

RESEARCH ARTICLE OPEN ACCESS

Detecting Changes in Space-Varying Parameters of Local Poisson Point Processes

Nicoletta D'Angelo 

Department of Economics, Business and Statistics, University of Palermo, Palermo, Italy

Correspondence: Nicoletta D'Angelo (nicoletta.dangelo@unipa.it)

Received: 10 April 2025 | **Revised:** 22 May 2025 | **Accepted:** 9 June 2025

Funding: The research work was supported by the Targeted Research Funds 2025 (FFR 2025) of the University of Palermo, GRINS PE00000018–CUP C93C22005270001 from the European Union-NextGenerationEU, and PRIN 2022: Spatio-temporal Functional Marked Point Processes for probabilistic forecasting of earthquakes 2022BN7CJP P. I. Giada Adelfio. CUP B53C24006340006.

Keywords: local analyses | point process | spatial segmentation | spatial statistics

ABSTRACT

Recent advances in local models for point processes have highlighted the need for flexible methodologies to account for the spatial heterogeneity of external covariates influencing process intensity. In this work, we introduce *tessellated spatial regression*, a novel framework that extends segmented regression models to spatial point processes, with the aim of detecting abrupt changes in the effect of external covariates on the process intensity. Our approach consists of two main steps. First, we apply a spatial segmentation algorithm to geographically weighted regression estimates, generating different tessellations that partition the study area into regions where model parameters can be assumed constant. Next, we fit log-linear Poisson models in which covariates interact with the tessellations, enabling region-specific parameter estimation and classical inferential procedures, such as hypothesis testing on regression coefficients. Unlike geographically weighted regression, our approach allows for discrete changes in regression coefficients, making it possible to capture abrupt spatial variations in the effect of real-valued spatial covariates. Furthermore, the method naturally addresses the problem of locating and quantifying the number of detected spatial changes. We validate our methodology through simulation studies and applications to two examples where a model with region-wise parameters seems appropriate and to an environmental dataset of earthquake occurrences in Greece.

1 | Introduction

Interest in methods for analyzing spatial point processes has increased across many fields of science, notably in ecology, epidemiology, geoscience, astronomy, econometrics, and crime research (Baddeley et al. 2015; P. J. Diggle 2013). When the structure of a given point pattern is observed, it is assumed as a realization of an underlying generating process, whose properties are estimated and then used to describe the structure of the observed pattern. Analyzing a spatial point process, the first step is to learn about its first-order characteristics, studying the

relationship of the points with the underlying environmental variables that describe the observed heterogeneity. The aim is typically to learn about the mechanism that generates these events (Møller et al. 1998; P. J. Diggle 2013; Illian et al. 2008).

The most common point process model taking into account external covariates is the log-linear Poisson point process model, typically globally defined with the process properties and estimated parameters assumed to be constant throughout the entire study area. However, a model with constant parameters may not adequately represent detailed local variations in the data since the

This is an open access article under the terms of the [Creative Commons Attribution](https://creativecommons.org/licenses/by/4.0/) License, which permits use, distribution and reproduction in any medium, provided the original work is properly cited.

© 2025 The Author(s). *Environmetrics* published by John Wiley & Sons Ltd.

pattern may present spatial variation in the influence of covariates, in the scale or spacing between points, and the abundance of points. Indeed, a different way of analyzing a point pattern is commonly based on local techniques identifying specific and undiscovered local structure, for instance, subregions characterized by different interactions among points, intensity, and influence of covariates. Throughout the paper, we shall distinguish between ‘global’ models, in which the parameters are assumed constant as in regular regression models, and ‘local’ models, in which the parameters are allowed to vary with location.

For spatial point processes, Baddeley (2017) presented a general *geographically weighted regression* framework based on the local composite likelihood to detect and model gradual spatial variation in any parameter of a spatial point process model (such as Poisson, Gibbs, and Cox processes). In particular, the parameters in the model that govern the intensity, the dependence of the intensity on the covariates, and the spatial interaction between points, are estimated locally. In the context of spatial and spatiotemporal point process modeling, the necessity of capturing local variations in the effects of the model’s parameters has led to the development of many advanced methodologies that allow for spatially varying parameters (Siino et al. 2017, 2018; D’Angelo et al. 2023, 2025; D’Angelo and Adelfio 2024, 2025; Tarantino et al. 2024). In this paper, we only deal with spatial Poisson point processes. For these processes, local likelihood is well-developed, because of its close relationship to probability density estimation from i.i.d. observations (Hjort and Jones 1996; Loader 1996; Loader 1999). Geographically weighted logistic regression for pixel presence–absence data (Osborne et al. 2007) is equivalent to maximum local likelihood for Poisson processes where the intensity is a log-linear function of the regression parameters (see also Fan et al. (2009); Monir Hossain and Lawson (2006); Ogata and Katsura (1988)).

However, these local approaches often rely on non-parametric techniques, which can complicate interpretation and standard statistical inference. In this work, we propose a novel method for formalizing and fitting a *tessellated* spatial log-linear Poisson point process model, specifically designed to provide clear-cut regions where covariate effects change while maintaining a fixed-effects modeling framework. Indeed, a tessellation is a division of a window into non-overlapping regions, called *tiles*. The main idea behind our proposal is to define a tessellation of the study area that partitions the space into distinct regions, each characterized by a different set of parameter estimates. Unlike geographically weighted regression, our approach allows for discrete changes in regression coefficients, making it possible to capture abrupt spatial variations in the effect of real-valued spatial covariates. Therefore, it is advisable to employ tessellated regression when the interest is not on estimating smoothly varying spatial effects but rather identifying and quantifying abrupt changes in model parameters. The main objective of *tessellated regression* is, therefore, the identification of these abrupt changes, represented by tessellations of the analyzed region. These delimitations between areas where the effect of a variable changes in explaining the process intensity may be informative of the underlying process generating the events. Indeed, region-wise parameters may be the effect of unobserved characteristics not taken into account in the fitted model. Moreover, our approach has the advantage of detecting and modeling abrupt spatial

variation in covariates’ effects within the formal likelihood framework, providing region-wise parameter estimates, confidence intervals, and hypothesis tests. This makes it particularly useful for applications requiring localized estimation while preserving interpretability within the classical Poisson log-linear modeling framework. Indeed, the motivation of this research came from the previous work of D’Angelo et al. (2022), where the authors fitted a local model to earthquake data, accounting for external geological information. That work proved the presence of a clear separation of the analyzed region into subregions where the model parameters completely change. The delimitation of such regions, however, was not possible employing the standard local models, as well as the straightforward assessment of the “locality” of a specific covariate.

Our proposed strategy can be seen as the spatial point processes extension of the most well-known segmented regression model (Lerman 1980), commonly employed in time series or, in general, when a covariate effect is assumed to affect the response variable in a piece-wise manner. The main advantage of these models lies in the results’ interpretability while also achieving a good trade-off with flexibility, typically achieved by non-parametric approaches. Our work represents the first attempt in this direction for point process models. Indeed, in the realm of spatial statistics, Hollaway and Killick (2024) recently presented a spatiotemporal changepoint method that utilizes a generalized additive model (GAM) dependent on the 2D spatial location and the observation time to account for the underlying spatiotemporal geostatistical process. However, to the best of our knowledge, no methods have yet been proposed to study the changes in the effect of covariates in point processes. For instance, Altieri et al. (2015) proposed a method to find multiple unknown changepoints over time in the inhomogeneous intensity of a spatiotemporal point process, allowing for spatial and temporal dependence within segments but without specifically addressing the search for changepoints in the spatial covariates.

Our methodology consists of a two-step procedure: tessellation identification and tessellated model fitting. The first step involves a segmentation algorithm applied to geographically weighted estimates from an assumed local model. We employ the Simple Linear Iterative Clustering (SLIC) algorithm (Achanta et al. 2012), followed by a cluster analysis to merge superpixels with similar values, preventing over-segmentation. Superpixels are an effective tool for spatial segmentation, grouping cells with similar values into regions that maintain boundaries and structures while providing a desired level of homogeneity. Among the various methods available, SLIC stands out as one of the most prominent. In the second step, the resulting tessellation is treated as a categorical spatial variable in a log-linear Poisson point process model, encoded via dummy variables that indicate the tile to which each point belongs. This enables us to obtain region-wise covariate effects while maintaining compatibility with standard Poisson log-linear models.

An important advantage of our methodology is that it naturally addresses some common challenges in segmented regression models: the selection of the number and position of changepoints, here represented by the tiles of the tessellation. Indeed, the first step of the segmentation and clustering procedure automatically determines their number and position before the

second step of model fitting. Although the proposed method does not explicitly address the problem of testing for the existence of a threshold, it does not require any prior knowledge or assumption regarding the number of abrupt spatial changes. In classical changepoint analysis, most fitting procedures focus on estimating the changepoint location and the associated covariate effects, typically under the assumption that a changepoint exists. This issue has been extensively studied (Feder 1975; Beckman and Cook 1979; Ulm 1991), highlighting that the distribution of the test statistic is often complex and depends on the alternative hypotheses, even in a nonspatial setting (Muggeo 2003). In contrast, our method relies on a clustering approach to determine the number of tiles in the tessellation by means of the higher silhouette value. Consequently, although the results do not provide an explicit measure of uncertainty, the procedure will prefer a model with no changes in the parameters whenever the algorithm selects a single cluster of parameter values.

We consider different formalizations of the proposed approach. The most general tessellated model assumes that each covariate is allowed to vary independently of the others across space, leading to potentially different tessellations for different predictors. Additionally, our proposed procedure, applied to the maps of the solely space-varying intercept (i.e., “constant” intensity) faces the problem of feature detection. Indeed, one of the main interests of spatial point pattern analysis is identifying features surrounded by clutter. The conventional terminology is that a feature is a point of the pattern or process of interest, and clutter (also called noise) consists of extraneous points that are not proper to the pattern of interest. For example, the detection of surface minefields in an image captured by a reconnaissance aircraft involves processing to generate a list of objects, among which some could be mines while others may belong to different object categories (Allard and Fraley 1997). Similarly to Byers and Raftery (1998), we are able to estimate this separation without making any assumptions about the shape or number of features. In other words, our approach represents a simple and intuitive method for estimating regions of different intensities in a point process. Furthermore, we introduce a simpler version of the tessellated model, where a single tessellation is shared among all parameters, ensuring that the entire set of covariate effects changes coherently across the identified regions. The motivation for this further simplification stems from the need for simplification in the interpretation of the fitted model.

The proposed framework supports multiple applications, and it can be conceptually extendable to geostatistics, where the aim is not to model the point pattern intensity but a generic spatial response variable. Furthermore, our methodological proposal lies the basis for many future extensions, including multitype Poisson point process models, more complex models based on the Poisson one, like the log-Gaussian Cox process models, and all of their spatiotemporal extensions.

Through simulation studies, we assess the performance of our method, highlighting its dependence on the number of observed points and the magnitude of parameter changes. Applications to two examples where a model with region-wise parameters could be appropriate illustrate the proposal’s potential for real-world spatial analysis. Furthermore, the analysis of the environmental

problem of earthquake occurrence is addressed by taking into account the data of D’Angelo et al. (2022).

The structure of the paper is as follows. After the preliminaries in Section 2, Sections 3 and 4 present two examples and the earthquake data that motivated this work. Then, Section 5 introduces the proposed methodology, and Section 6 explores various simulation scenarios to understand the performance of the method in terms of tessellation identification, parameter estimation, and goodness-of-fit. Section 7 provides the applications to the data introduced in Section 3. Moreover, the Greek seismic data are analyzed through a tessellated model in Section 8. Finally, new avenues for further methodological developments are addressed in Section 9.

2 | Preliminaries

2.1 | Spatial Point Processes

We consider a (simple) point process $X = \{x_i\}_{i=1}^N$ (Daley and Vere-Jones (2007)), with points x_i in the 2-dimensional Euclidean space domain $W \in \mathbb{R}^2$. Formally, X is a random element in the measurable space of locally finite point configurations/patterns $\mathbf{x} = \{(x_1, \dots, x_n)\}$, $n \geq 0$. A point location in the two-dimensional plane is denoted by a lowercase letter like u . Any location u can be specified by its Cartesian coordinates $u = (u_1, u_2)$ in such a way that we do not need to mention the coordinates explicitly. The first-order property of X is described by the intensity function, defined as

$$\lambda(u) = \lim_{|du| \rightarrow 0} \frac{\mathbb{E}[N(du)]}{|du|}$$

where du is an infinitesimal region that contains the point $u \in W$, $|du|$ is its area and $\mathbb{E}[N(du)]$ denotes the expected number of events in du . When the intensity is constant, the process is called homogeneous. In the inhomogeneous case, the intensity is not constant in the study area but may depend, for instance, on the coordinates of points. A point process model, assuming independence, is completely described by its intensity function $\lambda(u)$ (Daley and Vere-Jones 2007).

