_\lambda(a_{ij})$ is

$$C_{T_\lambda} = S_{\lambda'_1}(a_{11}, a_{21}, \ldots, a_{\lambda'_1 1}) \times \cdots \times S_{\lambda'_r}(a_{1\lambda_1}, \ldots, a_{\lambda'_s \lambda_r}),$$

where $\lambda' = (\lambda'_1, \ldots, \lambda'_s)$ is the conjugate partition of λ. Then C_{T_λ} is the subgroup of S_n consisting of all the permutation stabilizing the columns of T_λ.

Definition 2.4.11. For a given tableau T_λ, we define

$$e_{T_\lambda} = \sum_{\sigma \in R_{T_\lambda}, \tau \in C_{T_\lambda}} (sgn\tau)\sigma\tau.$$

It can be proved that $e_{T_\lambda}^2 = a e_{T_\lambda}$, where $a = \frac{n!}{d_\lambda} = \Pi_{i,j} h_{ij}$ is a non-zero integer, i.e., e_{T_λ} is an essential idempotent of FS_n.

Given a partition $\lambda \vdash n$, the symmetric group S_n acts on the set of Young tableaux of shape λ as follows: if $\sigma \in S_n$ and $T_\lambda = D_\lambda(a_{ij})$, then $\sigma T_\lambda = D_\lambda(\sigma(a_{ij}))$. This action has the property that $R_{\sigma T_\lambda} = \sigma R_{T_\lambda} \sigma^{-1}$ and $C_{\sigma T_\lambda} = \sigma C_{T_\lambda} \sigma^{-1}$. It follows that $\sigma e_{T_\lambda} \sigma^{-1} = e_{\sigma T_\lambda}$.

Proposition 2.4.3. *For every Young tableau T_λ of shape $\lambda \vdash n$, the element e_{T_λ} is a minimal essential idempotent of FS_n and $FS_n e_{T_\lambda}$ is a left minimal ideal of FS_n with character χ_λ. If T_λ and T^*_λ are Young tableaux with the same shape, then e_{T_λ} and $e_{T^*_\lambda}$ are conjugated in FS_n for some $\sigma \in S_n$; moreover $\sigma e_{T_\lambda} \sigma^{-1} = e_{\sigma T_\lambda}$.*

It means that given two tableaux T_λ and T^*_λ of the same shape λ, $FS_n e_{T_\lambda} \cong FS_n e_{T^*_\lambda}$, as S_n-modules.

Proposition 2.4.4. *If $T_1, \ldots, T_{d_\lambda}$ are standard tableaux of shape λ, then I_λ, the minimal two-sided ideal of FS_n, corresponding to λ, has the decomposition $I_\lambda = \bigoplus_{i=1}^{d_\lambda} FS_n e_{T_i}$.*

Now we are ready to introduce an action of the symmetric group S_n on the space of multilinear polynomials in n fixed variables.

Lemma 2.4.1. *Let M be an irreducible left S_n-module with character $\chi(M) = \chi_\lambda$, $\lambda \vdash n$. Then M can be generated as an S_n-module by an element of the form $e_{T_\lambda} f$, for some $f \in M$ and some Young tableau T_λ of shape λ. Moreover, given a Young tableau T^*_λ of shape λ, there exists $f' \in M$ such that $M = FS_n e_{T^*_\lambda} f'$.*

Proof. We remind that $FS_n = \bigoplus_{\mu \vdash n} I_\mu$, where I_μ is the two sided ideal FS_n correspondent to μ and

$$FS_n = \bigoplus_{\mu \vdash n, T_\mu standard} FS_n e_{T_\mu},$$

by Proposition 2.4.4. Since $M = FS_n M$, there exist $\mu \vdash n$, T_μ standard and $f \in M$ such that $0 \neq FS_n e_{T_\mu} f \subseteq M$. By the irreducibility of M we get $FS_n e_{T_\mu} f = M$. Since $\chi(M) = \chi_\lambda$, we obtain that $\lambda = \mu$. Finally, if T^*_λ is a Young tableau of the same shape, then $e_{T_\lambda} = \sigma e_{T^*_\lambda} \sigma^{-1}$ and $g = \sigma e_{T^*_\lambda} f'$, where $f' = \sigma^{-1} f$. □

By the previous lemma, we get that, given a partition $\lambda \vdash n$ and a Young tableau T_λ of shape λ, an irreducible S_n-module M such that $\chi(M) = \chi_\lambda$ can be generated by an element of the type $e_{T_\lambda} f$, for some $f \in M$. By the definition of R_{T_λ}, $\forall\, \sigma \in R_{T_\lambda}$, we get that $\sigma e_{T_\lambda} f = e_{T_\lambda} f$, i.e., $e_{T_\lambda} f$ is stable under the action of R_{T_λ}.

We introduce

$$P_n = span\{x_{\sigma(1)} \cdots x_{\sigma(n)} : \sigma \in S_n\},$$

the vector space of multilinear polynomials in $x_1, \ldots, x_n$ in the free algebra $F\langle X \rangle$.

In particular, it is possible to define a left action of S_n on P_n defined in the following way: if $f(x_1, \dots, x_n) \in P_n$, $\sigma \in S_n$,

$$\sigma f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)}),$$

that is σ acting by permuting the variables.

Now, consider a PI-algebra A and its T-ideal $\mathrm{Id}(A)$. In the previous section we prove that $\mathrm{Id}(A)$ is determined by its multilinear polynomials if $charF = 0$. Hence it suffices to study the multilinear identities of A, that is the space $P_n \cap \mathrm{Id}(A)$.

We remark that T-ideals are invariant under the permutation of the variables, then $P_n \cap \mathrm{Id}(A)$ is a left S_n-submodule of P_n. Therefore

$$P_n(A) = \frac{P_n}{P_n \cap \mathrm{Id}(A)}$$

has an induced structure of S_n-module.

If $F\langle X \rangle$ is the free algebra of countable rank on $X = \{x_1, x_2, \dots\}$, $P_n(A)$ is the space of multilinear elements in the first n variables of the free relatively algebra $F\langle X \rangle / \mathrm{Id}(A)$. If $\mathcal{V} = var(A)$, we write $P_n(\mathcal{V}) = P_n(A)$.

We describe the structure of multilinear identities in the language of the action of S_n. The next theorem follows from Lemma 2.4.1.

Theorem 2.4.3. *For any multilinear $f \in P_n$, there exists a finite set of polynomials $g_1, \dots, g_r \in P_n$ and partitions $\lambda(1), \dots, \lambda(r)$ of n such that we write $FS_n f = FS_n e_{T_{\lambda(1)}} g_1 + \dots + FS_n e_{T_{\lambda(r)}} g_r$.*

Proof. We write $M = FS_n f$ and we consider the decomposition $M = M_1 \oplus \dots \oplus M_r$ in the sum of irreducible S_n-modules. By Lemma 2.4.1, there exist $g_1 \in M_1, \dots, g_r \in M_r$ and Young tableaux $T_{\lambda(1)}, \dots, T_{\lambda(r)}$ such that we have that $M_1 = FS_n e_{T_{\lambda(1)}} g_1, \dots, M_r = FS_n e_{T_{\lambda(r)}} g_r$. □

2.5 Sequences associated to a T-ideal

In 1987 Kemer [30] proved in the affirmative the Specht conjecture [45]: in characteristic zero, every proper T-ideal of the free algebra is finitely generated. However the generators are known only in very few cases and the problem of finding them is still open. So, we study three sequences associated to a T-ideal: the cocharacter sequence, the codimension sequence and the colength sequence. Indeed, the explicit computation of these sequences for a given algebra gives us some information about the T-ideal $\mathrm{Id}(A)$ of a given algebra A. In this section we introduce these sequences and finally we present the explicit computation of the T-ideal and its sequences for two particular PI-algebras.

2.5.1 Cocharacters

Definition 2.5.1. For $n \geq 1$, the S_n-character of $P_n(A) = \frac{P_n}{P_n \cap \mathrm{Id}(A)}$ is called the *n-th cocharacter of A* and it is denoted by $\chi_n(A)$. Moreover, we write $\{\chi_n(A)\}_{n \geq 1}$ the sequence of cocharacters of the algebra A.

If we decompose the n-th cocharacter into irreducibles, we obtain

$$\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda,$$

where χ_λ is the irreducible S_n-cocharacter associated to the partition $\lambda \vdash n$ and $m_\lambda \geq 0$ is the corresponding multiplicity.

Theorem 2.5.1. *Let A be a PI-algebra with n-th cocharacter $\chi_n(A)$, given by*

$$\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda. \tag{2.2}$$

For a partition $\mu \vdash n$, the multiplicity m_μ is equal to zero if and only if for any Young tableau T_μ of shape μ and for any polynomial $f = f(x_1, \dots, x_n) \in P_n$, the algebra A satisfies the identity $e_{T_\mu} f \equiv 0$.

Proof. Consider the decomposition $FS_n = \bigoplus_{\lambda \vdash n} I_\lambda$, $P_n = Q \oplus J$, where $Q = P_n \cap \mathrm{Id}(A)$ and $J \cong P_n(A) = \frac{P_n}{P_n \cap \mathrm{Id}(A)}$. We fix some $\mu \vdash n$. Then $m_\mu = 0$ in (2.2) if and only if $I_\mu J = 0$. On the other end, the equality $I_\mu J = 0$ is equivalent to the inclusion $I_\mu P_n \subseteq Q$. Since I_μ is the sum of all left ideals $FS_n e_{T_\mu}$, the inclusion $I_\mu P_n \subseteq Q$ takes place if and only if $e_{T_\mu} f \in Q$, for all $f \in P_n$, i.e., $e_{T_\mu} f \equiv 0$ is an identity of A. □

2.5.2 Codimensions and colengths

We remark, by Corollary 2.3.1, that $\mathrm{Id}(A)$ is generated, as a T-ideal, by its multilinear polynomials. Then $\mathrm{Id}(A)$ is generated by the subspace $(P_1 \cap \mathrm{Id}(A)) \oplus (P_2 \cap \mathrm{Id}(A)) \oplus \cdots \oplus (P_n \cap \mathrm{Id}(A)) \oplus \dots$ in the free associative algebra $F\langle X \rangle$, where P_k is the space of multilinear polynomials in the first k variables $x_1, \dots, x_k$, for every k. If A satisfies all the identities of some PI-algebra B, then $P_n \cap \mathrm{Id}(A) \supseteq P_n \cap \mathrm{Id}(B)$ and $\dim(P_n \cap \mathrm{Id}(A)) \geq \dim(P_n \cap \mathrm{Id}(B))$, $\forall\, n = 1, 2, \dots$. Then, the dimensions of the spaces $P_n \cap \mathrm{Id}(A)$ give us some information about the growth of the identities of the algebra A.

Definition 2.5.2. The non negative integer

$$c_n(A) = \dim \frac{P_n}{P_n \cap \mathrm{Id}(A)} \tag{2.3}$$

is called the *n-th codimension of the algebra A*. Moreover, we denote by $\{c_n(A)\}_{n \geq 1}$ the codimension sequence of the algebra A.

It follows that $\dim\,(P_n \cap \mathrm{Id}(A)) = n! - c_n(A)$.

We remark that if A is an algebra, A is a PI-algebra if and only if $c_n(A) < n!$, for some $n \geq 1$.

If $\mathcal{V}$ is a variety of algebras and $\mathcal{V} = var(A)$, then we define $c_n(\mathcal{V}) = c_n(A)$.

Example 2.5.1. Let A be a nilpotent algebra with $A^m = 0$. Then $c_n(A) = 0$, $\forall\, n \geq m$.

Example 2.5.2. If A is a commutative algebra, then $c_n(A) \leq 1$, $\forall\, n \geq 1$.

Finally, we define the following sequence.

Definition 2.5.3. If $charF = 0$ and $\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$ is the decomposition of $\chi_n(A)$ in irreducible characters of S_n, then

$$l_n(A) = \sum_{\lambda \vdash n} m_\lambda \tag{2.4}$$

is called the *n-th colength*. We denote by $\{l_n(A)\}_{n \geq 1}$ the sequence of the colength of the algebra A.

In particular, $l_n(A)$ counts the number of irreducible S_n-modules appearing in the decomposition of $P_n(A)$.

As done for the codimensions, if $\mathcal{V}$ is a variety of algebras, we write $l_n(\mathcal{V}) = l_n(A)$, where A is an algebra generating $\mathcal{V}$.

Example 2.5.3. Let A be a non nilpotent commutative algebra. Then $c_n(A) = 1$ and $l_n(A) = 1, \forall\, n$.

2.5.3 Particular examples

Now we present two important theorem which present the generators of the T-ideal and the explicit computation of the cocharacter, codimension and colength sequences of two algebras.

The first theorem is about the algebra of upper triangular matrices of size two.

Theorem 2.5.2 ([35, 7]). *For $A = UT_2(F)$, the algebra of 2×2 upper triangular matrices over a field F of characteristic zero, the following hold:*

1. *The T ideal of the polynomial identities of $UT_2(F)$ is generated by the polynomial $[x_1, x_2][x_3, x_4]$.*
2. $c_n(A) = 2^{n-1}(n-2) + 2$.
3. $\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$, *where the only nonzero multiplicities are* $m_{(n)} = 1$, $m_{(\lambda_1,\lambda_2)} = m_{(\lambda_1,\lambda_2,1)} = \lambda_1 - \lambda_2 + 1$.
4. $l_n(A) = \frac{1}{2}n^2 + \frac{5}{2}n + 4$.

The following theorem concerns the Grassmann algebra G.

First, we define the infinite hook.

Definition 2.5.4. Given the integers $d, l \geq 0$, the *infinite hook* $H(d,l)$ is defined as

$$H(d,l) = \bigcup_{n \geq 1} \{\lambda = (\lambda_1, \lambda_2, \dots) \vdash n \mid \lambda_{d+1} \leq l\}.$$

We say that d is the *hand* and l is the *foot* of the hook.

We write $\lambda \in H(d,l)$, if the Young tableau corresponding to the diagram D_λ is contained in the hook of hand d e foot l. Moreover, if V is an S_n-module with $\chi(V) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$, we write $\chi(V) \subseteq H(d,l)$ if $\lambda \in H(d,l)$, for any partition λ such that $m_\lambda \neq 0$.

Theorem 2.5.3 ([33, 41]). *For the infinite dimensional Grassmann algebra G over a field of characteristic zero, the following are true:*

1. *The T-ideal of the identities of G is generated by the polynomial $[[x_1, x_2], x_3]$.*
2. $c_n(G) = 2^{n-1}$.
3. $\chi_n(G) = \sum_{\lambda \vdash n,\ \lambda \in H(1,1)} \chi_\lambda$.
4. $l_n(G) = n$.

Chapter 3

Polynomial identities satisfied by the algebra of 3×3 matrices with orthosymplectic superinvolution

This chapter is devoted to the results obtained during my PhD which can be found in [1]. In particular, it deals with the study of the polynomial identities satisfied by the algebra $M_3(F)$ endowed with the orthosymplectic superinvolution $*$. The representation theory of the group $\mathbb{H}_n = (\mathbb{Z}_2 \times \mathbb{Z}_2) \sim S_n$ has a prominent role, since we decompose the space of multilinear $*$-identities of degree n into the sum of irreducibles under the $\mathbb{H}_n$-action in order to study the irreducible characters appearing in this decomposition with non-zero multiplicity. Furthermore, by using the representation theory of the general linear group, we determine all the $*$-polynomial identities of $M_{1,2}(F)$ up to degree 3.

3.1 Superalgebras with superinvolution

In this section we present the superalgebras endowed with a superinvolution and we give all the definitions concerning the PI-theory regarding this particular structure. Throughout the chapter F will denote a field of characteristic zero.

Definition 3.1.1. An associative algebra A is a $\mathbb{Z}_2$*-graded algebra* or a *superalgebra* if it has a vector space decomposition $A = A_0 \oplus A_1$ such that $A_0A_0 + A_1A_1 \subseteq A_0$ and $A_0A_1 + A_1A_0 \subseteq A_1$.

The elements of A_0 and A_1 are called *homogeneous of degree zero* (or even degree) and of *degree one* (or odd degree), respectively.

Definition 3.1.2. Let $A = A_0 \oplus A_1$ be a superalgebra. A superinvolution on A is a graded linear map $* : A \longrightarrow A$, i.e., a map preserving the grading, such that $(a^*)^* = a$ and $(ab)^* = (-1)^{|a||b|}b^*a^*$ where $|c|$ denotes the homogeneous degree of the element $c \in A$.

A superalgebra endowed with a superinvolution is called a $*$*-superalgebra*. Since $\mathrm{char}F = 0$, the $*$-superalgebra can be written as

$$A = A_0^+ \oplus A_0^- \oplus A_1^+ \oplus A_1^-,$$

where, for $i = 0, 1$, $A_i^+ = \{a \in A_i \mid a^* = a\}$ and $A_i^- = \{a \in A \mid a^* = -a\}$ denote the sets of homogeneous symmetric and skew elements of A_i, respectively.

Let $X = \{x_1, x_2, \dots\}$ be a countable set of non-commuting variables. We write $X = Y \cup Z$ as disjoint union of two infinite homogeneous subsets $Y = \{y_1, y_2, \dots\}$ and $Z = \{z_1, z_2, \dots\}$ of degree 0 and 1, respectively.

We denote by $F\langle Y \cup Z\rangle = \mathcal{F}_0 \oplus \mathcal{F}_1 = \langle y_1, z_1, y_2, z_2, \dots\rangle$ the free associative superalgebra on the countable set $Y \cup Z$ over F, where the variables of Y and Z are even and odd, respectively. Here $\mathcal{F}_0$ and $\mathcal{F}_1$ are the subspaces of $F\langle Y \cup Z\rangle$ spanned by all monomials having an even or an odd number of variables of Z, respectively.

Moreover, if we write each set as the disjoint union of two infinite sets of symmetric and skew elements, respectively, then we have the free $*$-superalgebra

$$F\langle Y \cup Z, *\rangle = F\langle y_1^+, y_1^-, z_1^+, z_1^-, y_2^+, y_2^-, z_2^+, z_2^-, \dots\rangle,$$

where $y_i^+ = y_i + y_i^*$ denotes a symmetric variable of even degree, $y_i^- = y_i - y_i^*$ a skew symmetric variable of even degree, $z_i^+ = z_i + z_i^*$ a symmetric variable of odd degree and $z_i^- = z_i - z_i^*$ a skew variables of odd degree.

We say that $f(y_1^+, \dots, y_n^+, y_1^-, \dots, y_m^-, z_1^+, \dots, z_t^+, z_1^-, \dots, z_s^-) \in F\langle Y \cup Z, *\rangle$ is a $*$-identity of A, and we write $f \equiv 0$, if

$$f(u_1^+, \dots, u_n^+, u_1^-, \dots, u_m^-, v_1^+, \dots, v_t^+, v_1^-, \dots, v_s^-) = 0,$$

for all $u_1^+, \dots, u_n^+ \in A_0^+$, $u_1^-, \dots, u_m^- \in A_0^-$, $v_1^+, \dots, v_t^+ \in A_1^+$, $v_1^-, \dots, v_s^- \in A_1^-$.

We consider the set of all $*$-identities of A

$$Id_2^*(A) = \{f \in F\langle Y \cup Z, *\rangle : f \equiv 0 \text{ on } A\}$$

which is a T_2^*-ideal of $F\langle Y \cup Z, *\rangle$, i.e., an ideal invariant under all graded endomorphisms of the free superalgebra commuting with the superinvolution $*$.

It is well known that in characteristic zero every $*$-identity is equivalent to a system of multilinear $*$-identities. We denote by

$$P_n^* = span_F\{w_{\sigma(1)} \dots w_{\sigma(n)} \mid \sigma \in S_n,\ w_i \in \{y_i^+, y_i^-, z_i^+, z_i^-\},\ i = 1, \dots, n\}$$

the space of all multilinear $*$-polynomials of degree n in y_1^+, y_1^-, z_1^+, z_1^-, $\dots$, y_n^+, y_n^-, z_n^+, z_n^-. Then, the study of $Id_2^*(A)$ is equivalent to the study of $P_n^* \cap Id_2^*(A)$, $\forall\, n \geq 1$.

Now, we consider the group

$$\mathbb{H}_n = (\mathbb{Z}_2 \times \mathbb{Z}_2) \sim S_n = \{((g_1, h_1), \dots, (g_n, h_n);\ \sigma) : (g_i, h_i) \in (\mathbb{Z}_2 \times \mathbb{Z}_2),\ \sigma \in S_n\},$$

with multiplication given by

$$((g_1, h_1), \dots, (g_n, h_n);\ \sigma)((a_1, b_1), \dots, (a_n, b_n);\ \tau) = ((\bar{g}_1, \bar{h}_1), \dots, (\bar{g}_n, \bar{h}_n);\ \sigma\tau),$$

where $\bar{g}_i = g_i a_{\sigma^{-1}(i)}$ and $\bar{h}_i = h_i b_{\sigma^{-1}(i)}$, for all $1 \leq i \leq n$.

If we write $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, *\} \times \{1, \zeta\} = \{1, *, \zeta, *\zeta\}$, we get $\mathbb{H}_n$ acting on the left on P_n^* by setting, for any $h = (a_1, \dots, a_n; \sigma) \in \mathbb{H}_n$:

$$hy_i^+ = y_{\sigma(i)}^+, \qquad hy_i^- = \begin{cases} y_{\sigma(i)}^-, & \text{if } a_{\sigma(i)} \in \{1, \zeta\} \\ -y_{\sigma(i)}^-, & \text{if } a_{\sigma(i)} \in \{*, *\zeta\} \end{cases},$$

$$hz_i^+ = \begin{cases} z_{\sigma(i)}^+, & \text{if } a_{\sigma(i)} \in \{1, *\} \\ -z_{\sigma(i)}^+, & \text{if } a_{\sigma(i)} \in \{\zeta, *\zeta\} \end{cases}, \qquad hz_i^- = \begin{cases} z_{\sigma(i)}^-, & \text{if } a_{\sigma(i)} \in \{1, *\zeta\} \\ -z_{\sigma(i)}^-, & \text{if } a_{\sigma(i)} \in \{*, \zeta\} \end{cases},$$

for any $i = 1, \dots, n$. Since $P_n^* \cap Id_2^*(A)$ is invariant under this action,

$$P_n^*(A) = \frac{P_n^*}{P_n^* \cap Id_2^*(A)}$$

has a structure of $\mathbb{H}_n$-module, for all $n \geq 1$. Its character, $\chi_n^*(A)$, is called the *nth $*$-cocharacter of A* and the sequence $\{\chi_n^*(A)\}_{n\geq 1}$ is the *$*$-cocharacter sequence of A*.

Let $n \geq 1$ and write $n = n_1 + n_2 + n_3 + n_4$. We define $\langle n \rangle = (n_1, n_2, n_3, n_4)$, a composition of n, as a sum of four non-negative integers. We say that $\langle \lambda \rangle$ is a multipartition of $n = n_1 + n_2 + n_3 + n_4$, and we write $\langle \lambda \rangle \vdash \langle n \rangle$ (or $\langle \lambda \rangle \vdash n$), if $\langle \lambda \rangle = (\lambda(1), \lambda(2), \lambda(3), \lambda(4))$, with $\lambda(i) \vdash n_i$, $i = 1, \ldots, 4$.

Since char$F = 0$, there is a one-to-one correspondence between the irreducible $\mathbb{H}_n$-characters and the multipartitions $\langle \lambda \rangle \vdash n$. More precisely,

$$\chi_n^*(A) = \sum_{\langle \lambda \rangle \vdash n} m_{\langle \lambda \rangle} \chi_{\langle \lambda \rangle}, \tag{3.1}$$

where $\chi_{\langle \lambda \rangle}$ is the irreducible $\mathbb{H}_n$-character associated to the multipartition $\langle \lambda \rangle$ with corresponding multiplicity $m_{\langle \lambda \rangle} \geq 0$.

For $\langle n \rangle = (n_1, n_2, n_3, n_4)$ fixed, let $P_{n_1,n_2,n_3,n_4} \subseteq P_n^*$ be the vector space of the multilinear $*$-polynomials in which the first n_1 variables are symmetric of degree zero, the next n_2 variables are skew of degree zero, the next n_3 variables are symmetric of degree one and the last n_4 variables are skew of degree one. The group $S_{n_1} \times S_{n_2} \times S_{n_3} \times S_{n_4}$ acts on the left on the vector space P_{n_1,n_2,n_3,n_4} by permuting the variables of the same homogeneous degree which are all symmetric or all skew at the same time. So S_{n_1} permutes the even symmetric variables, S_{n_2} permutes the even skew variables, S_{n_3} permutes the odd symmetric variables and S_{n_4} permutes the odd skew variables. In this way, P_{n_1,n_2,n_3,n_4} becomes an $S_{n_1} \times S_{n_2} \times S_{n_3} \times S_{n_4}$-module. Since $P_{n_1,n_2,n_3,n_4} \cap Id^*(A)$ is invariant under this action, we get that

$$P_{n_1,n_2,n_3,n_4}(A) = \frac{P_{n_1,n_2,n_3,n_4}}{P_{n_1,n_2,n_3,n_4} \cap Id_2^*(A)}$$

has a induced structure of $S_{n_1} \times S_{n_2} \times S_{n_3} \times S_{n_4}$-module. We denote by $\chi_{n_1,n_2,n_3,n_4}(A)$ its character, which is called the *(n_1, n_2, n_3, n_4)-th cocharacter* of A.

Since char$F = 0$, then by complete reducibility, we write it as a sum of irreducible characters:

$$\chi_{n_1,n_2,n_3,n_4}(A) = \sum_{\langle \lambda \rangle \vdash n} \bar{m}_{\langle \lambda \rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}, \tag{3.2}$$

where $\bar{m}_{\langle \lambda \rangle} \geq 0$ is the multiplicity of $\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}$ in $\chi_{n_1,n_2,n_3,n_4}(A)$.

By a generalization of [11, Theorem 1.3], we obtain the following.

Theorem 3.1.1. *In the decompositions given in* (3.1) *and* (3.2), $m_{\langle \lambda \rangle} = \bar{m}_{\langle \lambda \rangle}$, *for all* $\langle \lambda \rangle \vdash n$.

In particular, if A is a finite dimensional algebra with dim$A_0^+ = d_1$, dim$A_0^- = d_2$, dim$A_1^+ = d_3$, dim$A_1^- = d_4$,

$$\chi_n^*(A) = \sum_{\substack{\langle \lambda \rangle \vdash n, \\ h(\lambda(1)) \leq d_1,\ h(\lambda(2)) \leq d_2, \\ h(\lambda(3)) \leq d_3,\ h(\lambda(4)) \leq d_4}} m_{\langle \lambda \rangle} \chi_{\langle \lambda \rangle}, \tag{3.3}$$

where, for $i = 1, \ldots, 4$, $h(\lambda(i))$ denotes the height of the partition $\lambda(i)$ (see [11, Lemma 1.2]).

Now, for $m \geq 1$, let $F_m = F_m\langle Y \cup Z, * \rangle$ be the space of $*$-polynomials in the variables $y_1^+, \ldots, y_m^+, y_1^-, \ldots, y_m^-, z_1^+, \ldots, z_m^+, z_1^-, \ldots, z_m^-$. Let $V_1 = span_F\{y_1^+, \ldots, y_m^+\}$, $V_2 = span_F\{y_1^-, \ldots, y_m^-\}$, $V_3 = span_F\{z_1^+, \ldots, z_m^+\}$ and $V_4 = span_F\{z_1^-, \ldots, z_m^-\}$. Then the group $GL(V_1) \times GL(V_2) \times GL(V_3) \times GL(V_4) \cong GL_m^4 = GL_m \times GL_m \times GL_m \times$

GL_m acts naturally on the left on $V_1 \oplus V_2 \oplus V_3 \oplus V_4$ and this action can be diagonally extended to an action on F_m. Here $GL(V_i)$ is the group of all the automorphisms of the vector space V_i and GL_m denotes the general linear group of degree m.

Consider F_m^n the subspace of all homogeneous $*$-polynomials of F_m of degree $n \geq m$ which is a GL_m^4-submodule of F_m. Since F_m^n is a GL_m^4-module and $F_m^n \cap Id_2^*(A)$ is invariant under this action,

$$F_m^n(A) = \frac{F_m^n}{F_m^n \cap Id_2^*(A)}$$

has a structure of GL_m^4-module and we denote by $\psi_n^*(A)$ its character.

There exists a one-to-one correspondence between the irreducible GL_m^4-characters and the multipartitions $\langle \lambda \rangle = (\lambda(1), \dots, \lambda(4))$ of n, where the $\lambda(i)$'s are partitions with at most m parts, $i = 1, \dots, 4$ (see [10, Theorem 12.4.4]).

Hence denoted by $\psi_{\langle \lambda \rangle}$ the irreducible GL_m^4-character corresponding to the multipartition $\langle \lambda \rangle$, we have:

$$\psi_n^*(A) = \sum_{\substack{\langle \lambda \rangle \vdash n, \\ h(\langle \lambda \rangle) \leq m}} \bar{\bar{m}}_{\langle \lambda \rangle} \psi_{\langle \lambda \rangle}, \tag{3.4}$$

where $\bar{\bar{m}}_{\langle \lambda \rangle} \geq 0$ and $h(\langle \lambda \rangle) = max\{h(\lambda(i)),\ i = 1, \dots, 4\}$.

For an extension of the result in the involution case (see [13, Theorem 3]), we get the following.

Theorem 3.1.2. *In the decompositions given in* (3.1) *and* (3.4), $m_{\langle \lambda \rangle} = \bar{\bar{m}}_{\langle \lambda \rangle}$, *for all* $\langle \lambda \rangle \vdash n$ *such that* $h(\langle \lambda \rangle) \leq m$.

We recall that an irreducible GL_m^4-submodule $W^{\langle \lambda \rangle}$ of $F_m^n(A)$ is generated by a non-zero $*$-polynomial $f_{\langle \lambda \rangle}$ called *highest weight vector* associated to the multipartition $\langle \lambda \rangle$ (see [10, Theorem 12.4.12]).

A multitableau $T_{\langle \lambda \rangle} = (T_{\lambda(1)}, T_{\lambda(2)}, T_{\lambda(3)}, T_{\lambda(4)})$ is a 4-tuple of Young tableaux $T_{\lambda(i)}$, $1 \leq i \leq 4$. The multitableau $\bar{T}_{\langle \lambda \rangle} = (\bar{T}_{\lambda(1)}, \dots, \bar{T}_{\lambda(4)})$ such that $1, \dots, n$ are inserted, in this order, from top to bottom, from left to right, column by column, from the tableau $\bar{T}_{\lambda(1)}$ to the tableau $\bar{T}_{\lambda(4)}$ is called initial multitableau of shape $\langle \lambda \rangle$. The initial multitableau is a standard multitableau, that is, each $\bar{T}_{\lambda(i)}$, $i = 1, \dots, 4$ is a standard Young tableau. The highest weight vector associated is called initial highest weight vector and it is given by

$$f_{\bar{T}_{\langle \lambda \rangle}} = \prod_{i=1}^{\lambda(1)_1} St_{h_i(\lambda(1))}(y_1^+, \dots, y_{h_i(\lambda(1))}^+) \prod_{i=1}^{\lambda(2)_1} St_{h_i(\lambda(2))}(y_1^-, \dots, y_{h_i(\lambda(2))}^-) \tag{3.5}$$

$$\prod_{i=1}^{\lambda(3)_1} St_{h_i(\lambda(3))}(z_1^+, \dots, z_{h_i(\lambda(3))}^+) \prod_{i=1}^{\lambda(4)_1} St_{h_i(\lambda(4))}(z_1^-, \dots, z_{h_i(\lambda(4))}^-),$$

where $h_i(\lambda(j))$ is the height of the ith column of the Young diagram corresponding to the partition $\lambda(j)$, $\lambda(j)_1$ is the first element of the partition $\lambda(j)$, for all $j = 1, \dots, 4$, and $St_r(x_1, \dots, x_r) = \sum_{\theta \in S_r} sgn(\theta) x_{\theta(1)} \dots x_{\theta(r)}$ is the *standard polynomial of degree* r.

For a fixed multitableau $T_{\langle \lambda \rangle}$ we denote by $f_{T_{\langle \lambda \rangle}} = f_{\bar{T}_{\langle \lambda \rangle}} \sigma^{-1}$ the highest weight vector associated to $T_{\langle \lambda \rangle}$, where σ is the only element of S_n transforming $\bar{T}_{\langle \lambda \rangle}$ in $T_{\langle \lambda \rangle}$ and S_n acts on the right on F_m^n by permuting places in which the variables occur.

From [37, Proposition 15], we have the following

Proposition 3.1.1. *For all* $\langle\lambda\rangle \vdash n$, $f_{\langle\lambda\rangle}$ *can be expressed uniquely as a linear combination of vectors* $f_{T_{\langle\lambda\rangle}}$, *where* $T_{\langle\lambda\rangle}$ *is a standard multitableau.*

Theorem 3.1.3. [10, Theorem 12.4.4] *In the decomposition* (3.4), $\bar{\bar{m}}_{\langle\lambda\rangle} \neq 0$ *if and only if there exists a multitableau* $T_{\langle\lambda\rangle}$ *such that* $f_{T_{\langle\lambda\rangle}} \notin Id_2^*(A)$. *Moreover,* $\bar{\bar{m}}_{\langle\lambda\rangle}$ *is equal to the maximal number of highest weight vectors* $f_{T_{\langle\lambda\rangle}} \notin Id_2^*(A)$ *which are linearly independent in* $F_m^n(A)$.

3.1.1 Algebra of 3 × 3 matrices with orthosymplectic superinvolution

In this subsection we introduce the particular algebra which is the object of our studies.

Let $M_n(F)$ be the algebra of $n \times n$ matrices over a field F of characteristic zero. It is well known that, up to isomorphism, a $\mathbb{Z}_2$-grading on $M_n(F)$ is given by

$$M_{k,h}(F) := \left\{ \begin{pmatrix} X & 0 \\ 0 & T \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix} \right\},$$

where $n = k + h$ and X, Y, Z, T are $k \times k$, $k \times h$, $h \times k$ and $h \times h$ matrices, respectively.

In case $h = 2l$ it is possible to define a superinvolution osp, called orthosymplectic superinvolution, as follows:

$$\begin{pmatrix} X & Y \\ Z & T \end{pmatrix}^{osp} = \begin{pmatrix} I_k & 0 \\ 0 & Q \end{pmatrix}^{-1} \begin{pmatrix} X & -Y \\ Z & T \end{pmatrix}^{t} \begin{pmatrix} I_k & 0 \\ 0 & Q \end{pmatrix} = \begin{pmatrix} X^t & Z^tQ \\ QY^t & -QT^tQ \end{pmatrix},$$

where I_k is the $k \times k$ identity matrix, $Q = \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix}$ and t is the usual transpose. We consider the particular case $k = 1$ and $l = 1$:

$$M_{1,2}(F) = \left\{ \left(\begin{array}{c|cc} a & b & c \\ \hline d & e & f \\ g & h & i \end{array} \right) | a,b,c,d,e,f,g,h,i \in F \right\}$$

and

$$\left(\begin{array}{c|cc} a & b & c \\ \hline d & e & f \\ g & h & i \end{array} \right)^{osp} = \left(\begin{array}{c|cc} a & -g & d \\ \hline c & i & -f \\ -b & -h & e \end{array} \right).$$

In this case, if we denote by $e_{i,j}$ the usually elementary matrix,

$$(M_{1,2}(F))_0^+ = span_F\{e_{11}, e_{22} + e_{33}\}, \quad (M_{1,2}(F))_0^- = span_F\{e_{22} - e_{33}, e_{23}, e_{32}\},$$

$$(M_{1,2}(F))_1^+ = span_F\{e_{12} - e_{31}, e_{13} + e_{21}\}, \quad (M_{1,2}(F))_1^- = span_F\{e_{12} + e_{31}, e_{13} - e_{21}\}.$$

Since

$$\dim(M_{1,2}(F))_0^+ = 2, \qquad \dim(M_{1,2}(F))_1^+ = 2,$$

$$\dim(M_{1,2}(F))_0^- = 3, \qquad \dim(M_{1,2}(F))_1^- = 2,$$

then, by (3.3) we obtain

$$\chi_n^*(M_{1,2}(F)) = \sum_{\substack{\langle\lambda\rangle \vdash n, \\ h(\lambda(1)) \leq 2,\, h(\lambda(2)) \leq 3, \\ h(\lambda(3)) \leq 2,\, h(\lambda(4)) \leq 2}} m_{\langle\lambda\rangle} \chi_{\langle\lambda\rangle}. \tag{3.6}$$

3.2 Classifying the $*$-identities of degree ≤ 3

The first result concerns the $*$-identities of $M_{1,2}(F)$ of degree ≤ 3. In this section we show how we determine them all through the representation theory.

We recall that, if we consider an algebra A with involution ϕ, (A, ϕ), we denote by $F\langle X, \phi\rangle$ the free associative algebra with involution generated by $X = \{x_1, x_2, \dots\}$ over F. So, an element $f(x_1, x_1^{\phi}, \dots, x_n, x_n^{\phi}) \in F\langle X, \phi\rangle$ is a ϕ-polynomial identity for A if $f(a_1, a_1^{\phi}, \dots, a_n, a_n^{\phi}) = 0$ for all substitutions $a_1, \dots, a_n \in A$. Then, we define also as $Id(A, \phi)$, the set of all ϕ-polynomial identities of A, that is a T^{ϕ}-ideal of $F\langle X, \phi\rangle$, i.e., an ideal invariant under all endomorphisms of $F\langle X, \phi\rangle$ commuting with ϕ.

Notice that $((M_{1,2}(F))_0, osp) \cong (F \oplus M_2(F), \phi_s)$, as algebras with involution, where ϕ_s is defined as $\phi_s(b + B) = b + B^s$ with s the symplectic involution on $M_2(F)$, since $((M_{1,2}(F))_0, osp)$ is in particular an algebra with involution. So we obtain that

Remark 3.2.1. $Id((M_{1,2}(F))_0, osp) = Id(M_2(F), s)$.

Given $*$-polynomials $f_1, \dots, f_l \in F\langle Y \cup Z, *\rangle$, we denote by $\langle f_1, \dots, f_t\rangle_{T_2^*}$ the T_2^*-ideal of $F\langle Y \cup Z, *\rangle$ generated by $f_1, \dots, f_l$. We recall that a $*$-identity g is a consequence of the $*$-polynomial identities f_i, with $i = 1, \dots, l$, if $g \in \langle f_1, \dots, f_l\rangle_{T_2^*}$.

By [34, Theorem 1], we know that $Id((M_2(F), s))$ is generated, as a T^{ϕ}-ideal, by $[y, x] := yx - xy$, where y denotes a symmetric variable and x denotes any variable in X. Hence we get.

Remark 3.2.2. Every $*$-identity on variables of degree zero follows from the $*$-identity $[y^+, y]$, where y denotes a variable of homogeneous degree zero.

Now, we prove the following.

Remark 3.2.3. If f is a $*$-identity of $M_{1,2}(F)$ of degree 2, then $f = \alpha[y_1^+, y_2^+]$ or $f = \alpha[y^+, y^-]$, with $\alpha \in F$.

Proof. Since F is an infinite field, every T_2^*-ideal is generated by its multihomogeneous $*$-polynomials (see [20, Theorem 1.3.2]). Hence, we may assume that f is a multihomogeneous $*$-polynomial of degree 2. If f is a $*$-identity on variables of degree zero, the result follows from Remark 3.2.2. Now, assume that at least one variable of degree one appears in f. It is easy to see, by making suitable evaluations of the variables of all possible $*$-polynomials, that f must be the zero $*$-polynomial. □

We will establish a relation between $*$-identities of $M_{1,2}(F)$ in terms of variables in Z^+ and in Z^-.

We define the algebra isomorphism $\tilde{\varphi} : F\langle Y \cup Z, *\rangle \longrightarrow F\langle Y \cup Z, *\rangle$ given by

$$\tilde{\varphi}(y^+) = y^+, \quad \tilde{\varphi}(y^-) = y^-, \quad \tilde{\varphi}(z^+) = z^-, \quad \tilde{\varphi}(z^-) = z^+.$$

We let $\varphi : M_{1,2}(F) \longrightarrow M_{1,2}(F)$ be the linear map defined as the extension of

$$e_{11} \mapsto e_{11},\ e_{12} \mapsto e_{12},\ e_{21} \mapsto -e_{21},\ e_{22} \mapsto e_{22},\ e_{13} \mapsto e_{13},\ e_{31} \mapsto -e_{31},\ e_{23} \mapsto e_{23},\ e_{32} \mapsto e_{32},\ e_{33} \mapsto e_{33}.$$

Remark 3.2.4. We observe that φ is a linear isomorphism such that

$$\varphi[(M_{1,2}(F))_0^+] = (M_{1,2}(F))_0^+; \quad \varphi[(M_{1,2}(F))_0^-] = (M_{1,2}(F))_0^-;$$

$$\varphi[(M_{1,2}(F))_1^+] = (M_{1,2}(F))_1^-; \quad \varphi[(M_{1,2}(F))_1^-] = (M_{1,2}(F))_1^+.$$

Now, the following result holds.

Theorem 3.2.1. *Let $f \in F\langle Y \cup Z, *\rangle$. Then $f \in Id_2^*(M_{1,2}(F))$ if and only if $\tilde{\varphi}(f) \in Id_2^*(M_{1,2}(F))$.*

Proof. It can be proved as an extension of [9, Corollary 3.6]. □

Remark 3.2.5. From now on, we write the symbols $\tilde{}$ and $\tilde{\tilde{}}$ in order to indicate alternation on a specific set of variables.

For example, if we write $\tilde{x}_1\tilde{\tilde{x}}_1 x_4 \tilde{x}_2 \tilde{x}_3 \tilde{\tilde{x}}_2$, its corresponding $*$-polynomial is the following:

$$\sum_{\sigma \in S_3, \tau \in S_2} sgn(\sigma) sgn(\tau) x_{\sigma(1)} x_{\tau(1)} x_4 x_{\sigma(2)} x_{\sigma(3)} x_{\tau(2)}.$$

For any multipartition $(n_1, n_2, n_3, n_4) \vdash 3$, we define $W_{(n_1,n_2,n_3,n_4)}$ as the subspace of F_3^3, formed by multihomogeneous $*$-polynomials of total degree n_1 in the variables y_1^+, y_2^+, y_3^+, of total degree n_2 in the variables y_1^-, y_2^-, y_3^-, of total degree n_3 in the variables z_1^+, z_2^+, z_3^+ and of total degree n_4 in the variables z_1^-, z_2^-, z_3^-. We can act on $Id_2^*(M_{1,2}(F)) \cap W_{(n_1,n_2,n_3,n_4)}$ with $GL_{n_1} \times GL_{n_2} \times GL_{n_3} \times GL_{n_4}$ and let $Id_2^*(M_{1,2}(F)) \cap W_{(n_1,n_2,n_3,n_4)} \cong \bigoplus_{\langle\lambda\rangle \vdash \langle 3 \rangle} m_{\langle\lambda\rangle} W^{\langle\lambda\rangle}$ be the decomposition into irreducible submodules, where $W^{\langle\lambda\rangle}$ is the irreducible submodule corresponding to $\langle\lambda\rangle$ generated by the highest weight vector $f_{\langle\lambda\rangle}$.

Next we shall determine the exact value of $m_{\langle\lambda\rangle}$ as follows. According to Proposition 3.1.1, any highest highest weight vector $f_{\langle\lambda\rangle}$ can be written uniquely as a linear combination of highest weight vectors $f_{T_{\langle\lambda\rangle}}$ corresponding to standard multitableaux. So, we write this linear combination explicitly and we evaluate it into 3×3 generic matrices with superinvolution by imposing that it must be a $*$-polynomial identity of the algebra $M_{1,2}(F)$. We obtain a system where the coefficients of the linear combination are the unknowns. It can be completely solved by making evaluations of the generic matrices with superinvolution in $M_{1,2}(F)$. Then we prove that we obtain $*$-identities of the algebra. If we obtain different highest weight vectors corresponding to the same multipartition, we check that they are linearly independent. The maximal number of linearly independent highest weight vectors will be the multiplicity of the corresponding $GL_{n_1} \times GL_{n_2} \times GL_{n_3} \times GL_{n_4}$-module.

Let us explain with an example. Consider the composition $(2,1,0,0) \vdash 3$. We have the multipartitions $\langle\lambda_1\rangle = ((2),(1),\emptyset,\emptyset)$ and $\langle\lambda_2\rangle = ((1,1),(1),\emptyset,\emptyset)$.

For $\langle\lambda_1\rangle = ((2),(1),\emptyset,\emptyset)$, we have the following standard multitableaux:

$$\left(\boxed{1}\boxed{2}, \boxed{3}, \emptyset, \emptyset\right), \left(\boxed{1}\boxed{3}, \boxed{2}, \emptyset, \emptyset\right) \text{ and } \left(\boxed{2}\boxed{3}, \boxed{1}, \emptyset, \emptyset\right),$$

with highest weight vectors given by:

$$f_{\bar{T}_{\langle\lambda_1\rangle}} = (y_1^+)^2 y_1^-, \quad f_{\bar{T}_{\langle\lambda_1\rangle}}(23)^{-1} = y_1^+ y_1^- y_1^+ \quad \text{and} \quad f_{\bar{T}_{\langle\lambda_1\rangle}}(123)^{-1} = y_1^-(y_1^+)^2.$$

For $\langle\lambda_2\rangle = ((1,1),(1),\emptyset,\emptyset)$, we have the following standard multitableaux:

$$\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}, \boxed{3}, \emptyset, \emptyset\right), \left(\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}, \boxed{2}, \emptyset, \emptyset\right) \text{ and } \left(\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}, \boxed{1}, \emptyset, \emptyset\right),$$

with highest weight vectors given by:

$$g_{\bar{T}_{\langle\lambda_2\rangle}} = \tilde{y}_1^+ \tilde{y}_2^+ y_1^-, \quad g_{\bar{T}_{\langle\lambda_2\rangle}}(23)^{-1} = \tilde{y}_1^+ y_1^- \tilde{y}_2^+ \quad \text{and} \quad g_{\bar{T}_{\langle\lambda_2\rangle}}(123)^{-1} = y_1^- \tilde{y}_1^+ \tilde{y}_2^+.$$

We consider the decomposition

$$Id_2^*(M_{1,2}(F)) \cap W_{(2,1,0,0)} \cong \beta_1 W^{((2),(1),\emptyset,\emptyset)} \oplus \beta_2 W^{((1,1),(1),\emptyset,\emptyset)}$$

and now we compute the multiplicities β_1 and β_2.

The highest weight vector which generates $W^{((2),(1),\emptyset,\emptyset)}$ is $f_{\langle\lambda_1\rangle} = \alpha_1(y_1^+)^2 y_1^- + \alpha_2 y_1^+ y_1^- y_1^+ + \alpha_3 y_1^-(y_1^+)^2$ that belongs to $Id_2^*(M_{1,2}(F))$, for some $\alpha_1, \alpha_2, \alpha_3 \in F$. Considering the substitutions $y_1^+ = e_{22} + e_{33}$ and $y_1^- = e_{23}$, we get $(\alpha_1 + \alpha_2 + \alpha_3)e_{23} = 0$, which implies that $\alpha_3 = -\alpha_1 - \alpha_2$. It follows that $f_{\langle\lambda_1\rangle} = \alpha_1[(y_1^+)^2, y_1^-] + \alpha_2[y_1^+, y_1^- y_1^+]$. We notice that $[(y_1^+)^2, y_1^-]$ and $[y_1^+, y_1^- y_1^+]$ are $*$-identities and that they are linearly independent, since, if we write $\alpha_1[(y_1^+)^2, y_1^-] + \alpha_2[y_1^+, y_1^- y_1^+] = 0$, it follows that $\alpha_1(y_1^+)^2 y_1^- + (-\alpha_1 - \alpha_2)(y_1^-(y_1^+)^2) + \alpha_2 y_1^+ y_1^- y_1^+ = 0$, which implies that $\alpha_1 = \alpha_2 = 0$. So we get $\beta_1 = 2$.

The highest weight vector which generates $W^{((1,1),(1),\emptyset,\emptyset)}$ is $f_{\langle\lambda_2\rangle} = \alpha_4 \tilde{y}_1^+ \tilde{y}_2^+ y_1^- + \alpha_5 \tilde{y}_1^+ y_1^- \tilde{y}_2^+ + \alpha_6 y_1^- \tilde{y}_1^+ \tilde{y}_2^+ \in Id_2^*(M_{1,2}(F))$, for some $\alpha_4, \alpha_5, \alpha_6 \in F$. We observe that $\tilde{y}_1^+ \tilde{y}_2^+ y_1^- = [y_1^+, y_2^+] y_1^- \equiv 0$ and $y_1^- \tilde{y}_1^+ \tilde{y}_2^+ = y_1^-[y_1^+, y_2^+] \equiv 0$, because $[y_1^+, y_2^+] \equiv 0$. It is clear that also $\tilde{y}_1^+ y_1^- \tilde{y}_2^+ \equiv 0$. Analogously to the previous case, these three identities are also linearly independent, so $\beta_2 = 3$.
In conclusion:

$$Id_2^*(M_{1,2}(F)) \cap W_{(2,1,0,0)} \cong 2W^{((2),(1),\emptyset,\emptyset)} \oplus 3W^{((1,1),(1),\emptyset,\emptyset)}$$

with

$$f'_{\langle\lambda_1\rangle} = [(y_1^+)^2, y_1^-], \quad f''_{\langle\lambda_1\rangle} = [y_1^+, y_1^- y_1^+]$$

and

$$f'_{\langle\lambda_2\rangle} = [y_1^+, y_2^+] y_1^-, \quad f''_{\langle\lambda_2\rangle} = \tilde{y}_1^+ y_1^- \tilde{y}_2^+, \quad f'''_{\langle\lambda_2\rangle} = y_1^-[y_1^+, y_2^+].$$

Proceeding in the same way for the remaining compositions of 3 and their corresponding multipartitions, we get the following.

Theorem 3.2.2. *The following decompositions are valid:*

(1) $Id_2^*(M_{1,2}(F)) \cap W_{(3,0,0,0)} \cong 2W^{((2,1),\emptyset,\emptyset,\emptyset)} \oplus W^{((1^3),\emptyset,\emptyset,\emptyset)}$ *with highest weight vectors*

$$f'_{((2,1),\emptyset,\emptyset,\emptyset)} = [y_1^+, y_2^+] y_1^+, \quad f''_{((2,1),\emptyset,\emptyset,\emptyset)} = \tilde{y}_1^+ y_1^+ \tilde{y}_2^+ = [(y_1^+)^2, y_2^+],$$
$$f_{((1^3),\emptyset,\emptyset,\emptyset)} = \tilde{y}_1^+ \tilde{y}_2^+ \tilde{y}_3^+ = St_3(y_1^+, y_2^+, y_3^+);$$

(2) $Id_2^*(M_{1,2}(F)) \cap W_{(0,3,0,0)} \cong W^{(\emptyset,(2,1),\emptyset,\emptyset)}$ *with highest weight vectors*

$$f_{(\emptyset,(2,1),\emptyset,\emptyset)} = \tilde{y}_1^- y_1^- \tilde{y}_2^- = [(y_1^-)^2, y_2^-];$$

(3) $Id_2^*(M_{1,2}(F)) \cap W_{(2,1,0,0)} \cong 2W^{((2),(1),\emptyset,\emptyset)} \oplus 3W^{((1^2),(1),\emptyset,\emptyset)}$ *with highest weight vectors*

$$f'_{((2),(1),\emptyset,\emptyset)} = [(y^+)^2, y^-], \quad f''_{((2),(1),\emptyset,\emptyset)} = [y^+, y^- y^+],$$
$$f'_{((1^2),(1),\emptyset,\emptyset)} = [y_1^+, y_2^+] y^-, \quad f''_{((1^2),(1),\emptyset,\emptyset)} = \tilde{y}_1^+ y^- \tilde{y}_2^+,$$
$$f'''_{((1^2),(1),\emptyset,\emptyset)} = y^-[y_1^+, y_2^+];$$

(4) $Id_2^*(M_{1,2}(F)) \cap W_{(1,2,0,0)} \cong 2W^{((1),(2),\emptyset,\emptyset)} \oplus 2W^{((1),(1^2),\emptyset,\emptyset)}$ *with highest weight vectors*

$$f'_{((1),(2),\emptyset,\emptyset)} = [y^+,(y^-)^2], \quad f''_{((1),(2),\emptyset,\emptyset)} = y^-[y^+,y^-],$$
$$f'_{((1),(1^2),\emptyset,\emptyset)} = [y^+,\tilde{y}_1^-\tilde{y}_2^-], \quad f''_{((1),(1^2),\emptyset,\emptyset)} = \tilde{y}_1^-[y^+,\tilde{y}_2^-];$$

(5) $Id_2^*(M_{1,2}(F)) \cap W_{(2,0,1,0)} \cong 2W^{((1^2),\emptyset,(1),\emptyset)}$ *with highest weight vectors*

$$f'_{((1^2),\emptyset,(1),\emptyset)} = [y_1^+,y_2^+]z^+, \quad f''_{((1^2),\emptyset,(1),\emptyset)} = z^+[y_1^+,y_2^+];$$

(6) $Id_2^*(M_{1,2}(F)) \cap W_{(2,0,0,1)} \cong 2W^{((1^2),\emptyset,\emptyset,(1))}$ *with highest weight vectors*

$$f'_{((1^2),\emptyset,\emptyset,(1))} = [y_1^+,y_2^+]z^-, \quad f''_{((1^2),\emptyset,\emptyset,(1))} = z^-[y_1^+,y_2^+];$$

(7) $Id_2^*(M_{1,2}(F)) \cap W_{(1,1,1,0)} \cong 2W^{((1),(1),(1),\emptyset)}$ *with highest weight vectors*

$$f'_{((1),(1),(1),\emptyset)} = [y^+,y^-]z^+, \quad f''_{((1),(1),(1),\emptyset)} = z^+[y^+,y^-];$$

(8) $Id_2^*(M_{1,2}(F)) \cap W_{(1,1,0,1)} \cong 2W^{((1),(1),\emptyset,(1))}$ *with highest weight vectors*

$$f'_{((1),(1),\emptyset,(1))} = [y^+,y^-]z^-, \quad f''_{((1),(1),\emptyset,(1))} = z^-[y^+,y^-];$$

(9) $Id_2^*(M_{1,2}(F)) \cap W_{(1,0,2,0)} \cong W^{((1),\emptyset,(2),\emptyset)} \oplus W^{((1),\emptyset,(1^2),\emptyset)}$ *with highest weight vectors*

$$f_{((1),\emptyset,(2),\emptyset)} = [y^+,(z^+)^2],$$
$$f_{((1),\emptyset,(1^2),\emptyset)} = [y^+,\tilde{z}_1^+\tilde{z}_2^+];$$

(10) $Id_2^*(M_{1,2}(F)) \cap W_{(1,0,0,2)} \cong W^{((1),\emptyset,\emptyset,(2))} \oplus W^{((1),\emptyset,\emptyset,(1^2))}$ *with highest weight vectors*

$$f_{((1),\emptyset,\emptyset,(2))} = [y^+,(z^-)^2],$$
$$f_{((1),\emptyset,\emptyset,(1^2))} = [y^+,\tilde{z}_1^-\tilde{z}_2^-];$$

(11) $Id_2^*(M_{1,2}(F)) \cap W_{(1,0,1,1)} \cong 2W^{((1),\emptyset,(1),(1))}$ *with highest weight vectors*

$$f'_{((1),\emptyset,(1),(1))} = [y^+,z^+z^-], \quad f''_{((1),\emptyset,(1),(1))} = [y^+,z^-z^+];$$

(12) $Id_2^*(M_{1,2}(F)) \cap W_{(0,0,3,0)} \cong W^{(\emptyset,\emptyset,(3),\emptyset)} \oplus W^{(\emptyset,\emptyset,(1^3),\emptyset)}$ *with highest weight vectors*

$$f_{(\emptyset,\emptyset,(3),\emptyset)} = (z^+)^3,$$
$$f_{(\emptyset,\emptyset,(1^3),\emptyset)} = \tilde{z}_1^+\tilde{z}_2^+\tilde{z}_3^+ = St_3(z_1^+, z_2^+, z_3^+);$$

(13) $Id_2^*(M_{1,2}(F)) \cap W_{(0,0,0,3)} \cong W^{(\emptyset,\emptyset,\emptyset,(3))} \oplus W^{(\emptyset,\emptyset,\emptyset,(1^3))}$ *with highest weight vectors*

$$f_{(\emptyset,\emptyset,\emptyset,(3))} = (z^-)^3,$$
$$f_{(\emptyset,\emptyset,\emptyset,(1^3))} = \tilde{z}_1^-\tilde{z}_2^-\tilde{z}_3^- = St_3(z_1^-, z_2^-, z_3^-);$$

(14) $Id_2^*(M_{1,2}(F)) \cap W_{(0,2,1,0)} \cong W^{(\emptyset,(2),(1),\emptyset)} \oplus W^{(\emptyset,(1^2),(1),\emptyset)}$ *with highest weight vectors*

$$f_{(\emptyset,(2),(1),\emptyset)} = y^-z^+y^-,$$
$$f_{(\emptyset,(1^2),(1),\emptyset)} = \tilde{y}_1^-z^+\tilde{y}_2^-;$$

(15) $Id_2^*(M_{1,2}(F)) \cap W_{(0,2,0,1)} \cong W^{(\emptyset,(2),\emptyset,(1))} \oplus W^{(\emptyset,(1^2),\emptyset,(1))}$ *with highest weight vectors*

$$f_{(\emptyset,(2),\emptyset,(1))} = y^-z^-y^-,$$
$$f_{(\emptyset,(1^2),\emptyset,(1))} = \tilde{y}_1^-z^-\tilde{y}_2^-;$$

(16) $Id_2^*(M_{1,2}(F)) \cap W_{(0,1,2,0)} \cong 2W^{(\emptyset,(1),(1^2),\emptyset)}$ *with highest weight vectors*

$$f'_{(\emptyset,(1),(1^2),\emptyset)} = \tilde{z}_1^+y^-\tilde{z}_2^+, \quad f''_{(\emptyset,(1),(1^2),\emptyset)} = [y^-, \tilde{z}_1^+\tilde{z}_2^+];$$

(17) $Id_2^*(M_{1,2}(F)) \cap W_{(0,1,0,2)} \cong 2W^{(\emptyset,(1),\emptyset,(1^2))}$ *with highest weight vectors*

$$f'_{(\emptyset,(1),\emptyset,(1^2))} = \tilde{z}_1^-y^-\tilde{z}_2^-, \quad f''_{(\emptyset,(1),\emptyset,(1^2))} = [y^-, \tilde{z}_1^-\tilde{z}_2^-];$$

(18) $Id_2^*(M_{1,2}(F)) \cap W_{(0,0,2,1)} \cong W^{(\emptyset,\emptyset,(2),(1))} \oplus W^{(\emptyset,\emptyset,(1^2),(1))}$ *with highest weight vectors*

$$f_{(\emptyset,\emptyset,(2),(1))} = (z^+)^2z^- - z^+z^-z^+ + z^-(z^+)^2,$$
$$f_{(\emptyset,\emptyset,(1^2),(1))} = \tilde{z}_1^+\tilde{z}_2^+z^- + \tilde{z}_1^+z^-\tilde{z}_2^+ + z^-\tilde{z}_1^+\tilde{z}_2^+;$$

(19) $Id_2^*(M_{1,2}(F)) \cap W_{(0,0,1,2)} \cong W^{(\emptyset,\emptyset,(1),(2))} \oplus W^{(\emptyset,\emptyset,(1),(1^2))}$ *with highest weight vectors*

$$f_{(\emptyset,\emptyset,(1),(2))} = (z^-)^2 z^+ - z^- z^+ z^- + z^+ (z^-)^2,$$
$$f_{(\emptyset,\emptyset,(1),(1^2))} = \tilde{z}_1^- \tilde{z}_2^- z^+ + \tilde{z}_1^- z^+ \tilde{z}_2^- + z^+ \tilde{z}_1^- \tilde{z}_2^-;$$

(20) $Id_2^*(M_{1,2}(F)) \cap 2W_{(0,1,1,1)} \cong W^{(\emptyset,(1),(1),(1))}$ *with highest weight vectors*

$$f'_{(\emptyset,(1),(1),(1))} = z^+ y^- z^- + z^- y^- z^+, \quad f''_{(\emptyset,(1),(1),(1))} = [y^-, z^+ \circ z^-].$$

Now, we consider the following set:

$$\mathcal{I} = \{[y^+, y], \ \ y_1^- z^+ y_2^-, \ \ y_1^- z^- y_2^-, \ \ z_1^+ z_2^+ z_3^+ + z_2^+ z_3^+ z_1^+ + z_3^+ z_1^+ z_2^+, \ \ z_1^- z_2^- z_3^- + z_2^- z_3^- z_1^- + z_3^- z_1^- z_2^-, \ \ \tilde{z}_1^+ y^- \tilde{z}_2^+, \ \ \tilde{z}_1^- y^- \tilde{z}_2^-, \ \ z_1^+ z_2^+ z^- - z_2^+ z^- z_1^+ + z^- z_1^+ z_2^+, \ \ z_1^- z_2^- z^+ - z_2^- z^+ z_1^- + z^+ z_1^- z_2^-, \ \ z^+ y^- z^- + z^- y^- z^+\}.$$

It is easy to prove that each $*$-polynomial of this set is a $*$-identity of $M_{1,2}(F)$, then $\mathcal{I} \subseteq Id_2^*(M_{1,2}(F))$.

Theorem 3.2.3. *Let $f \in Id_2^*(M_{1,2}(F))$ of degree ≤ 3. Then f is a consequence of $*$-polynomials in the set $\mathcal{I}$.*

Proof. It is obvious that $M_{1,2}(F)$ does not satisfy any $*$-identity of degree 1. If f has degree 2, by Remark 3.2.3, any $*$-identity follows from $[y_1^+, y_2^+]$ and $[y^+, y^-]$ which are consequences of $[y^+, y]$ and we are done in this case. Now assume that f has degree 3 and that f is multihomogeneous, as we may. So, $f \in W_{(n_1,n_2,n_3,n_4)}$, for some composition (n_1, n_2, n_3, n_4) of 3. In order to complete the proof we shall show that any highest weight vector corresponding to the multipartition (n_1, n_2, n_3, n_4) given in Theorem 3.2.2 is a consequence of the $*$-polynomials in $\mathcal{I}$. We enumerate such highest weight vectors in this way:

(a) $[y^+, y]$,

(b) $z_1^+ z_2^+ z_3^+ + z_2^+ z_3^+ z_1^+ + z_3^+ z_1^+ z_2^+$,

(c) $z_1^- z_2^- z_3^- + z_2^- z_3^- z_1^- + z_3^- z_1^- z_2^-$,

(d) $y_1^- z^+ y_2^-$,

(e) $y_1^- z^- y_2^-$,

(f) $\tilde{z}_1^+ y^- \tilde{z}_2^+$,

(g) $\tilde{z}_1^- y^- \tilde{z}_2^-$,

(h) $z_1^+ z_2^+ z^- - z_2^+ z^- z_1^+ + z^- z_1^+ z_2^+$,

(i) $z_1^- z_2^- z^+ - z_2^- z^+ z_1^- + z^+ z_1^- z_2^-$,

(j) $z^+ y^- z^- + z^- y^- z^+$.

Clearly, if f is a $*$-identity in variables of degree zero, by Remark 3.2.2, it is a consequence of $[y^+, y]$. Hence, the highest weight vectors in (1), (2), (3) and (4) follow from (a). Also the highest weight vectors in (5), (6), (7) and (8) follow from (a). Moreover, since the product of two variables of degree one is a variable of degree zero, also the highest weight vectors in (9), (10) and (11) follow from (a). It is immediate to see that the highest weight vectors in (12), (13), (14) and (15) follow from (b), (c), (d) and (e), respectively. In (16) and (17) we find the $*$-polynomials (f) and (g) and consequences of (a). The highest weight vectors in (18) and (19) follow from (h) and (i), respectively. Finally, the $*$-polynomials in (20) follow from (j) and (a). □

3.3 On the $*$-cocharacter of $M_{1,2}(F)$

In this last section we present the results about the decomposition of the $*$-cocharacter of $M_{1,2}(F)$ with multiplicity $m_{\langle\lambda\rangle} \neq 0$. In the end, motivated by some examples, we give a conjecture regarding the complete characterization of the $*$-cocharacter of $M_{1,2}(F)$.

First we recall that

$$\chi_n^*(M_{1,2}(F)) = \sum_{\substack{\langle\lambda\rangle\vdash n,\\ h(\lambda(1))\leq 2,\ h(\lambda(2))\leq 3,\\ h(\lambda(3))\leq 2,\ h(\lambda(4))\leq 2}} m_{\langle\lambda\rangle}\chi_{\langle\lambda\rangle}, \tag{3.7}$$

where $m_{\langle\lambda\rangle} \geq 0$ is the multiplicity corresponding to the irreducible $\mathbb{H}_n$-character $\chi_{\langle\lambda\rangle}$, with $\langle\lambda\rangle = (\lambda(1), \dots, \lambda(4)) \vdash n$.

In [11, Theorem 4.1b], Drensky and Giambruno determined the decomposition of the cocharacter of the algebra $M_2(F)$ endowed with the simplectic involution. They proved the following result.

Theorem 3.3.1. *The $\mathbb{Z}_2 \sim S_n$-cocharacter of $M_2(F)$ endowed with the symplectic involution s is*

$$\chi_n(M_2(F), s) = \sum_{\substack{n=n_1+n_2,\\ \lambda\vdash n_1,\ \mu\vdash n_2}} a_{\lambda,\mu}\chi_{\lambda,\mu}, \tag{3.8}$$

where $\lambda = (\lambda_1)$, $\mu = (\mu_1, \mu_2, \mu_3)$ and $a_{\lambda,\mu} = 1$.

An immediate consequence of Theorem 3.3.1 and Remark 3.2.1 is the following.

Proposition 3.3.1. *If $\langle\lambda\rangle = (\lambda(1), \lambda(2), \emptyset, \emptyset)$ in (3.7), then $m_{\langle\lambda\rangle} \neq 0$ if and only if $h(\lambda(1)) \leq 1$.*

Now we consider only the multiplicities $m_{\langle\lambda\rangle}$ in (3.7) with $\lambda(2) = \lambda(4) = \emptyset$.