A general spatial log-linear Poisson model generalizes both homogeneous and inhomogeneous models can be expressed as

$$\lambda(u) = \lambda(u, \theta) = \exp \left\{ \theta_0 + \sum_{j=1}^p \theta_j Z_j(u) \right\} \quad (1)$$

where $u \in W$, $\mathbf{Z}(u) = \{Z_1(u), \dots, Z_p(u)\}$ are p spatial covariates and $\theta = (\theta_0, \theta_1, \dots, \theta_p)$ are unknown parameters. The estimation of the point process parameters is carried out through the maximization of the log-likelihood, defined as

$$\log L(\theta) = \sum_{x_i \in \mathbf{x}} \log \lambda(x_i; \theta) - \int_W \lambda(u; \theta) du. \quad (2)$$

2.2 | Local Poisson Point Process Models

When spatial log-linear models are used, the model parameters are usually assumed to be constant across the entire study region.

This assumption may be too simplistic in contexts characterized by multiscale and fractal features, like the seismic one, where the relationship between the intensity of earthquakes and other possible characteristics of the area where events occur varies spatially. In this context, the features can be described by a *local* Poisson process with intensity

$$\lambda(u) = \lambda(u; \theta(u)) = \exp \left\{ \theta_0(u) + \sum_{j=1}^p \theta_j(u) Z_j(u) \right\}, \quad (3)$$

with local coefficients being functions of u . This is the local version of the global log-linear intensity (1). For estimating local models as in Equation (3), the likelihood at u is used, being a weighted version of the original likelihood, with the greatest weight on locations close to u . Therefore, in the local context, the local log-likelihood associated with location s (Loader (1999)) is

$$\log L(s; \theta) = \sum_{i=1}^n w_h(\mathbf{x}_i - s) \log \lambda(\mathbf{x}_i; \theta) - \int_W \lambda(u; \theta) w_h(u - s) du, \quad (4)$$

where $w_h(u) = h^{-d} w(u/h)$ is a weight non-parametric function, and $h > 0$ is a smoothing bandwidth. It is not necessary to assume that w_h is a probability density. Throughout the paper, we consider a kernel of fixed bandwidth h . In the local likelihood in Equation (4), this is usually chosen by the cross-validation criterion (Loader 1999). The optimal bandwidth h_{opt} maximizes $LCV(h) = \sum_i \log \lambda(\mathbf{x}_i; \hat{\theta}_i(\mathbf{x}_i)) - \int_D \lambda(u; \hat{\theta}(u)) du$ where $\hat{\theta}(u) = \hat{\theta}(u, h)$ is the local estimate of θ at location u using bandwidth $h > 0$, and $\hat{\theta}_i(\mathbf{x}_i) = \hat{\theta}_i(\mathbf{x}_i; h)$ is the corresponding ‘leave-one-out’ estimate at the location $\mathbf{x}_i$ computed from the data omitting $\mathbf{x}_i$.

Maximizing the local likelihood for each fixed u gives local parameter estimates $\hat{\theta}(u)$, also known as *geographically weighted estimates*. These fitted coefficients can be plotted as a function of spatial location u . However, geographically weighted regression presents a major limitation. Its continuous nature makes it difficult to identify abrupt changes in the effect of covariates, as it imposes smooth transitions across space.

The alternative we propose in this paper and introduce in Section 5 is *tessellated regression*, which partitions the study region into distinct tiles and estimates regression coefficients separately for each of them. In other words, we assume that the covariates influencing the intensity of the observed point pattern have region-wise effects. This method aggregates data within tiles rather than estimating parameters continuously across space. Unlike geographically weighted regression, this approach allows for discrete changes in regression coefficients, making it possible to capture abrupt spatial variations in the effect of covariates. Therefore, while geographically weighted regression is useful for modeling smoothly varying spatial effects, it may fail when abrupt changes occur. In particular, *tessellated regression* improves interpretability in spatial point process modeling.

The key assumption in geographically weighted regression is that, for each spatial location $u \in W$, there is an unobserved spatial tessellation $W(u)$ containing u where the global model is *exactly* true with parameters $\theta(u)$. If the support of the kernel centered at u falls entirely inside $W(u)$, then the statistical properties of $\hat{\theta}(u)$ are the same as they would be if the entire process followed the global model with constant parameter value $\theta = \theta(u)$ (Baddeley (2017)). This motivates us to define the *tessellated model* (Section 5) by means of the tessellations obtained from the maps of local estimates through a spatial segmentation algorithm.

3 | Examples

In this Section, we present two examples of point patterns for which a model with region-wise parameters could be more appropriate than both global and local Poisson models.

3.1 | Gordon Square Data

The simplest case occurs when the aim is to detect changes in the space-varying homogeneous intensity of a local Poisson point

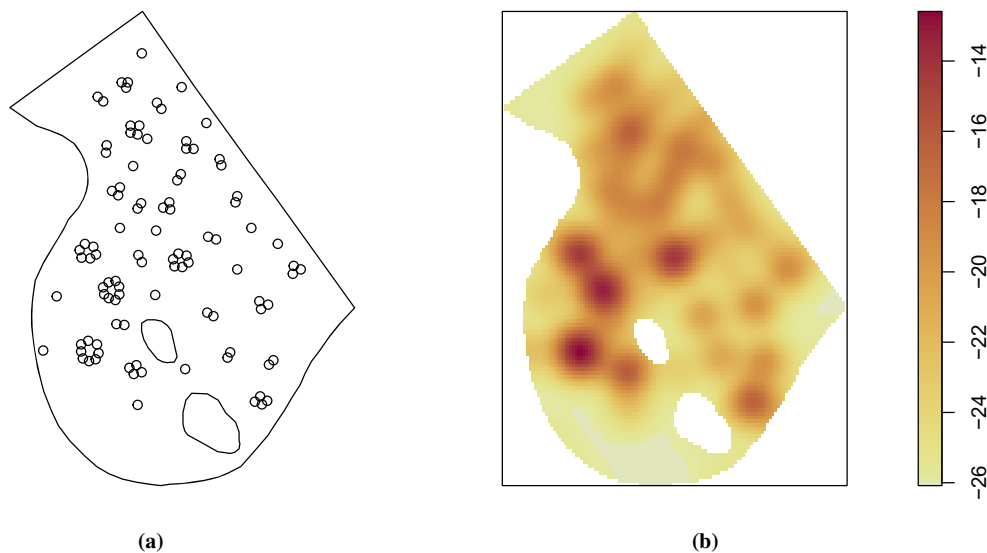


FIGURE 1 | (a) Gordon Square data. Locations of 99 people (circles) sitting on grass (gray shading) in Gordon Square, London, UK on a sunny afternoon (Baddeley et al. 2013). (b) Spatially varying intercept estimated from model (5).

process model. For instance, Figure 1a shows the spatial locations of people sitting on the grass in Gordon Square, London, on a sunny afternoon (Baddeley et al. 2013). The pattern appears to show spatial organization at several different scales. It would be possible to model the point pattern as a realization of a cluster or inhibitive process. However, the interest might lie in the will to estimate the locations of the clusters and treat them as *feature* patterns. With this aim in mind, the proposed approach could try segmenting the spatially varying “homogeneous” intensity estimated by the following local model

$$\hat{\lambda}_{gordon}(u) = \exp\{\hat{\theta}_0(u)\} \quad (5)$$

to obtain the two (or more) constant intensities in the subregion clearly defined by the higher intensity, that is, where at least two points form a spatial cluster. Indeed, Figure 1b shows the spatially varying intercept estimated from model (5). The fitting of a model with different region-wise intensities could help in discriminating the points of the pattern in clutter and feature.

3.2 | Bronze Filter Data

To illustrate a case where the aim is to detect changes in the space-varying parameters of a covariate of a fitted local Poisson point process, Figure 2 shows centers of grains in a bronze filter (Bernhardt et al. 1997). The pattern is clearly inhomogeneous with a negative effect of the x-coordinate. This effect, however, is not constant, nor smoothly changing along the coordinate. The estimation of region-wise effects of the x-coordinate could help in finding the specific places where the effect of the coordinate changes.

To have an idea of the possible changes in the coordinate effect, Figure 3 depicts the local coefficients estimated from the model

$$\hat{\lambda}_{bronze}(u) = \exp\{\hat{\theta}_0(u) + \hat{\theta}_1(u)Long(u)\} \quad (6)$$

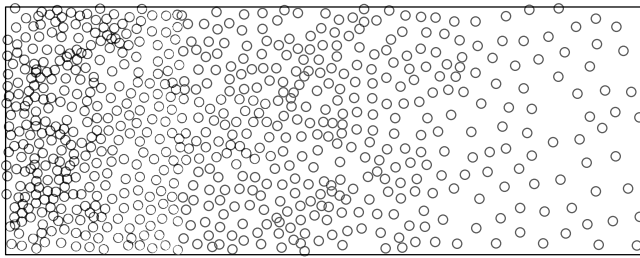


FIGURE 2 | Spatially inhomogeneous pattern of circular section profiles of particles, observed in a longitudinal plane section through a gradient sinter filter made from bronze powder (Bernhardt et al. 1997).

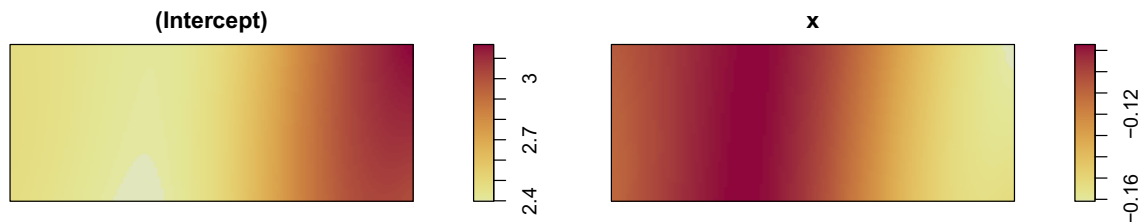


FIGURE 3 | Spatially varying coefficients of model (6) fitted to the Bronze filter data.

clearly displaying a sharp change in both the space-varying intercept and slope coefficient of the x-coordinate ($Long(u)$). Ideally, our methods should detect the different types of patterns and estimate the boundary between them using only the point location data.

4 | Motivating Problem: Greek Seismicity Data

As anticipated, the motivation of this work arose from the will to describe the abrupt changes in the effects of seismic sources on the earthquake phenomenon. In particular, we consider the same data analyzed in Siino et al. (2017); D'Angelo et al. (2022, 2023); D'Angelo and Adelfio (2024), related to 1111 earthquakes that occurred in Greece between 2005 and 2014 with a magnitude greater than 4, coming from the Hellenic Unified Seismic Network (H.U.S.N.). These are in Figure 4, with faults in blue and plate boundaries in red.

The aim is to understand whether the two spatial variables *Distance from the plate boundary* ($D_p(u)$) and *Distance from the faults* ($D_f(u)$) computed as the Euclidean distances from the spatial location u of events and the map of geological information, affect the intensity of the seismic process. This

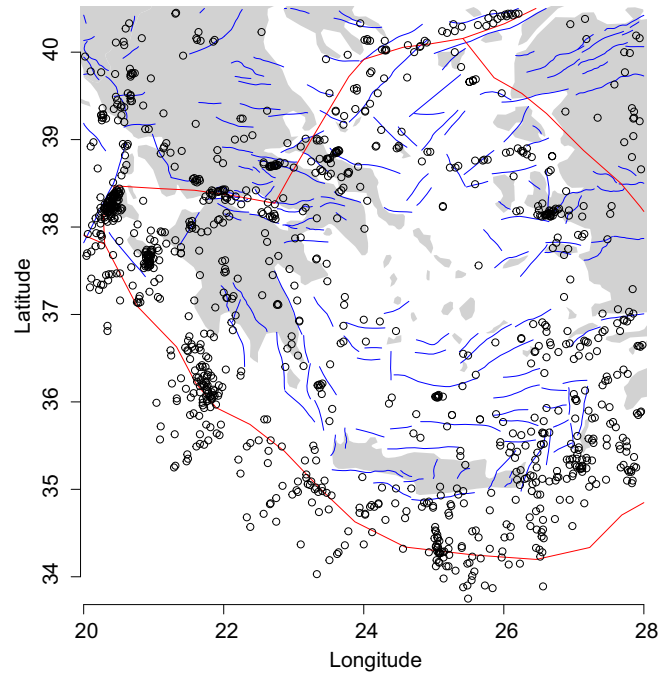


FIGURE 4 | Seismic point pattern.

application is particularly interesting as it also deals with changes in the space-varying parameters of an external covariate in local Poisson point processes. Note that D'Angelo et al. (2022) already proved the variables' effects to be local through the local model

$$\hat{\lambda}_{grecce}(u) = \exp\{\hat{\theta}_0(u) + \hat{\theta}_1(u)D_p(u) + \hat{\theta}_2(u)D_f(u)\}. \quad (7)$$

Indeed, Figure 5 shows the spatial varying coefficients $\hat{\theta}_1(u)$, $\hat{\theta}_2(u)$ and $\hat{\theta}_3(u)$ of the local Poisson Model in Equation (7), that is, the coefficients of the intercept and the variables $D_f(u)$ and $D_{pb}(u)$, respectively. Filling the gap in D'Angelo et al. (2022), we are interested in assessing whether a model with space-varying parameters could be appropriate while also clearly defining the regions where these effects change.