Proposition 3.3.2. *If $\langle\lambda\rangle = (\lambda(1), \emptyset, \lambda(3), \emptyset)$ in (3.7), with $\lambda(3) = (w_1 + w_2, w_2) \neq \emptyset$, then $m_{\langle\lambda\rangle} \neq 0$ if and only if $w_1 \leq 2$.*

Proof. In order to prove that $m_{\langle\lambda\rangle} \neq 0$, by Theorem 3.1.2 and Theorem 3.1.3, we just need to show that there exists a Young multitableau $T_{(\lambda(1),\emptyset,\lambda(3),\emptyset)}$ such that the corresponding highest weight vector $f_{T_{(\lambda(1),\emptyset,\lambda(3),\emptyset)}}$ is not a $*$-identity for the algebra. Consider the elements $R_1 = e_{11}$, $R_2 = e_{22} + e_{33}$, $N_1 = e_{12} - e_{31}$, $N_2 = e_{13} + e_{21}$.

First, we suppose $h(\lambda(1)) \leq 1$ and we consider the initial standard multitableau $\bar{T}_{(\lambda(1),\emptyset,\lambda(3),\emptyset)}$ and the corresponding highest weight vector

$$f_{\bar{T}_{(\lambda(1),\emptyset,\lambda(3),\emptyset)}}(y_1^+, z_1^+, z_2^+) = (y_1^+)^{\alpha_1}[z_1^+, z_2^+]^{w_2}(z_1^+)^{w_1}.$$

Then

$$f_{\bar{T}_{(\lambda(1),\emptyset,\lambda(3),\emptyset)}}(R_2, N_1, N_2) =$$
$$(e_{22} + e_{33})^{\alpha_1}(2^{w_2}e_{11} + (-1)^{w_2}e_{22} + (-1)^{w_2}e_{33})(e_{12} - e_{31})^{w_1} \neq 0.$$

Now assume that $h(\lambda(1)) = 2$ and distinguish three different cases.

- $w_1 = 0$.
  We consider the following standard multitableaux

$$T_{(\lambda(1),\emptyset,\lambda(3),\emptyset)} =$$

$$\left(\begin{array}{|c|c|c|c|c|}\hline 1 & \dots & \alpha_2 & \dots & \alpha_2+\alpha_1 \\ \hline \alpha_2+\alpha_1+2 & \dots & 2\alpha_2+\alpha_1+1 & & \\ \hline \end{array}\, , \emptyset , \begin{array}{|c|c|c|c|}\hline \alpha_2+\alpha_1+1 & 2\alpha_2+\alpha_1+3 & \dots & n-1 \\ \hline 2\alpha_2+\alpha_1+2 & 2\alpha_2+\alpha_1+4 & \dots & n \\ \hline \end{array}\, , \emptyset\right)$$

with

$$f_{T_{(\lambda(1),\emptyset,\lambda(3),\emptyset)}}(y_1^+, y_2^+, z_1^+, z_2^+) = \underbrace{\tilde{y}_1^+ \dots \tilde{\tilde{y}}_1^+}_{\alpha_2} (y_1^+)^{\alpha_1} \tilde{z}_1^+ \underbrace{\tilde{y}_2^+ \dots \tilde{\tilde{y}}_2^+}_{\alpha_2} \tilde{\tilde{z}}_2^+ [z_1^+, z_2^+]^{w_2 - 1}.$$

Then

$$f_{T_{(\lambda(1),\emptyset,\lambda(3),\emptyset)}}(R_2, R_1, N_1, N_2) = \pm\beta e_{11} \pm e_{22} \pm e_{33} \neq 0,$$

where $\beta \in \{0, 2^{w_2}\}$.

- $w_1 = 1$.
  If $\lambda(3) \vdash r$, we denote by $\bar{T}_{\lambda(3)}$ the initial standard tableau on the integers $\alpha_2 + \alpha_1 + 1, \dots, \alpha_2 + \alpha_1 + r$. Consider the following multitableau

$$T_{(\lambda(1),\emptyset,\lambda(3),\emptyset)} = \left(\begin{array}{|c|c|c|c|c|}\hline 1 & \dots & \alpha_2 & \dots & \alpha_2+\alpha_1 \\ \hline \alpha_2+\alpha_1+r+1 & \dots & n & & \\ \hline \end{array}\, , \emptyset \, , \bar{T}_{\lambda(3)} \, , \emptyset\right)$$

with corresponding highest weight vector

$$f_{T_{(\lambda(1),\emptyset,\lambda(3),\emptyset)}}(y_1^+, y_2^+, z_1^+, z_2^+) = \underbrace{\tilde{y}_1^+ \dots \tilde{\tilde{y}}_1^+}_{\alpha_2} (y_1^+)^{\alpha_1} f_{\bar{T}_{\lambda(3)}}(z_1^+, z_2^+) \underbrace{\tilde{y}_2^+ \dots \tilde{\tilde{y}}_2^+}_{\alpha_2} =$$

$$\underbrace{\tilde{y}_1^+ \dots \tilde{\tilde{y}}_1^+}_{\alpha_2} (y_1^+)^{\alpha_1} [z_1^+, z_2^+]^{w_2} z_1^+ \underbrace{\tilde{y}_2^+ \dots \tilde{\tilde{y}}_2^+}_{\alpha_2}.$$

Then

$$f_{T_{(\lambda(1),\emptyset,\lambda(3),\emptyset)}}(R_2, R_1, N_1, N_2) = \pm e_{31} \pm \beta e_{12} \neq 0,$$

where $\beta \in \{0, 2^{w_2}\}$.

- $w_1 = 2$.
  If $\lambda(3) \vdash r$, we consider the following standard multitableaux

$$T_{(\lambda(1),\emptyset,\lambda(3),\emptyset)} =$$

$$\left(\begin{array}{|c|c|c|c|c|}\hline 1 & \dots & \alpha_2 & \dots & \alpha_2+\alpha_1 \\ \hline \alpha_2+\alpha_1+r & \dots & n-1 & & \\ \hline \end{array}\, , \emptyset , \begin{array}{|c|c|c|c|c|}\hline \alpha_2+\alpha_1+1 & \dots & \alpha_2+\alpha_1+r-3 & \alpha_2+\alpha_1+r-1 & n \\ \hline \alpha_2+\alpha_1+2 & \dots & \alpha_2+\alpha_1+r-2 & & \\ \hline \end{array}\, , \emptyset\right)$$

with

$$f_{T_{(\lambda(1),\emptyset,\lambda(3),\emptyset)}}(y_1^+, y_2^+, z_1^+, z_2^+) = \underbrace{\tilde{y}_1^+ \dots \tilde{\tilde{y}}_1^+}_{\alpha_2} (y_1^+)^{\alpha_1} [z_1^+, z_2^+]^{w_2} z_1^+ \underbrace{\tilde{y}_2^+ \dots \tilde{\tilde{y}}_2^+}_{\alpha_2} z_1^+.$$

Hence

$$f_{T_{(\lambda(1),\emptyset,\lambda(3),\emptyset)}}(R_2, R_1, N_1, N_2) = \pm e_{32} \neq 0.$$

Conversely, we shall prove that if $w_1 \geq 3$ then $m_{\langle\lambda\rangle} = 0$. By Theorem 3.1.2 and Theorem 3.1.3, we have to show that for every Young multitableau $T_{\langle\lambda\rangle}$, the corresponding highest weight vector $f_{T_{\langle\lambda\rangle}}$ is a $*$-identity for the algebra. We distinguish two cases.

Case 1. $\lambda(1) = \emptyset$.

By the hypothesis, any monomial of $f_{T_{(\emptyset,\emptyset,\lambda(3),\emptyset)}}$ has $w_2 + w_1$ variables z_1^+ and w_2 variables z_2^+, so w_1 is the difference between the number of variables z_1^+ and the number of variables z_2^+. For any monomial m in z_1^+ and z_2^+, we define $l(m)$ as the number of z_1^+ in m, $k(m)$ as the number of variables z_2^+ in m and $d(m) = l(m) - k(m)$. It is enough to prove that any monomial m with $d(m) \geq 3$ is a $*$-identity for the algebra. We shall prove it by induction on $k(m)$. If $k(m) = 0$, then m is of the type $(z_1^+)^{l(m)}$, $l(m) = d(m) \geq 3$, and it is a $*$-identity, because it is a consequence of the $*$-identity (b) of Theorem 3.2.3. If $k(m) = 1$, the possible monomials m are of the type $(z_1^+)^i z_2^+ (z_1^+)^{l(m)-i}$, with $i = 0, \ldots, l(m)$. We observe that if $i \geq 3$ or $l(m) - i \geq 3$, then m is a $*$-identity, because it contains $(z_1^+)^3$ that is a $*$-identity as we have seen before. If $i \leq 2$ and $l(m) - i \leq 2$, since $l(m) = d(m) + 1 \geq 4$, we have to consider only the case $i = 2$ and $l(m) = 4$. Hence $m = (z_1^+)^2 z_2^+ (z_1^+)^2$ and it is a $*$-identity because it is a consequence of the $*$-identity (d) of Theorem 3.2.3.

If $k(m) > 1$, we assume that any monomial p with $k(p) < k(m)$ and $d(p) \geq 3$ is a $*$-identity and we shall prove that also m is a $*$- identity. Let $m = z_2^+ m'$ or $m = m' z_2^+$, where m' is a monomial with $k(m') = k(m) - 1$, $d(m') = l(m) - k(m) + 1 = d(m) + 1 \geq 4$. Since $k(m') < k(m)$, by the induction hypothesis $m' \equiv 0$ and so $m \equiv 0$ is a $*$-identity. Suppose $m = z_1^+ m' z_1^+$. Then if $m = z_1^+ z_2^+ \underbrace{m'' z_1^+}_{r}$ or $m = \underbrace{z_1^+ m''}_{r} z_2^+ z_1^+$, since r is a monomial with $k(r) = k(m) - 1$ and $d(r) = d(m) \geq 3$, by induction hypothesis, $r \equiv 0$ and so $m \equiv 0$ is a $*$-identity. Otherwise, if $m = z_1^+ z_1^+ m'' z_1^+ z_1^+$, with $m'' = z_1^+ m'''$ or $m'' = m''' z_1^+$, since $(z_1^+)^3 \equiv 0$, we get that $m \equiv 0$.

So we are left to consider the case when $m = z_1^+ z_1^+ z_2^+ \underbrace{m''' z_2^+ z_1^+ z_1^+}_{r}$, where r is a monomial with $k(r) = k(m) - 1$ and $d(r) = l(m) - 2 - k(m) + 1 = d(m) - 1 \geq 2$. If $d(r) \geq 3$, since $k(r) < k(m)$, by the induction hypothesis, $r \equiv 0$ and so $m \equiv 0$. If $d(r) = d(m) - 1 = 2$ then $d(m) = 3$ and the number of variables in m'' is $l(m'') + k(m'') = l(m) - 4 + k(m) = d(m) + k(m) - 4 + k(m) = 2k(m) - 1$. Hence there is an odd number of variables z_i^+, with $i \in \{1, 2\}$. So, since $z_1^+ z_1^+ z z_1^+ z_1^+ \equiv 0$, as a consequence of the $*$-identities (d) and (e) of Theorem 3.2.3, we get that $m \equiv 0$ is a $*$-identity.

Case 2. $\lambda(1) \neq \emptyset$.

More in general, we shall get that any monomial m in even symmetric variables and in z_1^+, z_2^+ with $d(m) \geq 3$ is a $*$-identity. Consider the monomial $m(y^+, z_1^+, z_2^+) = z_1^+ y^+ z_2^+$ and let $b_1, b_2 \in (M_{1,2}(F))_1^+$ and $a_1 = \beta e_{11} + \gamma(e_{22} + e_{33}) \in (M_{1,2}(F))_0^+$. It is easily checked that $m(a_1, b_1, b_2) = m'(a_1', b_1, b_2)$, where $m'(y^+, z_1^+, z_2^+) = z_1^+ z_2^+ y^+$ and $a_1' = \gamma e_{11} + \beta(e_{22} + e_{33})$. Hence $m \equiv 0$ if and only if $m' \equiv 0$. As a consequence, we also get that $z_1^+ y_1^+ \ldots y_n^+ z_2^+ \equiv 0$ if and only if $z_1^+ z_2^+ y_1^+ \ldots y_n^+ \equiv 0$. By using this property, in order to establish whether a monomial in variables y^+ and z^+ is a $*$-identity, we may assume that it is of the type $y_{i_1}^+ \ldots y_{i_{k_1}}^+ z_{j_1}^+ \ldots z_{j_r}^+ y_{l_1}^+ \ldots y_{l_{k_2}}^+$, with $k_1, k_2, r \geq 0$. Now, let $m = m(y_1^+, y_2^+, z_1^+, z_2^+)$ with $d(m) \geq 3$, then m is a $*$-identity by the previous case and the proof is completed. □

By Theorem 3.2.1, we get the following result.

Proposition 3.3.3. *If $\langle\lambda\rangle = (\lambda(1), \emptyset, \emptyset, \lambda(4))$ in (3.7), with $\lambda(4) = (\rho_1 + \rho_2, \rho_2) \neq \emptyset$, then $m_{\langle\lambda\rangle} \neq 0$ if and only if $\rho_1 \leq 2$.*

Next we consider the case $\lambda(3) \neq \emptyset$ and $\lambda(4) \neq \emptyset$.

Proposition 3.3.4. *If $\langle\lambda\rangle = (\lambda(1), \emptyset, \lambda(3), \lambda(4))$ in (3.7), with $\lambda(3) = (w_1 + w_2, w_2) \neq \emptyset$, $\lambda(4) = (\rho_1 + \rho_2, \rho_2) \neq \emptyset$ and $|w_1 - \rho_1| \leq 2$, then $m_{\langle\lambda\rangle} \neq 0$.*

Proof. Consider the elements $R_1 = e_{11}$, $R_2 = e_{22} + e_{33}$, $N_1 = e_{12} - e_{31}$, $N_2 = e_{13} + e_{21}$, $P_1 = e_{13} - e_{21}$, $P_2 = e_{12} + e_{31}$. First suppose that $w_1 \geq \rho_1$ and distinguish some cases.

- $w_1 \leq 2$.
  We consider the multitableau

$$T_{(\lambda(1),\emptyset,\lambda(3),\lambda(4))} = (T_{\lambda(1)}, \emptyset, T_{\lambda(3)}, \bar{T}_{\lambda(4)}),$$

where $\bar{T}_{\lambda(4)}$ is the initial standard tableau on the integers $2\alpha_2 + \alpha_2 + 2w_2 + w_1 + 1, \dots, n$ and $T_{\lambda(1)}$ and $T_{\lambda(3)}$ are the tableaux we constructed in the proof of Proposition 3.3.2 on the integers $1, \dots, 2\alpha_2 + \alpha_2 + 2w_2 + w_1$. Hence

$$f_{T_{(\lambda(1),\emptyset,\lambda(3),\lambda(4))}}(y_1^+, y_2^+, z_1^+, z_2^+, z_1^-, z_2^-) = f_{T_{(\lambda(1),\emptyset,\lambda(3),\emptyset)}}(y_1^+, y_2^+, z_1^+, z_2^+) f_{\bar{T}_{\lambda(4)}}(z_1^-, z_2^-),$$

where $f_{T_{(\lambda(1),\emptyset,\lambda(3),\emptyset)}}(y_1^+, y_2^+, z_1^+, z_2^+)$ are the highest weight vectors obtained in Proposition 3.3.2. Then, if $w_1 = 0$,

$$f_{T_{(\lambda(1),\emptyset,\lambda(3),\lambda(4))}}(R_2, R_1, N_1, N_2, P_1, P_2) =$$

$$(\pm\beta e_{11} \pm e_{22} \pm e_{33})(2^{\rho_2} e_{11} \pm e_{22} \pm e_{33}) \neq 0;$$

if $w_1 = 1$,

$$f_{T_{(\lambda(1),\emptyset,\lambda(3),\lambda(4))}}(R_2, R_1, N_1, N_2, P_1, P_2) =$$

$$(\pm e_{31} \pm \beta e_{12})(2^{\rho_2} e_{11} \pm e_{22} \pm e_{33})(e_{13} - e_{21})^{\rho_1} \neq 0,$$

where $\beta \in \{0, 2^{w_2}\}$;
if $w_1 = 2$,

$$f_{T_{(\lambda(1),\emptyset,\lambda(3),\lambda(4))}}(R_2, R_1, N_1, N_2, P_1, P_2) = \pm e_{32}(2^{\rho_2} e_{11} \pm e_{22} \pm e_{33})(e_{13} - e_{21})^{\rho_1} \neq 0.$$

- $w_1 \geq 3$.
  Let $\lambda(1) = \emptyset$. First, let $w_1 - \rho_1 = 0$ and consider the following multitableau

$$T_{(\emptyset,\emptyset,\lambda(3),\lambda(4))} = (\emptyset, \emptyset, T_{\lambda(3)}, T_{\lambda(4)}),$$

where

$T_{\lambda(3)} =$

1	...	$2w_2-1$	$2w_2+1$	$2w_2+2$	$2w_2+2\rho_2+4$	$2w_2+2\rho_2+6$	...	$n-2$
2	...	$2w_2$						

and

$T_{\lambda(4)} =$

$2w_2+3$	...	$2w_2+2\rho_2+1$	$2w_2+2\rho_2+3$	$2w_2+2\rho_2+5$	...	$n-3$	$n-1$	n
$2w_2+4$	...	$2w_2+2\rho_2+2$						

.

Then

$$f_{T_{(\emptyset,\emptyset,\lambda(3),\lambda(4))}}(z_1^+, z_2^+, z_1^-, z_2^-) = [z_1^+, z_2^+]^{w_2}(z_1^+)^2[z_1^-, z_2^-]^{\rho_2}(z_1^- z_1^+)^{\rho_1 - 2}(z_1^-)^2$$

and

$$f_{T_{(\emptyset,\emptyset,\lambda(3),\lambda(4))}}(N_1, N_2, P_1, P_2) = \pm e_{33} \neq 0.$$

If $w_1 - \rho_1 = 1$, then we consider the following multitableau

$$T_{(\emptyset,\emptyset,\lambda(3),\lambda(4))} = (\emptyset, \emptyset, T_{\lambda(3)}, T_{\lambda(4)}),$$

where

$$T_{\lambda(3)} = \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 1 & \ldots & 2w_2{-}1 & 2w_2{+}1 & 2w_2{+}2 & 2w_2{+}2\rho_2{+}4 & 2w_2{+}2\rho_2{+}6 & \ldots & n{-}3 & n \\ \hline 2 & \ldots & 2w_2 \\ \cline{1-3}\end{array}$$

and

$$T_{\lambda(4)} = \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 2w_2{+}3 & \ldots & 2w_2{+}2\rho_2{+}1 & 2w_2{+}2\rho_2{+}3 & 2w_2{+}2\rho_2{+}5 & \ldots & n{-}4 & n{-}2 & n{-}1 \\ \hline 2w_2{+}4 & \ldots & 2w_2{+}2\rho_2{+}2 \\ \cline{1-3}\end{array}.$$

Then

$$f_{T_{(\emptyset,\emptyset,\lambda(3),\lambda(4))}}(z_1^+, z_2^+, z_1^-, z_2^-) = [z_1^+, z_2^+]^{w_2}(z_1^+)^2[z_1^-, z_2^-]^{\rho_2}(z_1^- z_1^+)^{\rho_1 - 2}(z_1^-)^2 z_1^+$$

and

$$f_{T_{(\emptyset,\emptyset,\lambda(3),\lambda(4))}}(N_1, N_2, P_1, P_2) = \pm e_{31} \neq 0.$$

Finally, if $w_1 - \rho_1 = 2$, then we consider the following multitableau

$$T_{(\emptyset,\emptyset,\lambda(3),\lambda(4))} = (\emptyset, \emptyset, T_{\lambda(3)}, T_{\lambda(4)}),$$

where

$$T_{\lambda(3)} = \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 1 & \ldots & 2w_2{-}1 & 2w_2{+}1 & 2w_2{+}2 & 2w_2{+}2\rho_2{+}4 & 2w_2{+}2\rho_2{+}6 & \ldots & n{-}2 & n \\ \hline 2 & \ldots & 2w_2 \\ \cline{1-3}\end{array}$$

and

$$T_{\lambda(4)} = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 2w_2{+}3 & \ldots & 2w_2{+}2\rho_2{+}1 & 2w_2{+}2\rho_2{+}3 & 2w_2{+}2\rho_2{+}5 & \ldots & n{-}3 & n{-}1 \\ \hline 2w_2{+}4 & \ldots & 2w_2{+}2\rho_2{+}2 \\ \cline{1-3}\end{array}.$$

Then

$$f_{T_{(\emptyset,\emptyset,\lambda(3),\lambda(4))}}(z_1^+, z_2^+, z_1^-, z_2^-) = [z_1^+, z_2^+]^{w_2}(z_1^+)^2[z_1^-, z_2^-]^{\rho_2}(z_1^- z_1^+)^{\rho_1}$$

and

$$f_{T_{(\emptyset,\emptyset,\lambda(3),\lambda(4))}}(N_1, N_2, P_1, P_2) = \pm e_{32} \neq 0.$$

Now, let $\lambda(1) \neq \emptyset$. First suppose $w_1 - \rho_1 = 0$ and consider the multitableau $T_{(\lambda(1),\emptyset,\lambda(3),\lambda(4))}$ such that

$$f_{T_{(\lambda(1),\emptyset,\lambda(3),\lambda(4))}}(y_1^+, y_2^+, z_1^+, z_2^+, z_1^-, z_2^-) =$$

$$\underbrace{\tilde{y}_1^+ \ldots \tilde{y}_1^+}_{\alpha_2}(y_1^+)^{\alpha_1}[z_1^+, z_2^+]^{w_2}(z_1^+)^2[z_1^-, z_2^-]^{\rho_2}(z_1^- z_1^+)^{\rho_1 - 2} z_1^- \underbrace{\tilde{y}_2^+ \ldots \tilde{y}_2^+}_{\alpha_2} z_1^-.$$

Then

$$f_{T_{(\lambda(1),\emptyset,\lambda(3),\lambda(4))}}(R_2, R_1, N_1, N_2, P_1, P_2) = \pm e_{33} \neq 0.$$

If $w_1 - \rho_1 = 1$, then we consider the following multitableau

$$T_{(\lambda(1),\emptyset,\lambda(3),\lambda(4))} = (T_{\lambda(1)}, \emptyset, T_{\lambda(3)}, T_{\lambda(4)}),$$

where

$$T_{\lambda(1)} = \begin{array}{|c|c|c|c|c|} \hline 1 & \dots & \alpha_2 & \dots & \alpha_1+\alpha_2 \\ \hline n-\alpha_2+1 & \dots & n \\ \cline{1-3} \end{array}$$

and $T_{\lambda(3)}$ and $T_{\lambda(4)}$ are the tableaux we considered in the previous case on the integers $\alpha_1 + \alpha_2 + 1, \dots, n - \alpha_2$. Hence

$$f_{T_{(\lambda(1),\emptyset,\lambda(3),\lambda(4))}}(y_1^+, y_2^+, z_1^+, z_2^+, z_1^-, z_2^-) =$$

$$\underbrace{\tilde{y}_1^+ \dots \tilde{y}_1^+}_{\alpha_2} (y_1^+)^{\alpha_1} f_{T_{(\emptyset,\emptyset,\lambda(3),\lambda(4))}}(z_1^+, z_2^+, z_1^-, z_2^-) \underbrace{\tilde{y}_2^+ \dots \tilde{y}_2^+}_{\alpha_2}$$

where $f_{T_{(\emptyset,\emptyset,\lambda(3),\lambda(4))}}(z_1^+, z_2^+, z_1^-, z_2^-)$ is the highest weight vector obtained in the previous case. Then

$$f_{T_{(\lambda(1),\emptyset,\lambda(3),\lambda(4))}}(R_2, R_1, N_1, N_2, P_1, P_2) = \pm e_{31} \neq 0.$$

Finally, if $w_1 - \rho_1 = 2$, then we consider the multitableau $T_{\langle\lambda\rangle}$ such that

$$f_{T_{(\lambda(1),\emptyset,\lambda(3),\lambda(4))}}(y_1^+, y_2^+, z_1^+, z_2^+, z_1^-, z_2^-) =$$

$$\underbrace{\tilde{y}_1^+ \dots \tilde{y}_1^+}_{\alpha_2} (y_1^+)^{\alpha_1} [z_1^+, z_2^+]^{w_2} (z_1^+)^2 [z_1^-, z_2^-]^{\rho_2} (z_1^- z_1^+)^{\rho_1 - 1} z_1^- \underbrace{\tilde{y}_2^+ \dots \tilde{y}_2^+}_{\alpha_2} z_1^+.$$

Then

$$f_{T_{(\lambda(1),\emptyset,\lambda(3),\lambda(4))}}(R_2, R_1, N_1, N_2, P_1, P_2) = \pm e_{32} \neq 0.$$

By Theorem 3.2.1, the same approach applies in case $\rho_1 \geq w_1$. Now the proof is completed. □

Now we can ask: is it true that if the multiplicity is different than zero then $|w_1 - \rho_1| \leq 2$? We consider an easy case in which $|w_1 - \rho_1| \geq 3$.

Example 3.3.1. Let $\langle\lambda\rangle = (\emptyset, \emptyset, (1,1), (3))$ in (3.7). Notice that $|w_1 - \rho_1| = 4$. It is easy to see that for any $T_{(\emptyset,\emptyset,(1,1),(3))}$ the corresponding highest weight vector $f_{T_{(\emptyset,\emptyset,(1,1),(3))}}$ is a $*$-identity being a consequence of the set of $*$-identities

$$\{z_1^+ z_2^+ z_3^+ + z_2^+ z_3^+ z_1^+ + z_3^+ z_1^+ z_2^+,\ \tilde{z}_1^+ y^- \tilde{z}_2^+,\ \tilde{z}_1^+ z^- \tilde{z}_2^+ z^- z^-,\ z^- \tilde{z}_1^+ z^- \tilde{z}_2^+ z^-,\ z^- z^- \tilde{z}_1^+ z^- \tilde{z}_2^+\}.$$

Then $m_{\langle\lambda\rangle} = 0$ and the same result holds if $\langle\lambda\rangle = (\emptyset, \emptyset, (3), (1,1))$, by Theorem 3.2.1.

Motivated by this example we are led to think that the following conjecture holds.

Conjecture 3.3.1. *Let $(\lambda(1), \emptyset, \lambda(3), \lambda(4))$ in (3.7), with $\lambda(3) = (w_1 + w_2, w_2) \neq \emptyset$, $\lambda(4) = (\rho_1 + \rho_2, \rho_2) \neq \emptyset$. Then $|w_1 - \rho_1| \leq 2$ if and only if $m_{\langle\lambda\rangle} \neq 0$.*

Given a real number c, let $\lceil c \rceil$ denote the ceiling of c, i.e., the smallest integer greater than or equal to c.

Now we consider the case $\lambda(2) \neq \emptyset$ and $\lambda(3) \neq \emptyset$.

Proposition 3.3.5. *Let* $\langle\lambda\rangle = (\lambda(1),\lambda(2),\lambda(3),\emptyset)$ *in* (3.7)*, with* $\lambda(1) = (\alpha_1+\alpha_2,\alpha_2)$*,* $\lambda(2) = (\gamma_1+\gamma_2+\gamma_3,\gamma_2+\gamma_3,\gamma_3)\neq\emptyset$*,* $\lambda(3) = (w_1+w_2,w_2)\neq\emptyset$*. If*

- $w_1 \leq 2$ *or*
- $w_1 \geq 3$ *and* $\gamma_1+\gamma_2 \geq \lceil \frac{w_1}{2}\rceil - 1$

then $m_{\langle\lambda\rangle}\neq 0$.

Proof. Consider the elements $R_1 = e_{11}$, $R_2 = e_{22}+e_{33}$, $M_1 = e_{23}+e_{32}$, $M_2 = e_{23}-e_{32}$, $M_3 = e_{22}-e_{33}$, $N_1 = e_{12}-e_{31}$, $N_2 = e_{13}+e_{21}$. First, suppose $w_1\leq 2$ and we distinguish two cases.
Let $\lambda(1)=\emptyset$ and consider the initial standard multitableau. Hence

$$f_{\bar{T}_{(\emptyset,\lambda(2),\lambda(3),\emptyset)}}(y_1^-,y_2^-,y_3^-,z_1^+,z_2^+) = f_{\bar{T}_{\lambda(2)}}(y_1^-,y_2^-,y_3^-)f_{\bar{T}_{\lambda(3)}}(z_1^+,z_2^+)$$

and

$$f_{\bar{T}_{(\emptyset,\lambda(2),\lambda(3),\emptyset)}}(M_1,M_2,M_3,N_1,N_2) = \begin{cases} r(\pm e_{22}\pm e_{33})(e_{12}-e_{31})^{w_1}\neq 0, & \text{if } \gamma_1 \text{ is even,}\\ r(\pm e_{23}\pm e_{32})(e_{12}-e_{31})^{w_1}\neq 0, & \text{if } \gamma_1 \text{ is odd,}\end{cases}$$

where $r=\pm 6^{\gamma_3}2^{\gamma_2}$.
Then, let $\lambda(1)\neq\emptyset$. We consider the multitableau

$$T_{(\lambda(1),\lambda(2),\lambda(3),\emptyset)} = (T_{\lambda(1)},\bar{T}_{\lambda(2)},T_{\lambda(3)},\emptyset),$$

where $\bar{T}_{\lambda(2)}$ is the initial standard tableau on the integers $1,\dots,3\gamma_3+2\gamma_2+\gamma_1$ and $T_{\lambda(1)}$ and $T_{\lambda(3)}$ are the tableaux we constructed in the proof of Proposition 3.3.2 on the integers $3\gamma_3+2\gamma_2+\gamma_1+1,\dots,n$. Hence

$$\begin{gathered} f_{T_{(\lambda(1),\lambda(2),\lambda(3),\emptyset)}}(y_1^+,y_2^+,y_1^-,y_2^-,y_3^-,z_1^+,z_2^+) = \\ f_{\bar{T}_{\lambda(2)}}(y_1^-,y_2^-,y_3^-)f_{T_{(\lambda(1),\emptyset,\lambda(3),\emptyset)}}(y_1^+,y_2^+,z_1^+,z_2^+),\end{gathered}$$

where $f_{T_{(\lambda(1),\emptyset,\lambda(3),\emptyset)}}(y_1^+,y_2^+,z_1^+,z_2^+)$ are the highest weight vectors obtained in Proposition 3.3.2.
If $w_1=0$, then

$$f_{T_{(\lambda(1),\lambda(2),\lambda(3),\emptyset)}}(R_2,R_1,M_1,M_2,M_3,N_1,N_2) = \begin{cases} r(e_{22}\pm e_{33})\neq 0, & \text{if } \gamma_1 \text{ is even,}\\ r(e_{23}\pm e_{32})\neq 0, & \text{if } \gamma_1 \text{ is odd,}\end{cases}$$

if $w_1=1$, then

$$f_{T_{(\lambda(1),\lambda(2),\lambda(3),\emptyset)}}(R_2,R_1,M_1,M_2,M_3,N_1,N_2) = \begin{cases} re_{31}\neq 0, & \text{if } \gamma_1 \text{ is even,}\\ re_{21}\neq 0, & \text{if } \gamma_1 \text{ is odd,}\end{cases}$$

if $w_1=2$, then

$$f_{T_{(\lambda(1),\lambda(2),\lambda(3),\emptyset)}}(R_2,R_1,M_1,M_2,M_3,N_1,N_2) = \begin{cases} re_{32}\neq 0, & \text{if } \gamma_1 \text{ is even,}\\ re_{22}\neq 0, & \text{if } \gamma_1 \text{ is odd,}\end{cases}$$

where $r=\pm 6^{\gamma_3}2^{\gamma_2}$.

Now, suppose $w_1\geq 3$ and $\gamma_1+\gamma_2\geq k-1$, with $\lceil\frac{w_1}{2}\rceil = k$. We consider the Young diagram $D_{\lambda(2)} = D_\mu|D_\nu$, corresponding to the partition $\lambda(2)$, as obtained by gluing two diagrams D_μ and D_ν, where D_μ is the subdiagram of $D_{\lambda(2)}$ made up of $\gamma_1+\gamma_2+\gamma_3-(k-1)$ columns and D_ν is the diagram made up of the last $k-1$ columns. We also consider the Young diagram $D_{\lambda(3)} = D_\beta|D_\delta$ obtained by gluing the diagram D_β that has w_2 columns of height two and two columns of height one and the diagram D_δ that has w_1-2 columns of height one. We define t_i as the number of columns of height i in D_μ, with $t_i \le \gamma_i$, for $i \in \{1,2\}$, such that $t_1+t_2 = \gamma_1+\gamma_2-(k+1)$. We distinguish the following cases.

- $w_1 = 2k$.
  If $\gamma_1 \ge k-1$, we consider the multitableau $T_{(\lambda(1),\lambda(2),\lambda(3),\emptyset)}$ such that

$$f_{T_{(\lambda(1),\lambda(2),\lambda(3),\emptyset)}}(y_1^+,y_2^+,y_1^-,y_2^-,y_3^-,z_1^+,z_2^+) =$$

$$\underbrace{\tilde{y}_1^+\dots\tilde{y}_1^+}_{\alpha_2}(y_1^+)^{\alpha_1} f_{\bar{T}_\mu}(y_1^-,y_2^-,y_3^-) f_{\bar{T}_\beta}(z_1^+,z_2^+)(y_1^-(z_1^+)^2)^{k-2} y_1^- z_1^+ \underbrace{\tilde{y}_2^+\dots\tilde{y}_2^+}_{\alpha_2} z_1^+.$$

  Then

$$f_{T_{(\lambda(1),\lambda(2),\lambda(3),\emptyset)}}(R_2,R_1,M_1,M_2,M_3,N_1,N_2) = \begin{cases} re_{32} \neq 0, & \text{if } \gamma_1-k+1 \text{ is even,} \\ re_{22} \neq 0, & \text{if } \gamma_1-k+1 \text{ is odd,} \end{cases}$$

  where $r = \pm 6^{\gamma_3} 2^{\gamma_2}$.
  If $0 \le \gamma_1 < k-1$, we consider the multitableau $T_{(\lambda(1),\lambda(2),\lambda(3),\emptyset)}$ such that

$$f_{T_{(\lambda(1),\lambda(2),\lambda(3),\emptyset)}}(y_1^+,y_2^+,y_1^-,y_2^-,y_3^-,z_1^+,z_2^+) =$$

$$\underbrace{\tilde{y}_1^+\dots\tilde{y}_1^+}_{\alpha_2}(y_1^+)^{\alpha_1} f_{\bar{T}_\mu}(y_1^-,y_2^-,y_3^-) f_{\bar{T}_\beta}(z_1^+,z_2^+)([y_1^-,y_2^-](z_1^+)^2)^{k-2-\gamma_1}[y_1^-,y_2^-]z_1^+$$

$$\underbrace{\tilde{y}_2^+\dots\tilde{y}_2^+}_{\alpha_2} z_1^+(y_1^-(z_1^+)^2)^{\gamma_1}.$$

  Then

$$f_{T_{(\lambda(1),\lambda(2),\lambda(3),\emptyset)}}(R_2,R_1,M_1+M_3,M_2,M_3,N_1,N_2) =$$

$$\begin{cases} \pm 6^{\gamma_3} 2^{\frac{1}{2}(s+1)} e_{32} \neq 0, & \text{if } t_1+t_2 \text{ is even,} \\ \pm 6^{\gamma_3} 2^{\frac{1}{2}s}(e_{22}+e_{32}) \neq 0, & \text{if } t_1+t_2 \text{ is odd,} \end{cases}$$

  where $s = 3\gamma_2 - k + \gamma_1$.

- $w_1 = 2k-1$.
  We consider the multitableau $T_{(\lambda(1),\lambda(2),\lambda(3),\emptyset)}$ such that, if $\gamma_1 \neq 0$,

$$f_{T_{(\lambda(1),\lambda(2),\lambda(3),\emptyset)}}(y_1^+,y_2^+,y_1^-,y_2^-,y_3^-,z_1^+,z_2^+) =$$

$$\underbrace{\tilde{y}_1^+\dots\tilde{y}_1^+}_{\alpha_2}(y_1^+)^{\alpha_1} f_{\bar{T}_\mu}(y_1^-,y_2^-,y_3^-) f_{\bar{T}_\beta}(z_1^+,z_2^+)([y_1^-,y_2^-](z_1^+)^2)^{\gamma_2-t_2}(y_1^-(z_1^+)^2)^{\gamma_1-t_1-1}$$

$$y_1^- z_1^+ \underbrace{\tilde{y}_2^+\dots\tilde{y}_2^+}_{\alpha_2}$$

  and, if $\gamma_1 = 0$,

$$f_{T_{(\lambda(1),\lambda(2),\lambda(3),\emptyset)}}(y_1^+,y_2^+,y_1^-,y_2^-,y_3^-,z_1^+,z_2^+) =$$

$$\underbrace{\tilde{y}_1^+ \dots \tilde{y}_1^+}_{\alpha_2}(y_1^+)^{\alpha_1} f_{\bar{T}_\mu}(y_1^-, y_2^-, y_3^-) f_{\bar{T}_\beta}(z_1^+, z_2^+)([y_1^-, y_2^-](z_1^+)^2)^{\gamma_2 - t_2 - 1}$$

$$[y_1^-, y_2^-] z_1^+ \underbrace{\tilde{y}_2^+ \dots \tilde{y}_2^+}_{\alpha_2}.$$

So, if $\gamma_1 \geq k-1$,

$$f_{T_{(\lambda(1),\lambda(2),\lambda(3),\emptyset)}}(R_2, R_1, M_1, M_2, M_3, N_1, N_2) = \begin{cases} re_{31} \neq 0, & \text{if } \gamma_1 - k + 1 \text{ is even,} \\ re_{21} \neq 0, & \text{if } \gamma_1 - k + 1 \text{ is odd} \end{cases}$$

and, if $0 \leq \gamma_1 < k-1$,

$$f_{T_{(\lambda(1),\lambda(2),\lambda(3),\emptyset)}}(R_2, R_1, M_1 + M_3, M_2, M_3, N_1, N_2) =$$

$$\begin{cases} \pm 6^{\gamma_3} 2^{\frac{1}{2}(s+1)} e_{31} \neq 0, & \text{if } t_1 + t_2 \text{ is even,} \\ \pm 6^{\gamma_3} 2^{\frac{1}{2}s}(e_{21} + e_{31}) \neq 0, & \text{if } t_1 + t_2 \text{ is odd,} \end{cases}$$

where $r = \pm 6^{\gamma_3} 2^{\gamma_2}$ and $s = 3\gamma_2 - k + \gamma_1$.

□

Consider the following example.

Example 3.3.2. Let $\langle \lambda \rangle = (\emptyset, \lambda(2), (w_1), \emptyset)$, with $\lambda(2) = (\gamma_1 + \gamma_2 + \gamma_3, \gamma_2 + \gamma_3, \gamma_3) \vdash s$, $w_1 \geq 5$ and $s < \lceil \frac{w_1}{2} \rceil - 1$. Then $m_{\langle \lambda \rangle} = 0$, since, for any $T_{\langle \lambda \rangle}$, each monomial of $f_{T_{\langle \lambda \rangle}}$ contains at least three consecutive odd symmetric variables and, so, it is a $*$-identity (see Theorem 3.2.3 (b)).
Notice that, if $\langle \lambda \rangle = (\emptyset, (1,1,1), (3), \emptyset)$ we also get that $m_{\langle \lambda \rangle} = 0$, since any highest weight vector $f_{T_{\langle \lambda \rangle}}$ is a consequence of the following set of $*$-identities

$$\{[y^+, y], \quad z_1^+ z_2^+ z_3^+ + z_2^+ z_3^+ z_1^+ + z_3^+ z_1^+ z_2^+, \quad y_1^- z^+ y_2^-, \quad \tilde{y}_1^- \tilde{y}_2^- z^+ z^+ \tilde{y}_3^- z^+,$$

$$\tilde{y}_1^- z^+ z^+ \tilde{y}_2^- \tilde{y}_3^- z^+, \quad z^+ \tilde{y}_1^- \tilde{y}_2^- z^+ z^+ \tilde{y}_3^-, \quad z^+ \tilde{y}_1^- z^+ z^+ \tilde{y}_2^- \tilde{y}_3^- \}.$$

Motivated by the above example, we make the following conjecture.

Conjecture 3.3.2. *Let* $\langle \lambda \rangle = (\lambda(1), \lambda(2), \lambda(3), \emptyset)$ *in (3.7), with* $\lambda(1) = (\alpha_1 + \alpha_2, \alpha_2)$, $\lambda(2) = (\gamma_1 + \gamma_2 + \gamma_3, \gamma_2 + \gamma_3, \gamma_3) \neq \emptyset$, $\lambda(3) = (w_1 + w_2, w_2) \neq \emptyset$. *Then*

- $w_1 \leq 2$ *or*
- $w_1 \geq 3$ *and* $\gamma_1 + \gamma_2 \geq \lceil \frac{w_1}{2} \rceil - 1$

if and only if $m_{\langle \lambda \rangle} \neq 0$.

According to Theorem 3.2.1, if $\lambda(2) \neq \emptyset$ and $\lambda(4) \neq \emptyset$, we obtain the following result.

Proposition 3.3.6. *If* $\langle \lambda \rangle = (\lambda(1), \lambda(2), \emptyset, \lambda(4))$ *in (3.7), with* $\lambda(2) = (\gamma_1 + \gamma_2 + \gamma_3, \gamma_2 + \gamma_3, \gamma_3) \neq \emptyset$ $\lambda(4) = (\rho_1 + \rho_2, \rho_2) \neq \emptyset$ *and*

- $\rho_1 \leq 2$ *or*
- $\rho_1 \geq 3$ *and* $\gamma_1 + \gamma_2 \geq \lceil \frac{\rho_1}{2} \rceil - 1$,

then $m_{\langle \lambda \rangle} \neq 0$.

Now we consider the case $\lambda(2) \neq \emptyset$, $\lambda(3) \neq \emptyset$ and $\lambda(4) \neq \emptyset$.

Proposition 3.3.7. *Let* $\langle\lambda\rangle = (\emptyset, \lambda(2), \lambda(3), \lambda(4))$ *in* (3.7), *with* $\lambda(2) = (\gamma_1 + \gamma_2 + \gamma_3, \gamma_2 + \gamma_3, \gamma_3) \neq \emptyset$, $\lambda(3) = (w_1 + w_2, w_2) \neq \emptyset$, $\lambda(4) = (\rho_1 + \rho_2, \rho_2) \neq \emptyset$ *and let* $l = max\{w_1, \rho_1\}$. *If*

- $|w_1 - \rho_1| \leq 2$ *or*
- $|w_1 - \rho_1| \geq 3$ *and* $\gamma_1 + \gamma_2 \geq \lceil \frac{l}{2} \rceil - 1$,

then $m_{\langle\lambda\rangle} \neq 0$.

Proof. Consider the elements $M_1 = e_{23} + e_{32}$, $M_2 = e_{23} - e_{32}$, $M_3 = e_{22} - e_{33}$, $N_1 = e_{12} - e_{31}$, $N_2 = e_{13} + e_{21}$, $P_1 = e_{13} - e_{21}$, $P_2 = e_{12} + e_{31}$. Because of Theorem 3.2.1, we may assume $w_1 \geq \rho_1$. First suppose $w_1 - \rho_1 \leq 2$ and we distinguish two cases.

- $w_1 \leq 2$.
  If $\rho_1 \in \{0, 1\}$, we consider the initial standard multitableau, then

$$f_{T_{(\emptyset,\lambda(2),\lambda(3),\lambda(4))}}(y_1^-, y_2^-, y_3^-, z_1^+, z_2^+, z_1^-, z_2^-) =$$

$$f_{\bar{T}_{\lambda(2)}}(y_1^-, y_2^-, y_3^-) f_{\bar{T}_{\lambda(3)}}(z_1^+, z_2^+) f_{\bar{T}_{\lambda(4)}}(z_1^-, z_2^-).$$

  So,

$$f_{T_{(\emptyset,\lambda(2),\lambda(3),\lambda(4))}}(M_1, M_2, M_3, N_1, N_2, P_1, P_2) =$$

$$\begin{cases} \pm 6^{\gamma_3} 2^{\gamma_2} (\pm e_{22} \pm e_{33})(e_{12} - e_{31})^{w_1} (2^{\rho_2} e_{11} \pm e_{22} \pm e_{33})(e_{13} - e_{21})^{\rho_1} \neq 0, & \text{if } \gamma_1 \text{ is even,} \\ \pm 6^{\gamma_3} 2^{\gamma_2} (\pm e_{23} \pm e_{32})(e_{12} - e_{31})^{w_1} (2^{\rho_2} e_{11} \pm e_{22} \pm e_{33})(e_{13} - e_{21})^{\rho_1} \neq 0, & \text{if } \gamma_1 \text{ is odd.} \end{cases}$$

  If $\rho_1 = 2$, we consider the following multitableau

$$T_{(\emptyset,\lambda(2),\lambda(3),\lambda(4))} = (\emptyset, \bar{T}_{\lambda(2)}, \bar{T}_{\lambda(3)}, \bar{T}_{\lambda(4)}),$$

  where $\bar{T}_{\lambda(2)}$ is the initial standard tableau on the integers $2w_2 + w_1 + 2\rho_2 + \rho_1 + 1, \ldots, n$, $\bar{T}_{\lambda(3)}$ is the initial standard tableau on the integers $1, \ldots, 2w_2 + w_1$ and $\bar{T}_{\lambda(4)}$ is the initial standard tableau on the integers $2w_2 + w_1 + 1, \ldots, 2w_2 + w_1 + 2\rho_2 + \rho_1$. Then

$$f_{T_{(\emptyset,\lambda(2),\lambda(3),\lambda(4))}}(y_1^-, y_2^-, y_3^-, z_1^+, z_2^+, z_1^-, z_2^-) =$$

$$f_{\bar{T}_{\lambda(3)}}(z_1^+, z_2^+) f_{\bar{T}_{\lambda(4)}}(z_1^-, z_2^-) f_{\bar{T}_{\lambda(2)}}(y_1^-, y_2^-, y_3^-)$$

  and so,

$$f_{T_{(\emptyset,\lambda(2),\lambda(3),\lambda(4))}}(M_1, M_2, M_3, N_1, N_2, P_1, P_2) = \begin{cases} re_{33} \neq 0, & \text{if } \gamma_1 \text{ is even,} \\ re_{32} \neq 0, & \text{if } \gamma_1 \text{ is odd,} \end{cases}$$

  with $r = \pm 6^{\gamma_3} 2^{\gamma_2}$.

- $w_1 \geq 3$.
  We consider the multitableau

$$T_{(\emptyset,\lambda(2),\lambda(3),\lambda(4))} = (\emptyset, \bar{T}_{\lambda(2)}, T_{\lambda(3)}, T_{\lambda(4)}),$$

  where $\bar{T}_{\lambda(2)}$ is the initial standard tableau on the integers $1, \ldots, 3\gamma_3 + 2\gamma_2 + \gamma_1$ and $T_{\lambda(3)}$ and $T_{\lambda(4)}$ are the tableaux we considered in the proof of Proposition 3.3.4 on the integers $3\gamma_3 + 2\gamma_2 + \gamma_1 + 1, \ldots, n$. Then

$$f_{T_{(\emptyset,\lambda(2),\lambda(3),\lambda(4))}} = f_{\bar{T}_{\lambda(2)}}(y_1^-, y_2^-, y_3^-) f_{T_{(\emptyset,\emptyset,\lambda(3),\lambda(4))}}(z_1^+, z_2^+, z_1^-, z_2^-),$$

where $f_{T_{(\emptyset,\emptyset,\lambda(3),\lambda(4))}}(z_1^+,z_2^+,z_1^-,z_2^-)$ is the highest weight vector obtained in Pro-position 3.3.4.
So, if $\rho_1 = w_1 - 2$,

$$f_{T_{(\emptyset,\lambda(2),\lambda(3),\lambda(4))}}(M_1,M_2,M_3,N_1,N_2,P_1,P_2) = \begin{cases} re_{32} \neq 0, & \text{if } \gamma_1 \text{ is even,} \\ re_{22} \neq 0, & \text{if } \gamma_1 \text{ is odd;} \end{cases}$$

if $\rho_1 = w_1 - 1$,

$$f_{T_{(\emptyset,\lambda(2),\lambda(3),\lambda(4))}}(M_1,M_2,M_3,N_1,N_2,P_1,P_2) = \begin{cases} re_{31} \neq 0, & \text{if } \gamma_1 \text{ is even,} \\ re_{21} \neq 0, & \text{if } \gamma_1 \text{ is odd;} \end{cases}$$

if $\rho_1 = w_1$,

$$f_{T_{(\emptyset,\lambda(2),\lambda(3),\lambda(4))}}(M_1,M_2,M_3,N_1,N_2,P_1,P_2) = \begin{cases} re_{33} \neq 0, & \text{if } \gamma_1 \text{ is even,} \\ re_{23} \neq 0, & \text{if } \gamma_1 \text{ is odd,} \end{cases}$$

with $r = \pm 6^{\gamma_3} 2^{\gamma_2}$.

Now, consider the case $w_1 - \rho_1 \geq 3$, so $l = w_1$ and $\gamma_1 + \gamma_2 \geq k-1$, with $\lceil \frac{l}{2} \rceil = k$. We consider the Young diagrams $D_{\lambda(2)} = D_\mu | D_\nu$ and $D_{\lambda(3)} = D_\beta | D_\delta$ as in the proof of Proposition 3.3.5 and $D_{\lambda(4)} = D_\epsilon | D_\sigma$, such that D_ϵ is the subdiagram of $D_{\lambda(4)}$ made up of the first ρ_2 columns and D_σ is the diagram obtained considering the last ρ_1 columns. We define the integer a as

$$a := \begin{cases} \rho_1, & \text{if } \rho_1 \text{ is even,} \\ \rho_1 - 1, & \text{if } \rho_1 \text{ is odd.} \end{cases}$$

We distinguish these different cases.

- $w_1 = 2k$.
  We consider the multitableau $T_{\langle\lambda\rangle}$ such that:
  if $\gamma_1 \geq k-1$,
  $$f_{T_{(\emptyset,\lambda(2),\lambda(3),\lambda(4))}}(y_1^-,y_2^-,y_3^-,z_1^+,z_2^+,z_1^-,z_2^-) = g_1(y_1^-,y_2^-,y_3^-,z_1^+,z_2^+,z_1^-,z_2^-)(y_1^-(z_1^+)^2)^{k-1-\frac{a}{2}}(z_1^-)^{\rho_1-a},$$
  where
  $$g_1(y_1^-,y_2^-,y_3^-,z_1^+,z_2^+,z_1^-,z_2^-) = f_{\bar{T}_\mu}(y_1^-,y_2^-,y_3^-) f_{\bar{T}_\beta}(z_1^+,z_2^+) f_{\bar{T}_\epsilon}(z_1^-,z_2^-)(y_1^-(z_1^+)^2(z_1^-)^2)^{\frac{a}{2}};$$
  if $0 < \gamma_1 < k-1$ and $\frac{a}{2} \leq k - 1 - \gamma_1$,
  $$f_{T_{(\lambda(1),\lambda(2),\lambda(3),\lambda(4))}}(y_1^-,y_2^-,y_3^-,z_1^+,z_2^+,z_1^-,z_2^-) = g_2(y_1^-,y_2^-,y_3^-,z_1^+,z_2^+,z_1^-,z_2^-)(y_1^-(z_1^+)^2)^{\gamma_1}(z_1^-)^{\rho_1-a},$$
  where
  $$g_2(y_1^-,y_2^-,y_3^-,z_1^+,z_2^+,z_1^-,z_2^-) = f_{\bar{T}_\mu}(y_1^-,y_2^-,y_3^-) f_{\bar{T}_\beta}(z_1^+,z_2^+) f_{\bar{T}_\epsilon}(z_1^-,z_2^-)([y_1^-,y_2^-](z_1^+)^2(z_1^-)^2)^{\frac{a}{2}} ([y_1^-,y_2^-](z_1^+)^2)^{k-1-\gamma_1-\frac{a}{2}};$$

if $0 < \gamma_1 < k-1$ and $\frac{a}{2} > k-1-\gamma_1$,

$$f_{T_{(\emptyset,\lambda(2),\lambda(3),\lambda(4))}}(y_1^-,y_2^-,y_3^-,z_1^+,z_2^+,z_1^-,z_2^-) =$$

$$g_3(y_1^-,y_2^-,y_3^-,z_1^+,z_2^+,z_1^-,z_2^-)(y_1^-(z_1^+)^2)^{k-1-\frac{a}{2}}(z_1^-)^{\rho_1-a},$$

where

$$g_3(y_1^-,y_2^-,y_3^-,z_1^+,z_2^+,z_1^-,z_2^-) =$$

$$f_{\bar{T}_\mu}(y_1^-,y_2^-,y_3^-)f_{\bar{T}_\beta}(z_1^+,z_2^+)f_{\bar{T}_\epsilon}(z_1^-,z_2^-)([y_1^-,y_2^-](z_1^+)^2(z_1^-)^2)^{k-1-\gamma_1}$$

$$(y_1^-(z_1^+)^2(z_1^-)^2)^{\frac{a}{2}-k+1+\gamma_1};$$

if $\gamma_1 = 0$,

$$f_{T_{(\emptyset,\lambda(2),\lambda(3),\lambda(4))}}(y_1^-,y_2^-,y_3^-,z_1^+,z_2^+,z_1^-,z_2^-) =$$

$$g_4(y_1^-,y_2^-,y_3^-,z_1^+,z_2^+,z_1^-,z_2^-)([y_1^-,y_2^-](z_1^+)^2)^{k-1-\frac{a}{2}}(z_1^-)^{\rho_1-a},$$

where

$$g_4(y_1^-,y_2^-,y_3^-,z_1^+,z_2^+,z_1^-,z_2^-) =$$

$$f_{\bar{T}_\mu}(y_1^-,y_2^-,y_3^-)f_{\bar{T}_\beta}(z_1^+,z_2^+)f_{\bar{T}_\epsilon}(z_1^-,z_2^-)([y_1^-,y_2^-](z_1^+)^2(z_1^-)^2)^{\frac{a}{2}}.$$

So, if $\gamma_1 \geq k-1$,

$$f_{T_{(\emptyset,\lambda(2),\lambda(3),\lambda(4))}}(M_1,M_2,M_3,N_1,N_2,P_1+P_2,P_2) =$$

$$\begin{cases} re_{32}(-e_{21})^{\rho_1-a} \neq 0, & \text{if } \gamma_1-k+1 \text{ is even,} \\ re_{22}(-e_{21})^{\rho_1-a} \neq 0, & \text{if } \gamma_1-k+1 \text{ is odd,} \end{cases}$$

where $r = \pm 6^{\gamma_3}2^{\gamma_2}$.
And, if $0 \leq \gamma_1 < k-1$,

$$f_{T_{(\emptyset,\lambda(2),\lambda(3),\lambda(4))}}(M_1+M_3,M_2,M_3,N_1,N_2,P_1,P_2) =$$

$$\begin{cases} \pm 6^{\gamma_3}2^{\frac{1}{2}(s+1)}e_{32}(-e_{21})^{\rho_1-a} \neq 0, & \text{if } t_1+t_2 \text{ is even,} \\ \pm 6^{\gamma_3}2^{\frac{1}{2}s}(e_{22}+e_{32})(-e_{21})^{\rho_1-a} \neq 0, & \text{if } t_1+t_2 \text{ is odd,} \end{cases}$$

where $s = 3\gamma_2 - k + \gamma_1$.

- $w_1 = 2k-1$.
  We consider the multitableau $T_{(\emptyset,\lambda(2),\lambda(3),\lambda(4))}$ such that:
  if $\gamma_1 \geq k-1$,

$$f_{T_{(\emptyset,\lambda(2),\lambda(3),\lambda(4))}}(y_1^-,y_2^-,y_3^-,z_1^+,z_2^+,z_1^-,z_2^-) =$$

$$g_1(y_1^-,y_2^-,y_3^-,z_1^+,z_2^+,z_1^-,z_2^-)(y_1^-(z_1^+)^2)^{k-2-\frac{a}{2}}y_1^-z_1^+(z_1^-)^{\rho_1-a};$$

if $0 < \gamma_1 < k-1$ and $\frac{a}{2} \leq k-1-\gamma_1$,

$$f_{T_{(\emptyset,\lambda(2),\lambda(3),\lambda(4))}}(y_1^-,y_2^-,y_3^-,z_1^+,z_2^+,z_1^-,z_2^-) =$$

$$g_2(y_1^-,y_2^-,y_3^-,z_1^+,z_2^+,z_1^-,z_2^-)(y_1^-(z_1^+)^2)^{\gamma_1-1}y_1^-z_1^+(z_1^-)^{\rho_1-a};$$

if $0 < \gamma_1 < k-1$ and $\frac{a}{2} > k-1-\gamma_1$,

$$f_{T_{(\emptyset,\lambda(2),\lambda(3),\lambda(4))}}(y_1^-,y_2^-,y_3^-,z_1^+,z_2^+,z_1^-,z_2^-) =$$

$$g_3(y_1^-, y_2^-, y_3^-, z_1^+, z_2^+, z_1^-, z_2^-)(y_1^-(z_1^+)^2)^{k-2-\frac{a}{2}} y_1^- z_1^+ (z_1^-)^{\rho_1 - a};$$

if $\gamma_1 = 0$

$$f_{T_{(\emptyset,\lambda(2),\lambda(3),\lambda(4))}}(y_1^-, y_2^-, y_3^-, z_1^+, z_2^+, z_1^-, z_2^-) =$$

$$g_4(y_1^-, y_2^-, y_3^-, z_1^+, z_2^+, z_1^-, z_2^-)([y_1^-, y_2^-](z_1^+)^2)^{k-2-\frac{a}{2}}[y_1^-, y_2^-] z_1^+ (z_1^-)^{\rho_1 - a}.$$

Then, if $\gamma_1 \geq k-1$,

$$f_{T_{(\emptyset,\lambda(2),\lambda(3),\lambda(4))}}(M_1, M_2, M_3, N_1, N_2, P_1 + P_2, P_2) =$$

$$\begin{cases} re_{31}(e_{13} + e_{12})^{\rho_1 - a} \neq 0, & \text{if } t_1 + t_2 \text{ is even,} \\ re_{21}(e_{13} + e_{12})^{\rho_1 - a} \neq 0, & \text{if } t_1 + t_2 \text{ is odd,} \end{cases}$$

where $r = \pm 6^{\gamma_3} 2^{\gamma_2}$.
And, if $0 \leq \gamma_1 < k-1$,

$$f_{T_{(\emptyset,\lambda(2),\lambda(3),\lambda(4))}}(M_1 + M_3, M_2, M_3, N_1, N_2, P_1, P_2) =$$

$$\begin{cases} \pm 6^{\gamma_3} 2^{\frac{1}{2}(s+1)} e_{31}(e_{13})^{\rho_1 - a} \neq 0, & \text{if } t_1 + t_2 \text{ is even,} \\ \pm 6^{\gamma_3} 2^{\frac{1}{2}s}(e_{21} + e_{31})(e_{13})^{\rho_1 - a} \neq 0, & \text{if } t_1 + t_2 \text{ is odd,} \end{cases}$$

where $s = 3\gamma_2 - k + \gamma_1$.

□

Now we are ready to prove the main theorem of this section.

Theorem 3.3.2. *Consider*

$$\chi_n^*(M_{1,2}(F)) = \sum_{\substack{\langle\lambda\rangle \vdash n, \\ h(\lambda(1)) \leq 2,\, h(\lambda(2)) \leq 3, \\ h(\lambda(3)) \leq 2,\, h(\lambda(4)) \leq 2}} m_{\langle\lambda\rangle} \chi_{\langle\lambda\rangle}$$

the n-th $*$-cocharacter of $M_{1,2}(F)$. Let $\langle\lambda\rangle = (\lambda(1), \lambda(2), \lambda(3), \lambda(4))$, with $\lambda(2) = (\gamma_1 + \gamma_2 + \gamma_3, \gamma_2 + \gamma_3, \gamma_3)$, $\lambda(3) = (w_1 + w_2, w_2)$, $\lambda(4) = (\rho_1 + \rho_2, \rho_2)$ and $l = max\{w_1, \rho_1\}$. The following statements hold.

If $\lambda(3) = \lambda(4) = \emptyset$, then $m_{\langle\lambda\rangle} \neq 0$ if and only if $h(\lambda(1)) \leq 1$.

If $m_{\langle\lambda\rangle} \neq 0$, $\lambda(2) = \emptyset$, $\lambda(i) = \emptyset$ and $\lambda(j) \neq \emptyset$, with $i, j \in \{3,4\}$, $i \neq j$, then $|w_1 - \rho_1| \leq 2$.

If $\lambda(j) \neq \emptyset$ for some $j \in \{3,4\}$ and

- *$|w_1 - \rho_1| \leq 2$ or*
- *$|w_1 - \rho_1| \geq 3$ and $\lambda(2) \neq \emptyset$, with $\gamma_1 + \gamma_2 \geq \lceil \frac{l}{2} \rceil - 1$*

then $m_{\langle\lambda\rangle} \neq 0$.

Proof. The first two items follow from Propositions 3.3.1, 3.3.2 and 3.3.3.

Let $\lambda(j) \neq \emptyset$ for some $j \in \{3,4\}$. If $\lambda(2) = \emptyset$ and $|w_1 - \rho_1| \leq 2$, then $m_{\langle\lambda\rangle} \neq 0$ by Propositions 3.3.2, 3.3.3 and 3.3.4. We may assume, since the previous cases hold, that $\lambda(2) \neq \emptyset$. If $\lambda(i) = \emptyset$, for $i \in \{3,4\}$, $i \neq j$, $|w_1 - \rho_1| \leq 2$ or $|w_1 - \rho_1| \geq 3$ and $\gamma_1 + \gamma_2 \geq \lceil \frac{l}{2} \rceil - 1$, then we get the result by Propositions 3.3.5 and 3.3.6. Now, let $\lambda(i) \neq \emptyset$, for $i \in \{3,4\}$, $i \neq j$. If $\lambda(1) = \emptyset$, $|w_1 - \rho_1| \leq 2$ or $|w_1 - \rho_1| \geq 3$ and $\gamma_1 + \gamma_2 \geq \lceil \frac{l}{2} \rceil - 1$, then $m_{\langle\lambda\rangle} \neq 0$ by Proposition 3.3.7.

We are left to prove the case $\lambda(i) \neq \emptyset$ for all $i \in \{1,2,3,4\}$, $|w_1 - \rho_1| \leq 2$ or $|w_1 - \rho_1| \geq 3$ and $\gamma_1 + \gamma_2 \geq \lceil \frac{l}{2} \rceil - 1$. Consider the elements $R_1 = e_{11}$, $R_2 = e_{22} + e_{33}$, $M_1 = e_{23} + e_{32}$, $M_2 = e_{23} - e_{32}$, $M_3 = e_{22} - e_{33}$, $N_1 = e_{12} - e_{31}$, $N_2 = e_{13} + e_{21}$, $P_1 = e_{13} - e_{21}$, $P_2 = e_{12} + e_{31}$.

Because of Theorem 3.2.1, we may suppose $w_1 \geq \rho_1$.

If $w_1 - \rho_1 \leq 2$, we distinguish two cases:

- $w_1 \leq 2$.
  If $\rho_1 \in \{0,1\}$, we consider the multitableau

$$T_{(\lambda(1),\lambda(2),\lambda(3),\lambda(4))} = (T_{\lambda(1)}, \bar{T}_{\lambda(2)}, T_{\lambda(3)}, T_{\lambda(4)}),$$

where $\bar{T}_{\lambda(2)}$ is the initial standard tableau on the integers $1, \ldots, 3\gamma_3 + 2\gamma_2 + \gamma_1$ and $T_{\lambda(1)}$, $T_{\lambda(3)}$ and $T_{\lambda(4)}$ are the tableaux we considered in the proof of Proposition 3.3.4. Then

$$f_{T_{(\lambda(1),\lambda(2),\lambda(3),\lambda(4))}}(y_1^+, y_2^+, y_1^-, y_2^-, y_3^-, z_1^+, z_2^+, z_1^-, z_2^-) =$$

$$f_{\bar{T}_{\lambda(2)}}(y_1^-, y_2^-, y_3^-) f_{T_{(\lambda(1),\emptyset,\lambda(3),\lambda(4))}}(y_1^+, y_2^+, z_1^+, z_2^+, z_1^-, z_2^-),$$

where $f_{T_{(\lambda(1),\emptyset,\lambda(3),\lambda(4))}}(y_1^+, y_2^+, z_1^+, z_2^+, z_1^-, z_2^-)$ is the highest weight vector obtained in Proposition 3.3.4. So,

$$f_{T_{(\lambda(1),\lambda(2),\lambda(3),\lambda(4))}}(R_2, R_1, M_1, M_2, M_3, N_1, N_2, P_1, P_2) =$$

$$\begin{cases} \pm 6^{\gamma_3} 2^{\gamma_2} (\pm e_{22} \pm e_{33})(e_{12} - e_{31})^{w_1} (2^{\rho_2} e_{11} \pm e_{22} \pm e_{33})(e_{13} - e_{21})^{\rho_1}) \neq 0, & \text{if } \gamma_1 \text{ is even,} \\ \pm 6^{\gamma_3} 2^{\gamma_2} (\pm e_{23} \pm e_{32})(e_{12} - e_{31})^{w_1} (2^{\rho_2} e_{11} \pm e_{22} \pm e_{33})(e_{13} - e_{21})^{\rho_1}) \neq 0, & \text{if } \gamma_1 \text{ is odd.} \end{cases}$$

If $\rho_1 = 2$, we consider the following multitableau

$$T_{(\lambda(1),\lambda(2),\lambda(3),\lambda(4))} = (T_{\lambda(1)}, \bar{T}_{\lambda(2)}, T_{\lambda(3)}, T_{\lambda(4)}),$$

where $\bar{T}_{\lambda(2)}$ is the initial standard tableau on the integers $n - 3\gamma_3 - 2\gamma_2 - \gamma_1 + 1, \ldots, n$ and $T_{\lambda(1)}$, $T_{\lambda(3)}$ and $T_{\lambda(4)}$ are the tableaux we considered in the proof of Proposition 3.3.4 on the integers $1, \ldots, n - 3\gamma_3 - 2\gamma_2 - \gamma_1$.
Then

$$f_{T_{(\lambda(1),\lambda(2),\lambda(3),\lambda(4))}}(y_1^+, y_2^+, y_1^-, y_2^-, y_3^-, z_1^+, z_2^+, z_1^-, z_2^-) =$$

$$f_{T_{(\lambda(1),\emptyset,\lambda(3),\lambda(4))}}(y_1^+, y_2^+, z_1^+, z_2^+, z_1^-, z_2^-) f_{\bar{T}_{\lambda(2)}}(y_1^-, y_2^-, y_3^-),$$

where $f_{T_{(\lambda(1),\emptyset,\lambda(3),\lambda(4))}}(y_1^+, y_2^+, z_1^+, z_2^+, z_1^-, z_2^-)$ is the highest weight vector obtained in Proposition 3.3.4.
Then,

$$f_{T_{(\lambda(1),\lambda(2),\lambda(3),\lambda(4))}}(R_2, R_1, M_1, M_2, M_3, N_1, N_2, P_1, P_2) = \begin{cases} re_{33} \neq 0, & \text{if } \gamma_1 \text{ is even,} \\ re_{32} \neq 0, & \text{if } \gamma_1 \text{ is odd,} \end{cases}$$

with $r = \pm 6^{\gamma_3} 2^{\gamma_2}$.