5 | Tessellated Spatial Poisson Point Process Models

The *tessellated* model we propose is, therefore, expressed as

$$\lambda(u) = \lambda(u, \beta, \gamma) = \exp \left\{ \beta_0 + \sum_{j=1}^p \beta_j Z_j(u) + \sum_{k=1}^{q-1} \gamma_{0k} W_k(u) + \sum_{j=1}^p \sum_{k=1}^{q-1} \gamma_{jk} Z_j(u) W_{Z_j, k}(u) \right\}, \quad (8)$$

where the β parameters are the effects of the non-tessellated spatial covariates and the γ parameters are those referred to the change in the effects of the covariates in the tiles identified by the tessellation $W(u)$. A tessellation refers to the division of space into non-overlapping regions or cells, often used to analyze the spatial structure of point patterns. The tessellation $W(u)$ of the variable $Z_j(u)$ is indicated by $W_{Z_j}(u)$, and it has q tiles, encoded in the model using $q - 1$ dummy variables. They can be interpreted as categorical variables indicating which tile of the tessellation u belongs to. Inference on γ_j , and on the tessellation of the j th covariate in turn, can be carried out, for instance, through the Likelihood Ratio Test (LRT), since model (1) and (8) are nested. We decided to denote the non-tessellated effects in model (8) by β to avoid confusion with θ of model (1) in the applications.

A special case of (8), is defined as follows

$$\lambda(u) = \lambda(u, \beta, \gamma) = \exp \left\{ \beta_0 + \sum_{j=1}^p \beta_j Z_j(u) + \sum_{k=1}^{q-1} \gamma_{0k} W_k(u) + \sum_{j=1}^p \sum_{k=1}^{q-1} \gamma_{jk} Z_j(u) W_k(u) \right\}. \quad (9)$$

Basically, this represents the case of the tessellated model (8) with $W(u)$ common to every variable, ensuring that the entire set of covariate effects changes coherently across the identified regions. This model is even easier to interpret with respect to the tessellated one, and therefore, if the data are better described by this model, one further gain in interpretability of the phenomenon under analysis.

We employ the *supercells* (Nowosad and Stepinski (2022)) R (R Core Team (2019)) package for obtaining the tessellation and *spatstat* (Baddeley and Turner (2005)) for the model fitting.

Of course, the first step and main challenge here is the identification of the tessellations $W(u)$, discussed below.

5.1 | Tessellation Identification

Superpixels are an effective tool for spatial segmentation, grouping cells with similar values into regions that maintain boundaries and structures while providing a desired level of homogeneity. These groupings not only enrich the information content compared to single cells but also accelerate subsequent processing. Among the various methods available, the SLIC algorithm (Achanta et al. 2012) stands out as one of the most prominent.

SLIC begins by initializing cluster centers at regular intervals and assigning each cell to the nearest cluster center based on a combined distance metric that incorporates both spectral and spatial proximity. The distance D between a cell and a cluster center is defined as

$$D = \sqrt{d_c^2 + g \frac{d_s^2}{S^2}}, \quad (10)$$

where d_c is the color (spectral) distance, d_s is the spatial (Euclidean) distance, S is the interval between initial cluster

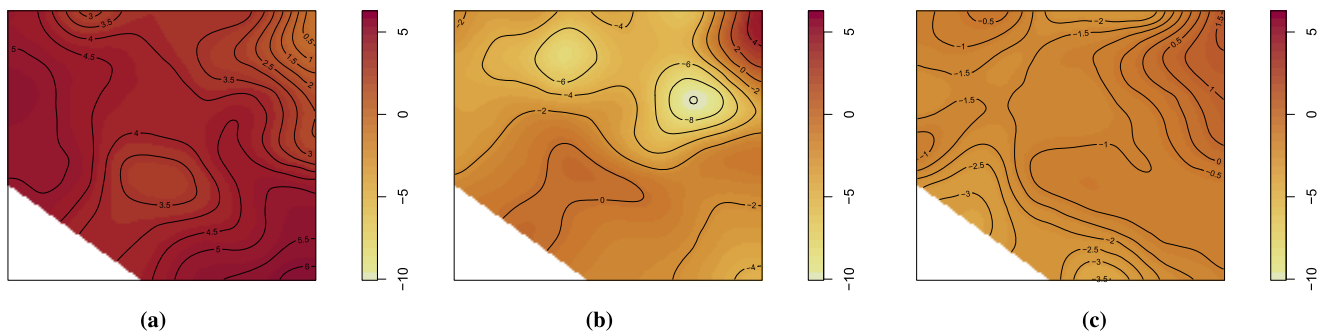


FIGURE 5 | Spatial varying coefficients $\hat{\theta}_1(u)$, $\hat{\theta}_2(u)$ and $\hat{\theta}_3(u)$ of the local Poisson Model in Equation (7), that is, the coefficients of the intercept and the variables $D_f(u)$ and $D_{pb}(u)$, respectively.

centers, and g is the compactness parameter that balances the influence of spectral and spatial distances. In our context, spectral bands refer to the different layers (or channels) of the raster data representing measurements from distinct portions of the electromagnetic spectrum, such as visible (e.g., red, green, blue) or near-infrared bands. Higher values of g produce regularly shaped superpixels, whereas lower values result in more spatially adapted, irregular shapes.

The color distance is calculated as

$$d_c = \sqrt{\sum_{p \in B} (I(x_i, y_i, sp) - I(x_j, y_j, sp))^2},$$

where B is the set of spectral bands, and $I(x_i, y_i, sp)$ represents the intensity value of the spectral band sp at location (x_i, y_i) . For example, if $B = \{\text{Red, Green, Blue}\}$, then d_c captures the overall difference in color between the two cells across these three channels. The spatial distance is given by

$$d_s = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}.$$

The algorithm iteratively updates cluster centers and assignments until convergence, typically achieved after 4–10 iterations.

The SLIC algorithm provides an efficient and effective method for generating superpixels, but it may result in over-segmentation when the number of superpixels is too high. Indeed, if the number of tiles is too small, the model may not capture all the changes in the relationship between the variables, resulting in bias and reduced model fit. On the other hand, if the number of tiles is too large, the model may overfit the data and may not generalize well to new data. Surely, this would be a problem common to many different methods. Nevertheless, the proposed cluster analysis helps in reducing the number of tiles within a consolidated statistical procedure framework. We, therefore, propose to carry out hierarchical clustering for the j th covariate as follows.

1. Compute a distance matrix using the distance metric (10) on the maps of the estimated local parameters $\hat{\theta}_j(u)$ coming from the local model (3).
2. Perform hierarchical clustering using a method like Ward's linkage on the resulting superpixels' values.
3. Cut the dendrogram to define the desired number of clusters according to the highest silhouette value.
4. Assign cluster labels to the superpixels to form the tiles to obtain the tessellation $W_{Z_j}(u)$.

For the simplest case of model (9), the SLIC algorithm is applied to the multiple rasters of the maps of all the parameters in the corresponding local model. These rasters may include multiple layers, each representing a different estimated parameter surface. On the other hand, the main perk of model (8) over model (9) is that the interval between initial cluster centers S and the compactness parameter g can be tuned separately for each covariate. Finally, note that the proposed tessellation identification procedure allows for the identification of noncontiguous or geographically separated areas (i.e., tiles) that share the same parameter values and therefore similar model behavior.

5.2 | Model Fitting

Once the tessellation for each covariate is obtained, the *tessellated* model (8) can be fitted.

Any tessellated model can be fitted through the established Berman-Turner technique (Berman and Turner 1992), also called *quadrature scheme*. Renaming the data points as $x_1, \dots, x_n$ with $(u_i, t_i) = x_i$ for $i = 1, \dots, n$, then generate m additional “dummy points” $(u_{n+1}, t_{n+1}), \dots, (u_{n+m}, t_{n+m})$ to form a set of $n + m$ cubature points (where $m > n$). Then we determine cubature weights $a_1, \dots, a_m$ so that the integral in (2) can be approximated by a Riemann sum $\int_W \lambda(u; \theta) du \approx \sum_{k=1}^{n+m} a_k \lambda(u_k; \theta)$ where a_k are the cubature weights such that $\sum_{k=1}^{n+m} a_k = l(W)$ where l is the Lebesgue measure.

The log-likelihood (2) of the template model can be approximated by

$$\begin{aligned} \log L(\theta) &\approx \sum_k \log \lambda(u_k; \theta) + \sum_k (1 - \lambda(u_k; \theta)) a_k \\ &= \sum_k e_k \log \lambda(u_k; \theta) + (1 - \lambda(u_k; \theta)) a_k \end{aligned} \quad (11)$$

where e_k is the indicator that equals 1 if (u_k, t_k) is a data point. Writing $y_k = e_k / a_k$ (11) becomes $\log L(\theta) \approx \sum_k a_k (y_k \log \lambda(u_k, t_k; \theta) - \lambda(u_k, t_k; \theta)) + \sum_k a_k$.

Apart from the constant $\sum_k a_k$, this expression is formally equivalent to the weighted log-likelihood of a Poisson regression model with responses y_k and weights a_k . This can be maximized using standard GLM software. Another option not detailed in this work is the logistic spatial regression (Baddeley et al. 2014).

6 | Simulation Study

This section is devoted to some numerical studies to understand the performance of the method in terms of tessellation identification, parameter estimation, and goodness-of-fit.

6.1 | Tessellation Identification

We consider point processes with region-wise constant intensity as follows

$$\lambda = \exp\{\beta_0 + \gamma_0 W(u)\} \quad (12)$$

with

$$W(u) = W(x, y) = \begin{cases} 1 & \text{if } x > y \\ 0 & \text{otherwise} \end{cases}$$

meaning that we assume a constant intensity and only two tiles where this changes. To understand the ability of our proposal to correctly identify $W(u)$, we simulate 100 patterns from intensity (12) and compute the percentage of times the algorithm correctly identifies the two tiles.

For illustrative purposes, Figure 6 shows the true intensity of the form $\lambda = \exp\{4.61 + 5.99W(u)\}$, a simulated pattern according to

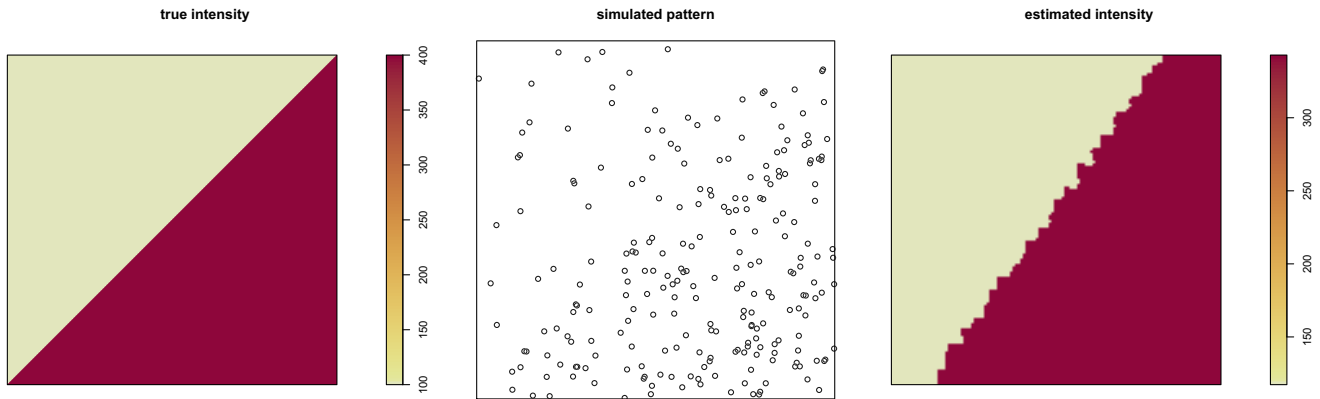


FIGURE 6 | Graphical representation of the intensity $\lambda = \exp\{4.61 + 5.99W(u)\}$ (a), a pattern simulated from such intensity (b), and an estimated intensity through our proposed method (c).

TABLE 1 | Percentage of times the algorithm correctly identifies the two tiles in (12), over 100 simulations.

$\frac{\exp(\beta_0)}{\exp(\gamma_0)}$	β_0	γ_0	$\mathbb{E}[N]$	Percentage
1	3.91	3.91	100	0.84
	5.52	5.52	500	0.74
0.66	3.69	4.09	100	0.78
	5.30	5.70	500	0.84
0.25	3.00	4.38	100	0.73
	4.61	5.99	500	0.82
0.01	2.30	4.50	100	0.83
	3.91	6.11	500	0.84

this intensity function, and its estimated intensity following our proposal.