- $w_1 \geq 3$.
  We consider the multitableau

$$T_{(\lambda(1),\lambda(2),\lambda(3),\lambda(4))} = (T_{\lambda(1)}, \bar{T}_{\lambda(2)}, T_{\lambda(3)}, T_{\lambda(4)}),$$

where $\bar{T}_{\lambda(2)}$ is the initial standard tableau on the integers $1, \ldots, 3\gamma_3 + 2\gamma_2 + \gamma_1$ and $T_{\lambda(1)}$, $T_{\lambda(3)}$ and $T_{\lambda(4)}$ are the tableaux we considered in the proof of Proposition 3.3.4

on the integers $3\gamma_3 + 2\gamma_2 + \gamma_1 + 1, \dots, n$.
Then

$$f_{T_{(\lambda(1),\lambda(2),\lambda(3),\lambda(4))}} = f_{\bar{T}_{\lambda(2)}}(y_1^-, y_2^-, y_3^-) f_{T_{(\lambda(1),\emptyset,\lambda(3),\lambda(4))}}(y_1^+, y_2^+, z_1^+, z_2^+, z_1^-, z_2^-),$$

where $f_{T_{(\lambda(1),\emptyset,\lambda(3),\lambda(4))}}(y_1^+, y_2^+, z_1^+, z_2^+, z_1^-, z_2^-)$ is the highest weight vector obtained in Proposition 3.3.4.
If $\rho_1 = w_1 - 2$,

$$f_{T_{(\lambda(1),\lambda(2),\lambda(3),\lambda(4))}}(R_2, R_1, M_1, M_2, M_3, N_1, N_2, P_1, P_2) = \begin{cases} re_{32} \neq 0, & \text{if } \gamma_1 \text{ is even}, \\ re_{22} \neq 0, & \text{if } \gamma_1 \text{ is odd}; \end{cases}$$

if $\rho_1 = w_1 - 1$,

$$f_{T_{(\lambda(1),\lambda(2),\lambda(3),\lambda(4))}}(M_1, M_2, M_3, N_1, N_2, P_1, P_2) = \begin{cases} re_{31} \neq 0, & \text{if } \gamma_1 \text{ is even}, \\ re_{21} \neq 0, & \text{if } \gamma_1 \text{ is odd}; \end{cases}$$

if $\rho_1 = w_1$,

$$f_{T_{(\lambda(1),\lambda(2),\lambda(3),\lambda(4))}}(M_1, M_2, M_3, N_1, N_2, P_1, P_2) = \begin{cases} re_{33} \neq 0, & \text{if } \gamma_1 \text{ is even}, \\ re_{23} \neq 0, & \text{if } \gamma_1 \text{ is odd}, \end{cases}$$

with $r = \pm 6^{\gamma_3} 2^{\gamma_2}$.

Now, if $w_1 - \rho_1 \geq 3$ and $\gamma_1 + \gamma_2 \geq k - 1$, where $k = \lceil \frac{w_1}{2} \rceil$, we define the integer a as in Proposition 3.3.7,

$$a := \begin{cases} \rho_1, & \text{if } \rho_1 \text{ is even}, \\ \rho_1 - 1, & \text{if } \rho_1 \text{ is odd} \end{cases}$$

and we consider the $*$-polynomials $g_i(y_1^-, y_2^-, y_3^-, z_1^+, z_2^+, z_1^-, z_2^-)$ constructed in Proposition 3.3.7.
We distinguish these different cases.

- $w_1 = 2k$.
  We consider the multitableau $T_{(\lambda(1),\lambda(2),\lambda(3),\lambda(4))}$ such that:
  if $\gamma_1 \geq k - 1$,

$$f_{T_{(\lambda(1),\lambda(2),\lambda(3),\lambda(4))}}(y_1^+, y_2^+, y_1^-, y_2^-, y_3^-, z_1^+, z_2^+, z_1^-, z_2^-) =$$

$$\underbrace{\tilde{y}_1^+ \dots \tilde{y}_1^+}_{\alpha_2} (y_1^+)^{\alpha_1} g_1(y_1^-, y_2^-, y_3^-, z_1^+, z_2^+, z_1^-, z_2^-) (y_1^-(z_1^+)^2)^{k-2-\frac{a}{2}} y_1^- z_1^+ \underbrace{\tilde{y}_2^+ \dots \tilde{y}_2^+}_{\alpha_2}$$

$$z_1^+ (z_1^-)^{\rho_1 - a};$$

  if $0 < \gamma_1 < k - 1$ and $\frac{a}{2} \leq k - 1 - \gamma_1$,

$$f_{T_{(\lambda(1),\lambda(2),\lambda(3),\lambda(4))}}(y_1^+, y_2^+, y_1^-, y_2^-, y_3^-, z_1^+, z_2^+, z_1^-, z_2^-) =$$

$$\underbrace{\tilde{y}_1^+ \dots \tilde{y}_1^+}_{\alpha_2} (y_1^+)^{\alpha_1} g_2(y_1^-, y_2^-, y_3^-, z_1^+, z_2^+, z_1^-, z_2^-) (y_1^-(z_1^+)^2)^{\gamma_1 - 1} y_1^- z_1^+ \underbrace{\tilde{y}_2^+ \dots \tilde{y}_2^+}_{\alpha_2} z_1^+ (z_1^-)^{\rho_1 - a};$$

  if $0 < \gamma_1 < k - 1$ and $\frac{a}{2} > k - 1 - \gamma_1$,

$$f_{T_{(\lambda(1),\lambda(2),\lambda(3),\lambda(4))}}(y_1^+, y_1^+, y_1^-, y_2^-, y_3^-, z_1^+, z_2^+, z_1^-, z_2^-) =$$

$$\underbrace{\tilde{y}_1^+ \dots \tilde{y}_1^+}_{\alpha_2} (y_1^+)^{\alpha_1} g_3(y_1^-, y_2^-, y_3^-, z_1^+, z_2^+, z_1^-, z_2^-)(y_1^-(z_1^+)^2)^{k-2-\frac{a}{2}} y_1^- z_1^+ \underbrace{\tilde{y}_2^+ \dots \tilde{y}_2^+}_{\alpha_2}$$

$$z_1^+(z_1^-)^{\rho_1 - a}$$

and if $\gamma_1 = 0$

$$f_{T_{(\lambda(1),\lambda(2),\lambda(3),\lambda(4))}}(y_1^+, y_2^+, y_1^-, y_2^-, y_3^-, z_1^+, z_2^+, z_1^-, z_2^-) =$$

$$\underbrace{\tilde{y}_1^+ \dots \tilde{y}_1^+}_{\alpha_2} (y_1^+)^{\alpha_1} g_4(y_1^-, y_2^-, y_3^-, z_1^+, z_2^+, z_1^-, z_2^-)([y_1^-, y_2^-](z_1^+)^2)^{k-2-\frac{a}{2}} [y_1^-, y_2^-] z_1^+$$

$$\underbrace{\tilde{y}_2^+ \dots \tilde{y}_2^+}_{\alpha_2} z_1^+(z_1^-)^{\rho_1 - a}.$$

So, if $\gamma_1 \geq k-1$,

$$f_{T_{(\lambda(1),\lambda(2),\lambda(3),\lambda(4))}}(R_2, R_1, M_1, M_2, M_3, N_1, N_2, P_1 + P_2, P_2) =$$

$$\begin{cases} re_{32}(-e_{21})^{\rho_1 - a} \neq 0, & \text{if } \gamma_1 - k + 1 \text{ is even,} \\ re_{22}(-e_{21})^{\rho_1 - a} \neq 0, & \text{if } \gamma_1 - k + 1 \text{ is odd,} \end{cases}$$

where $r = \pm 6^{\gamma_3} 2^{\gamma_2}$
and, if $0 \leq \gamma_1 < k-1$,

$$f_{T_{(\lambda(1),\lambda(2),\lambda(3),\lambda(4))}}(R_2, R_1, M_1 + M_3, M_2, M_3, N_1, N_2, P_1, P_2) =$$

$$\begin{cases} \pm 6^{\gamma_3} 2^{\frac{1}{2}(s+1)} e_{32}(-e_{21})^{\rho_1 - a} \neq 0, & \text{if } t_1 + t_2 \text{ is even,} \\ \pm 6^{\gamma_3} 2^{\frac{1}{2}s}(e_{22} + e_{32})(-e_{21})^{\rho_1 - a} \neq 0, & \text{if } t_1 + t_2 \text{ is odd,} \end{cases}$$

where $s = 3\gamma_2 - k + \gamma_1$.

- $w_1 = 2k - 1$.
  We consider the multitableau $T_{(\lambda(1),\lambda(2),\lambda(3),\lambda(4))}$ such that

$$f_{T_{(\lambda(1),\lambda(2),\lambda(3),\lambda(4))}}(y_1^+, y_2^+, y_1^-, y_2^-, y_3^-, z_1^+, z_2^+, z_1^-, z_2^-) =$$

$$\underbrace{\tilde{y}_1^+ \dots \tilde{y}_1^+}_{\alpha_2} (y_1^+)^{\alpha_1} f^{\eta}(y_1^+, y_2^+, y_1^-, y_2^-, y_3^-, z_1^+, z_2^+, z_1^-, z_2^-) \underbrace{\tilde{y}_2^+ \dots \tilde{y}_2^+}_{\alpha_2} (z_1^-)^{\rho_1 - a},$$

where $f^{\eta}(y_1^+, y_2^+, y_1^-, y_2^-, y_3^-, z_1^+, z_2^+, z_1^-, z_2^-)$ is such that

$$f_{T_{(\emptyset,\lambda(2),\lambda(3),\lambda(4))}} = f^{\eta}(y_1^+, y_2^+, y_1^-, y_2^-, y_3^-, z_1^+, z_2^+, z_1^-, z_2^-)(z_1^-)^{\rho_1 - a}$$

and $f_{T_{(\emptyset,\lambda(2),\lambda(3),\lambda(4))}}$ is the highest weight vector constructed in Proposition 3.3.7.
Then, if $\gamma_1 \geq k-1$,

$$f_{T_{(\lambda(1),\lambda(2),\lambda(3),\lambda(4))}}(R_2, R_1, M_1, M_2, M_3, N_1, N_2, P_1 + P_2, P_2) =$$

$$\begin{cases} \pm re_{31}(e_{13} + e_{12})^{\rho_1 - a} \neq 0, & \text{if } \gamma_1 - k + 1 \text{ is even,} \\ \pm re_{21}(e_{13} + e_{12})^{\rho_1 - a} \neq 0, & \text{if } \gamma_1 - k + 1 \text{ is odd,} \end{cases}$$

where $r = 6^{\gamma_3} 2^{\gamma_2}$.
If $0 \leq \gamma_1 < k-1$,

$$f_{T_{(\lambda)}}(R_2, R_1, M_1 + M_3, M_2, M_3, N_1, N_2, P_1, P_2) =$$

$$\begin{cases} \pm 6^{\gamma_3} 2^{\frac{1}{2}(s+1)} e_{31}(e_{13})^{\rho_1 - a} \neq 0, & \text{if } t_1 + t_2 \text{ is even,} \\ \pm 6^{\gamma_3} 2^{\frac{1}{2}s}(e_{21} + e_{31})(e_{13})^{\rho_1 - a} \neq 0, & \text{if } t_1 + t_2 \text{ is odd,} \end{cases}$$

where $s = 3\gamma_2 - k + \gamma_1$.

So, the proof is complete. □

By Conjectures 3.3.1, 3.3.2 and Theorem 3.3.2, we are led to conjecture the following.

Conjecture 3.3.3. *Consider*

$$\chi_n^*(M_{1,2}(F)) = \sum_{\substack{\langle\lambda\rangle \vdash n, \\ h(\lambda(1)) \leq 2,\ h(\lambda(2)) \leq 3, \\ h(\lambda(3)) \leq 2,\ h(\lambda(4)) \leq 2}} m_{\langle\lambda\rangle} \chi_{\langle\lambda\rangle}$$

the n-th $*$-cocharacter of $M_{1,2}(F)$. Let $\langle\lambda\rangle = (\lambda(1), \lambda(2), \lambda(3), \lambda(4))$, with $\lambda(2) = (\gamma_1 + \gamma_2 + \gamma_3, \gamma_2 + \gamma_3, \gamma_3)$, $\lambda(3) = (w_1 + w_2, w_2)$, $\lambda(4) = (\rho_1 + \rho_2, \rho_2)$ and $l = max\{w_1, \rho_1\}$. Then $m_{\langle\lambda\rangle} \neq 0$ if and only if $\lambda(3) = \lambda(4) = \emptyset$ and $h(\lambda(1)) \leq 1$ or $\lambda(j) \neq \emptyset$ for some $j \in \{3,4\}$ and $|w_1 - \rho_1| \leq 2$ or $\lambda(j) \neq \emptyset$ for some $j \in \{3,4\}$, $|w_1 - \rho_1| \geq 3$ and $\lambda(2) \neq \emptyset$, with $\gamma_1 + \gamma_2 \geq \lceil \frac{l}{2} \rceil - 1$.

Chapter 4

On the polynomial codimension growth of superalgebras with superautomorphism

This chapter present further results obtained during my PhD on associative algebras with additional structure, focusing specifically on superalgebras with superautomorphism. One can find these results in [2]. We characterize the superalgebras endowed with a superautomorphism of order ≤ 2 with multiplicities of the cocharacter bounded by a constant. Moreover, we determine a characterization of such superalgebras with polynomial growth of the codimensions and we give a classification of the subvarieties of the varieties of almost polynomial growth. Finally, we characterize superalgebras with superautomorphism with linear codimension growth.

4.1 Superalgebras with superautomorphism

In this section we give all the definitions useful in order to understand the main results of this chapter.

Let $A = A_0 \oplus A_1$ be an associative superalgebra over F, a field of characteristic zero, and assume that A is endowed with a superautomorphism that is a graded linear map $\varphi : A \longrightarrow A$, i.e., a map preserving the grading, such that for any homogeneous elements $a, b \in A_0 \cup A_1$,

$$(a^\varphi)^\varphi = a,$$

$$(ab)^\varphi = (-1)^{|a||b|} a^\varphi b^\varphi.$$

Here $|c|$ denotes the homogeneous degree of the element $c \in A_0 \cup A_1$.

Since char$F = 0$, we write $A = A_0^+ \oplus A_0^- \oplus A_1^+ \oplus A_1^-$, where, for $i = 0, 1$, $A_i^+ = \{a \in A_i \mid a^\varphi = a\}$ and $A_i^- = \{a \in A_i \mid a^\varphi = -a\}$ denote the sets of symmetric and skew elements of A_i, respectively.

From now on we shall refer to a superalgebra with superautomorphism φ as a φ-algebra.

We notice that there is a one-to-one correspondence between $\mathbb{Z}_4$-gradings and superautomorphisms. Let $G = \{1_G = g^0, g = g^1, g^2, g^3\} \cong \mathbb{Z}_4$. Consider A as a G-graded algebra, i.e., $A = A_{1_G} \oplus A_g \oplus A_{g^2} \oplus A_{g^3}$ such that $A_{g^i} A_{g^j} \subseteq A_{g^i g^j}$, $i, j \in \{0, \ldots, 3\}$. We can see A as a $\mathbb{Z}_2$-graded algebra, $A = A_0 \oplus A_1$, where $A_0 = A_{1_G} \oplus A_{g^2}$ and $A_1 = A_g \oplus A_{g^3}$. Then A can be endowed with a superautomorphism $\varphi : A \to A$ such that $\varphi(a_{1_G} + a_g + a_{g^2} + a_{g^3}) = a_{1_G} + a_g - a_{g^2} - a_{g^3}$, for all $a_{1_G} \in A_{1_G}$, $a_g \in A_g$, $a_{g^2} \in A_{g^2}$ and $a_{g^3} \in A_{g^3}$. Conversely, let $A = A_0 \oplus A_1$ be endowed with a superautomorphism φ and let $A = A_0^+ \oplus A_0^- \oplus A_1^+ \oplus A_1^-$, where $A_i^+ = \{a \in A_i \mid a^\varphi = a\}$ and $A_i^- = \{a \in A_i \mid a^\varphi = -a\}$, for $i = 0,1$. Then A is a G-graded algebra with grading $A_{1_G} = A_0^+$, $A_g = A_1^+$, $A_{g^2} = A_0^-$ and $A_{g^3} = A_1^-$.

As in the previous chapter, we write $F\langle Y \cup Z\rangle = \mathcal{F}_0 \oplus \mathcal{F}_1$ as the free associative superalgebra on the countable set $Y \cup Z$ over F.

One can define in a natural way a superautomorphism φ on the free associative superalgebra $F\langle Y \cup Z\rangle$. We shall write $F\langle Y \cup Z, \varphi\rangle$ for the free associative φ-algebra on the countable set $Y \cup Z$ over F. If we let $y_i^+ = y_i + y_i^\varphi$, $y_i^- = y_i - y_i^\varphi$, $z_i^+ = z_i + z_i^\varphi$ and $z_i^- = z_i - z_i^\varphi$, for $i = 1, 2, \dots$, then

$$F\langle Y \cup Z, \varphi\rangle = F\langle y_1^+, y_1^-, z_1^+, z_1^-, y_2^+, y_2^-, z_2^+, z_2^-, \dots\rangle.$$

A polynomial $f(y_1^+, \dots, y_m^+, y_1^-, \dots y_n^-, z_1^+, \dots, z_r^+, z_1^-, \dots, z_s^-) \in F\langle Y \cup Z, \varphi\rangle$ is a φ-polynomial identity of A (or a φ-identity) and we write $f \equiv 0$, if

$$f(u_1^+, \dots, u_m^+, u_1^-, \dots, u_n^-, v_1^+, \dots, v_r^+, v_1^-, \dots, v_s^-) = 0,$$

for all $u_1^+, \dots, u_m^+ \in A_0^+$, $u_1^-, \dots, u_n^- \in A_0^-$, $v_1^+, \dots, v_r^+ \in A_1^+$, $v_1^-, \dots, v_s^- \in A_1^-$.

The set of all φ-identities of A, $\mathrm{Id}^\varphi(A)$, is a T_2^φ-ideal of $F\langle Y \cup Z, \varphi\rangle$, i.e., it is an ideal invariant under all graded endomorphisms of $F\langle Y \cup Z, \varphi\rangle$ commuting with the superautomorphism φ.

It is known that in characteristic zero every φ-identity is equivalent to a system of multilinear ones. Hence, if we denote by

$$P_n^\varphi = \mathrm{span}_F\{w_{\sigma(1)} \cdots w_{\sigma(n)} \mid \sigma \in S_n,\ w_i \in \{y_i^+, y_i^-, z_i^+, z_i^-\},\ i = 1, \dots, n\}$$

the space of multilinear polynomials of degree n in the variables $y_i^+, y_i^-, z_i^+, z_i^-$, for $i = 1, \dots, n$, the study of $\mathrm{Id}^\varphi(A)$ is equivalent to the study of $P_n^\varphi \cap \mathrm{Id}^\varphi(A)$, for all $n \geq 1$.

The non-negative integer

$$c_n^\varphi(A) = dim_F \frac{P_n^\varphi}{P_n^\varphi \cap \mathrm{Id}^\varphi(A)},\ n \geq 1,$$

is called the n-th φ-codimension of A.

Let $n \geq 1$ and write $n = n_1 + n_2 + n_3 + n_4$ as a sum of four non-negative integers. We denote by $P_{n_1,\dots,n_4} \subseteq P_n^\varphi$ the vector space of the multilinear polynomials in which the first n_1 variables are even symmetric, the next n_2 are even skew, the next n_3 are odd symmetric and the last n_4 are odd skew.

If $c_{n_1,\dots,n_4}(A) = dim_F \frac{P_{n_1,\dots,n_4}}{P_{n_1,\dots,n_4} \cap \mathrm{Id}^\varphi(A)}$, we have

$$c_n^\varphi(A) = \sum_{n_1+\cdots+n_4=n} \binom{n}{n_1, \dots, n_4} c_{n_1,\dots,n_4}(A), \tag{4.1}$$

where $\binom{n}{n_1,\dots,n_4} = \frac{n!}{n_1! \cdots n_4!}$ is the multinomial coefficient.

The group $S_{n_1} \times \cdots \times S_{n_4}$ acts on the left on the vector space $P_{n_1,\dots,n_4}$ by permuting the variables of the same homogeneous degree which are all symmetric or all skew at the same time. So S_{n_1} permutes the even symmetric variables, S_{n_2} permutes the even skew variables, S_{n_3} permutes the odd symmetric variables and S_{n_4} permutes the odd skew variables. In this way $P_{n_1,\dots,n_4}$ becomes an $S_{n_1} \times \cdots \times S_{n_4}$-module.

Since $P_{n_1,\dots,n_4} \cap \mathrm{Id}^\varphi(A)$ is invariant under this action, we get that the vector space

$$P_{n_1,\dots,n_4}(A) = \frac{P_{n_1,\dots,n_4}}{P_{n_1,\dots,n_4} \cap \mathrm{Id}^\varphi(A)}$$

has an induced structure of $S_{n_1} \times \cdots \times S_{n_4}$-module. We denote by $\chi^\varphi_{n_1,\dots,n_4}(A)$ its character, which is called the $(n_1,\dots,n_4)$*-th cocharacter* of A.

It is well known that there is a one-to-one correspondence between partitions of n and irreducible S_n-characters. So, if $\lambda \vdash n$ is a partition of n, we denote by χ_λ the corresponding irreducible S_n-character. Since $\mathrm{char} F = 0$, then, by complete reducibility, $\chi^\varphi_{n_1,\dots,n_4}(A)$ can be written as a sum of irreducible characters:

$$\chi^\varphi_{n_1,\dots,n_4}(A) = \sum_{\langle\lambda\rangle \vdash n} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}, \tag{4.2}$$

where $\langle\lambda\rangle = (\lambda(1), \lambda(2), \lambda(3), \lambda(4)) \vdash (n_1,\dots,n_4)$ or $\langle\lambda\rangle \vdash n$, is a multipartition of $n = n_1 + \cdots + n_4$, with $\lambda(i) \vdash n_i$ partitions, $i = 1,\dots,4$, and $m_{\langle\lambda\rangle} \geq 0$ is the multiplicity of $\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}$ in $\chi^\varphi_{n_1,\dots,n_4}(A)$.

For $n \geq 1$ we define the n-th φ-colength of A as

$$l^\varphi_n(A) = \sum_{\substack{\langle\lambda\rangle \vdash (n_1,\dots,n_4) \\ n_1+\cdots+n_4=n}} m_{\langle\lambda\rangle}.$$

We conclude this section by recalling the classification of simple φ-algebras.

If A is a superalgebra, a subset (subalgebra, ideal) $S \subseteq A$ is a graded subset (subalgebra, ideal) of A if $S = (S \cap A_0) \oplus (S \cap A_1)$. Furthermore, an ideal (subalgebra) I of A is a φ-ideal (φ-subalgebra) of A if it is a graded ideal (subalgebra) and $I^\varphi = I$. So, we are ready for the following definition.

Definition 4.1.1. Let A be a φ-algebra such that $A^2 \neq 0$. We say that A is

- simple, as an ordinary algebra, if it has no non-trivial ideals;
- simple, as a superalgebra or super simple, if it has no non-trivial graded ideals;
- simple, as a φ-algebra or φ-simple, if it has no non-trivial φ-ideals.

We recall the Wedderburn-Malcev Theorem for φ-algebras.

Theorem 4.1.1. *[25, Theorem 9] Let A be a finite dimensional φ-algebra over a field F of characteristic zero. Then there exists a semisimple φ-subalgebra A' such that $A = A' + J(A)$ and $J(A)$ is a φ-ideal of A. Moreover $A' = A_1 \oplus \cdots \oplus A_m$ where $A_1,\dots,A_m$ are φ-simple algebras.*

In order to present the classification of φ-simple algebras, given a superalgebra B, we need to define $\bar{B}$ as the superalgebra with the same graded vector space structure as B and product $\circ$ given on homogeneous elements a, b by the formula $a \circ b := (-1)^{|a||b|}ab$.

The algebra $B \oplus \bar{B}$ is a superalgebra with superautomorphism $ex : B \oplus \bar{B} \longrightarrow B \oplus \bar{B}$ defined as $(a,b)^{ex} = (b,a)$.

Given two superalgebras with superautomorphism (A,φ) and (C,ψ) we say that they are isomorphic (as φ-algebras) if there exists a graded isomorphism of algebras $\tau : A \longrightarrow C$ such that $\tau(a^\varphi) = \tau(a)^\psi$, for any $a \in A$.

Here is the classification of φ-simple algebras.

Theorem 4.1.2. *[25, Theorem 20] Let A be a finite dimensional simple superalgebra with superautomorphism of order ≤ 2 over an algebraically closed field F of characteristic zero. Then A is isomorphic to one of the following:*

(1) $M_{k,h}(F)$, *with superautomorphism φ defined as*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{\varphi} = \begin{pmatrix} PAP & PBQ \\ -QCP & QDQ \end{pmatrix},$$

where $P = \begin{pmatrix} I_{k_1} & 0 \\ 0 & -I_{k_2} \end{pmatrix}$, $Q = \begin{pmatrix} I_{h_1} & 0 \\ 0 & -I_{h_2} \end{pmatrix}$, I_{k_1}, I_{k_2}, I_{h_1}, I_{h_2} *are the identity matrices of orders* k_1, k_2, h_1, h_2, *respectively,* $k = k_1 + k_2$, $h = h_1 + h_2$, $k_1 \geq k_2$ *and* $h_1 \geq h_2$.

(2) $M_{k,h}(F) \oplus \overline{M_{k,h}(F)}$ *with the exchange superautomorphism ex.*

(3) $Q(n) \oplus \overline{Q(n)}$ *with the exchange superautomorphism ex.*

Here $M_{k,h}(F) = \left\{ \begin{pmatrix} X & 0 \\ 0 & T \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix} \right\}$ *denotes the* $\mathbb{Z}_2$*-graded algebra of* $n \times n$ *matrices,* $n = k + h$, $k \geq h \geq 0$, *where* X, Y, Z, T *are* $k \times k$, $k \times h$, $h \times k$, $h \times h$ *matrices, respectively.*
And $Q(n) = M_n(F \oplus cF) = Q(n)_0 \oplus Q(n)_1$, *where* $Q(n)_0 = M_n(F)$ *and* $Q(n)_1 = cM_n(F)$, *with* $c^2 = 1$.

4.2 Varieties of almost polynomial growth generated by φ-algebras

The purpose of this section is to introduce the φ-algebras generating varieties of almost polynomial growth (see [25]). We shall present their T_2^{φ}-ideals, the multiplicities of their cocharacters and φ-colengths.

We start with the following definition.

Definition 4.2.1. Given a variety of φ-algebras $\mathcal{V}$, the growth of $\mathcal{V}$ is the growth of the sequence of φ-codimensions of any φ-algebra A generating $\mathcal{V}$. Furthermore, we say that $\mathcal{V}$ has polynomial growth if $c_n^{\varphi}(\mathcal{V})$ is polynomially bounded and that $\mathcal{V}$ has almost polynomial growth if $c_n^{\varphi}(\mathcal{V})$ is not polynomially bounded but every proper subvariety of $\mathcal{V}$ has polynomial growth.

We recall the following [25, Remark 2].

Remark 4.2.1. Let $A = A_0 \oplus A_1$ be a superalgebra.

1. If $A_1^2 = 0$ then the superautomorphisms on A coincide with the graded automorphisms on A, i.e., automorphisms preserving the grading. In particular, if $A_1 = 0$ then the superautomorphisms on A coincide with the automorphisms on A.

2. If A is commutative then the superautomorphisms on A of order ≤ 2 coincide with the superinvolutions on A.

Given polynomials $f_1, \ldots, f_n \in F\langle Y \cup Z, \varphi \rangle$, we write $\mathrm{Id}^{\varphi}(A) = \langle f_1, \ldots, f_n \rangle_{T_2^{\varphi}}$ if the T_2^{φ}-ideal is generated by $f_1, \ldots, f_n$.

The two dimensional commutative algebra $F \oplus F$ with trivial grading and exchange superautomorphism φ given by $(a,b)^\varphi = (b,a)$ is a φ-algebra, with

$$\mathrm{Id}^\varphi(F \oplus F) = \langle [x_1, x_2], z^+, z^- \rangle_{T_2^\varphi},$$

for any $x_1, x_2 \in Y \cup Z$. Now, if $\chi^\varphi_{n_1,\dots,n_4}(F \oplus F) = \sum_{\langle\lambda\rangle \vdash n} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}$ is the $(n_1, \dots, n_4)$-th cocharacter of $F \oplus F$, $n_1 + \cdots + n_4 = n$, then (see [18])

$$m_{\langle\lambda\rangle} = \begin{cases} 1, & \text{if } \langle\lambda\rangle = ((n_1),(n_2),\emptyset,\emptyset)),\ n_1 + n_2 = n \\ 0, & \text{otherwise} \end{cases}.$$

As a consequence,

$$l_n^\varphi(F \oplus F) = n + 1, \text{ for all } n \geq 1. \tag{4.3}$$

Now we consider the algebra

$$UT_2 = UT_2(F) = \left\{ \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \mid a, b, c \in F \right\}$$

of 2×2 upper triangular matrices over F.

We consider two non-isomorphic $\mathbb{Z}_2$-gradings in order to see UT_2 as a superalgebra:

$$\text{Trivial grading: } UT_2 = \left\{ \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\};$$

$$\text{Natural grading: } UT_2^{gr} = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \right\}.$$

Moreover, we define the following two automorphisms on UT_2, that are graded automorphisms on UT_2 and UT_2^{gr}:

$$\text{Trivial automorphism } id\text{: } \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}^{id} = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix};$$

$$\text{Natural automorphism } sup\text{: } \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}^{sup} = \begin{pmatrix} a & -c \\ 0 & b \end{pmatrix}.$$

Let $A = A_0 \oplus A_1$ be a superalgebra. We say that A is endowed with the trivial superautomorphism φ, if $A_1 = 0$ and φ is the identity map.

Since in both gradings $(UT_2)_1^2 = 0$, by Remark 4.2.1, we get four φ-algebras:

- UT_2 is the algebra UT_2 with trivial grading and trivial superautomorphism;
- UT_2^{sup} is the algebra UT_2 with trivial grading and natural superautomorphism;
- UT_2^{gr} is the algebra UT_2 with natural grading and trivial superautomorphism;
- $UT_2^{gr,sup}$ is the algebra UT_2 with natural grading and natural superautomorphism.

Definition 4.2.2. Given two φ-algebras A and B, we say that A is T_2^φ-equivalent to B, and we write $A \sim_{T_2^\varphi} B$, if $\mathrm{Id}^\varphi(A) = \mathrm{Id}^\varphi(B)$.