Table 1 reports the percentage of correct identification of the two tiles in system (12) over 100 simulated datasets, for different combinations of the baseline intensities β_0 and γ_0 , and corresponding values of the expected total number of points $\mathbb{E}[N]$. The results indicate that the performance of the algorithm remains relatively stable across varying values of the intensity ratio $\exp(\beta_0)/\exp(\gamma_0)$, with slightly higher accuracy when the expected number of points increases. Notably, even when the ratio is equal to one, the percentage of correct identifications does not deteriorate, suggesting that the algorithm is robust to small differences in baseline intensities across tiles.

6.2 | Parameter Estimation

Assuming now the tessellation $W(u)$ to be known, we study the performance of the algorithm in terms of point estimation. To this aim, we explore both the cases of a region-wise constant intensity and the case of an inhomogeneous intensity depending on a spatial region-wise covariate. We illustrate the performance of the method by simulating patterns from such intensities and by fitting our proposed tessellated model. We report the mean and standard deviations of the estimated parameters, together with their Mean Squared Error (MSE), averaged over 100 simulations.

In the first scenario, the intensity is the same as in Equation (12), but with varying expected number of points and ratios between β_0 and γ_0 , that is, the difference between the expected number of points within the two tiles. Table 2 presents the results.

As $\mathbb{E}[N]$ increases, the estimates of both parameters become more accurate, with means approaching the true values and the standard deviations and MSE decreasing. The ratio between the true values of β_0 and γ_0 also plays a role in the estimation accuracy. When the two parameters are closer in value, both parameters are estimated precisely with very low MSE. However, as the difference between β_0 and γ_0 increases, the estimate of β_0 shows more variability, resulting in a higher MSE for $\hat{\beta}_0$, while $\hat{\gamma}_0$ remains more stable. Overall, the difference does not appear to be very relevant. We can, therefore, conclude that increasing $\mathbb{E}[N]$ leads to a reduction in estimation error, and a more balanced ratio between the true parameters could yield more precise estimates with lower MSE, even though the improvement is quite small.

We now proceed with the second scenario, that is, the one where the intensity function has the following form

$$\lambda(u) = \exp\{\beta_0 + \beta_1 Z(u) + \gamma_1 Z(u)W(u)\}, \quad (13)$$

with $Z(u)$ a spatial covariate generated from a Gaussian Random Field with zero mean. Equation (13) implies that $Z(u)$ has region-wise constant effect, and, therefore, has effect β_1 on the intensity when $x < y$ and $\beta_1 + \gamma_1$ when $x > y$. Table 3 displays the mean, standard deviations, and MSE of the estimated parameters $\hat{\beta}_0$, $\hat{\beta}_1$, and $\hat{\gamma}_1$, averaged over 100 simulated point patterns generated according to the intensity function in Equation (13).

As in the previous scenario, the results show that increasing $\mathbb{E}[N]$ leads to more accurate estimates. Also in this scenario, when the true values of β_1 and γ_1 are close, the estimates for $\hat{\beta}_1$ are relatively precise even for smaller $\mathbb{E}[N]$, and the MSE is lower compared to cases where β_1 and γ_1 differ more evidently. However, when the values of β_1 and γ_1 differ more, the estimates for $\hat{\gamma}_1$ become less accurate, with higher MSE and greater variability in the estimates, especially for smaller values of $\mathbb{E}[N]$. This suggests that a greater disparity between β_1 and γ_1 makes estimation more challenging, particularly in smaller samples. As $\mathbb{E}[N]$ increases, the estimates become more stable, and the MSEs decrease, but the

TABLE 2 | Mean (sd) and MSE of the parameters, averaged over the 100 simulated point patterns generated assuming intensity (12).

True parameters				$\hat{\beta}_0$		$\hat{\gamma}_0$	
$\frac{\exp(\beta_0)}{\exp(\gamma_0)}$	β_0	γ_0	$\mathbb{E}[N]$	mean (sd)	MSE	mean (sd)	MSE
1	3.91	3.91	100	3.85 (0.19)	0.042	3.93 (0.21)	0.041
	5.52	5.52	500	5.47 (0.09)	0.011	5.52 (0.09)	0.008
	6.21	6.21	1000	6.18 (0.07)	0.006	6.21 (0.07)	0.004
0.66	3.69	4.09	100	3.6 (0.29)	0.090	4.07 (0.21)	0.042
	5.30	5.70	500	5.26 (0.10)	0.012	5.71 (0.08)	0.006
	5.99	6.40	1000	5.95 (0.08)	0.008	6.40 (0.05)	0.002
0.25	3.00	4.38	100	2.89 (0.34)	0.128	4.36 (0.16)	0.025
	4.61	5.99	500	4.58 (0.15)	0.022	5.99 (0.08)	0.007
	5.30	6.68	1000	5.28 (0.10)	0.011	6.68 (0.05)	0.003
0.01	2.30	4.50	100	2.23 (0.41)	0.171	4.47 (0.17)	0.031
	3.91	6.11	500	3.94 (0.20)	0.040	6.12 (0.07)	0.004
	4.61	6.80	1000	4.62 (0.14)	0.020	6.81 (0.04)	0.002

TABLE 3 | Mean (sd) and MSE of the parameters, averaged over the 100 simulated point patterns generated assuming intensity (13).

True parameters					$\hat{\beta}_0$		$\hat{\beta}_1$		$\hat{\gamma}_1$	
$\frac{\beta_1}{\gamma_1}$	β_0	β_1	γ_1	$\mathbb{E}[N]$	mean (sd)	MSE	mean (sd)	MSE	mean (sd)	MSE
0.5	3.15	1	2	100	3.34 (0.16)	0.061	0.93 (0.17)	0.033	1.56 (0.07)	0.201
	4.755	1	2	500	4.96 (0.09)	0.052	0.87 (0.08)	0.023	1.58 (0.04)	0.178
	5.5	1	2	1000	5.71 (0.06)	0.045	0.89 (0.05)	0.016	1.61 (0.03)	0.156
0.5	1.15	1.5	3	100	1.95 (0.28)	0.718	1.29 (0.34)	0.155	2.16 (0.10)	0.718
	2.75	1.5	3	500	3.15 (0.14)	0.176	1.52 (0.12)	0.015	2.45 (0.05)	0.304
	3.45	1.5	3	1000	3.92 (0.09)	0.226	1.46 (0.09)	0.009	2.43 (0.04)	0.329
0.33	1.2	1	3	100	1.99 (0.27)	0.692	1.10 (0.38)	0.153	2.15 (0.10)	0.724
	2.775	1	3	500	3.17 (0.15)	0.176	1.30 (0.18)	0.125	2.45 (0.06)	0.302
	3.5	1	3	1000	3.93 (0.11)	0.196	1.27 (0.11)	0.083	2.44 (0.04)	0.319

TABLE 4 | Mean (sd) and MSE of the parameters, averaged over the 100 simulated point patterns generated assuming intensity (14).

Parameters					$\hat{\beta}_0$		$\hat{\beta}_1$		$\hat{\gamma}_0$		$\hat{\gamma}_1$	
β_0	β_1	γ_0	γ_1	$\mathbb{E}[N]$	mean (sd)	MSE	mean (sd)	MSE	mean (sd)	MSE	mean (sd)	MSE
2.5	0.50	4.95	0.75	100	2.51 (0.44)	0.192	5.02 (0.12)	0.019	0.50 (0.40)	0.161	0.61 (0.08)	0.026
3.40	0.50	6.64	0.75	500	3.52 (0.26)	0.081	6.69 (0.05)	0.005	0.50 (0.24)	0.058	0.60 (0.03)	0.023
3.74	0.50	7.31	0.75	1000	3.99 (0.20)	0.100	7.34 (0.03)	0.002	0.51 (0.16)	0.025	0.63 (0.02)	0.014

influence of the ratio between the parameters remains slightly visible in the results.

Finally, we also explore the case of all model parameters changing within the same tiles, assuming the following in intensity

$$\lambda(u) = \exp\{\beta_0 + \beta_1 Z(u) + \gamma_0 W(u) + \gamma_1 Z(u)W(u)\} \quad (14)$$

basically signifying that all the parameters of an inhomogeneous model change according to the tiles of $W(u)$. Formally, we are considering the intensity of the form of Equation (9). With respect

to intensity (13), we are now also assuming different intercepts (β_0 and γ_0) in the two tiles.

Table 4 provides the mean, standard deviations, and MSE of the estimated parameters $\hat{\beta}_0$, $\hat{\beta}_1$, $\hat{\gamma}_0$, and $\hat{\gamma}_1$, averaged over 100 simulated point patterns generated according to the intensity function in Equation (14).

As in the previous scenarios, increasing the expected number of points $\mathbb{E}[N]$ results in improved estimates. The MSE for both $\hat{\beta}_0$

and $\hat{\gamma}_0$ are relatively low even for $\mathbb{E}[N] = 100$, with the mean values being very close to the true values. The estimates for $\hat{\beta}_0$ and $\hat{\gamma}_0$ improve further as $\mathbb{E}[N]$ increases, although the MSE remains relatively stable across the larger sample sizes. This suggests that the parameters β_0 and γ_0 are relatively easy to estimate, even with smaller samples, when the values are reasonably close to each other. The estimates for $\hat{\beta}_1$ and $\hat{\gamma}_1$ show a similar trend, with their MSE decreasing as $\mathbb{E}[N]$ increases. The estimation of $\hat{\beta}_1$ looks less sensitive to sample size compared to $\hat{\gamma}_1$, but both parameters are more stable with larger sample sizes.

6.2.1 | Goodness-Of-Fit

This section is devoted to the study of the performance of the proposed method in terms of overall goodness-of-fit. We compare the proposed method to both the global and the local models in terms of Mean Integrated Squared Error (MISE).

Let $\lambda(\cdot)$ denote the intensity function of the generating process. Given an intensity estimator $\hat{\lambda}(\cdot) = \hat{\lambda}(\cdot; X)$, when the true intensity is known, the goodness-of-fit of $\hat{\lambda}(\cdot)$ may be measured by

$$\begin{aligned} \text{MISE}_{\hat{\lambda}} &= \mathbb{E} \left[\int_W (\hat{\lambda}(u) - \lambda(u))^2 du \right] \\ &= \int_W \text{Var}(\hat{\lambda}(u)) du + \int_W \text{Bias}(\hat{\lambda}(u))^2 du \end{aligned}$$

where $\text{Bias}(\hat{\lambda}(u)) = \mathbb{E}[\hat{\lambda}(u)] - \lambda(u)$. Since it is hard to compute $\text{MISE}_{\hat{\lambda}}$ analytically for a given intensity estimator $\hat{\lambda}$, we estimate the MISE by

$$\text{MISE}_{\hat{\lambda}} = \frac{1}{n} \sum_{i=1}^n \int_W (\hat{\lambda}(u; \mathbf{x}_i) - \lambda(u))^2 du$$

For each of the scenarios explored previously, we fit the tessellated model and compare its MISE with those of the corresponding global and local parameters. For the former, we fit the model with fixed-effects coefficients and without any tessellation. For the latter, we fit the corresponding local model with space-varying parameters. The aim is to prove the best goodness-of-fit of the tessellated model when the change in the model parameters is actually abrupt.

Table 5 presents the MISE for three different models—global, local, and tessellated—averaged over 100 simulated point patterns generated with the intensity function as defined in Equation (12).

The table clearly shows the better performance of the tessellated models in all the scenarios. The only exception is given when the ratio between β_0 and γ_0 is equal to 1, actually corresponding to a constant intensity. Indeed, in such cases, the lower MISE is correctly obtained by the global model. In all of the remaining scenarios, however, the tessellated model proves its superiority compared to both the global and the local model. This result is even more evident as the expected number of points increases.

Tables 6 and 7 report the results for the other scenarios previously considered, always showing the best fitting performance

TABLE 5 | MISE for the global, local, and tessellated fitted models, averaged over the 100 simulated point patterns generated assuming intensity (12).

True parameters				MISE		
$\frac{\exp(\beta_0)}{\exp(\gamma_0)}$	β_0	γ_0	$\mathbb{E}[N]$	global	local	tessellated
1	3.91	3.91	100	43	92	77
	5.52	5.52	500	242	384	518
0.66	3.69	4.09	100	165	144	123
	5.30	5.70	500	2793	1543	600
0.25	3.00	4.38	100	963	494	130
	4.61	5.99	500	23250	10852	1525
0.01	2.30	4.50	100	1692	847	185
	3.91	6.11	500	18662	41041	2062

TABLE 6 | MISE for the global, local, and tessellated fitted models, averaged over the 100 simulated point patterns generated assuming intensity (13).

True parameters					MISE		
$\frac{\beta_1}{\gamma_1}$	β_0	β_1	γ_1	$\mathbb{E}[N]$	Global	Local	Tessellated
0.5	3.15	1	2	100	152361	246243	116831
	4.755	1	2	500	3787483	4783515	2674396
0.5	1.15	1.5	3	100	1681857	2009251	1595638
	2.75	1.5	3	500	35991375	44708693	30626456
0.33	1.2	1	3	100	1873354	2225262	1778719
	2.775	1	3	500	37496379	47889891	32178178

TABLE 7 | MISE for the global, local, and tessellated models, averaged over the 100 simulated point patterns generated assuming intensity (14).

True parameters					MISE		
β_0	β_1	γ_0	γ_1	$\mathbb{E}[N]$	global	local	tessellated
2.5	0.50	4.95	0.75	100	13087	19816	1608
3.40	0.50	6.64	0.75	500	409464	457716	45097

achieved by the tessellated model, always followed by the global one, and then by the local one.

7 | Applications

This section applies the tessellated model to the two examples introduced in Section 3. Of course, since the fitting of the tessellated model depends on the initial step of the tessellation identification, this is always performed in advance, given that no information on the true tessellation is known for real data applications.

7.1 | Gordon Square Data

Consider now the point pattern of the Gordon Square example. We fit the following tessellated model

$$\hat{\lambda}_{gordon}(u) = \exp\{\hat{\beta}_0 + \hat{\gamma}_0 W(u)\} \quad (15)$$

which basically consists of the estimation of two constant intensities in the identified tessellation, resulting in having only two tiles. As also in the subsequent analysis, the number of tiles has resulted from the application of the tessellation identification algorithm, involving also the cluster analysis to avoid over-segmentation and too many tiles. Table 8 compares the goodness-of-fit of the tessellated model (15) and its global and local counterparts,

$$\hat{\lambda}_{gordon}(u) = \exp\{\hat{\theta}_0\} \quad \text{and} \quad \hat{\lambda}_{gordon}(u) = \exp\{\hat{\theta}_0(u)\}$$

For the global model, we compare the AIC, while for the local model, we have to resort to the MISE. For an observed point pattern, of course the true intensity is not known, so in the same spirit as the smoothed raw residuals (Alm 1998), we compute the $MISE_{\hat{\lambda}}$ as by $\frac{1}{n} \sum_{i=1}^n \int_W (\hat{\lambda}(u; \mathbf{x}_i) - \tilde{\lambda}(u))^2 du$, where $\tilde{\lambda}(u) = e(u) \sum_{i=1}^{n(\mathbf{x})} \kappa(u - v_i)$ is the non-parametric, kernel estimate of the fitted intensity $\hat{\lambda}(u)$. Here, κ is the smoothing kernel, and $e(u)$ is the edge correction. The smoothing bandwidth for the kernel estimation of the raw residuals is selected by cross-validation as the value that minimizes the MSE criterion defined by P. Diggle (1985), by the method of Berman and Diggle (1989). See P. J. Diggle (2013) for further details. Given the above, this formulation of the MISE favors non-parametric models like the local ones.

Table 8 reports a lower AIC of the tessellated model if compared to the global one and also the lowest MISE among the

TABLE 8 | AIC and MISE of the models fitted to the Gordon Square data.

Model	AIC	MISE
global	811	25
tessellated	721	22
local	—	27

three fitted models. This proves the best fitting of the tessellated model. Figure 7a shows $\hat{\beta}_0 + \hat{\gamma}_0 W(u)$. From this, we know that the detected tessellation consists of two tiles only, basically distinguishing two main types of area, one characterized by a higher intensity of $\exp(\hat{\gamma}_0) = \exp(2.47) = 11.82$ and another characterized by a lower intensity of $\exp(\hat{\beta}_0) = \exp(-4.87) = 0.008$. Figure 7b further reports the fitted intensity, of course reflecting the identified tessellation and the estimated parameters. Note that a standard kernel smoothing of the points, depicted in Figure 7c, would oversmooth the points' location while also providing no interpretation of the phenomenon through the parameters' estimates.

Table 9 provides further information on the estimates, such as their significance and confidence intervals.

Moreover, we compared the feature detection with the results obtained from the algorithm of Byers and Raftery (1998). That method assumes only two clusters, feature and noise, even though the algorithm could be extended to consider multiple features in a fractal framework or with just different intensities. Figure 8 shows the classification of the points into clutter and features (black and red points, respectively). The tessellation identified by our method is superimposed on the graphs even though it did not have a role in the feature detection with Byers and Raftery (1998)'s method. As regards the classification obtained from the tessellated model (left panel), the points are just classified according to their place of occurrence with respect to the identified tiles. Of course, the points belonging to the central tile with the highest intensity are classified as features, and the rest are classified as clutter. For Byers and Raftery (1998)'s method (right panel), the number of nearest neighbors to be considered in the EM algorithm based on the points' distances is 8. This has been selected through D'Angelo (2024)'s proposal, which considered the nearest neighbor K as that value which leads to the highest entropy measure, after which no further improvement is found.

As evident from Figure 8, our method better succeeds in detecting all of the clusters with at least two points. On the contrary, Byers and Raftery (1998)'s method leaves some spatial clusters belonging to the clutter pattern. This results in the two estimated intensities being really close to each other, if considering Byers

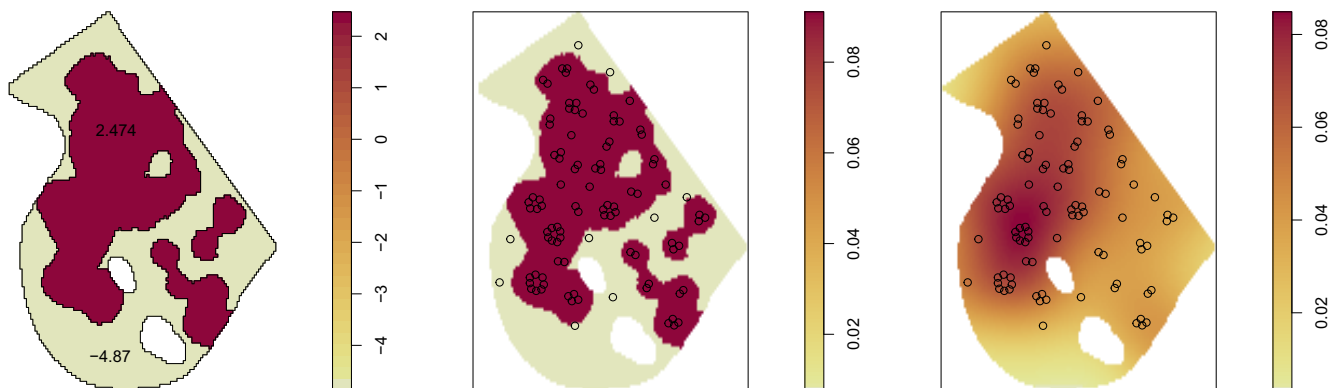
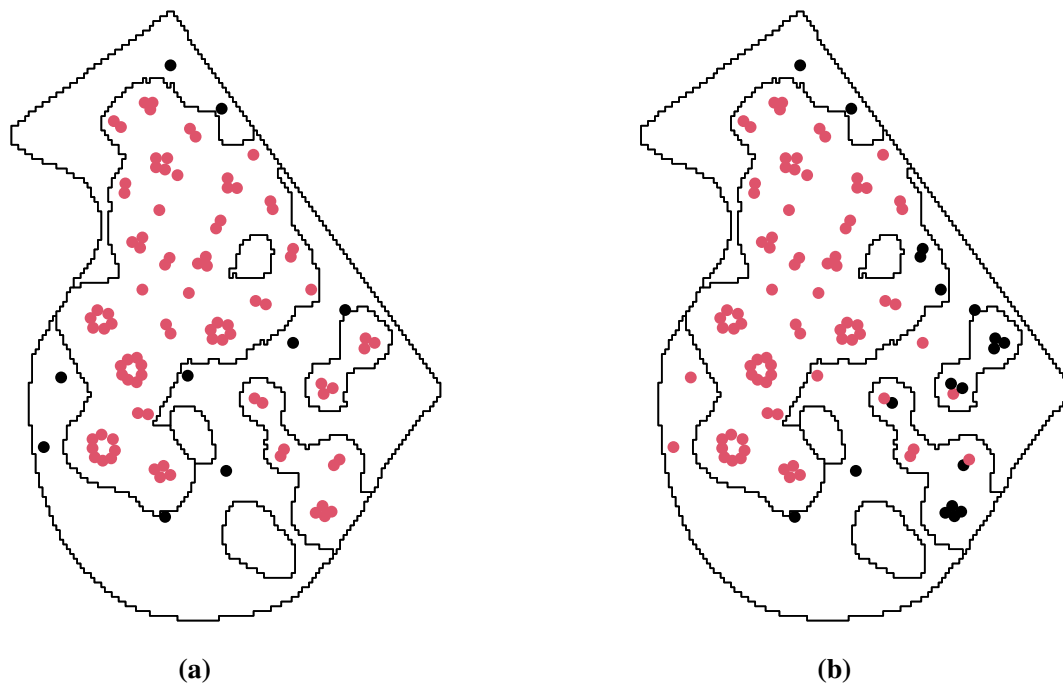
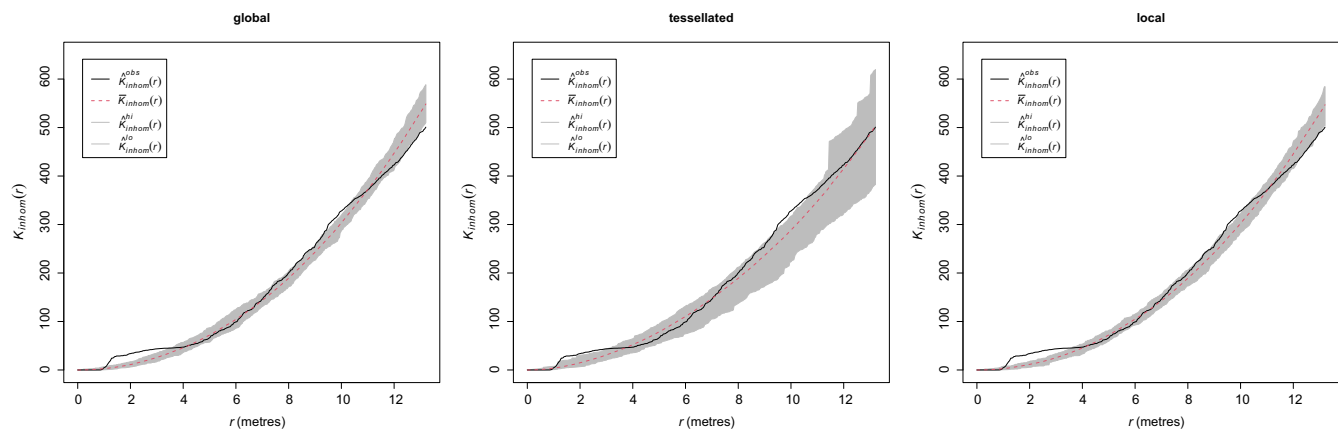


FIGURE 7 | Spatially varying intercept estimated from model (15) fitted to the Gordon Square data (a). Fitted tessellated intensity according to (15) (b) and kernel smoothing of the points (c), with the points of the pattern overlapped.

TABLE 9 | Estimated parameters of model (15).

Variable	Parameter	Estimate	S.E.	CI95.lo	CI95.hi	Ztest	Zval
Intercept	$\hat{\beta}_0$	-4.87	0.33	-5.52	-4.22	***	-14.61
$W(u)$	$\hat{\gamma}_0$	2.47	0.35	1.79	3.16	***	7.08

**FIGURE 8** | Classification of the points by the tessellated regression model (a) and the feature detection algorithm of Byers and Raftery (1998) (b). Clutter points are in black, and feature points are in red.**FIGURE 9** | Inhomogeneous K -function for the three models fitted to the Gordon Square data. The solid black line represents the inhomogeneous K -function $\hat{K}_{inhom}(r)$ estimated with the intensity of the fitted model. The corresponding 100 envelopes are in gray. The red line corresponds to the mean envelope.

and Raftery (1998)'s method: 0.01 for the clutter pattern and 2.29 for the feature pattern.