The algebras defined before are not T_2^{φ}-equivalent, indeed:
$\mathrm{Id}^{\varphi}(UT_2) = \langle [y_1^+, y_2^+][y_3^+, y_4^+], y^-, z^+, z^- \rangle_{T_2^{\varphi}}$,
$\mathrm{Id}^{\varphi}(UT_2^{sup}) = \langle [y_1^+, y_2^+], y_1^- y_2^-, z^+, z^- \rangle_{T_2^{\varphi}}$,
$\mathrm{Id}^{\varphi}(UT_2^{gr}) = \langle [y_1^+, y_2^+], y^-, z_1^+ z_2^+, z^- \rangle_{T_2^{\varphi}}$,
$\mathrm{Id}^{\varphi}(UT_2^{gr,sup}) = \langle [y_1^+, y_2^+], y^-, z^+, z_1^- z_2^- \rangle_{T_2^{\varphi}}$.
Now, if $\chi^{\varphi}_{n_1,\dots,n_4}(UT_2) = \sum_{\langle\lambda\rangle\vdash n} m_{\langle\lambda\rangle}\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}$ is the $(n_1,\dots,n_4)$-th cocharacter of UT_2, $n_1 + \cdots + n_4 = n$, then, by [7],

$$m_{\langle\lambda\rangle} = \begin{cases} 1, & \text{if } \langle\lambda\rangle = ((n),\emptyset,\emptyset,\emptyset)) \\ p-r+1, & \text{if } \langle\lambda\rangle = ((p,r),\emptyset,\emptyset,\emptyset),\ p+r=n \\ p-r+1, & \text{if } \langle\lambda\rangle = ((p,r,1),\emptyset,\emptyset,\emptyset),\ p+r=n-1 \\ 0, & \text{otherwise} \end{cases}. \tag{4.4}$$

As a consequence,

$$l_n^{\varphi}(UT_2) = \frac{1}{2}n^2 + \frac{5}{2}n + 4. \tag{4.5}$$

If $\chi^{\varphi}_{n_1,\dots,n_4}(UT_2^{sup}) = \sum_{\langle\lambda\rangle\vdash n} m_{\langle\lambda\rangle}\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}$ is the $(n_1,\dots,n_4)$-th cocharacter of UT_2^{sup}, $n_1 + \cdots + n_4 = n$, then, by [46],

$$m_{\langle\lambda\rangle} = \begin{cases} 1, & \text{if } \langle\lambda\rangle = ((n),\emptyset,\emptyset,\emptyset) \\ r+1, & \text{if } \langle\lambda\rangle = ((p+r,p),(1),\emptyset,\emptyset),\ 2p+r=n-1 \\ 0, & \text{otherwise} \end{cases}. \tag{4.6}$$

Then,

$$l_n^{\varphi}(UT_2^{sup}) = \begin{cases} \frac{n^2-2n+9}{4}, & \text{if } n \text{ is odd} \\ \frac{n^2-2n+8}{4}, & \text{if } n \text{ is even} \end{cases}. \tag{4.7}$$

By [46], if $\chi^{\varphi}_{n_1,\dots,n_4}(UT_2^{gr}) = \sum_{\langle\lambda\rangle\vdash n} m_{\langle\lambda\rangle}\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}$ is the $(n_1,\dots,n_4)$-th cocharacter of UT_2^{gr}, $n_1 + \cdots + n_4 = n$, then

$$m_{\langle\lambda\rangle} = \begin{cases} 1, & \text{if } \langle\lambda\rangle = ((n),\emptyset,\emptyset,\emptyset) \\ r+1, & \text{if } \langle\lambda\rangle = ((p+r,p),\emptyset,(1),\emptyset),\ 2p+r=n-1 \\ 0, & \text{otherwise} \end{cases} \tag{4.8}$$

and

$$l_n^{\varphi}(UT_2^{gr}) = \begin{cases} \frac{n^2-2n+9}{4}, & \text{if } n \text{ is odd} \\ \frac{n^2-2n+8}{4}, & \text{if } n \text{ is even} \end{cases}. \tag{4.9}$$

Finally, similarly to the previous cases, if $\chi^{\varphi}_{n_1,\dots,n_4}(UT_2^{gr,sup}) = \sum_{\langle\lambda\rangle\vdash n} m_{\langle\lambda\rangle}\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}$ is the $(n_1,\dots,n_4)$-th cocharacter of $UT_2^{gr,sup}$, $n_1 + \cdots + n_4 = n$, then

$$m_{\langle\lambda\rangle} = \begin{cases} 1, & \text{if } \langle\lambda\rangle = ((n),\emptyset,\emptyset,\emptyset) \\ r+1, & \text{if } \langle\lambda\rangle = ((p+r,p),\emptyset,\emptyset,(1)),\ 2p+r=n-1 \\ 0, & \text{otherwise} \end{cases} \tag{4.10}$$

and

$$l_n^{\varphi}(UT_2^{sup}) = \begin{cases} \frac{n^2-2n+9}{4}, & \text{if } n \text{ is odd} \\ \frac{n^2-2n+8}{4}, & \text{if } n \text{ is even} \end{cases}. \tag{4.11}$$

The above φ-algebras characterize the varieties of polynomial growth.

Theorem 4.2.1. *[25, Theorem 23] Let A be a finite dimensional φ-algebra over a field F of characteristic zero. Then the sequence $c_n^{\varphi}(A)$, $n = 1, 2, \dots$, is polynomially bounded if and only if UT_2, UT_2^{sup}, UT_2^{gr}, $UT_2^{gr,sup}$, $F \oplus F \notin var^{\varphi}(A)$.*

As a consequence, we have the following.

Corollary 4.2.1. *[25, Corollary 24] The algebras UT_2, UT_2^{sup}, UT_2^{gr}, $UT_2^{gr,sup}$, $F \oplus F$ are the only finite dimensional φ-algebras generating varieties of almost polynomial growth.*

4.3 Characterizing φ-algebras with polynomial codimension growth

In this section we present some results about the characterization of φ-algebras with polynomial growth of the codimensions.

In order to prove the first characterization of this section, we need the following lemma.

Lemma 4.3.1. *Let $\bar{F}$ be the algebraic closure of the field F and let A be a finite dimensional φ-algebra over $\bar{F}$ such that $dim_{\bar{F}} A/J(A) \leq 1$. Then $A \sim_{T_2^{\varphi}} B$ for some finite dimensional φ-algebra B over F with $dim_{\bar{F}} A/J(A) = dim_F B/J(B)$.*

Proof. Since $dim_{\bar{F}} A/J(A) \leq 1$, then $A \cong \bar{F} + J(A)$ or $A \cong J(A)$ is a nilpotent algebra. Let $\{w_1, \dots, w_p\}$ be a φ-basis (i.e., consisting of even and odd symmetric and even and odd skew elements) of $J(A)$ over $\bar{F}$ and let B be the φ-algebra generated by $\mathcal{B} = \{1_{\bar{F}}, w_1, \dots w_p\}$ or by $\mathcal{B} = \{w_1, \dots, w_p\}$ according as $A \cong \bar{F} + J(A)$ or $A \cong J(A)$, respectively. Then, $dim_F B/J(B) = dim_{\bar{F}} A/J(A)$ and $\mathrm{Id}^{\varphi}(A) \subseteq \mathrm{Id}^{\varphi}(B)$. While, if f is a multilinear φ-identity of B, then f vanishes on $\mathcal{B}$. But $\mathcal{B}$ is also a basis of A over $\bar{F}$. So, $\mathrm{Id}^{\varphi}(B) \subseteq \mathrm{Id}^{\varphi}(A)$. Then we conclude that $A \sim_{T_2^{\varphi}} B$. □

Theorem 4.3.1. *Let A be a finite dimensional φ-algebra over a field F of characteristic zero. Then $c_n^{\varphi}(A)$, $n = 1, 2, \dots$, is polynomially bounded if and only if $A \sim_{T_2^{\varphi}} B$, where $B = B_1 \oplus \cdots \oplus B_m$, with $B_1, \dots, B_m$ finite dimensional φ-algebras over F and $dim_F B_i/J(B_i) \leq 1$, for all $i = 1, \dots, m$.*

Proof. Suppose first that $A \sim_{T_2^{\varphi}} B$, with $B = B_1 \oplus \cdots \oplus B_m$, with $B_1, \dots, B_m$ finite dimensional φ-algebras over F and $dim Bi/J(B_i) \leq 1$, for all $i = 1, \dots, m$. Then $c_n^{\varphi}(A) = c_n^{\varphi}(B) \leq c_n^{\varphi}(B_1) + \cdots + c_n^{\varphi}(B_m)$ and the claim follows since $c_n^{\varphi}(B_i)$ is polynomially bounded, for all $i = 1, \dots, m$, by Theorem 4.1.1, [25, Lemma 22] and Theorem 4.2.1.

Conversely, let $c_n^{\varphi}(A)$ be polynomially bounded. Suppose that F is algebraically closed. By Theorem 4.1.1, $A = A_1 \oplus \cdots \oplus A_l + J$ and, by the hypothesis, for all $i = 1, \dots, l$, $A_i \cong F$ is endowed with the trivial superautomorphism and $A_i J A_k = 0$, for all $1 \leq i, k \leq l$, $i \neq k$ (see the proof of [32, Theorem 2.2]). Set $B_1 = A_1 + J, \dots, B_l = A_l + J$, so $dim B_i/J(B_i) \leq 1$. We notice that $\mathrm{Id}^{\varphi}(A) \subseteq \mathrm{Id}^{\varphi}(B_1 \oplus \cdots \oplus B_l)$. Now we consider $f \in \mathrm{Id}^{\varphi}(B_1 \oplus \cdots \oplus B_l)$ and we suppose $f \notin \mathrm{Id}^{\varphi}(A)$. We may assume f is multilinear. We consider a basis of A as the union of a basis of $A_1 \oplus \cdots \oplus A_l$ and a basis of J, let $u_1, \dots, u_t$ elements of this basis such that $f(u_1, \dots, u_t) \neq 0$. But $f \in \mathrm{Id}^{\varphi}(J)$, then there exists at least an element $u_k \notin J$, so $u_k \in A_i$, for some i. We remark that $A_i A_j = A_j A_i = A_i J A_j = A_j J A_i = 0$, for all $j \neq i$, then $u_1, \dots, u_t \in A_i \cup J$, then $u_1, \dots, u_t \in A_i + J = B_i$ and this contradicts the fact that $f \in \mathrm{Id}^{\varphi}(B_i)$. Then

$A \sim_{T_2^\varphi} B_1 \oplus \cdots \oplus B_l + J$.
Now suppose F is arbitrary. Let $\bar{A} = A \otimes_F \bar{F}$, where $\bar{F}$ is the algebraic closure of F and $\bar{A}$ is a superalgebra with the induced superautomorphism $(a \otimes \alpha)^\varphi = a^\varphi \otimes \alpha$, for all $a \in A$ and $\alpha \in \bar{F}$. Clearly, $A \sim_{T_2^\varphi} \bar{A}$ and $c_n^\varphi(A) = c_n^\varphi(\bar{A})$, $n = 1, 2, \ldots$. Then also $c_n(\bar{A})$ is polynomially bounded. Then, by the first part of the proof, $\bar{A} = B_1 \oplus \cdots \oplus B_m$, where $B_1, \ldots, B_m$ are finite dimensional φ-algebras over $\bar{F}$ and $dim_{\bar{F}} B_i / J(B_i) \leq 1$, for all $i = 1, \ldots, m$. By Lemma 4.3.1 there exist finite dimensional φ-algebras $C_1, \ldots, C_m$ over F such that $C_i \sim_{T_2^\varphi} B_i$ and $dim_F C_i / J(C_i) = dim_{\bar{F}} B_i / J(B_i) \leq 1$, for all $i = 1, \ldots, m$. Then $\mathrm{Id}^\varphi(A) = \mathrm{Id}^\varphi(\bar{A}) = \mathrm{Id}^\varphi(B_1 \oplus \cdots \oplus B_m) = \mathrm{Id}^\varphi(C_1 \oplus \cdots \oplus C_m)$ and we are done. □

We recall that given a partition $\lambda = (\lambda_1, \ldots, \lambda_m) \vdash n$, we denote by T_λ the associated tableau and by d_λ the degree of the corresponding irreducible character. If $f(n)$ and $g(n)$ are sequences (or functions $\mathbb{N} \longrightarrow \mathbb{R}$) we write $f \approx g$ if there exist positive constants a and b such that $ag(n) \leq f(n) \leq bg(n)$, for all (large enough) n.

Now we recall two technical results of [32].

Proposition 4.3.1. *[32, Proposition 2.1] Let $\lambda = (\lambda_1, \ldots, \lambda_m) \vdash n$ be a partition of n. If $\lambda_1 = n - r$ then $d_\lambda \approx cn^r$ for some constant c.*

Proposition 4.3.2. *[32, Proposition 2.2] Let $n = n_1 + \cdots + n_4 \geq 1$ and denote by $t = n - n_1$. Then there exists a constant c such that $\binom{n}{n_1, \ldots, n_4} \approx cn^t$.*

We say that a polynomial $f \in P_{n_1, \ldots, n_4}$ corresponds to the multitableau $T_{\langle\lambda\rangle}$ associated to the multipartition $\langle\lambda\rangle \vdash n$ if $f = e_{T_{\langle\lambda\rangle}} f_0$, for some polynomial $f_0 \in P_{n_1, \ldots, n_4}$, where $e_{T_{\langle\lambda\rangle}}$ is an essential idempotent of $F(S_{n_1} \times \cdots \times S_{n_4})$ corresponding to the multitableau $T_{\langle\lambda\rangle}$ (see [20, Chapter 10]).
If $dim_F A_0^+ = d_1$, $dim_F A_0^- = d_2$, $dim_F A_1^+ = d_3$ and $dim_F A_1^- = d_4$, then, in (4.2), we get $m_{\langle\lambda\rangle} \neq 0$ only if $h(\lambda(i)) \leq d_i$, for all $i = 1, \ldots, 4$, where $h(\lambda(i))$ stands for the height of the partition $\lambda(i)$, i.e., the number of the rows of $\lambda(i)$ (see [19, Theorem 5.8]).

We are ready to prove the following theorem concerning the polynomial growth of the codimension and the cocharacter.

Theorem 4.3.2. *Let A be a φ-algebra over a field F of characteristic zero. Then $c_n^\varphi(A)$, $n = 1, 2, \ldots$, is polynomially bounded if and only if, for all $n_1, \ldots, n_4$, with $n_1 + \cdots + n_4 = n$,*

$$\chi^\varphi_{n_1, \ldots, n_4}(A) = \sum_{\substack{\langle\lambda\rangle \vdash (n_1, \ldots, n_4) \\ n - \lambda(1)_1 < q}} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)},$$

where q is such that $J(A)^q = 0$.

Proof. We remark that neither the decomposition of $\chi^\varphi_{n_1, \ldots, n_4}(A)$ into irreducible characters nor $c_n^\varphi(A)$ change under extensions of the base field (see [20, Theorem 4.1.9] for the ordinary case, it can be proved in a similar way). Also if $\bar{F}$ is the algebraic closure of F and $J(A)^q = 0$, then $J(A \otimes_F \bar{F})^q = 0$. Therefore we may assume, without loss of generality, that F is algebraically closed.
Suppose first that

$$\chi^\varphi_{n_1, \ldots, n_4}(A) = \sum_{\substack{\langle\lambda\rangle \vdash (n_1, \ldots, n_4) \\ n - \lambda(1)_1 < q}} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}.$$

If $m_{\langle\lambda\rangle} \neq 0$ for some multipartition $\langle\lambda\rangle = (\lambda(1), \ldots, \lambda(4)) \vdash (n_1, \ldots, n_4)$, we have $t = n - \lambda(1)_1 < q$. Hence $\lambda(1)_1 = n - t$ and by Proposition 4.3.1, for some constant

a, we obtain that $d_{\lambda(1)} = \deg \chi_{\lambda(1)} \leq an_1^t \leq an^t \leq an^q$. Moreover, $\deg \chi_{\lambda(i)} \leq q!$, for any $i = 1, \dots, 4$ and $n_2 + n_3 + n_4 < q$, then, by Proposition 4.3.2, $\binom{n}{n_1,\dots,n_4} < n^q$. Since the multiplicities are polynomially bounded (see [3, Lemma 2.1]) and since there are finitely many multipartitions $\langle\lambda\rangle \vdash (n_1, \dots, n_4)$ satisfying the condition $n - \lambda(1)_1 < q$, then by (4.1) we have that the codimensions are polynomially bounded.
Now suppose that $c_n^\varphi(A)$, $n = 1, 2, \dots$, is polynomially bounded. By Theorem 4.3.1, we have that $A \sim_{T_2^\varphi} B$, where B is a finite dimensional φ-algebra. Therefore, we may assume that A is a finite dimensional φ-algebra. By Theorem 4.1.1, we can write $A = A' + J = A_1 \oplus \cdots \oplus A_m + J$, where A' is a maximal semisimple subalgebra of A, $J = J(A)$ is the Jacobson radical of A and the A_i's are φ-simple algebras, for $i = 1, \dots, m$. Since $c_n^\varphi(A)$ is polynomially bounded, then $A_i J A_j = 0$, for all $i \neq j$, and $A_i \cong F$ (see the proof of [32, Theorem 2.2]). Hence $A_0^- \oplus A_1^+ \oplus A_1^- \subseteq J$ and, if q is the least positive integer such that $J^q = 0$, then $A_0^- \oplus A_1^+ \oplus A_1^-$ generates a nilpotent ideal of A of index of nilpotence $\leq q$. Let $\langle\lambda\rangle = (\lambda(1), \dots, \lambda(4))$ be a multipartition of $n = n_1 + \cdots + n_4$ such that $n - \lambda(1)_1 \geq q$. We claim that every multilinear φ-polynomial $f = e_{T_{\langle\lambda\rangle}} f_0$ corresponding to $T_{\langle\lambda\rangle} = (T_{\lambda(1)}, \dots, T_{\lambda(4)})$ vanishes on A, where $e_{T_{\langle\lambda\rangle}} = e_{T_{\lambda(1)}} \cdots e_{T_{\lambda(4)}}$ is an essential idempotent of $F(S_{n_1} \times \cdots \times S_{n_4})$ corresponding to $T_{\langle\lambda\rangle}$. Since $f \in F\langle Y \cup Z, \varphi\rangle$ is a polynomial in the variables of the disjoint infinite sets Y^+, Y^-, Z^+, Z^-, where Y^+ denotes the set of even symmetric variables, Y^- denotes the set of even skew variables, Z^+ denotes the set of odd symmetric variables and Z^- denotes the set of odd skew variables, we write $f = f(Y^+, Y^-, Z^+, Z^-)$. We denote by $\lambda(1)' = (\lambda(1)'_1, \dots, \lambda(1)'_d)$ the conjugate partition of $\lambda(1)$. By [17, Lemma 4], if $\lambda(1)'_1 > 1$, then there exists a subset Y_1 of Y^+ such that $Y_1 = Y^1 \cup \cdots \cup Y^d$, where $|Y^i| = \lambda(1)'_i$ and for some $r \in F(S_{n_1} \times \cdots \times S_{n_4})$, the element $rf \neq 0$ is alternating on Y^i, for all $1 \leq i \leq d$. Then f generates an irreducible left $S_{n_1} \times \cdots \times S_{n_4}$-module and $F(S_{n_1} \times \cdots \times S_{n_4})f = F(S_{n_1} \times \cdots \times S_{n_4})rf$. In order to prove that $f \in \mathrm{Id}^\varphi(A)$, we just need to show that $rf \in \mathrm{Id}^\varphi(A)$. Since rf is alternating on each set Y^i, in order to get a non-zero value, no two variables of Y^i can take values in the same $A_i \cong F$. But $A_i A_j = 0$, if $i \neq j$. Hence, we must substitute in rf at least $n_1 - \lambda(1)_1$ variables for elements of the radical J. Since $A_0^- \oplus A_1^+ \oplus A_1^- \subseteq J$, at least $n_1 - \lambda(1)_1 \geq q$ variables must be evaluated on J. But $J^q = 0$, so rf vanishes on A. In this way all the irreducible characters appearing in $\chi^\varphi_{n_1,\dots,n_4}(A)$ with non-zero multiplicities correspond to multipartitions $\langle\lambda\rangle$ with $n - \lambda(1)_1 < q$. □

4.4 On multiplicities of cocharacters bounded by a constant

In this section we shall give a characterization, up to T_2^φ-equivalence, of the φ-algebras with multiplicities of the cocharacter bounded by a constant.

First we prove some useful results.

Lemma 4.4.1. *Let A be a finite dimensional φ-algebra over an algebraically closed field F of characteristic zero. If UT_2, UT_2^{sup}, UT_2^{gr}, $UT_2^{gr,sup} \notin var^\varphi(A)$ then $A = A_1 \oplus \cdots \oplus A_m + J$, where $A_i \cong F$ with trivial superautomorphism or $A_i \cong (F \oplus F, ex)$ or $A_i \cong (Q(1) \oplus \overline{Q(1)}, ex)$.*

Proof. By Theorem 4.1.1, $A = A_1 \oplus \cdots A_m + J$, where $A_1, \dots, A_m$ are finite dimensional φ-simple algebras and J is the Jacobson radical of A. According to Theorem 4.1.2, we have to consider four cases.

Case 1. $A_i \cong (M_{k,0}(F), \varphi)$, $k > 1$.

By definition, the superautomorphism φ is determined by the decomposition $k =$

$k_1 + k_2$. Consider the elements $a = e_{11}$, $b = e_{1,k_1+k_2}$ and $c = e_{k_1+k_2,k_1+k_2}$. The subalgebra $C = \langle a, b, c \rangle$ of A_i, generated by a, b and c is graded and endowed with the superautomorphism induced by φ. We get that C is isomorphic as φ-algebra to UT_2 if $k_2 = 0$ and to UT_2^{sup} if $k_2 > 0$. Then UT_2 and UT_2^{sup} belong to $var^{\varphi}(A_i) \subseteq var^{\varphi}(A)$ that is a contradiction.

Case 2. $A_i \cong (M_{k,h}(F), \varphi)$, $h > 0$.
Consider the subalgebra $C = \langle a, b, c \rangle$ of A_i, generated by the elements $a = e_{11}$, $b = e_{1,k+h}$ and $c = e_{k+h,k+h}$. Then C is a graded subalgebra with the superautomorphism induced by φ. We notice that, by definition, $\varphi(e_{1,k+h}) = \pm e_{1,k+h}$. Then C is isomorphic as φ-algebras to UT_2^{gr} or $UT_2^{gr,sup}$ via the isomorphism f such that $f(a) = e_{11}$, $f(b) = e_{12}$ and $f(c) = e_{22}$. In this way, UT_2^{gr} and $UT_2^{gr,sup}$ belong to $var^{\varphi}(A_i) \subseteq var^{\varphi}(A)$, a contradiction.

Case 3. $A_i \cong (M_{k,h}(F) \oplus \overline{M_{k,h}(F)}, ex)$.
If $k > 1$ and $h = 0$, consider the elements $a = (e_{11}, e_{11})$, $b = (e_{kk}, e_{kk})$ and $c = (e_{1k}, e_{1k})$ and define the subalgebra $C = \langle a, b, c \rangle$. Clearly C, with trivial grading and endowed with the induced superautomorphism, is isomorphic to UT_2 as φ-algebra.
If $h > 0$, we define the subalgebra $C = \langle a, b, c \rangle$, generated by the elements $a = (e_{11}, e_{11})$, $b = (e_{k+h,k+h})$ and $c = (e_{1,k+h}, e_{1,k+h})$. By the isomorphism f such that $f(a) = e_{11}$, $f(b) = e_{22}$ and $f(c) = e_{12}$, C is isomorphic to UT_2^{gr} as φ-algebra.
So, we get that UT_2 and UT_2^{gr} belong to $var^{\varphi}(A_i) \subseteq var^{\varphi}(A)$ and this is a contradiction.

Case 4. $A_i \cong (Q(n) \oplus \overline{Q(n)}, ex)$, with $n > 1$.
Consider the elements $a = (e_{11}, e_{11})$, $b = (e_{1n}, e_{1n})$, $c = (e_{nn}, e_{nn})$ and the subalgebra $C = \langle a, b, c \rangle$ of A_i. The linear map $f : C \to UT_2$ such that $f(a) = e_{11}$, $f(b) = e_{12}$ and $f(c) = e_{22}$ is an isomorphism of φ-algebras. Hence $UT_2 \in var^{\varphi}(A_i) \subseteq var^{\varphi}(A)$, a contradiction.

Hence, for every $i = 1, \dots, m$, $A_i \cong F$ with trivial grading and trivial superautomorphism or $A_i \cong (F \oplus F, ex)$ or $A_i \cong (Q(1) \oplus \overline{Q(1)}, ex)$. □

Theorem 4.4.1. *Let A be a finite dimensional φ-algebra such that UT_2, UT_2^{sup}, UT_2^{gr}, $UT_2^{gr,sup} \notin var^{\varphi}(A)$. Then $var^{\varphi}(A) = var^{\varphi}(B_1 \oplus \cdots \oplus B_m)$, where, for each $i = 1, \dots, m$, B_i is isomorphic to one of the following:*

1. *$F + J_i$, with trivial grading and trivial superautomorphism;*
2. *$F \oplus F + J_i$, with superautomorphism ex on $F \oplus F$;*
3. *$Q(1) \oplus \overline{Q(1)} + J_i$, with superautomorphism ex on $Q(1) \oplus \overline{Q(1)}$.*

Proof. By Lemma 4.4.1, we can decompose $A = A_1 \oplus \cdots \oplus A_m + J$, where, for each $i = 1, \dots, m$, A_i is isomorphic either to F with trivial superautomorphism or to $F \oplus F$ with exchange superautomorphism or to $Q(1) \oplus \overline{Q(1)}$ with exchange superautomorphism.
Suppose by contradiction that $A_i J A_k \neq 0$ for some $i \neq k$. Hence $a_1 j a_2 \neq 0$ for some $a_1 \in A_i$, $j \in J$ and $a_2 \in A_k$. It clearly follows that $e_1 j e_2 \neq 0$, where e_1 and e_2 are the unit elements of A_i and A_k, respectively. Then $e_1^2 = e_1 = e_1^{\varphi}$ and $e_2^2 = e_2 = e_2^{\varphi}$. Without loss of generality, we may assume that j is an homogeneous element either symmetric or skew of even or odd degree. Let C be the subalgebra of A generated by e_1, e_2 and $e_1 j e_2$, which is a graded superalgebra with induced superautomorphism. We consider the map $f : C \to UT_2$ such that $f(e_1) = e_{11}$, $f(e_2) = e_{22}$ and $f(e_1 j e_2) = e_{12}$ that is an isomorphism of ordinary algebras. But f can be regarded

as an isomorphism of φ-algebras between C and UT_2, UT_2^{sup}, UT_2^{gr} and $UT_2^{gr,sup}$, according as j is even symmetric, even skew, odd symmetric or odd skew, respectively. We reach a contradiction, hence we get that

$$A_i J A_k = A_i A_k = 0, \tag{4.12}$$

for all $i, k \in \{1, \dots m\}$, $i \neq m$.
We write $B_i = A_i + J$, $i = 1, \dots, m$, then $A = A_1 \oplus \cdots \oplus A_m + J = (A_1 + J) \oplus \cdots \oplus (A_m + J) = B_1 \oplus \cdots \oplus B_m$. For each $i = 1, \dots, m$, $J_i \subseteq B_i$ is the Jacobson radical of B_i and $B_i / J_i \cong A_i$. Hence, each B_i is isomorphic to one of the algebras 1., 2. or 3.
Now we want to prove that $\mathrm{Id}^\varphi(B_1 \oplus \cdots \oplus B_m) = \mathrm{Id}^\varphi(B_1) \cap \cdots \cap \mathrm{Id}^\varphi(B_m)$.
The inclusion $\mathrm{Id}^\varphi(B_1 \oplus \cdots \oplus B_m) \subseteq \mathrm{Id}^\varphi(B_1) \cap \cdots \cap \mathrm{Id}^\varphi(B_m)$ is obvious.
Then we are left to prove the other one.
We shall prove that if $f = f(y_1^+, \dots, y_{n_1}^+, y_1^-, \dots, y_{n_2}^-, z_1^+, \dots, z_{n_3}^+, z_1^-, \dots, z_{n_4}^-) \in P_{n_1,\dots,n_4}$ belongs to $\mathrm{Id}^\varphi(B_1) \cap \cdots \cap \mathrm{Id}^\varphi(B_m)$, then $f \in \mathrm{Id}^\varphi(A) = \mathrm{Id}^\varphi(B_1 \oplus \cdots \oplus B_m)$. It suffices to check substitutions in $B_1 \cup \cdots \cup B_m$, that is, substitutions of the type $y_i^+ \to \tilde{y}_i^+ \in (B_1)_0^+ \cup \cdots \cup (B_m)_0^+$, $y_j^- \to \tilde{y}_j^- \in (B_1)_0^- \cup \cdots \cup (B_m)_0^-$, $z_l^+ \to \tilde{z}_l^+ \in (B_1)_1^+ \cup \cdots \cup (B_m)_1^+$ and $z_t^- \to \tilde{z}_t^- \in (B_1)_1^- \cup \cdots \cup (B_m)_1^-$.
If $\tilde{y}_1^+, \dots, \tilde{y}_{n_1}^+, \tilde{y}_1^-, \dots, \tilde{y}_{n_2}^-, \tilde{z}_1^+, \dots, \tilde{z}_{n_3}^+, \tilde{z}_1^-, \dots, \tilde{z}_{n_4}^- \in B_d$ for a single d, we get a zero value for f, since f belongs to $\mathrm{Id}^\varphi(B_d)$. Otherwise, since $B_i = A_i + J$ for all i, there exist k, l, with $k \neq l$, such that $\tilde{x}_k \in A_k$ and $\tilde{x}_l \in A_l$ for some elements $\tilde{x}_k, \tilde{x}_l \in \{\tilde{y}_i^+, \tilde{y}_j^-, \tilde{z}_l^+, \tilde{z}_t^-\}$. In all cases, since $A_i J A_k = A_i A_k = 0$, we get $\tilde{w}_{\sigma(1)} \dots \tilde{w}_{\sigma(n)} = 0$, for any monomial $w_{\sigma(1)} \dots w_{\sigma(n)}$ in f, with $\sigma \in S_n$, under the substitution $w_i \to \tilde{y}_i^+$, for all $1 \leq i \leq n_1$, $w_j \to \tilde{y}_j^-$, for all $n_1 + 1 \leq j \leq n_1 + n_2$, $w_l \to \tilde{z}_l^+$, for $n_1 + n_2 + 1 \leq l \leq n_1 + n_2 + n_3$ and $w_t \to \tilde{z}_t^-$ for $n_1 + n_2 + n_3 + 1 \leq t \leq n_4$.
Thus, since $A = B_1 \oplus \cdots \oplus B_m$ and $\mathrm{Id}^\varphi(B_1 \oplus \cdots \oplus B_m) = \mathrm{Id}^\varphi(B_1) \cap \cdots \cap \mathrm{Id}^\varphi(B_m)$, this implies that $\mathrm{Id}^\varphi(A) = \mathrm{Id}^\varphi(B_1 \oplus \cdots \oplus B_m)$. Hence $var^\varphi(A) = var^\varphi(B_1 \oplus \cdots \oplus B_m)$ and the proof is complete. □

Now we present two useful lemmas. The proof of the next result is analogous to the one of Lemma 7 in [42] for involution case and will be omitted.