Finally, a comparison is carried out in terms of the inhomogeneous K -function (Baddeley et al. 2000) $\hat{K}_{inhom}(r)$, weighted

by $\hat{\lambda}(u)$, an estimate of the intensity function $\lambda(u)$. If $\hat{\lambda}(u)$ is the intensity estimated through a fitted model, then the inhomogeneous version of the K -function can be used as a diagnostic tool (Adelfio et al. 2020). In Figure 9, we report the inhomogeneous K -function for the three models fitted. The solid black line

represents $\hat{K}_{inhom}(r)$ estimated on the observed point pattern, while the corresponding envelopes (in gray), are obtained by estimating the inhomogeneous K -function for a given number of patterns simulated assuming the intensity to diagnose. For their computation, we simulated 100 realizations from the fitted intensity. The red line corresponds to the mean envelope. The inhomogeneous K -function of all models lies almost entirely within the envelopes, even if one of the tessellated models stays between the envelopes for more spatial lags r , especially for smaller lags. Since a K -function lying completely within the envelopes would indicate a satisfactory goodness-of-fit, the tessellated model is confirmed to be preferred to both the global and the local ones. As further confirmation, we compute

$$T = \int_W \frac{(\hat{K}_{inhom}(r) - \bar{K}(r))^2}{\bar{K}(r)} dr \quad (16)$$

which is expected to result in small values for K -functions weighted by an intensity function proving a satisfactory goodness-of-fit. Indeed, the closer the observed K -function to the mean envelope $\bar{K}(r)$, the better the goodness-of-fit of the intensity. Therefore, the intensity leading to the smaller value of (16) should be preferred. The three models report values of 3618, 2332, and 3519, respectively, further leading to the choice of the tessellated model.

TABLE 10 | AIC and MISE of the models fitted to the bronze filter data.

Model	AIC	MISE
Global	-1117	140
Tessellated	-1119	101
Local	—	119

7.2 | Bronze Filter Data

For the bronze filter data, we fit the following tessellated model

$$\hat{\lambda}_{bronze}(u) = \exp \left\{ \hat{\beta}_0 + \hat{\beta}_1 Long(u) + \hat{\gamma}_0 W_{Intercept}(u) + \sum_{k=1}^4 \hat{\gamma}_{1k} Long(u) W_{Long,k}(u) \right\} \quad (17)$$

In this case, the tessellation obtained, through the application of the SLIC algorithm and the subsequent cluster analysis, for the intercept has only two tiles, while the tessellation for the x-coordinate covariate has five. Table 10 reports the AIC and MISE of the models fitted to the bronze filter data, proving also in this case the better goodness-of-fit of the tessellated model over the other competitors.

Figure 10a shows the estimated coefficients, indicating a decrease in the intensity along the x-axis and a peak in the x-coordinate effect in the middle of the region. Figure 10b–c further compares the intensity estimated by the tessellated model (17) to a kernel smoothing of the points.

Note that even though the estimates did not result significantly (see Table 11), the tessellated model still provides better prediction if compared to the competitor models. Furthermore, the LRT on the γ parameters showed a p -value of 0.03, confirming the necessity to include region-wise effects in the model.

In Figure 11, we report the inhomogeneous K -function for the three models fitted. For this application, even if the envelopes coming from the tessellated intensity are wider, the K -function lies within the envelopes of the three models for similar spatial ranges r . This is confirmed by the T values, all equal to 19 for the three models. Therefore, even though the tessellated model

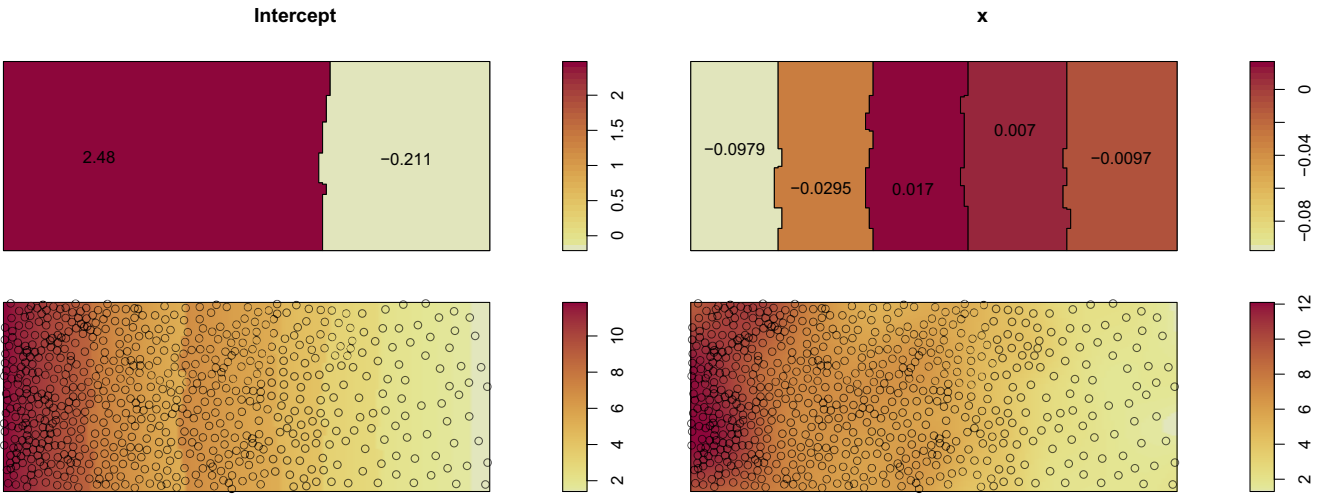


FIGURE 10 | Spatially varying coefficients of model (17) fitted to the Bronze filter data (a–b). Fitted tessellated intensity (c), and kernel smoothing of the points (d), with the points of the pattern overlapped.

TABLE 11 | Estimated parameters of model (17).

Variable	Parameter	Estimate	S.E.	CI95.lo	CI95.hi	Ztest	Zval
Intercept	$\hat{\beta}_0$	2.48	0.12	2.25	2.71	***	20.95
$Long(u)$	$\hat{\beta}_1$	-0.21	0.22	-0.65	0.22		-0.95
$W_{Intercept}(u)$	$\hat{\gamma}_0$	-0.10	0.07	-0.23	0.03		-1.46
$W_{Long,1}(u)$	$\hat{\gamma}_{11}$	-0.03	0.05	-0.13	0.07		-0.57
$W_{Long,2}(u)$	$\hat{\gamma}_{12}$	0.02	0.06	-0.09	0.13		0.30
$W_{Long,3}(u)$	$\hat{\gamma}_{13}$	0.01	0.06	-0.11	0.12		0.12
$W_{Long,4}(u)$	$\hat{\gamma}_{14}$	-0.01	0.06	-0.13	0.11		-0.16

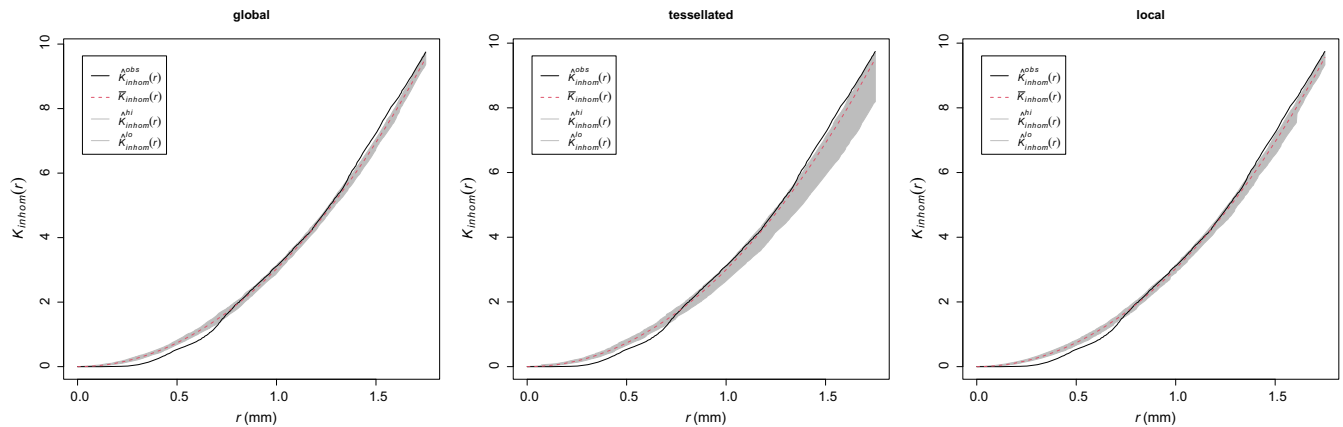


FIGURE 11 | Inhomogeneous K -function for the three models fitted to the Bronze filter data. The solid black line represents the inhomogeneous K -function $\hat{K}_{inhom}(r)$ estimated with the intensity of the fitted model. The corresponding 100 envelopes are in gray. The red line corresponds to the mean envelope.

resulted in the one to favor, the difference between this and the global and local models is modest.

8 | Analysis of the Greek Seismicity

For the Greek data, we fit the following tessellated model

$$\hat{\lambda}_{greece}(u) = \exp \left\{ \hat{\beta}_0 + \hat{\beta}_1 D_p(u) + \hat{\beta}_2 D_f(u) + \sum_{k=1}^4 \hat{\gamma}_{0k} W_{Intercept,k}(u) + \hat{\gamma}_1 D_p(u) W_{D_p}(u) + \hat{\gamma}_2 D_f(u) W_{D_f}(u) \right\} \quad (18)$$

and compare it to the global and local ones in Table 12. For this application, the tessellation identification algorithm selects five tiles for the intercept and two different sets of tiles for the effects of the distances from both the seismic sources available.

The tessellated model has a lower AIC than the global one. Fitting the (non-tessellated) global model, the parameter estimates $\hat{\theta} = \{3.96, -0.90, -0.91\}$ are obtained, all significantly different from zero. As expected, the distance from the seismic sources decreases the intensity of the earthquakes. We further aim to compare the tessellated model with the global one by conducting a formal test to assess the significance of γ . The LRT for γ shows a zero p -value, indicating that the tessellated model should be preferred over the global one.

TABLE 12 | AIC and MISE of the models fitted to the Greek seismicity data.

Model	AIC	MISE
global	-4485	431123
tessellated	-5093	425597
local	—	403759

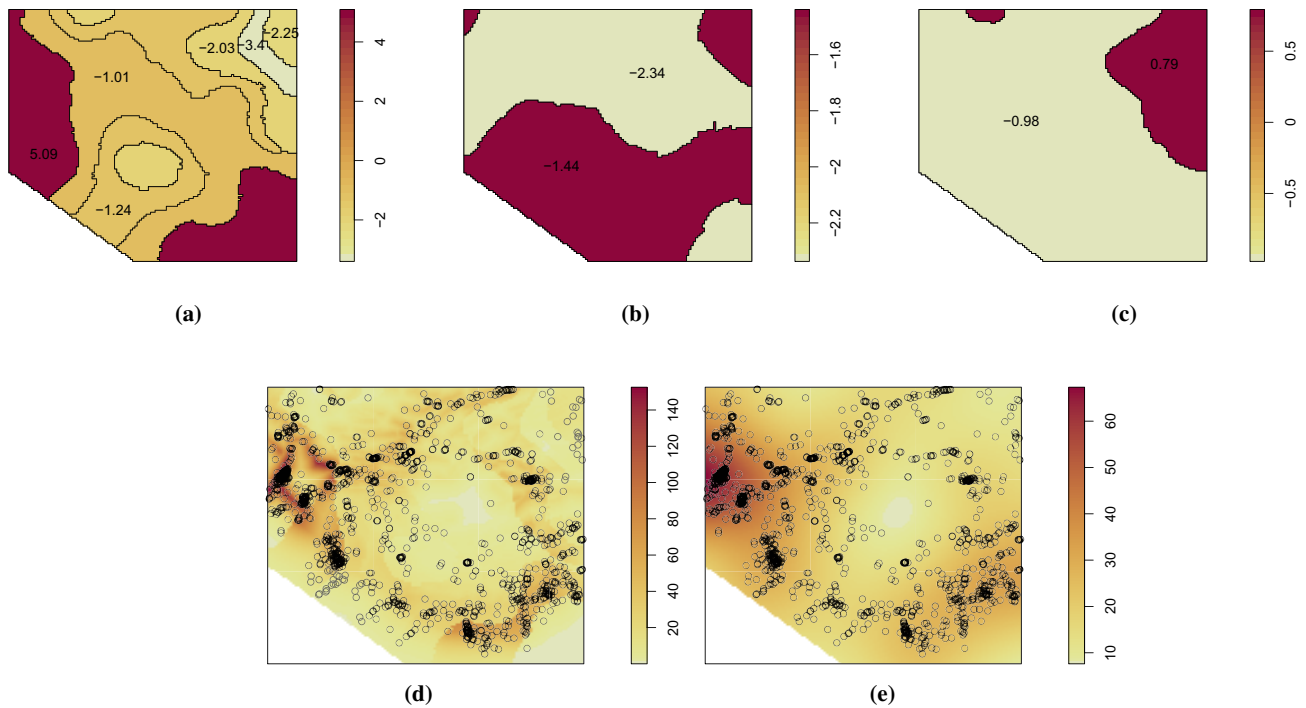
Then, from Table 12, we know that the lowest MISE is obtained by the local model, meaning that this provides better goodness-of-fit. However, we decided to proceed with the interpretation of the tessellated model, which further provides clear-cut spatial changes effects of the model parameters. Indeed, the regions identified may reveal similar characteristics of the underlying process that generated the events in those regions.