Lemma 4.4.2. *Let $A = C + J$ be a finite dimensional φ-algebra, where $J = J(A)$ is its Jacobson radical and C is a φ-simple subalgebra of A isomorphic to either F with trivial superautomorphism or $F \oplus F$ with superautomorphism ex. If the $(n_1, \dots, n_4)$-cocharacter of A has decomposition as in (4.2), then there exists a constant N such that $m_{\langle\lambda\rangle} \leq N$, for all $\langle\lambda\rangle \vdash (n_1, \dots, n_4)$.*

Lemma 4.4.3. *Let $A = C + J$ be a finite dimensional φ-algebra, where $J = J(A)$ is its Jacobson radical and $C \cong (Q(1) \oplus \overline{Q(1)}, ex)$. If the $(n_1, \dots, n_4)$-cocharacter of A has decomposition as in (4.2), then there exists a constant N such that $m_{\langle\lambda\rangle} \leq N$, for all $\langle\lambda\rangle \vdash (n_1, \dots, n_4)$.*

Proof. Let $d = dim_F A$ and choose

$$\{a_1, a_2, \dots, a_{d_1}\}, \quad \{b_1, b_2, \dots, b_{d_2}\}, \quad \{c_1, c_2, \dots, c_{d_3}\}, \quad \{e_1, e_2, \dots, e_{d_4}\},$$

basis of A_0^+, A_0^-, A_1^+ and A_1^-, respectively, such that $a_1 \in C_0^+$, $a_2, \dots, a_{d_1} \in J_0^+$, $b_1 \in C_0^-$, $b_2, \dots, b_{d_2} \in J_0^-$, $c_1 \in C_1^+$, $c_2, \dots, c_{d_3}$, $e_1 \in C_1^-$ and $e_2, \dots, e_{d_4} \in C_1^-$. Moreover, let q be the smallest positive integer such that $J^q = 0$. If $q = 1$, then $A \cong (Q(1) \oplus \overline{Q(1)}, ex)$. Since $\mathrm{Id}^\varphi(Q(1) \oplus \overline{Q(1)}) = \langle [x_1, x_2] \rangle_{T_2^\varphi}$, it is obvious that if $\chi^\varphi_{n_1,\dots,n_4}(Q(1) \oplus$

$\overline{Q(1)})$ as in (4.2) is the $(n_1, \dots, n_4)$-cocharacter of $(Q(1) \oplus \overline{Q(1)})$, then

$$m_{\langle\lambda\rangle} = \begin{cases} 1, & \text{if } \langle\lambda\rangle = ((n_1), (n_2), (n_3), (n_4)),\ n_1 + \cdots + n_4 = n \\ 0, & \text{otherwise} \end{cases}.$$

So, we get that the multiplicities in (4.2) are bounded by a constant.
Then we suppose $q \geq 2$ and prove that $m_{\langle\lambda\rangle} \leq N = d(q^d)^{d_1 d_2 d_3 d_4}$, for all $\langle\lambda\rangle \vdash (n_1, \dots, n_4)$.
We may assume that $h(\lambda(i)) \leq d_i$, for all $1 \leq i \leq 4$. Let $(T_{\lambda(1)}, \dots, T_{\lambda(4)})$ be Young tableaux corresponding to $\langle\lambda\rangle$ and define, for all $1 \leq i \leq 4$,

$$e_{T_{\lambda(i)}} = \Big(\sum_{\sigma \in R_{T_{\lambda(i)}}} \sigma \Big) \Big(\sum_{\sigma \in C_{T_{\lambda(i)}}} sgn(\sigma)\sigma \Big) = \tilde{R}_{T_{\lambda(i)}} \tilde{C}_{T_{\lambda(i)}},$$

where $R_{T_{\lambda(i)}}$ and $C_{T_{\lambda(i)}}$ are the row and column stabilizers of $T_{\lambda(i)}$, respectively.
For all $1 \leq i \leq 4$, it is well known that the element $e_{T_{\lambda(i)}}$ is an essential idempotent in the group algebra FS_{n_i}. Similarly, the element $e = e_{T_{\lambda(1)}} e_{T_{\lambda(2)}} e_{T_{\lambda(3)}} e_{T_{\lambda(4)}}$ is an essential idempotent in the group algebra $F(S_{n_1} \times S_{n_2} \times S_{n_3} \times S_{n_4})$. The minimality of the left ideal $F(S_{n_1} \times \cdots \times S_{n_4})e$ implies that for any multilinear φ-polynomial f the φ-identities $ef \equiv 0$ and $ref \equiv 0$ are PI-equivalent for any $r \in F(S_{n_1} \times \cdots \times S_{n_4})$, provided that $re \neq 0$.
By the definition of e, ef is symmetric in each set of variables corresponding to the numbers in the same rows in $T_{\lambda(i)}$, for each $i = 1, \dots, 4$. Denote by $Y_1^+, \dots, Y_{d_1}^+$ the sets of variables corresponding to the elements of the first, second, …, d_1th row of $T_{\lambda(1)}$, by $Y_1^-, \dots, Y_{d_2}^-$ the sets of variables corresponding to the elements of the first, second, …, d_2th row of $T_{\lambda(2)}$, by $Z_1^+, \dots, Z_{d_3}^+$ the sets of variables corresponding to the elements of the first, second, …, d_3th row of $T_{\lambda(3)}$ and by $Z_1^-, \dots, Z_{d_4}^-$ the sets of variables corresponding to the elements of the first, second, …, d_4th row of $T_{\lambda(4)}$. Then ef is symmetric on each set of variables $Y_k^+, Y_h^-, Z_r^+, Z_s^-$, for $k = 1, \dots, d_1$, $h = 1, \dots, d_2$, $r = 1, \dots, d_3$ and $s = 1, \dots, d_4$.
Let $ef \neq 0$. Then, by setting $\eta = (\sigma_1, \dots, \sigma_4) \in S_{n_1} \times \cdots \times S_{n_4}$, we get that ηef is PI-equivalent to ef. In particular, we can choose η such that ηef is symmetric on the first $\lambda(1)_1$ variables which are even symmetric, on the next $\lambda(1)_2$ variables and so on. Similarly, ηef is symmetric on the first $\lambda(2)_1$ variables which are odd symmetric, on the next $\lambda(2)_2$ variables and so on. Analogously, ηef is symmetric on the first $\lambda(3)_1$ ($\lambda(4)_1$) variables which are even (odd) skew, on the next $\lambda(3)_2$ ($\lambda(4)_2$) variables and so on.
Now we take any $f_1, \dots, f_m$ multilinear φ-polynomials generating different but isomorphic irreducible $(S_{n_1} \times \cdots \times S_{n_4})$-modules corresponding to the same multipartition $\langle\lambda\rangle$. By the above, we can choose permutations $\eta_1, \dots, \eta_m \in S_{n_1} \times \cdots \times S_{n_4}$ and a decomposition

$$X_{T_{\langle\lambda\rangle}} = Y_1^+ \cup \cdots \cup Y_{d_1}^+ \cup Y_1^- \cup \cdots \cup Y_{d_2}^- \cup Z_1^+, \cup \cdots \cup Z_{d_3}^+ \cup Z_1^- \cup \cdots \cup Z_{d_4}^-$$

such that $\eta_1 f_1, \dots, \eta_m f_m$ are simultaneously symmetric on Y_k^+, Y_h^-, Z_r^+ and Z_s^-, for $k = 1, \dots, d_1$, $h = 1, \dots, d_2$, $r = 1, \dots, d_3$ and $s = 1, \dots, d_4$. Thus, without loss of generality, we may assume that $f_1, \dots, f_m$ satisfy this condition.
Let us assume by contradiction that $m = m_{\langle\lambda\rangle} > N = d(q^d)^{d_1 d_2 d_3 d_4}$ and we will prove that A satisfies a φ-identity of the type $f = \mu_1 f_1 + \cdots + \mu_m f_m$, where $\mu_1, \dots, \mu_m \in F$ are not all zero. This would imply that $f_1, \dots, f_m$ are linearly dependent modulo

$\mathrm{Id}^{\varphi}(A)$, that is a contradiction.
Since $f_1, \ldots, f_m$ are multilinear, in order to prove that $f \equiv 0$, we can evaluate it only on elements of a basis of A. First, we define substitutions of a special kind. We consider non-negative integers

$$\alpha_{k1}, \ldots, \alpha_{kd_1}, \beta_{h1}, \ldots, \beta_{hd_2}, \gamma_{r1}, \ldots, \gamma_{rd_3}, \delta_{s1}, \ldots, \delta_{sd_4}$$

such that, for all $1 \leq k \leq d_1$, $1 \leq h \leq d_2$, $1 \leq r \leq d_3$ and $1 \leq s \leq d_4$,

$$\sum_{l=1}^{d_1} \alpha_{kl} = |Y_k^+|, \quad \sum_{l=1}^{d_2} \beta_{hl} = |Y_h^-|, \quad \sum_{l=1}^{d_3} \gamma_{rl} = |Z_r^+|, \quad \sum_{l=1}^{d_4} \delta_{sl} = |Z_s^-|.$$

We say that an evaluation ϕ has type

$$(\alpha_{k1}, \ldots, \alpha_{kd_1}, \beta_{h1}, \ldots, \beta_{hd_2}, \gamma_{r1}, \ldots, \gamma_{rd_3}, \delta_{s1}, \ldots, \delta_{sd_4}),$$

if we replace the variables in the following way: for any fixed k, h, r and s, we evaluate the first α_{k1} symmetric even variables from $X_{T_{\langle\lambda\rangle}}$ for a_1, the next α_{k2} symmetric even variables from $X_{T_{\langle\lambda\rangle}}$ for a_2 and so on up to the last α_{kd_1} symmetric even variables for a_{d_1}. Similarly, we replace the first β_{h1} skew even variables from $X_{T_{\langle\lambda\rangle}}$ for b_1 and so on up to the last β_{hd_2} skew even variables for b_{d_2}. Analogously, we make the evaluation of the other variables, taking into account the symmetric and skew odd variables and the basis $\{c_1, c_2, \ldots, c_{d_3}\}$ and $\{e_1, e_2, \ldots, e_{d_4}\}$ of A_1^+ and A_1^-, respectively.
In order to get a non-zero value of f, we have to consider the nilpotency of J. Thus we get the following conditions:

1) $\displaystyle\sum_{l=2}^{d_1} \alpha_{kl} \leq q-1,$

2) $\displaystyle\sum_{l=2}^{d_2} \beta_{hl} \leq q-1,$

3) $\displaystyle\sum_{l=2}^{d_3} \gamma_{rl}^{\lambda(3)} \leq q-1,$

4) $\displaystyle\sum_{l=2}^{d_4} \delta_{sl} \leq q-1.$

By definition, we add also these restrictions:

5) $\displaystyle\alpha_{k1} = |Y_k^+| - \sum_{l=2}^{d_1} \alpha_{kl},$

6) $\displaystyle\beta_{h1} = |Y_h^-| - \sum_{l=2}^{d_2} \beta_{hl},$

7) $\displaystyle\gamma_{r1} = |Z_r^+| - \sum_{l=2}^{d_3} \gamma_{rl},$

8) $\displaystyle\delta_{s1} = |Z_s^-| - \sum_{l=2}^{d_4} \delta_{sl}.$

Fixed $1 \leq k \leq d_1$, $1 \leq h \leq d_2$, $1 \leq r \leq d_3$ and $1 \leq s \leq d_4$, by taking into account conditions $1), \ldots, 8)$, it is clear that the number of distinct d_1-tuples $(\alpha_{k1}, \ldots, \alpha_{kd_1})$ is less that q^{d_1}, the number of distinct d_2-tuples $(\beta_{h1}, \ldots, \beta_{hd_1})$ is less that q^{d_2}, the number of distinct d_3-tuples $(\gamma_{r1}, \ldots, \gamma_{rd_3})$ is less that q^{d_3} and the number of distinct d_4-tuples $(\delta_{s1}, \ldots, \delta_{sd_4})$ is less that q^{d_4}. Hence the number of distinct special substitutions is at most $q^{d_1}q^{d_2}q^{d_3}q^{d_4} = q^{d_1+d_2+d_3+d_4} = q^d$, for given $1 \leq k \leq d_1$, $1 \leq h \leq d_2$, $1 \leq r \leq d_3$ and $1 \leq s \leq d_4$. Since the number of 4-tuples (k, h, r, s) is $d_1d_2d_3d_4$, it follows that the number $\tilde{N}$ of particular substitutions is less than $N_0 = (q^d)^{d_1d_2d_3d_4}$.
Now we consider all these $\tilde{N}$ particular substitutions $\phi_1, \ldots, \phi_{\tilde{N}}$ and we construct

the matrix (u_{ij}), where, for all $1 \le i \le m$ and $1 \le j \le \tilde{N}$,

$$\phi_{ij}(f_i) = u_{ij}.$$

This matrix has m rows and $\tilde{N}$ columns of elements of A. Since we are assuming that $m > N = dN_0 > \tilde{N}$, we have that the rows of (u_{ij}) are linearly dependent. Hence, there exist $\mu_1, \dots, \mu_m \in F$ not all zero such that

$$\sum_{i=1}^{m} \mu_i u_{ij} = 0, \text{ for all } 1 \le j \le \tilde{N}.$$

Thus

$$0 = \sum_{i=1}^{m} \mu_i(\phi_j(f_i)) = \phi_j(\sum_{i=1}^{m} \mu_i f_i),$$

for all $1 \le j \le \tilde{N}$. This means that the polynomial $f = \mu_1 f_1 + \cdots + \mu_m f_m$ is zero under all particular substitutions $\phi_1, \dots, \phi_{\tilde{N}}$.
Now it suffices to show that $f \in \mathrm{Id}^{\varphi}(A)$.
So let ρ be any substitutions of the variables of f in the elements of the basis of A_0^+, A_0^-, A_1^+ and A_1^-. Let α'_{k1} be the number of variables in Y_k^+ mapped by ρ to a_1, α'_{k2} be the number of variables in Y_k^+ mapped by ρ to a_2 and so on. Similarly, let β'_{h1} be the number of variables in Y_h^- mapped by ρ to b_1, β'_{h2} be the number of variables in Y_h^- mapped by ρ to b_2 and so on.
Analogously, let γ'_{rl} and δ'_{sp}, with $1 \le l \le d_3$ and $1 \le p \le d_4$, be the number of variables in Z_r^+ and Z_s^- mapped by ρ to c_l and e_p, respectively. Since f is symmetric on each $Y_1^+, \dots, Y_{d_1}^+, Y_1^-, \dots, Y_{d_2}^-, Z_1^+, \dots, Z_{d_3}^+, Z_1^-, \dots, Z_{d_4}^-$, we get that, for all $\theta = (\theta_1, \dots, \theta_4) \in S_{n_1} \times \cdots \times S_{n_4}$ such that $\theta_i \in R_{T_{\lambda(i)}}$, for all $1 \le i \le 4$,

$$\rho(f) = \rho(\theta f) = (\rho\theta) f.$$

In particular, we can choose θ such that $\rho\theta$ is the particular substitutions of the type

$$(\alpha'_{k1}, \dots, \alpha'_{kd_1}, \beta'_{h1}, \dots, \beta'_{hd_2}, \gamma'_{r1}, \dots, \gamma'_{rd_3}, \delta'_{s1}, \dots, \delta'_{sd_4}).$$

According to what was proved above, $\rho(f) = (\rho\theta)f = 0$ and $f \in \mathrm{Id}^{\varphi}(A)$, a contradiction. Hence we must have $m_{\langle\lambda\rangle} \le N$, for all $\langle\lambda\rangle \vdash (n_1, \dots, n_4)$, and we get the desired result. □

Finally, we are ready to prove the main theorem of this section.

Theorem 4.4.2. *Let A be a finite dimensional φ-algebra and let its $(n_1, \dots, n_4)$-cocharacter be as in (4.2). Then UT_2, UT_2^{sup}, UT_2^{gr}, $UT_2^{gr,sup} \notin var^{\varphi}(A)$ if and only if there exists a constant N such that $m_{\langle\lambda\rangle} \le N$, for all $\langle\lambda\rangle \vdash (n_1, \dots, n_4)$.*

Proof. By putting together Theorem 4.4.1 and Lemmas 4.4.2 and 4.4.3, we get the first implication.
On the other hand, if $UT_2 \in var^{\varphi}(A)$, $UT_2^{sup} \in var^{\varphi}(A)$, $UT_2^{gr} \in var^{\varphi}(A)$ or $UT_2^{gr,sup} \in var^{\varphi}(A)$, then by (4.4), (4.6), (4.8) and (4.10), we get a contradiction. □

By putting together Theorem 4.2.1 and Theorem 4.4.2 we get the following.

Theorem 4.4.3. *Let A be a finite dimensional φ-algebra, then the sequence $c_n^{\varphi}(A)$, $n = 1, 2, \dots,$ is polynomially bounded if and only if $UT_2, UT_2^{sup}, UT_2^{gr}, UT_2^{gr,sup} \notin var^{\varphi}(A)$ and there exists a constant N such that $m_{\langle\lambda\rangle} \le N$ in (4.2) for all $\langle\lambda\rangle \vdash (n_1, \dots, n_4)$.*

We conclude this section with the next theorem in which we find a condition ensuring the multiplicities in (4.2) are equal to zero.

Theorem 4.4.4. *Let A be a finite dimensional φ-algebra and let its $(n_1, \ldots, n_4)$-cocharacter be as in (4.2). Then UT_2, UT_2^{sup}, UT_2^{gr}, $UT_2^{gr,sup} \notin var^{\varphi}(A)$ if and only if there exists a constant q such that in (4.2) we have $m_{\langle\lambda\rangle} = 0$ whenever*

$$(|\lambda(1)| - \lambda(1)_1) + (|\lambda(2)| - \lambda(2)_1) + (|\lambda(3)| - \lambda(3)_1) + (|\lambda(4)| - \lambda(4)_1) \geq q.$$

Proof. First we suppose that UT_2, UT_2^{sup}, UT_2^{gr}, $UT_2^{gr,sup} \notin var^{\varphi}(A)$. Let q be the smallest positive integer such that $J^q = 0$, where J is the Jacobson radical of A. By contradiction, let us suppose that there exists $\langle\lambda\rangle \vdash (n_1, \ldots, n_4)$ such that $m_{\langle\lambda\rangle} \neq 0$ and

$$(|\lambda(1)| - \lambda(1)_1) + (|\lambda(2)| - \lambda(2)_1) + (|\lambda(3)| - \lambda(3)_1) + (|\lambda(4)| - \lambda(4)_1) \geq q.$$

Then, if $e = e_{T_{\lambda(1)}} e_{T_{\lambda(2)}} e_{T_{\lambda(3)}} e_{T_{\lambda(4)}}$ (see Lemma 4.4.3), there exist four Young tableaux $T_{\lambda(1)}, \ldots, T_{\lambda(4)}$ and $f \in P_{n_1,\ldots,n_4}$ such that $ef \notin \mathrm{Id}^{\varphi}(A)$ and $F(S_{n_1} \times \cdots \times S_{n_4})ef$ is a minimal left ideal of the group algebra $F(S_{n_1} \times \cdots \times S_{n_4})$.
Now we define $e' = \tilde{C}_{T_{\lambda(1)}} e_{T_{\lambda(1)}} \cdots \tilde{C}_{T_{\lambda(4)}} e_{T_{\lambda(4)}}$. Since, in general, $0 \neq \tilde{R}_{T_\lambda} \tilde{C}_{T_\lambda} h$ implies $\tilde{C}_{T_\lambda} h \neq 0$, where h is a multilinear polynomial, we immediately get that $e'f$ is not a φ-identity of A.
Moreover, it is clear that $e'f$ is alternating on each $\lambda(i)_1$ sets of variables corresponding to the columns of $T_{\lambda(i)}$, for all $1 \leq i \leq 4$. In order to get a contradiction, we shall prove that $g = e'f \in \mathrm{Id}^{\varphi}(A)$.
To this end, since UT_2, UT_2^{sup}, UT_2^{gr}, $UT_2^{gr,sup} \notin var^{\varphi}(A)$, by Lemma 4.4.1, we have that $A = A_1 \oplus \cdots \oplus A_m + J$, where, for each $1 \leq i \leq m$, either $A_i \cong F$ with trivial superautomorphism or $A_i \cong (F \oplus F, ex)$ or $A_i \cong (Q(1) \oplus \overline{Q(1)}, ex)$. Then, by (4.12) we have that

$$A_i J A_k = 0 \text{ and } A_i A_k = 0, \text{ for all } i \neq k.$$

Thus, in order to get a non-zero value of g, we must evaluate its variables with elements of J and elements of just a single component of A, say A_i. In each case, $dim_F(A_i)_0^+ = 1$, $dim_F(A_i)_0^- \leq 1$, $dim_F(A_i)_1^+ \leq 1$ and $dim_F(A_i)_1^- \leq 1$. Hence we can substitute at most one element of $(A_i)_0^+$ in each alternating set of even symmetric variables. A similar argument holds also for $(A_i)_0^-$, $(A_i)_1^+$ and $(A_i)_1^-$. Thus we evaluate at most $\lambda(1)_1$ elements from $(A_i)_0^+$, at most $\lambda(2)_1$ elements from $(A_i)_0^-$, at most $\lambda(3)_1$ elements from $(A_i)_1^+$ and at most $\lambda(4)_1$ elements from $(A_i)_1^-$. So we have at least

$$(|\lambda(1)| - \lambda(1)_1) + (|\lambda(2)| - \lambda(2)_1) + (|\lambda(3)| - \lambda(3)_1) + (|\lambda(4)| - \lambda(4)_1) \geq q$$

variables that must be evaluated in elements of J. In conclusion we obtain that $g \in \mathrm{Id}^{\varphi}(A)$, a contradiction.
Otherwise, suppose by contradiction that either $UT_2 \in var^{\varphi}(A)$ or $UT_2^{sup} \in var^{\varphi}(A)$ or $UT_2^{gr} \in var^{\varphi}(A)$ or $UT_2^{gr,sup} \in var^{\varphi}(A)$. In case $UT_2 \in var^{\varphi}(A)$, according to (4.4), if $\langle\lambda\rangle = ((p,r), \emptyset, \emptyset, \emptyset)$, with $p + r = n$, then $m_{\langle\lambda\rangle} = p - r + 1 > 0$. Thus $m_{\langle\lambda\rangle} \neq 0$, for any multipartition $\langle\lambda\rangle$ such that $|\lambda(1)| - \lambda(1)_1 = r$ arbitrary large and $\lambda(2) = \lambda(3) = \lambda(4) = \emptyset$. Hence, A does not satisfy the hypothesis.
In case $UT_2^{sup} \in var^{\varphi}(A)$, according to (4.6), if $\langle\lambda\rangle = ((p + r, p), (1), \emptyset, \emptyset)$, with $2p + r = n - 1$, then $m_{\langle\lambda\rangle} = r + 1 > 0$. Thus $m_{\langle\lambda\rangle} \neq 0$, for any multipartition $\langle\lambda\rangle$ such that $|\lambda(1)| - \lambda(1)_1 = p$ arbitrary large, $\lambda(2) = 1$ and $\lambda(3) = \lambda(4) = \emptyset$. Hence

A does not satisfy the hypothesis. A similar argument holds in case $UT_2^{gr} \in var^{\varphi}(A)$ or $UT_2^{gr,sup} \in var^{\varphi}(A)$. Then we are done. $\square$

4.5 Polynomial codimension growth and φ-colength

This section is devoted to the proof of another characterization of the polynomial growth of the codimension related to its φ-colength.

Theorem 4.5.1. *Let A be a finite dimensional φ-algebra over a field F of characteristic zero. Then $c_n^{\varphi}(A)$ is polynomially bounded if and only if $l_n^{\varphi}(A) \leq h$, for some constant h and for all $n \geq 1$.*

Proof. First, let us suppose that $c_n^{\varphi}(A)$, $n = 1, 2, \dots,$ is polynomially bounded. By Theorem 4.3.2, we obtain that

$$\chi^{\varphi}_{n_1,\dots,n_4}(A) = \sum_{\substack{\langle\lambda\rangle \vdash (n_1,\dots,n_4) \\ n-\lambda(1)_1 < q}} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)},$$

where q is such that $J(A)^q = 0$.. By Theorem 4.2.1 and Theorem 4.4.2, there exists a constant N such that $m_{\langle\lambda\rangle} \leq N$, for all $\langle\lambda\rangle \vdash (n_1, \dots, n_4)$ and for all $n_1 + \cdots + n_4 = n$. But there are finitely many multipartitions $\langle\lambda\rangle$ satisfying the condition $n - \lambda(1)_1 < q$. Hence it follows that $l_n^{\varphi}(A) \leq h$, for some constant h.
Conversely, assume that $l_n^{\varphi}(A) \leq h$, for some constant h and for all $n \geq 1$. It is clear that if $B \in var^{\varphi}(A)$, then $l_n^{\varphi}(B) \leq l_n^{\varphi}(A)$. By (4.3), (4.5), (4.7), (4.9) and (4.11), then $F \oplus F$, UT_2, UT_2^{gr}, UT_2^{sup}, $UT_2^{gr,sup} \notin var^{\varphi}(A)$. So, by Theorem 4.2.1, $c_n^{\varphi}(A)$, $n = 1, 2, \dots,$ is polynomially bounded. $\square$

In the following theorem we collect all the characterizations about polynomial codimension growth of φ-algebras.

Theorem 4.5.2. *Let A be a finite dimensional φ-algebra A, then the following conditions are equivalent:*

1. *$c_n^{\varphi}(A)$ is polynomially bounded;*
2. *UT_2, UT_2^{sup}, UT_2^{gr}, $UT_2^{gr,sup}$, $F \oplus F \notin var^{\varphi}(A)$;*
3. *$A \sim_{T_2^{\varphi}} B$, where $B = B_1 \oplus \cdots \oplus B_m$, with $B_1, \dots, B_m$ finite dimensional φ-algebras over F and, for all $i = 1, \dots, m$, $dim_F B_i / J(B_i) \leq 1$.*
4. *there exists a constant q such that, for all $n_1, \dots, n_4$, with $n_1 + \cdots + n_4 = n$,*
$$\chi^{\varphi}_{n_1,\dots,n_4}(A) = \sum_{\substack{\langle\lambda\rangle \vdash (n_1,\dots,n_4) \\ n-\lambda(1)_1 < q}} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)};$$
5. *$UT_2, UT_2^{sup}, UT_2^{gr}, UT_2^{gr,sup} \notin var^{\varphi}(A)$ and there exists a constant N such that $m_{\langle\lambda\rangle} \leq N$ in (4.2) for all $\langle\lambda\rangle \vdash (n_1, \dots, n_4)$;*
6. *$l_n^{\varphi}(A) \leq h$, for some constant h and for all $n \geq 1$.*

4.6 Classifying the subvarieties of $var^{\varphi}(F \oplus F)$

In this section we classify, up to T_2^{φ}-equivalence, all the φ-algebras contained in the variety generated by $F \oplus F$.

Since $F \oplus F$ is commutative and is endowed with trivial grading, by Remark 4.2.1 the superautomorphism ex is just a superinvolution and also a graded involution. Hence, the classification was given in [40, 36] and in what follows we recall such result.

For $k \geq 2$, let $UT_k = UT_k(F)$ be the algebra of $k \times k$ upper triangular matrices over F, let I_k be the $k \times k$ identity matrix and $E_1 = \sum_{i=1}^{k-1} e_{i,i+1} \in UT_k$. We denote by

$$C_k = \{\alpha I_k + \sum_{1 \leq i < k} \alpha_i E_1^i \mid \alpha, \alpha_i \in F\} \subseteq UT_k$$

the commutative subalgebra of UT_k with trivial grading and superautomorphism φ given by

$$(\alpha I_k + \sum_{1 \leq i < k} \alpha_i E_1^i)^{\varphi} = \alpha I_k + \sum_{1 \leq i < k} (-1)^i \alpha_i E_1^i.$$

We next state the following result characterizing the φ-identities and the φ-codimensions of C_k.

Theorem 4.6.1. *Let $k \geq 2$. Then*

1. $Id^{\varphi}(C_k) = \langle [x_1, x_2], y_1^- \cdots y_k^-, z^+, z^- \rangle_{T_2^{\varphi}}$;
2. $c_n^{\varphi}(C_k) = \sum_{j=0}^{k-1} \binom{n}{j} \approx \frac{1}{(k-1)!} n^{k-1}$.

The following result classifies all the subvarieties of the variety generated by $F \oplus F$.

Theorem 4.6.2. *[40, Theorem 8.3][36, Theorem 6.5] Let A be a φ-algebra such that $A \in var^{\varphi}(F \oplus F)$. Then either $A \sim_{T_2^{\varphi}} F \oplus F$ or $A \sim_{T_2^{\varphi}} N$ or $A \sim_{T_2^{\varphi}} C \oplus N$ or $A \sim_{T_2^{\varphi}} C_k \oplus N$, for some $k \geq 2$, where N is a nilpotent φ-algebra and C is a commutative superalgebra with trivial superautomorphism.*

As a consequence of the previous theorem, we can also classify all φ-algebras generating minimal varieties.

Definition 4.6.1. A variety $\mathcal{V}$ is minimal if it satisfies the property that $c_n^{\varphi}(\mathcal{V}) \approx qn^k$, for some $k \geq 1$ and $q > 0$, and for any proper subvariety $\mathcal{U} \subsetneq \mathcal{V}$, $c_n^{\varphi}(\mathcal{U}) \approx q'n^t$, with $t < k$.

Corollary 4.6.1. *[40, Corollary 8.2] A φ-algebra $A \in var^{\varphi}(F \oplus F)$ generates a minimal variety if and only if $A \sim_{T_2^{\varphi}} C_k$, for some $k \geq 2$.*

4.7 Classification of the subvarieties of $var^{\varphi}(UT_2)$, $var^{\varphi}(UT_2^{gr})$, $var^{\varphi}(UT_2^{sup})$ and $var^{\varphi}(UT_2^{gr,sup})$

In this section we classify, up to T_2^{φ}-equivalence, all the φ-algebras contained in the variety generated by the φ-algebras UT_2, UT_2^{gr}, UT_2^{sup} and $UT_2^{gr,sup}$.

Notice that these classifications are equivalent to the classifications of the superalgebras inside the varieties of superalgebras generated by UT_2 and UT_2^{gr}, for

Remark 4.2.1. Such a classification was given in [40]. In what follows we present such results without proof in the language of φ-algebras.

We start by constructing, for any fixed $k \geq 1$, φ-algebras belonging to the variety generated by UT_2, UT_2^{gr}, UT_2^{sup} and $UT_2^{gr,sup}$ whose φ-codimension sequence grows polynomially as n^k.

Now we define a grading and an automorphism φ on UT_k.

If $\mathbf{g} = (g_1, \ldots, g_k) \in \mathbb{Z}_2^k$ is an arbitrary k-tuple of elements of $\mathbb{Z}_2$, then $\mathbf{g}$ defines an elementary $\mathbb{Z}_2$-grading on UT_k by setting

$$(UT_k)_0 = span\{e_{ij} \mid g_i + g_j = 0\} \text{ and } (UT_k)_1 = span\{e_{ij} \mid g_i + g_j = 1\}.$$

If A is a subalgebra of UT_k, then the induced grading on A is also called elementary.

We define an automorphism φ on UT_k as follows:

$$\varphi : UT_k \longrightarrow UT_k$$

$$\alpha_{11}e_{11} + \sum_{\substack{i,j=2 \\ i \leq j}}^{k} \alpha_{ij}e_{ij} + \sum_{j=2}^{k} \alpha_{1j}e_{1j} \mapsto \alpha_{11}e_{11} + \sum_{\substack{i,j=2 \\ i \leq j}}^{k} \alpha_{ij}e_{ij} - \sum_{j=2}^{k} \alpha_{1j}e_{1j}.$$

We notice that this automorphism is graded on a subalgebra of UT_k with the elementary $\mathbb{Z}_2$-grading induced by $\mathbf{g} \in \mathbb{Z}_2^k$.

For $k \geq 2$, let

$$N_k = span\{I_k, E_1, E_1^2, \ldots, E_1^{k-2}; e_{12}, e_{13}, \ldots, e_{1k}\} \subseteq UT_k$$

and let

$$A_k = A_k(F) = span\{e_{11}, E_1, E_1^2, \ldots, E_1^{k-2}; e_{12}, e_{13}, \ldots, e_{k-1,k}\} \subseteq UT_k.$$

So, let us denote by

- N_k and A_k the algebras N_k and A_k, respectively, with trivial grading and trivial superautomorphism;
- N_k^{gr} and A_k^{gr} the algebras N_k and A_k, respectively, with elementary grading induced by $\mathbf{g} = (0, 1, \ldots, 1)$ and trivial superautomorphism;
- N_k^{sup} and A_k^{sup} the algebras N_k and A_k, respectively, with trivial grading and superautomorphism φ;
- $N_k^{gr,sup}$ and $A_k^{gr,sup}$ the algebras N_k and A_k, respectively, with elementary grading induced by $\mathbf{g} = (0, 1, \ldots, 1)$ and superautomorphism φ.

Let A_k^* be the subalgebra of UT_k obtained by flipping the matrices in A_k along their secondary diagonals. So,

$$A_k^* = span\{e_{kk}, E_1, E_1^2, \ldots, E_1^{k-2}; e_{1k}, e_{2k}, \ldots, e_{k-1,k}\}.$$

We define the automorphism φ^* on UT_k as

$$\varphi^* : UT_k \longrightarrow UT_k$$

$$\alpha_{kk}e_{kk} + \sum_{\substack{i,j=1 \\ i \leq j}}^{k-1} \alpha_{ij}e_{ij} + \sum_{j=1}^{k-1} \alpha_{jk}e_{jk} \mapsto \alpha_{kk}e_{kk} + \sum_{\substack{i,j=1 \\ i \leq j}}^{k-1} \alpha_{ij}e_{ij} - \sum_{j=1}^{k-1} \alpha_{jk}e_{jk},$$

which is graded on a subalgebra of UT_k with the elementary $\mathbb{Z}_2$-grading induced by $\mathbf{g} \in \mathbb{Z}_2^k$.

Hence, we denote by

- A_k^* the algebra A_k^* with trivial grading and trivial superautomorphism;
- $(A_k^{gr})^*$ the algebra A_k^* with elementary grading induced by $\mathbf{g} = (0, 1, \dots, 1)$ and trivial superautomorphism;
- $(A_k^{sup})^*$ the algebra A_k^* with trivial grading and superautomorphism φ^*;
- $(A_k^{gr,sup})^*$ the algebra A_k^* with elementary grading induced by $\mathbf{g} = (0, 1, \dots, 1)$ and superautomorphism φ^*.

We next state the following result characterizing the φ-identities and the φ-codimensions of these φ-algebras (see [40, Lemma 3.1, Lemma 3.2, Theorem 4.1, Theorem 4.2]).

We notice that given a polynomial $f \in F\langle Y \cup Z, \varphi \rangle$, f^* is the polynomial obtained by reversing the order of the variables in each monomial of f. Then f is a φ-identity of A_k if and only if f^* is a φ-identity of $(A_k)^*$. We get the following result (see [40, Lemma 3.2]).

Theorem 4.7.1. *If $k = 2$, then $Id^{\varphi}(N_2) = Id^{\varphi}(F)$.*

If $k \geq 3$, then $Id^{\varphi}(N_k) = \langle [y_1^+, \dots, y_k^+], [y_1^+, y_2^+][y_3^+, y_4^+], y^-, z^+, z^- \rangle_{T_2^{\varphi}}$.

Let $k \geq 2$. Then:

$Id^{\varphi}(A_k) = \langle [y_1^+, y_2^+][y_3^+, y_4^+], [y_1^+, y_2^+]y_3^+ \cdots y_{k+1}, y^-, z^+, z^- \rangle_{T_2^{\varphi}}$;

$Id^{\varphi}(A_k^*) = \langle [y_1^+, y_2^+][y_3^+, y_4^+], y_3^+ \cdots y_{k+1}[y_1^+, y_2^+], y^-, z^+, z^- \rangle_{T_2^{\varphi}}$;

$Id^{\varphi}(N_k^{gr}) = \langle [y_1^+, y_2^+], [z^+, y_1^+, \dots, y_{k-1}], y^-, z_1^+ z_2^+, z^- \rangle_{T_2^{\varphi}}$;

$Id^{\varphi}(A_k^{gr}) = \langle [y_1^+, y_2^+], z^+ y_1^+ \cdots y_{k-1}^+, y^-, z_1^+ z_2^+, z^- \rangle_{T_2^{\varphi}}$;

$Id^{\varphi}((A_k^{gr})^*) = \langle [y_1^+, y_2^+], y_1^+ \cdots y_{k-1}^+ z^+, y^-, z_1^+ z_2^+, z^- \rangle_{T_2^{\varphi}}$;

$Id^{\varphi}(N_k^{sup}) = \langle [y_1^+, y_2^+], [y^-, y_1^+, \dots, y_{k-1}], y_1^- y_2^-, z^+, z^- \rangle_{T_2^{\varphi}}$;

$Id^{\varphi}(A_k^{sup}) = \langle [y_1^+, y_2^+], y^- y_1^+ \cdots y_{k-1}^+, y_1^- y_2^-, z^+, z^- \rangle_{T_2^{\varphi}}$;

$Id^{\varphi}((A_k^{sup})^*) = \langle [y_1^+, y_2^+], y_1^+ \cdots y_{k-1}^+ y^-, y_1^- y_2^-, z^+, z^- \rangle_{T_2^{\varphi}}$;

$Id^{\varphi}(N_k^{gr,sup}) = \langle [y_1^+, y_2^+], [z^-, y_1^+, \dots, y_{k-1}], y^-, z^+, z_1^- z_2^- \rangle_{T_2^{\varphi}}$;

$Id^{\varphi}(A_k^{gr,sup}) = \langle [y_1^+, y_2^+], z^- y_1^+ \cdots y_{k-1}^+, y^-, z^+, z_1^- z_2^- \rangle_{T_2^{\varphi}}$;

$Id^{\varphi}((A_k^{gr,sup})^*) = \langle [y_1^+, y_2^+], y_1^+ \cdots y_{k-1}^+ z^-, y^-, z^+, z_1^- z_2^- \rangle_{T_2^{\varphi}}$.

Now we are ready to present the main results of this section.

The following result classifies all the subvarieties of the variety generated by UT_2 (see [40, Theorem 3.1]).

Theorem 4.7.2. *Let A be a φ-algebra such that $A \in var^{\varphi}(UT_2)$. Then A is T_2^{φ}-equivalent to one of the following φ-algebras: UT_2, N, $N_t \oplus N$, $N_t \oplus A_k \oplus N$, $N_t \oplus A_r^* \oplus N$, $N_t \oplus A_k \oplus A_r^* \oplus N$, where N is a nilpotent φ-algebra and $k, r, t \geq 2$.*

As a consequence of the previous theorem, we can classify all φ-algebras generating minimal varieties (see [40, Corollary 3.1]).

Corollary 4.7.1. *A φ-algebra $A \in var^{\varphi}(UT_2)$ generates a minimal variety if and only if either $A \sim_{T_2^{\varphi}} N_t$ or $A \sim_{T_2^{\varphi}} A_k$ or $A \sim_{T_2^{\varphi}} A_k^*$, for some $k \geq 2$, $t > 2$.*

Now we present the classification for the subvarieties of $var^{\varphi}(UT_2^{gr})$ (see [40, Theorem 6.1]).

Theorem 4.7.3. *Let A be a φ-algebra such that $A \in var^{\varphi}(UT_2^{gr})$. Then A is T_2^{φ}-equivalent to one of the following φ-algebras: $UT_2^{gr}, N, C \oplus N, N_t^{gr} \oplus N, A_k^{gr} \oplus N, (A_k^{gr})^* \oplus N, N_t^{gr} \oplus A_k^{gr} \oplus N, N_t^{gr} \oplus (A_r^{gr})^* \oplus N, A_k^{gr} \oplus (A_k^{gr})^* \oplus N, N_t^{gr} \oplus A_k^{gr} \oplus (A_r^{gr})^* \oplus N$, where N is a nilpotent φ-algebra, C is a commutative superalgebra with trivial superautomorphism and trivial grading and $k, r, t \geq 2$.*

As a consequence, we obtain the following corollary (see [40, Corollary 6.1]).

Corollary 4.7.2. *A φ-algebra $A \in var^{\varphi}(UT_2^{gr})$ generates a minimal variety if and only if either $A \sim_{T_2^{\varphi}} N_k^{gr}$ or $A \sim_{T_2^{\varphi}} A_k^{gr}$ or $A \sim_{T_2^{\varphi}} (A_k^{gr})^*$, for some $k \geq 2$.*

Here is the classification of the subvarieties of the variety generated by UT_2^{sup} (see [40, Theorem 6.1, Corollary 6.1]).

Theorem 4.7.4. *Let A be a φ-algebra such that $A \in var^{\varphi}(UT_2^{sup})$. Then A is T_2^{φ}-equivalent to one of the following φ-algebras: UT_2^{sup}, N, $C \oplus N$, $N_t^{sup} \oplus N$, $A_k^{sup} \oplus N$, $(A_k^{sup})^* \oplus N$, $N_t^{sup} \oplus A_k^{sup} \oplus N$, $N_t^{sup} \oplus (A_r^{sup})^* \oplus N$, $A_k^{sup} \oplus (A_k^{sup})^* \oplus N$, $N_t^{sup} \oplus A_k^{sup} \oplus (A_r^{sup})^* \oplus N$, where N is a nilpotent φ-algebra, C is a commutative superalgebra with trivial superautomorphism and trivial grading and $k, r, t \geq 2$.*

Corollary 4.7.3. *A φ-algebra $A \in var^{\varphi}(UT_2^{sup})$ generates a minimal variety if and only if either $A \sim_{T_2^{\varphi}} N_k^{sup}$ or $A \sim_{T_2^{\varphi}} A_k^{sup}$ or $A \sim_{T_2^{\varphi}} (A_k^{sup})^*$, for some $k \geq 2$.*

Finally, we present the classification of the subvarieties of the variety generated by $UT_2^{gr,sup}$ (see [40, Theorem 6.1, Corollary 6.1]).

Theorem 4.7.5. *Let A be a φ-algebra such that $A \in var^{\varphi}(UT_2^{gr,sup})$. Then A is T_2^{φ}-equivalent to one of the following φ-algebras: $UT_2^{gr,sup}$, N, $C \oplus N$, $N_t^{gr,sup} \oplus N$, $A_k^{gr,sup} \oplus N$, $(A_k^{gr,sup})^* \oplus N$, $N_t^{gr,sup} \oplus A_k^{gr,sup} \oplus N$, $N_t^{gr,sup} \oplus (A_r^{gr,sup})^* \oplus N$, $A_k^{gr,sup} \oplus (A_k^{gr,sup})^* \oplus N$, $N_t^{gr,sup} \oplus A_k^{gr,sup} \oplus (A_r^{gr,sup})^* \oplus N$, where N is a nilpotent φ-algebra, C is a commutative superalgebra with trivial superautomorphism and trivial grading and $k, r, t \geq 2$.*

Corollary 4.7.4. *A φ-algebra $A \in var^{\varphi}(UT_2^{gr,sup})$ generates a minimal variety if and only if either $A \sim_{T_2^{\varphi}} N_k^{gr,sup}$ or $A \sim_{T_2^{\varphi}} A_k^{gr,sup}$ or $A \sim_{T_2^{\varphi}} (A_k^{gr,sup})^*$, for some $k \geq 2$.*

4.8 Characterization of φ-algebras with linear growth of the codimensions

We conclude this paper with a characterization, up to T_2^{φ}-equivalence, of the finite dimensional φ-algebras generating varieties of at most linear growth, i.e., such that $c_n^{\varphi}(A) \leq an$, for some constant a. In order to do this, we follow the same arguments used in [27].

First we present some results inspired by Theorem 5.1, Corollary 5.1, 5.2 and 5.3 in [32].

Theorem 4.8.1. *Let A be a φ-algebra. Then $c_n^{\varphi}(A) \leq an^p$ for some constants a and p if and only if for every $n_1 + \cdots + n_4 = n$ it holds*

$$\chi^{\varphi}_{n_1,\dots,n_4}(A) = \sum_{\substack{\langle\lambda\rangle \vdash (n_1,\dots,n_4) \\ n-\lambda(1)_1 \leq p}} m_{\langle\lambda\rangle}\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}.$$

The summation runs over all multipartitions $\langle\lambda\rangle = (\lambda(1), \dots, \lambda(4)) \vdash (n_1, \dots, n_4)$ such that $n - \lambda(1)_1 \leq p$, $n = n_1 + \cdots + n_4$.

Proof. Suppose first that $c_n^\varphi(A) \le an^p$. According to Theorem 4.3.2, since $c_n^\varphi(A)$ is polynomially bounded, for all $n_1 + \cdots + n_4 = n$ one has

$$\chi^\varphi_{n_1,\dots,n_4}(A) = \sum_{\substack{\langle\lambda\rangle \vdash (n_1,\dots,n_4)\\ n-\lambda(1)_1 < q}} m_{\langle\lambda\rangle}\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)},$$

for some q. Moreover, since $c_n^\varphi(A) \le an^p$, by (4.1) it follows that

$$\binom{n}{n_1,\dots,n_4} c_{n_1,\dots,n_4} \le an^p.$$

Hence, if $\langle\lambda\rangle = (\lambda(1),\dots,\lambda(4)) \vdash (n_1,\dots,n_4)$ is a multipartition, we must have

$$\binom{n}{n_1,\dots,n_4} m_{\langle\lambda\rangle} \deg \chi_{\lambda(1)} \cdots \deg \chi_{\lambda(4)} \le an^p.$$

But the multiplicities $m_{\langle\lambda\rangle}$ are bounded by a constant and $\deg \chi_{\lambda(i)}$ is a constant for $i = 2,3,4$. This implies that $\binom{n}{n_1,\dots,n_4} \deg \chi_{\lambda(1)} \le a'n^p$ for some constant a'. By Propositions 4.3.1 and 4.3.2,

$$bn^{t+r} \le \binom{n}{n_1,\dots n_4} \deg \chi_{\lambda(1)} \le cn^{t+r}$$

where r is the number of boxes under the first row of $\lambda(1)$ and $t = n_2 + n_3 + n_4$. Hence we must have $t + r = n - \lambda(1)_1 \le p$ and this implies

$$\chi^\varphi_{n_1,\dots,n_4}(A) = \sum_{\substack{\langle\lambda\rangle \vdash (n_1,\dots,n_4)\\ n-\lambda(1)_1 \le p}} m_{\langle\lambda\rangle}\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}.$$

The converse is deduced by proceeding backwards through the proof. □

As a consequence we obtain the following corollaries.

Corollary 4.8.1. *Let A be a φ-algebra. Then $c_n^\varphi(A) \approx an^p$ if and only if there exists n_0 such that for every $n \ge n_0$*

1. $\chi^\varphi_{n_1,\dots,n_4}(A) = \sum_{\substack{\langle\lambda\rangle \vdash (n_1,\dots,n_4)\\ n-\lambda(1)_1 \le p}} m_{\langle\lambda\rangle}\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)}$, *for every $n_1 + \cdots + n_4 = n$;*
2. *there exist $n'_1 + \cdots + n'_4 = n$ and a multipartition $\langle\mu\rangle = (\mu(1),\cdots,\mu(4)) \vdash (n'_1,\dots,n'_4)$ such that $n - \mu(1)_1 = p$ and the corresponding multiplicity $m_{\langle\mu\rangle} \neq 0$.*

Corollary 4.8.2. *Let A be a φ-algebra. Then $c_n^\varphi(A) \le an$, for some constant a, if and only if, for every $n_1 + \cdots + n_4 = n$, it holds*

$$\chi^\varphi_{n_1,\dots,n_4}(A) = \sum_{\langle\lambda\rangle \vdash (n_1,\dots,n_4)} m_{\langle\lambda\rangle}\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(4)},$$

where $\langle\lambda\rangle = (\lambda(1),\dots,\lambda(4))$ is such that either $\lambda(1) = (n-1,1)$ and $\lambda(i) = \emptyset$, $i = 2,3,4$, or $\lambda(1) = (n)$ and $\lambda(i) = \emptyset$, $i = 2,3,4$, or $\lambda(1) = (n-1)$ and $\lambda(i) = (1)$ for some $i \in \{2,3,4\}$ and $\lambda(j) = \emptyset$ for all $j \neq i$.

Corollary 4.8.3. *Any φ-algebra A such that $c_n^\varphi(A) \le an$, for some constant a, satisfies the polynomial identities $x_1x_2 \equiv 0$ for all $x_1, x_2 \in X/Y^+$, with $Y^+ = \{y_1^+, y_2^+, \dots\}$.*

Then we get the following results inspired by Lemma 5.1, 5.2 and 5.3 in [32].

Lemma 4.8.1. *Let $A = F + J$ be a finite dimensional φ-algebras such that $c_n^\varphi(A) \leq an$, for some constant a. Then*

$$A \sim_{T_2^\varphi} (F + J_0^+) \oplus (F + J_0^-) \oplus (F + J_1^+) \oplus (F + J_1^-).$$

Proof. Since $c_n^\varphi(A) \leq an$, by Corollary 4.8.3, A satisfies the polynomial identities $x_1x_2 \equiv 0$, for all $x_1, x_2 \in X/Y^+$, with $Y^+ = \{y_1^+, y_2^+, \dots\}$. Hence $F + J_0^+$, $F + J_0^-$, $F + J_1^+$ and $F + J_1^-$ are subalgebras of A and so $\mathrm{Id}^\varphi(A) \subseteq \mathrm{Id}^\varphi((F + J_0^+) \oplus (F + J_0^-) \oplus (F + J_1^+) \oplus (F + J_1^-))$.
Conversely, let $f \in \mathrm{Id}^\varphi((F + J_0^+) \oplus (F + J_0^-) \oplus (F + J_1^+) \oplus (F + J_1^-))$ be a multilinear polynomial of degree n. By multihomogeneity of T_2^φ-ideals we may assume, modulo $\mathrm{Id}^\varphi(A)$, that either

$$f = \sum_{\sigma \in S_n} \alpha_\sigma y^+_{\sigma(1)} \cdots y^+_{\sigma(n)} \quad \text{or} \quad f = \sum_{\substack{i=1,\dots,n \\ \sigma \in S_n}} \beta_\sigma y^+_{\sigma(1)} \cdots y^+_{\sigma(i-1)} x_{\sigma(i)} y^+_{\sigma(i+1)} \cdots y^+_{\sigma(n)},$$

where $x_i \in X/Y^+$, $i = 1, \dots, n$. If f is of the first type, we should evaluate f on $F + J_0^+$. But $f \in \mathrm{Id}^\varphi(F + J_0^+)$ by the hypothesis, and so we get that $f \equiv 0$ on A. Similarly, if f is of the second type we get that $f \equiv 0$ on A. Hence $\mathrm{Id}^\varphi((F + J_0^+) \oplus (F + J_0^-) \oplus (F + J_1^+) \oplus (F + J_1^-)) \subseteq \mathrm{Id}^\varphi(A)$ and we are done. □

Corollary 4.8.4. *Let $A = F + J$ be a finite dimensional φ-algebra such that $c_n^\varphi(A) \leq an$, for some constant a. Then $A \sim_{T_2^\varphi} B_i \oplus N$, with $i \in \{1, \dots, 4\}$, or $A \sim_{T_2^\varphi} B_i \oplus B_j \oplus N$, with $i, j \in \{1, \dots, 4\}$ and $i \neq j$, or $A \sim_{T_2^\varphi} B_i \oplus B_j \oplus B_k \oplus N$ with $i, j, k \in \{1, \dots, 4\}$ and $i \neq j \neq k$, or $A \sim_{T_2^\varphi} B_1 \oplus B_2 \oplus B_3 \oplus B_4 \oplus N$, where $B_1 \in var^\varphi(UT_2)$, $B_2 \in var^\varphi(UT_2^{sup})$, $B_3 \in var^\varphi(UT_2^{gr})$ and $B_4 \in var^\varphi(UT_2^{gr,sup})$ and N is a nilpotent φ-algebra.*

Proof. Since $c_n^\varphi(A) \leq an$, by Corollary 4.8.3, A satisfies the polynomial identity $x_1x_2 \equiv 0$, for all $x_1, x_2 \in X/Y^+$. Hence, by Lemma 4.8.1, $A \sim_{T_2^\varphi} (F + J_0^+) \oplus (F + J_0^-) \oplus (F + J_1^+) \oplus (F + J_1^-)$.

Now we observe that $F + J_0^+ \in var^\varphi(UT_2)$.

Since $F + J_0^+$ has trivial grading and trivial superautomorphism, then $y^- \equiv 0$, $z^+ \equiv 0$ and $z^- \equiv 0$. So $F + J_0^+$ is an ordinary algebra and we write $F + J_0^+ = F + (J_0^+)_{10} + (J_0^+)_{01} + (J_0^+)_{00} + (J_0^+)_{11}$, where $(J_0^+)_{ij}$ is a left faithful module or a 0-left module according as $i = 1$ or $i = 0$, respectively, and $(J_0^+)_{ij}$ is a right faithful module or a 0-module according as $j = 1$ or $j = 0$, respectively. And for $i, j, l, m \in \{0, 1\}$, one has $(J_0^+)_{ij}(J_0^+)_{lm} \subseteq \delta_{kl}(J_0^+)_{im}$, where δ_{jl} is the Kronecker delta. We must have $(J_0^+)_{10}(J_0^+)_{01} = (J_0^+)_{01}(J_0^+)_{10} = (J_0^+)_{10}(J_0^+)_{00} = (J_0^+)_{00}(J_0^+)_{01} = 0$. Suppose that $(J_0^+)_{10}(J_0^+)_{01} \neq 0$ and let $a \in (J_0^+)_{10}$ and $b \in (J_0^+)_{01}$ such that $ab \neq 0$. Let

$$f_{((n-2,1,1),\emptyset,\emptyset,\emptyset)} = \sum_{\sigma \in S_3} (sgn\sigma) y^+_{\sigma(1)} (y_1^+)^{n-3} y^+_{\sigma(2)} y^+_{\sigma(3)}$$

be a highest weight vector corresponding to $\langle \lambda \rangle = ((n-2,1,1), \emptyset, \emptyset, \emptyset)$ (see [10, Theorem 12.4.12]). By making the evaluation $y_1^+ = 1_F$, $y_2^+ = a$, $y_3^+ = b$, we get $ab + ba \neq 0$ since $ab \in (J_0^+)_{11}$ and $ba \in (J_0^+)_{00}$. So, $f_{((n-2,1,1),\emptyset,\emptyset,\emptyset)}$ is not a φ-identity of A. Therefore $\chi_{((n-2,1,1),\emptyset,\emptyset,\emptyset)}$ appears with non-zero multiplicity in the decomposition of $P_n^\varphi / (P_n^\varphi \cap \mathrm{Id}^\varphi(A))$ into irreducible characters, but this is a contradiction to Corollary 4.8.2. As above, if $(J_0^+)_{01}(J_0^+)_{10} \neq 0$ or $(J_0^+)_{10}(J_0^+)_{00} \neq 0$ we reach a contradiction since the same polynomial $f_{((n-2,1,1),\emptyset,\emptyset,\emptyset)}$ is not a φ-identity for A.
At last, suppose $(J_0^+)_{00}(J_0^+)_{01} \neq 0$.

Then $f_{((n-2,1,1),\emptyset,\emptyset,\emptyset)} = \sum_{\sigma \in S_3}(sgn\sigma) y^+_{\sigma(1)} y^+_{\sigma(2)} (y_1^+)^{n-3} y^+_{\sigma(3)}$ is a highest weight vector corresponding to $\langle\lambda\rangle = ((n-2,1,1),\emptyset,\emptyset,\emptyset)$ and it is not a φ-identity, a contradiction.
Then one has

$$F + J_0^+ = (F + (J_0^+)_{10} + (J_0^+)_{01} + (J_0^+)_{11}) \oplus (J_0^+)_{00} \sim_{T_2^\varphi} A_1 \oplus A_2 \oplus N,$$

where $A_1 = F + (J_0^+)_{11} + (J_0^+)_{10}$, $A_2 = F + (J_0^+)_{11} + (J_0^+)_{01}$ and N is a nilpotent algebra.
Now we prove that $[(J_0^+)_{11}, (J_0^+)_{11}] = 0$. If not, let $a, b \in (J_0^+)_{11}$ such that $ab \neq ba$ and let $f_{((n-2,1,1),\emptyset,\emptyset,\emptyset)} = \sum_{\sigma \in S_3}(sgn\sigma) y^+_{\sigma(1)} y^+_{\sigma(2)} y^+_{\sigma(3)} (y_1^+)^{n-3}$ be a highest weight vector corresponding to $\langle\lambda\rangle = ((n-2,1,1),\emptyset,\emptyset,\emptyset)$. By evaluating $y_1^+ = 1_F$, $y_2^+ = a$ and $y_3^+ = b$, we get $f_{((n-2,1,1),\emptyset,\emptyset,\emptyset)} = ab - ba \neq 0$, which is a contradiction.
So, we get that $[y_1^+, y_2^+][y_3^+, y_4^+] \equiv 0$ is an identity of A_1 and A_2.
Finally, we claim that $F + J_0^- \in var^\varphi(UT_2^{sup})$. It is obvious, since $[y_1^+, y_2^+] \equiv 0$, $y_1^- y_2^- \equiv 0$ and $z^\theta \equiv 0$, for $\theta \in \{+,-\}$.
The same argument holds for the other cases and then the proof is complete. □

Now we are ready to present the main result of this section.

Theorem 4.8.2. *Let A be a finite dimensional φ-algebra such that $c_n^\varphi(A) \leq an$, for some constant a. Then*

$$A \sim_{T_2^\varphi} B_1 \oplus \cdots \oplus B_m \oplus N,$$

where $B_i \in var^\varphi(UT_2)$ or $B_i \in var^\varphi(UT_2^{sup})$ or $B_i \in var^\varphi(UT_2^{gr})$ or $B_i \in var^\varphi(UT_2^{gr,sup})$, for all $i = 1, \ldots, m$, and N is a nilpotent φ-algebra.

Proof. Since $c_n^\varphi(A) \leq an$, for some constant a, by Theorem 4.3.1, we may assume that $A = A_1 \oplus \cdots \oplus A_m$, where $A_1, \ldots, A_m$ are finite dimensional φ-algebras with $\dim_F A_i/J(A_i) \leq 1$, for all $1 \leq i \leq m$. Notice that this says that either $A_i \cong F + J(A_i)$ or $A_i \cong J(A_i)$ is a nilpotent φ-algebra. Since $c_n^\varphi(A_i) \leq c_n^\varphi(A)$, then $c_n^\varphi(A_i) \leq an$, for all $i = 1, \ldots, m$. Now the result follows by applying Corollary 4.8.4 to each non-nilpotent A_i. □

By putting together Theorems 4.7.2, 4.7.3, 4.7.4 and 4.7.5, we get a finer classification of the φ-algebras of at most linear growth.

Theorem 4.8.3. *Let A be a finite dimensional φ-algebra such that $c_n^\varphi(A) \leq an$, for some constant a. Then*

$$A \sim_{T_2^\varphi} B_1 \oplus \cdots \oplus B_m \oplus N,$$

where N is a nilpotent φ-algebra and, for all $i = 1, \ldots, m$, B_i is T_2^φ-equivalent to one of the following φ-algebras:

$$N^i, N_2 \oplus N^i, N_2 \oplus A_2 \oplus N^i, N_2 \oplus A_2^* \oplus N^i, N_2 \oplus A_2 \oplus A_2^* \oplus N^i,$$

$$C \oplus N^i, N_2^{sup} \oplus N^i, A_2^{sup} \oplus N^i, (A_2^{sup})^* \oplus N^i, N_2^{sup} \oplus A_2^{sup} \oplus N^i, N_2^{sup} \oplus (A_2^{sup})^* \oplus N^i,$$

$$A_2^{sup} \oplus (A_2^{sup})^* \oplus N^i, N_2^{sup} \oplus A_2^{sup} \oplus (A_2^{sup})^* \oplus N^i,$$

$$N_2^{gr} \oplus N^i, A_2^{gr} \oplus N^i, (A_2^{gr})^* \oplus N^i, N_2^{gr} \oplus A_2^{gr} \oplus N^i, N_2^{gr} \oplus (A_2^{gr})^* \oplus N^i,$$

$$A_2^{gr} \oplus (A_2^{gr})^* \oplus N^i, N_2^{gr} \oplus A_2^{gr} \oplus (A_2^{gr})^* \oplus N^i,$$

$$N_2^{gr,sup} \oplus N^i, A_2^{gr,sup} \oplus N^i, (A_2^{gr,sup})^* \oplus N^i, N_2^{gr,sup} \oplus A_2^{gr,sup} \oplus N^i,$$

$N_2^{gr,sup} \oplus (A_2^{gr,sup})^* \oplus N^i$, $A_2^{gr,sup} \oplus (A_2^{gr,sup})^* \oplus N^i$, $N_2^{gr,sup} \oplus A_2^{gr,sup} \oplus (A_2^{gr,sup})^* \oplus N^i$,

where C *is a commutative* φ*-algebra with trivial grading and trivial superautomorphism and* N^i *is a nilpotent* φ*-algebra.*

Bibliography

[1] S. Accomando. “On the identities and cocharacters of the algebra of 3×3 matrices with orthosymplectic superinvolution”. In: *J. Algebra* ***659 (13)*** (2024), pp. 482–515.

[2] S. Accomando. “Polynomial codimension growth of superalgebras with superautomorphism”. In: *Linear Algebra Appl.* ***710 (9)*** (2025), pp. 50–79.

[3] E. Aljadeff and A. Giambruno. “Multialternating graded polynomials and growth of polynomial identities”. In: *Proc. American Math. Soc.* ***41 (9)*** (2013), pp. 3055–3065.

[4] E. Aljadeff, A. Giambruno, and Y. Karasik. “Polynomial identities with involution, superinvolutions and the Grassmann envelope”. In: *Proc. American Math. Soc.* ***145*** (2017), pp. 1843–1857.

[5] F. Benanti. “On the cocharacter sequence of 3×3 matrices”. In: *Commun. Algebra* ***24 (13)*** (1996), pp. 4263–4279.

[6] F. Benanti and M. G. Campanella. “On the $*$-cocharacter sequence of 3×3 matrices”. In: *Linear Algebra Appl.* ***312*** (2000), pp. 101–114.

[7] F. Benanti, A. Giambruno, and I. Sviridova. “Asymptotics for the multiplicities in the cocharacters of some PI-algebras”. In: *Proc. American Math. Soc.* ***132 (3)*** (2004), pp. 669–679.

[8] W.D.S. Costa, A. Ioppolo, R. B. dos Santos, and A. C. Vieira. “Unitary superalgebras with graded involution or superinvolution of polynomial growth”. In: *J. Pure Appl. Algebra* ***225*** (2021), pp. 106–666.

[9] J. P. Cruz, R. B. dos Santos, and A. C. Vieira. “Identities and cocharacters of the algebra of 3×3 matrices with non-trivial grading and transpose graded involution”. In: *Linear Algebra Appl.* ***652*** (2022), pp. 173–195.

[10] V. Drensky. *Free algebras and PI-algebras, Graduate course in algebra*. Springer-Verlag, Singapore, 2000.

[11] V. Drensky and A. Giambruno. “Cocharacters, codimensions and Hilbert series of the polynomial identities for 2×2 matrices with involution”. In: *Canad. J. Math.* ***46*** (1994), pp. 718–733.

[12] A. Elduque and O. Villa. “The existence of superinvolutions”. In: *J. Algebra* ***319*** (2008), pp. 4338–4359.

[13] A. Giambruno. “$GL_m \times Gl_m$-representations and *-polynomial identities”. In: *Commun. Algebra* ***14*** (1986), pp. 787–796.

[14] A. Giambruno, A. Ioppolo, and F. Martino. “Standard polynomials and matrices with superinvolution”. In: *Linear Algebra Appl.* ***504*** (2016), pp. 272–291.

[15] A. Giambruno, A. Ioppolo, and D. La Mattina. “Superalgebras with Involution or Superinvolution and Almost Polynomial Growth of the Codimensions”. In: *Algebr. Represent. Theory* ***22*** (2019), pp. 961–976.

[16] A. Giambruno, A. Ioppolo, and D. La Mattina. "Varieties of algebras with superinvolution of almost polynomial growth". In: *Algebr. Represent. Theory* ***19 (3)*** (2016), pp. 599–611.

[17] A. Giambruno and D. La Mattina. "Graded polynomial identities and codimensions: Computing the exponential growth". In: *Advances in Mathematics* ***225*** (2010), pp. 859–881.

[18] A. Giambruno and S. Mishchenko. "Polynomial growth of the $*$-codimensions and Young diagrams". In: *Comm. Algebra* ***29 (1)*** (2001), pp. 277–284.

[19] A. Giambruno and A. Regev. "Wreath products and P.I. algebras". In: *J. Pure Appl. Algebra* ***35*** (1985), pp. 133–149.

[20] A. Giambruno and M. Zaicev. *Polynomial identities and asymptotic methods*. Mathematical Survey and Monographs Vol. **122**, American Mathematical Society, 2005.

[21] A. Ioppolo. "Some characterizations of algebras with involution with polynomial growth of their codimensions". In: *Linear and Multilinear Algebra* ***67 (6)*** (2019), pp. 1217–1230.

[22] A. Ioppolo. "Some results concerning the multiplicities of cocharacters of superalgebras with graded involution". In: *Linear Algebra and its Applications* ***594*** (2020), pp. 51–70.

[23] A. Ioppolo. "The exponent for superalgebras with superinvolution". In: *Linear Algebra Appl.* ***555*** (2018), pp. 1–20.

[24] A. Ioppolo and F. Martino. "On multiplicities of cocharacters for algebras with superinvolution". In: *J. Pure Appl. Algebra* ***225 (3)*** (2021), p. 106536.

[25] A. Ioppolo and D. La Mattina. "Algebras with superautomorphism: simple algebras and codimension growth". In: *Israel J. Math.* (to appear).

[26] A. Ioppolo and D. La Mattina. "Codimension growth of algebras with superautomorphism". In: *J. Pure Appl. Algebra* ***229 (2)*** (2025), p. 107871.

[27] A. Ioppolo and D. La Mattina. "Polynomial codimension growth of algebras with involutions and superinvolutions". In: *J. Algebra* ***472*** (2017), pp. 519–545.

[28] G. James and A. Kerber. *The Representation Theory of the Symmetric Group*. Encyclopedi of Mathematics and its Applications, Vol. **16**, Addison-Wesley, London, 1981.

[29] A. R. Kemer. "T-ideals with power growth of the codimensions are Specht". In: *Sibirsk. Mat. Zh.* ***19*** (1978), pp. 54–69 (in Russian), English translation in: *Sib. Math. J.* **19** (1978), pp- 37–48.

[30] A. R. Kemer. "Solution of the problem as to whether associative algebras have a finite basis of identities". Russian. In: *Dokl. Akad. Nauk SSSR* ***298 (2)*** (1988), pp. 273–277, English translation in: *Sov. Math. Dokl.* **37 (1)** (1988), pp. 60–64.

[31] A. R. Kemer. "Varieties of finite rank". In: *Proc. 15-th All the Union Algebraic Conf., Krasnoyarsk, Vol. 2 (in Russian)* (1979), p. 73.

[32] P. Koshlukov and D. La Mattina. "Graded algebras with polynomial growth of their codimensions". In: *Journal of Algebra* ***434*** (2015), pp. 115–137.

[33] D. Krakowski and A. Regev. "The polynomial identities of the Grassmann algebra". In: *Trans. Amer. Math. Soc.* ***181*** (1973), pp. 429–438.

[34] D. V. Levchenko. "Identities with involution of a second order matrix algebra". In: *C. R. Acad. Bulg. Sci.* ***33*** (1980), pp. 1043–1045.

[35] Yu. N. Malcev. “A basis for the identities of the algebra of upper triangular matrices”. Russian. In: *Algebra i Logika* ***10*** (1971), pp. 393–400, English translation in: *Algebra and Logic* **10**, pp. 242–247.

[36] D. La Mattina. “Almost polynomial growth: classifying varieties of graded algebras”. In: *Israel J. Math.* ***207*** (2015), pp. 53–75.

[37] D. La Mattina. “On the graded identities and cocharacters of the algebra of 3 $\times$ 3 matrices”. In: *Linear Algebra Appl.* ***384*** (2004), pp. 55–75.

[38] D. La Mattina. “Varieties of algebras of polynomial growth”. In: *Boll. Unione Mat. Ital. (9)* ***1 (3)*** (2008), pp. 525–538.

[39] D. La Mattina. “Varieties of almost polynomial growth: classifying their subvarieties”. In: *Manuscripta Math.* ***123*** (2007), pp. 185–203.

[40] D. La Mattina. “Varieties of superalgebras of almost polynomial growth”. In: *Journal of Algebra* ***336*** (2011), pp. 209–226.

[41] J. B. Olson and A. Regev. “An application of representation theory to PI-algebras”. In: *Proc. American Math. Soc.* ***55*** (1976), pp. 253–257.

[42] F. C. Otera. “Finitely generated PI-superalgebras with bounded multiplicities of the cocharacters”. In: *Commun. Algebra* ***33 (6)*** (2005), pp. 1693–1707.

[43] M. L. Racine. “Primitive superalgebras with superinvolution”. In: *J. Algebra* ***206 (2)*** (1998), pp. 588–614.

[44] A. Regev. “Existence of identities in $A \otimes B$”. In: *Isr. J. Math.* ***11*** (1972), pp. 131–152.

[45] W. Specht. “Gesetze in Ringen. I”. In: *Math. Z.* ***52*** (1950), pp. 557–589.

[46] A. Valenti. “The graded identities of upper triangular matrices of size two”. In: *J. Pure Appl. Algebra* **172** ***(2-3)*** (2002), pp. 325–335.