The estimated parameters of model (18) are reported in Table 13 and shown in Figure 12.

In particular, Figure 12a–c informs us of the spatial displacement of the estimated coefficients, aiding the interpretation. The spatially varying intercept shows regions at the west and south boundaries where the intensity is significantly higher with respect to the other identified regions. This high intensity may be attributable to some specific characteristics of those regions, not taken into account by the model. Then, the intensity gradually

TABLE 13 | Estimated parameters of model (18).

Variable	Parameter	Estimate	S.E.	CI95.lo	CI95.hi	Ztest	Zval
Intercept	β_0	5.09	0.07	4.95	5.23	***	70.00
$W_{\text{Intercept},1}(u)$	γ_{01}	-1.01	0.08	-1.17	-0.86	***	-12.96
$W_{\text{Intercept},2}(u)$	γ_{02}	-1.24	0.09	-1.42	-1.07	***	-14.10
$W_{\text{Intercept},3}(u)$	γ_{03}	-2.03	0.14	-2.31	-1.75	***	-14.12
$W_{\text{Intercept},4}(u)$	γ_{04}	-3.40	0.41	-4.21	-2.59	***	-8.23
$W_{\text{Intercept},5}(u)$	γ_{05}	-2.25	0.29	-2.81	-1.69	***	-7.88
$D_f(u)$	β_1	-1.44	0.11	-1.65	-1.23	***	-13.26
$D_p(u)$	β_2	-0.98	0.07	-1.12	-0.84	***	-13.88
$D_f(u)W_{D_f}(u)$	γ_2	-2.34	0.28	-2.90	-1.78	***	-8.21
$D_p(u)W_{D_p}(u)$	γ_1	0.79	0.10	0.60	0.98	***	8.25

**FIGURE 12** | Spatially varying coefficients of model (18) fitted to the Greek seismicity data (a–c). Tessellated intensity fitted according to model (18) (d) and kernel density estimate (e).

decreases along the northeast part of the region. However, the most crucial information that a local model would not be able to provide is the clear separation of the effects of the spatial covariates. Indeed, the distance from the plate boundary seems to negatively affect the intensity of the seismic process with different magnitudes in the two identified tiles/subregions. Most interestingly, the distance from the faults has a negative effect on the process intensity in the majority of the region, as expected. The only expectation is in the northeast region, where the effect becomes positive. This specific tile identifies a region where few earthquakes occurred in the presence of many faults. This finding could lead to a deeper study of the seismic phenomenon in the specific region. Moreover, Figure 12d–e compares the tessellated intensity fitted according to model (18) to a kernel

density estimate, further proving the better goodness-of-fit of the tessellated model, together with its advantageous possibility of interpreting the estimated parameters. In particular, it is evident that the dependence of the intensity function of the tessellated model on the distance from the seismic sources.

In Figure 13, we report the inhomogeneous K -function for the three models fitted. For this application, even if the K -functions of the three models seem equally close to their envelopes, the tessellated model one lies within the envelopes for a wider range of spatial lags r . This is confirmed by the T values, equal to 58 for the global and local models, and equal to 38 for the tessellated model. Therefore, the tessellated model is undoubtedly the model to be chosen among the three competitors.

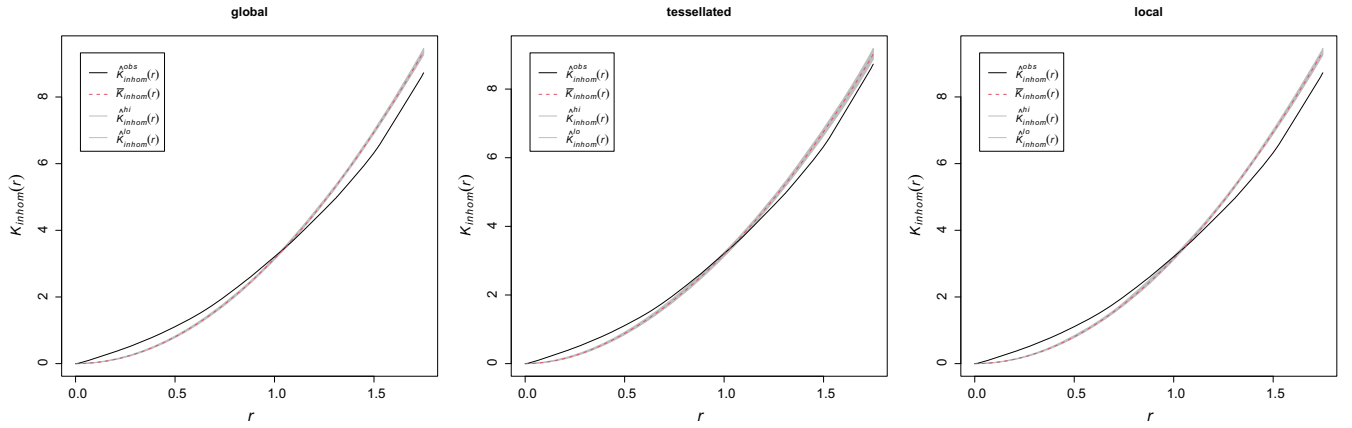


FIGURE 13 | Inhomogeneous K -function for the three models fitted to the Greek seismicity data. The solid black line represents the inhomogeneous K -function $\hat{K}_{inhom}(r)$ estimated with the intensity of the fitted model. The corresponding 100 envelopes are in gray. The red line corresponds to the mean envelope.

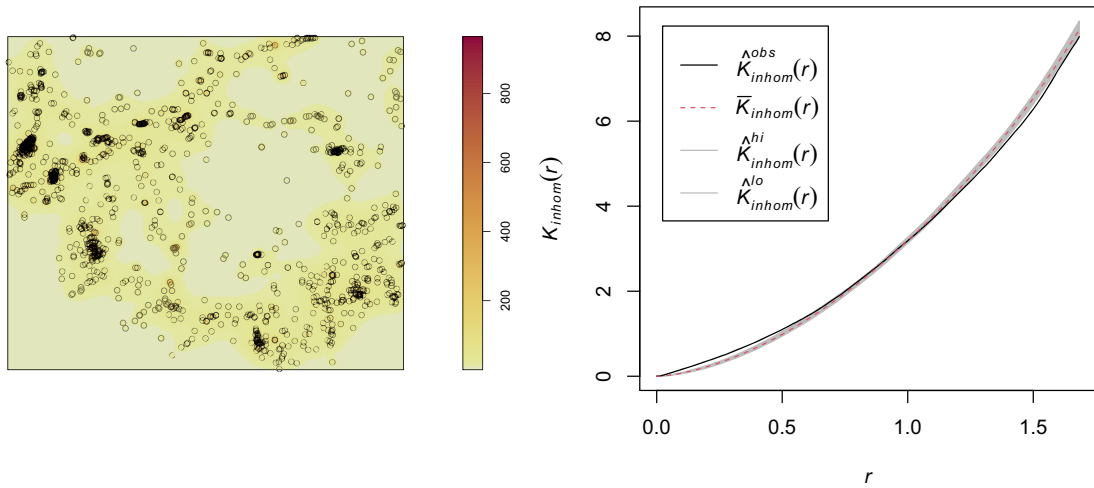


FIGURE 14 | Marginalized spatial intensity estimated by the ETAS model (19) (a) and its second-order based diagnostics (b).

However, the generator process of earthquakes is quite complex, since it is characterized by multidimensional and multiscale dynamics, dependence features of points both in space and time, spatial anisotropy and spatial variation of the intensity according to the seismogenic sources (D'Angelo et al. 2022). Several spatial and spatiotemporal models have been proposed in the literature for describing the evolution of the earthquake process in space, in time, and both dimensions. The temporal clustered structure of earthquake data has been described by self-exciting point processes, that is, the Hawkes model (Hawkes and Adamopoulos 1973) or the Epidemic Type Aftershock Sequence (ETAS) model (Ogata 1988; Adelfio and Chiodi 2015). According to a branching structure, the conditional intensity function of the self-exciting model is defined as the sum of a term describing the large-time scale variation (spontaneous activity or background, generally assumed homogeneous in time but not in space) and one relative to the small-time scale variation due to the interaction with the events in the past (induced or triggered activity). We therefore fit an ETAS model to the data and compare its marginalized intensity function to the Poisson ones fitted so far, in order to compare our proposal to more complex models typically employed for earthquake data.

In particular, we resort to the convenient specification of Adelfio and Chiodi (2021), which also includes the covariate effects on the triggering component of the overall intensity function. The ETAS model and its marginalized spatial intensity can be written as

$$\lambda_{\theta}(t, u | \mathcal{H}_t) = \mu f(u) + \sum_{t_j < t} \frac{\kappa_0 \exp(\eta_j)}{(t - t_j + c)^p} \{(u - u_j)^2 + d\}^{-q}$$

$$\lambda(u) = \int_{\mathcal{T}} \lambda_{\theta}(t, u | \mathcal{H}_t) dt, \quad (19)$$

where $\mathcal{H}_t$ is the past history of the process, μ is the background general intensity, and $f(u)$ is the spatial density. Concerning the triggered component, $\eta_j = \beta' \mathbf{Z}_j$ is a linear predictor, with $\mathbf{Z}_j$ the external known covariate vector, including the magnitude, and $\theta = (\mu, \kappa_0, c, p, d, q, \beta)$ are the parameters to be estimated. In the usual ETAS model, with the only external covariate representing the magnitude, $\beta = \alpha$, as $\eta_j = \beta' \mathbf{Z}_j = \alpha(m_j - m_0)$, where m_j is the magnitude of the j^{th} event and m_0 the threshold magnitude, that is, the lower bound for which earthquakes with higher values of magnitude are surely recorded in the catalogue.

We employ the `etasFLP` (Chiodi and Adelfio (2017)) R package for the model fitting. We fit the ETAS model of Equation (19) with $\eta_j = \alpha(m_j - m_0) + \beta_1 Dp(u_j) + \beta_2 Df(u_j)$ and obtain $\hat{\beta}_1 = -0.56$ and $\hat{\beta}_2 = -0.53$. We report its marginalized intensity and second-order base diagnostics in Figure 14.

From this, we know that the ETAS model achieves a better fit if compared to any of the Poisson models previously fitted, but resulting in an overly adaptive intensity estimate. Therefore, if the aim of the analysis is only predictive, the ETAS proves to be a better choice, while for interoperability aims, the tessellated model represents the best trade-off between goodness-of-fit and interpretation of the covariate effects. Another perk of the tessellated intensity over the ETAS one is that it is affected by the effect of the covariates, which is reflected in higher intensities along the faults and plate boundary (see Figure 12d). This modification of the intensity is not met in the ETAS model since the covariates are assumed to influence only the immigrant events but do not modify the background rate, that is, the independent rate of occurrence of mainshocks.

9 | Conclusions

This work has formalized and proposed a method to fit the novel tessellated spatial log-linear Poisson point process model, which is particularly useful in the local estimation context. The motivation of the work came from the necessity to obtain clear-cut regions where the parameters of the spatial covariates in the spatial Poisson point process model change, in the same philosophy as the models with spatial varying parameters, but with the advantage of sticking to the realm of models with fixed-effects parameters.

Our approach is based on a two-step procedure: tessellation identification and tessellated model fitting. For the first step, a segmentation algorithm is applied to the geographically weighted estimates of an assumed local model. We decided to resort to the SLIC algorithm for this first step, together with a cluster analysis to group superpixels with similar values, to avoid over-segmentation. The second step is straightforwardly achieved by treating the tessellation obtained at the first step as a categorical spatial variable, encoded in some dummy variables indicating which tile the point of the pattern belongs to. This allows to obtain region-wise effects of the covariates within the framework of the most classical Poisson log-linear point process models.

The main advantage of this proposal is the ability to retain both interpretation and standard statistical procedures while also achieving a higher degree of flexibility without specifically resorting to non-parametric approaches.

It is noteworthy to highlight that this proposal for the tessellation identification solves a common problem in “segmented” regression models, that is, the estimation of the number of change-points and their position. Here, the change-points’ counterparts are the tiles of the tessellations, and the SLIC algorithm, with the subsequent cluster analysis, already identifies both the number and the position of tiles prior to the actual model fitting.

With simulations, we have explored the performance of the methodology finding out, as expected, that the performance strongly relies on the number of points in the pattern but, most importantly, on the magnitude of the change in the parameters. In general, the tessellated model provides better goodness-of-fit when the effects of real-valued covariates on the process intensity are actually region-wise. The applications to three real data problems have illustrated potential applications of the proposed methodology. These have ranged from the simplest case of only space-varying homogeneous intensity to the most complex case of multiple external real-valued spatial covariates. We argue that the former case can serve as a feature detection/clutter removal algorithm, with the further advantage, with respect to standard methodologies, of not having to assume the number of features. Furthermore, the shape of the features does not represent a problem, and it is strongly dependent on the geographically weighted estimates obtained from the local model fitted in advance.

The major limitation of our proposal is the dependence on the geographically weighted regression parameters and on the fitting of a local model, which, in turn, could be computationally demanding when dealing with big data patterns. We, however, believe this is necessary in order to fully exploit the gradual estimated changes provided by the local models and to fully exploit standard inferential tools of Poisson likelihood.

This work could be enhanced in many ways. First, other segmentation algorithms could be employed to improve the overall performance and robustness of the results. Secondly, inference on the tessellation identification could be explored in order to provide an uncertainty measure.

We believe the formalization of this model opens the path for many future developments and applications to various contexts. First, the same reasoning is straightforwardly applicable to the geostatistical context, where the response of the model is not the point pattern intensity but rather a proper observed response variable. Secondly, regarding the model formalization, another option could be to treat the belonging to the specific tile as a random effect rather than a fixed effect through the tessellation. This would imply the specification of a multitype point process model, which we decided not to address in the current work. Of course, the extension to the spatiotemporal context would represent a relevant development, conceptually following the work carried out in the R (R Core Team (2019)) package `stopp` (D’Angelo and Adelfio (2025)). Finally, the methodology could be extended to a more complex model whose local version is already defined, like the log-Gaussian Cox processes (LGCPs). Indeed, D’Angelo et al. (2022, 2023) fitted also local LGCPs to earthquake data, finding that the seismic phenomenon displayed a spatially varying variance as well as scale parameters, meaning that points tend to cluster in smaller clusters in specific regions of the analyzed area. Segmenting such local effects is of course possible with the SLIC algorithm, but what remains for future work would be the formalization of a tessellated LGCP, which would imply a region-wise covariance function. Another interesting open problem in the literature is the one related to the definition of self-exciting models with space- and time-varying parameters. For earthquake data,

Ogata and Katsura (1988) and Ogata (1989) used a space-adaptive space-time point process model. In Ogata and Katsura (1988), a method for estimating both the spatial intensity of the point locations and the spatial variation of a characteristic parameter of the distributions for the attached marks was developed, analyzing seismological and ecological data. In Ogata (1989), the occurrence rate of earthquakes was extended to include multiple aftershock sequences. More recently, Zhuang (2015) proposed a weighted likelihood estimator for temporal and spatiotemporal point processes, allowing for the estimation of space-varying ETAS model parameters. A further challenge would be to extend the local parameters fitting procedure to the covariates related to the background component of a self-exciting point process (D'Angelo et al. 2024) or to the external covariate parameters of the triggering component (Adelfio and Chiodi 2021).

Acknowledgments

Open access publishing facilitated by Università degli Studi di Palermo, as part of the Wiley - CRUI-CARE agreement.

Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

References

- Achanta, R., A. Shaji, K. Smith, A. Lucchi, P. Fua, and S. Süsstrunk. 2012. "Slic Superpixels Compared to State-Of-The-Art Superpixel Methods." *IEEE Transactions on Pattern Analysis and Machine Intelligence* 34, no. 11: 2274–2282. <https://doi.org/10.1109/TPAMI.2012.120>.
- Adelfio, G., and M. Chiodi. 2015. "Flp Estimation of Semi-Parametric Models for Space-Time Point Processes and Diagnostic Tools." *Spatial Statistics* 14: 119–132.
- Adelfio, G., and M. Chiodi. 2021. "Including Covariates in a Space-Time Point Process With Application to Seismicity." *Statistical Methods & Applications* 30: 947–971.
- Adelfio, G., M. Siino, J. Mateu, and F. J. Rodríguez-Cortés. 2020. "Some Properties of Local Weighted Second-Order Statistics for Spatio-Temporal Point Processes." *Stochastic Environmental Research and Risk Assessment* 34: 149–168.
- Allard, D., and C. Fraley. 1997. "Nonparametric Maximum Likelihood Estimation of Features in Spatial Point Processes Using Voronoi Tessellation." *Journal of the American Statistical Association* 92, no. 440: 1485–1493.
- Alm, S. E. 1998. "Approximation and Simulation of the Distributions of Scan Statistics for Poisson Processes in Higher Dimensions." *Extremes* 1, no. 1: 111–126.
- Altieri, L., E. M. Scott, D. Cocchi, and J. B. Illian. 2015. "A Change-point Analysis of Spatio-Temporal Point Processes." *Spatial Statistics* 14: 197–207.
- Baddeley, A. 2017. "Local Composite Likelihood for Spatial Point Processes." *Spatial Statistics* 22: 261–295.
- Baddeley, A., J.-F. Coeurjolly, E. Rubak, and R. Waagepetersen. 2014. "Logistic Regression for Spatial Gibbs Point Processes." *Biometrika* 101, no. 2: 377–392.
- Baddeley, A., E. Rubak, and R. Turner. 2015. *Spatial Point Patterns: Methodology and Applications With R*. Chapman and Hall/CRC Press.
- Baddeley, A., and R. Turner. 2005. "Spatstat: An R Package for Analyzing Spatial Point Patterns." *Journal of Statistical Software* 12, no. 6: 1–42.
- Baddeley, A., R. Turner, J. Mateu, and A. Bevan. 2013. "Hybrids of Gibbs Point Process Models and Their Implementation." *Journal of Statistical Software* 55, no. 11: 1–43.
- Baddeley, A. J., J. Möller, and R. Waagepetersen. 2000. "Non-and Semi-Parametric Estimation of Interaction in Inhomogeneous Point Patterns." *Statistica Neerlandica* 54, no. 3: 329–350.
- Beckman, R., and R. Cook. 1979. "Testing for Two-Phase Regressions." *Technometrics* 21, no. 1: 65–69.
- Berman, M., and P. Diggle. 1989. "Estimating Weighted Integrals of the Second-Order Intensity of a Spatial Point Process." *Journal of the Royal Statistical Society, Series B: Statistical Methodology* 51, no. 1: 81–92.
- Berman, M., and T. R. Turner. 1992. "Approximating Point Process Likelihoods With Glim." *Journal of the Royal Statistical Society: Series C: Applied Statistics* 41, no. 1: 31–38.
- Bernhardt, R., F. Meyer-Olbersleben, and B. Kieback. 1997. "Fundamental Investigation on the Preparation of Gradient Structures by Sedimentation of Different Powder Fractions Under Gravity." In *Proceedings of the 4th International Conference on Composite Engineering*, 147–148. Springer.
- Byers, S., and A. E. Raftery. 1998. "Nearest-Neighbor Clutter Removal for Estimating Features in Spatial Point Processes." *Journal of the American Statistical Association* 93, no. 442: 577–584.
- Chiodi, M., and G. Adelfio. 2017. "Mixed Non-Parametric and Parametric Estimation Techniques in r Package Etasflp for Earthquakes' Description." *Journal of Statistical Software* 76: 1–29.
- Daley, D. J., and D. Vere-Jones. 2007. *An Introduction to the Theory of Point Processes*, General Theory and Structure. Vol. II. 2nd ed. Springer-Verlag.
- D'Angelo, N. 2024. "Advances in Kth Nearest-Neighbour Clutter Removal." *Environmental and Ecological Statistics* 31, no. 2: 537–554.
- D'Angelo, N., and G. Adelfio. 2024. "Minimum Contrast for the First-Order Intensity Estimation of Spatial and Spatio-Temporal Point Processes." *Statistical Papers* 65: 3651–3679.
- D'Angelo, N., and G. Adelfio. 2025. "Stopp: An r Package for Spatio-Temporal Point Pattern Analysis." *Journal of Statistical Software*.
- D'Angelo, N., G. Adelfio, and J. Mateu. 2023. "Locally Weighted Minimum Contrast Estimation for Spatio-Temporal Log-Gaussian Cox Processes." *Computational Statistics & Data Analysis* 180: 107679.
- D'Angelo, N., A. Albano, A. Gilardi, and G. Adelfio. 2025. "Non-Separable Spatio-Temporal Poisson Point Process Models for Fire Occurrences." *Environmental and Ecological Statistics* 32: 347–381.
- D'Angelo, N., D. Payares, G. Adelfio, and J. Mateu. 2024. "Self-Exciting Point Process Modelling of Crimes on Linear Networks." *Statistical Modelling* 24, no. 2: 139–168.
- D'Angelo, N., M. Siino, A. D'Alessandro, and G. Adelfio. 2022. "Local Spatial Log-Gaussian Cox Processes for Seismic Data." *Advances in Statistical Analysis* 106: 633–667. <https://doi.org/10.1007/s10182-022-00444-w>.
- Diggle, P. 1985. "A Kernel Method for Smoothing Point Process Data." *Journal of the Royal Statistical Society: Series C: Applied Statistics* 34, no. 2: 138–147.
- Diggle, P. J. 2013. *Statistical Analysis of Spatial and Spatio-Temporal Point Patterns*. CRC Press.
- Fan, J., Y. Wu, and Y. Feng. 2009. "Local Quasi-Likelihood With a Parametric Guide." *Annals of Statistics* 37, no. 6B: 4153–4183.
- Feder, P. I. 1975. "The Log Likelihood Ratio in Segmented Regression." *Annals of Statistics* 3, no. 1: 84–97.
- Hawkes, A., and L. Adamopoulos. 1973. "Cluster Models for Earthquakes-Regional Comparison." *Bulletin of the International Statistical Institute* 45, no. 3: 454–461.

- Hjort, N. L., and M. C. Jones. 1996. "Locally Parametric Nonparametric Density Estimation." *Annals of Statistics* 24: 1619–1647.
- Hollaway, M. J., and R. Killick. 2024. "Detection of Spatiotemporal Change-points: A Generalised Additive Model Approach." *Statistics and Computing* 34, no. 5: 162.
- Illian, J., A. Penttinen, H. Stoyan, and D. Stoyan. 2008. *Statistical Analysis and Modelling of Spatial Point Patterns*. Vol. 70. John Wiley & Sons.
- Lerman, P. 1980. "Fitting Segmented Regression Models by Grid Search." *Journal of the Royal Statistical Society: Series C: Applied Statistics* 29, no. 1: 77–84.
- Loader, C. R. 1996. "Local Likelihood Density Estimation." *Annals of Statistics* 24, no. 4: 1602–1618.
- Loader, C. R. 1999. "Bandwidth Selection: Classical or Plug-In?" *Annals of Statistics* 27, no. 2: 415–438.
- Møller, J., A. R. Syversveen, and R. P. Waagepetersen. 1998. "Log gaussian Cox Processes." *Scandinavian Journal of Statistics* 25, no. 3: 451–482.
- Monir Hossain, M., and A. B. Lawson. 2006. "Cluster Detection Diagnostics for Small Area Health Data: With Reference to Evaluation of Local Likelihood Models." *Statistics in Medicine* 25, no. 5: 771–786.
- Muggeo, V. M. 2003. "Estimating Regression Models With Unknown Break-Points." *Statistics in Medicine* 22, no. 19: 3055–3071.
- Nowosad, J., and T. F. Stepinski. 2022. "Extended Slic Superpixels Algorithm for Applications to Non-Imagery Geospatial Rasters." *International Journal of Applied Earth Observation and Geoinformation* 112: 102935.
- Ogata, Y. 1988. "Statistical Models for Earthquake Occurrences and Residual Analysis for Point Processes." *Journal of the American Statistical Association* 83, no. 401: 9–27.
- Ogata, Y. 1989. "Statistical Model for Standard Seismicity and Detection of Anomalies by Residual Analysis." *Tectonophysics* 169, no. 1–3: 159–174.
- Ogata, Y., and K. Katsura. 1988. "Likelihood Analysis of Spatial Inhomogeneity for Marked Point Patterns." *Annals of the Institute of Statistical Mathematics* 40: 29–39.
- Osborne, P. E., G. M. Foody, and S. Suárez-Seoane. 2007. "Non-Stationarity and Local Approaches to Modelling the Distributions of Wildlife." *Diversity and Distributions* 13, no. 3: 313–323.
- R Core Team. 2019. *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing.
- Siino, M., G. Adelfio, J. Mateu, M. Chiodi, and A. D'alessandro. 2017. "Spatial Pattern Analysis Using Hybrid Models: An Application to the Hellenic Seismicity." *Stochastic Environmental Research and Risk Assessment* 31: 1633–1648.
- Siino, M., A. D'alessandro, G. Adelfio, S. Scudero, and M. Chiodi. 2018. "Multiscale Processes to Describe the Eastern Sicily Seismic Sequences." *Annals of Geophysics* 61, no. 2: SE229.
- Tarantino, M., N. D'Angelo, and G. Adelfio. 2024. "Modelling Three-Dimensional Point Patterns." In *Scientific Meeting of the Italian Statistical Society*, 588–594. Springer.
- Ulm, K. 1991. "A Statistical Method for Assessing a Threshold in Epidemiological Studies." *Statistics in Medicine* 10, no. 3: 341–349.
- Zhuang, J. 2015. "Weighted Likelihood Estimators for Point Processes." *Spatial Statistics* 14: 166–178.